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STRUCTURE-PRESERVING EQUIVALENT MARTINGALE MEASURES FOR $\mathcal{H}$-SII MODELS

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Abstract

In this paper we relate the set of structure-preserving equivalent martingale measures $\mathcal{M}^{\mathrm{sp}}$ for financial models driven by semimartingales with conditionally independent increments to a set of measurable and integrable functions $\mathcal{Y}$. More precisely, we prove that $\mathcal{M}^{\mathrm{sp}} \neq \varnothing$ if and only if $\mathcal{Y} \neq \varnothing$, and connect the sets $\mathcal{M}^{\mathrm{sp}}$ and $\mathcal{Y}$ to the semimartingale characteristics of the driving process. As examples we consider integrated Lévy models with independent stochastic factors and time-changed Lévy models and derive mild conditions for $\mathcal{M}^{\mathrm{sp}} \neq \varnothing$.

Keywords: Equivalent martingale measure; stochastic volatility model; conditionally independent increments

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1. Introduction

A class of stochastic models which reflects many statistical observations and yet has good analytical properties is the class of so-called $\mathcal{H}$-SII models. The stock price process S is defined by $S = \exp(X)$, where X is a semimartingale with $\mathcal{H}$-conditionally independent increments ($\mathcal{H}$-SII). Examples of $\mathcal{H}$-SII models are exponential Lévy models and the stochastic volatility models suggested in [1], [5], [9], and [22].

We highlight that for pure-jump exponential Lévy models, Eberlein and Jacod [6] established a precise description of the set $\mathcal{M}^{\mathrm{sp}}$ of structure-preserving equivalent martingale measures in terms of a set of deterministic functions. This result is mathematically sharp and engages through its simple deterministic nature.

We show that such a result also holds for $\mathcal{H}$-SII models. More precisely, we prove that there exists a set of measurable and integrable functions $\mathcal{Y}$ such that for each element in $\mathcal{Y}$ there exists a corresponding measure in $\mathcal{M}^{\mathrm{sp}}$ and vice versa.

To the best of our current knowledge, for $\mathcal{H}$-SII models the set $\mathcal{M}^{\mathrm{sp}}$ was only studied for individual models (see, e.g. [10], [18], [20], and [21]) and not from a general perspective. We stress that some key techniques of previous approaches to do not apply to a general setting. For example, in the discussion of $\mathcal{M}^{\mathrm{sp}}$ for the Barndorf-Nielson and Shephard model in [20], the following fact was used. If ξ is a process independent of a Brownian motion W then conditioned on ξ the random variable $\int_0^T \xi_s \,\mathrm{d}W_s$ is Gaussian distributed. This claim relies on the fact that W stays a Brownian motion under the enlarged filtration which includes all informations on ξ; see Appendix B. Using the fact that $\int_0^T \xi_s \,\mathrm{d}W_s$ is Gaussian, the martingale property of a candidate density process for an element of $\mathcal{M}^{\mathrm{sp}}$ can be computed directly. In more general situations

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one cannot hope to perform this kind of computation. Hence, a more robust argumentation is necessary.

At the core of the proof of Eberlein and Jacod [6] is the fact that an exponential Lévy process is a martingale if and only if it is a local martingale. This observation is also true in the case of $\mathcal{H}$-SIIs with absolutely continuous characteristics; see [16]. By reducing the claim to semimartingales with independent increments (SIIs), for which the result was proven by Kallsen and Muhle-Karbe [15] exploiting a technique based on a change of measure, we generalize this observation to general $\mathcal{H}$-SIIs. In order to use this fact to construct a density process of a measure in $\mathcal{M}^{\text{sp}}$, one has to show that the logarithm of a candidate density process is an $\mathcal{H}$-SII. This requires in-depth measurability considerations; see Appendix A. On the other hand, to obtain necessary conditions for $\mathcal{M}^{\text{sp}} \neq \varnothing$, we benefit from Girsanov's theorem and results on local absolute continuity of laws of semimartingales as given in [12].

The paper is structured as follows. In Section 2.1 we introduce our mathematical setting. Our main result is given in Section 2.2. We discuss the simplified situation of a quasi-left-continuous driving process with continuous local martingale part in Section 2.3. In Section 3 we present examples such as a Black–Scholes-type model with independent stochastic volatility and an exponential Lévy model with independent stochastic time-change. The proof of our main result is presented in Section 4.

2. Structure-preserving equivalent martingale measures

Let $T > 0$ be a finite time horizon. All processes in this paper are indexed on $[0, T]$. We fix a, not necessarily right-continuous, filtration $(\mathcal{F}_t^o)_{t\in[0,T]}$ on a measurable space $(\Omega, \mathcal{F})$ and set $\mathcal{F}_t := \mathcal{F}_{t+}^o$. Throughout the paper, we let $(\Omega, \mathcal{F}, \boldsymbol{F} := (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be the underlying filtered probability space. Note that we do not assume the *usual conditions*. For a careful discussion of the general theory of stochastic processes without assuming the usual conditions, we refer the reader to [8] and [11].

Let $\mathcal{H} \subseteq \mathcal{F}$ and consider the enlarged filtration $\boldsymbol{G} := (\mathcal{G}_t)_{t\in[0,T]}$ given by $\mathcal{G}_t := \mathcal{G}_{t+}^o$, where $\mathcal{G}_t^o := \mathcal{F}_t^o \vee \mathcal{H}$. We impose the following assumption on the underlying filtered space.

Assumption 2.1. *The space Ω is Polish and $\mathcal{F}$ is its topological Borel σ-field. Moreover, for all $t \in [0, T]$, the σ-fields $\mathcal{H}$ and $\mathcal{F}_t^o$ are countably generated.*

In the following lemma we show that many σ-fields are countably generated.

Lemma 2.1. *Let $(Y_t)_{t\geq 0}$ be a right- or left-continuous process with values in a Polish space. Then, for $t \in [0, T]$, the σ-field $\sigma(Y_s,\ s \in [0, t])$ is countably generated.*

Proof. It suffices to note that $\sigma(Y_s,\ s \in [0, t]) = \sigma(Y_{s\wedge t},\ s \in \mathbb{Q}_+)$. □

An important consequence of Assumption 2.1 is the existence of a regular conditional probability $\mathbb{P}(\cdot \mid \mathcal{H})(\cdot)$ from $(\Omega, \mathcal{H})$ to $(\Omega, \mathcal{F})$; see, e.g. [23, Theorem 9.2.1]. More precisely, $\mathbb{P}(\cdot \mid \mathcal{H})(\cdot)$ satisfies the following:

(i) for all $\omega \in \Omega$, $A \mapsto \mathbb{P}(A \mid \mathcal{H})(\omega)$ is a probability measure on $(\Omega, \mathcal{F})$;

(ii) for all $A \in \mathcal{F}$, $\omega \mapsto \mathbb{P}(A \mid \mathcal{H})(\omega)$ is $\mathcal{H}$-measurable;

(iii) for all $A \in \mathcal{F}$, the random variable $\mathbb{P}(A \mid \mathcal{H})$ is a $\mathbb{P}$-version of $\mathbb{E}[\mathbf{1}_A \mid \mathcal{H}]$, where $\mathbf{1}_A$ is the indicator function of a set A;

(iv) there exists a $\mathbb{P}$-null set $N \in \mathcal{H}$ such that, for all $\omega \in N^c$ and all $G \in \mathcal{H}$, we have

$$\mathbb{P}(G \mid \mathcal{H})(\omega) = \mathbf{1}_G(\omega). \tag{2.1}$$

Part (iv) uses the assumption that the σ-field $\mathcal{H}$ is countably generated. We note two elementary observations.

Remark 2.1. (i) For all $\mathcal{F}$-measurable functions $Y\colon \Omega \to \mathbb{R}^+$, the random variable $\int Y(\omega)\mathbb{P}(\mathrm{d}\omega \mid \mathcal{H})$ is a $\mathbb{P}$-version of the conditional expectation $\mathbb{E}[Y \mid \mathcal{H}]$.

(ii) For all $\mathbb{P}$-almost surely ($\mathbb{P}$-a.s.) events $A \in \mathcal{F}$, there exists a $\mathbb{P}$-null set $N_A \in \mathcal{H}$ such that, for all $\omega \in N_A^c$, we have $\mathbb{P}(A \mid \mathcal{H})(\omega) = 1$.

2.1. $\mathcal{H}$-SIIs

As observed by Grigelionis [7], $\mathcal{H}$-SIIs can be characterized by measurability properties of their characteristics. Before we provide a precise statement, let us clarify some terminology. We say that $B \in \mathcal{V}$ has a $\mathcal{H}$-measurable version if, for each $t \in [0, T]$, the random variable B_t has a $\mathcal{H}$-measurable version. Denote by $\mathcal{I}$ the set of all Borel functions $g\colon \mathbb{R} \to \mathbb{R}$ with $|g(x)| \le 1 \wedge |x|^2$. We say that a compensator ν of a random measure of jumps has a $\mathcal{H}$-measurable version if, for all $t \in [0, T]$ and all $g \in \mathcal{I}$, the random variable $\nu([0, t] \times g)$ has a $\mathcal{H}$-measurable version.

In this paper we will fix a truncation function $h\colon \mathbb{R} \to \mathbb{R}$. Whenever we talk about (semimartingale) characteristics, we refer to the characteristics corresponding to h.

Definition 2.1. We call a real-valued $(\boldsymbol{G}, \mathbb{P})$-semimartingale which starts at 0 a $(\mathcal{H}, \boldsymbol{F}, \mathbb{P})$-SII if its $(\boldsymbol{G}, \mathbb{P})$-characteristics have a $\mathcal{H}$-measurable $\mathbb{P}$-version.

The following lemma can be used to deduce claims concerning $\mathcal{H}$-SIIs from results concerning semimartingales with independent increments. It is a consequence of [11, Lemma II.6.13, Corollary II.6.15].

Lemma 2.2. *A process Y is a $(\mathcal{H}, \boldsymbol{F}, \mathbb{P})$-SII if and only if there exists a $\mathbb{P}$-null set $N \in \mathcal{F}$ such that, for all $\omega \in N^c$, the process Y is an $(\{\Omega, \varnothing\}, \boldsymbol{G}, \mathbb{P}(\cdot \mid \mathcal{H})(\omega))$-SII. In this case, the $(\boldsymbol{G}, \mathbb{P}(\cdot \mid \mathcal{H})(\omega))$-characteristics of Y coincide with the $(\boldsymbol{G}, \mathbb{P})$-characteristics.*

2.2. Structure-preserving equivalent martingale measures

Let us now describe the class of financial models considered in this paper.

Assumption 2.2. *The process X is a $(\mathcal{H}, \boldsymbol{F}, \mathbb{P})$-SII and also an $(\boldsymbol{F}, \mathbb{P})$-semimartingale for which the $(\boldsymbol{F}, \mathbb{P})$- and $(\boldsymbol{G}, \mathbb{P})$-characteristics coincide.*

We discuss Assumption 2.2 in Appendix B and give examples. The characteristics of X are denoted by (B^X, C^X, ν^X). Thanks to [11, Proposition II.2.9] we may assume that without loss of generality

$$\{a \le 1\} = \Omega \times [0, T], \tag{2.2}$$

where $a_t := \nu^X(\{t\} \times \mathbb{R})$. The stock price process S of an $\mathcal{H}$-SII model is given by

$$S_t := \mathrm{e}^{X_t}, \qquad t \in [0, T].$$

Clearly, the assumption $S_0 = 1$ is no restriction and serves only for notational convenience. Let us now define our key objects of interest.

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Definition 2.2. We denote by $\mathcal{M}^{\mathrm{sp}}$ the set of structure-preserving equivalent martingale measures, i.e. all probability measures Q on $(\Omega, \mathcal{F})$ such that the following hold:

(i) $Q \sim \mathbb{P}$;

(ii) S is an $(\boldsymbol{F}, Q)$-martingale;

(iii) X is a $(\mathcal{H}, \boldsymbol{F}, Q)$-SII;

(iv) The $(\boldsymbol{F}, Q)$- and $(\boldsymbol{G}, Q)$-characteristics of X coincide.

In our setting we do not need to distinguish between structure-preserving equivalent *true, local,* or *sigma*-martingale measures, since all exponential $\mathcal{H}$-SIIs which are sigma-martingales are martingales; see Lemma 4.1 below.

Definition 2.3. We define $\mathcal{Y}$ to be the set of all tuple (β, U) which satisfy the following: β is a real-valued $\boldsymbol{F}$-predictable process and U is a $[0, \infty)$-valued $\mathcal{P}(\boldsymbol{F}) \otimes \mathcal{B}$-measurable function such that

(i) $\{U > 0\} = \{a' \leq 1\} = \Omega \times [0, T]$ and $\{a = 1\} = \{a' = 1\}$, where $a'_t := \int_{\mathbb{R}} U(t, x)\nu^X(\{t\} \times \mathrm{d}x)$;

(ii) $\mathbb{P}$-a.s. it holds that $|h(x)(U - 1)| * \nu^X_T < \infty$ and

$$H_T := \beta^2 \cdot C^X_T + (1 - \sqrt{U})^2 * \nu^X_T + \sum_{s \in [0,T]} (\sqrt{1 - a_s} - \sqrt{1 - a'_s})^2 < \infty, \tag{2.3}$$

where '$\cdot$' and '$*$' denote stochastic integration as defined in [11];

(iii) $\mathbb{P}$-a.s. it holds that $(\mathrm{e}^x - 1)U\mathbf{1}_{\{x>1\}} * \nu^X_T < \infty$ and that, for all $t \in [0, T]$,

$$B^X_t + \big(\beta + \tfrac{1}{2}\big) \cdot C^X_t + ((\mathrm{e}^x - 1)U - h(x)) * \nu^X_t + \sum_{s \in [0,t]} (\log(1 + \widetilde{V}_s) - \widetilde{V}_s) = 0, \tag{2.4}$$

where $\widetilde{V}_t := \int_{\mathbb{R}} (\mathrm{e}^x - 1)U(t, x)\nu^X(\{t\} \times \mathrm{d}x)$;

(iv) the modified characteristics

$$B := B^X + \beta \cdot C^X + h(x)(U - 1) * \nu^X, \qquad C := C^X, \qquad \nu := U \cdot \nu^X, \tag{2.5}$$

have a $\mathbb{P}$-version which is $\mathcal{H}$-measurable.

Motivated by Girsanov's theorem [11, Theorem III.3.24], the elements in $\mathcal{Y}$ are called Girsanov quantities. The function U is used to influence the jump structure of X and both U and β change the drift of X. If U is given by $U(t, x) = \mathrm{e}^{\beta_t x}/(1 + \widehat{W}_t)$, where $\widehat{W}_t := \int_{\mathbb{R}} (\mathrm{e}^{\beta_t x} - 1)\nu^X(\{t\} \times \mathrm{d}x)$, then (β, U) correspond to the famous *Esscher measure*; see, e.g. [17]. Equation (2.4) is often called the market price of risk equation (MPRE).

The set $\{a > 0\}$ is thin. Consequently, as a section of a thin set, $\{t \in [0, T] : a_t(\omega) > 0\}$ is at most countable and the sums in Definition 2.3(ii) and 2.3(iii) are well defined. Definition 2.3(ii) implies that $B \in \mathcal{V}(\boldsymbol{F}, \mathbb{P})$. Note that

$$(1 \wedge |x|^2)U * \nu^X_t \leq 4(1 \wedge |x|^2) * \nu^X_t + 4(1 - \sqrt{U})^2 * \nu^X_t.$$

Hence, $U \cdot \nu^X$ makes sense as a candidate for a compensator.

Now we are in a position to state our main result, which generalizes [6, Proposition 1] to $\mathcal{H}$-SII models. For a detailed proof, we refer the reader to Section 4.

Theorem 2.1. *We have*

$$\mathcal{Y} \neq \varnothing \quad \Longleftrightarrow \quad \mathcal{M}^{\mathrm{sp}} \neq \varnothing.$$

Moreover, the following hold:

(i) *for each $(\beta, U) \in \mathcal{Y}$, there exists a $Q \in \mathcal{M}^{\mathrm{sp}}$ such that the $(\boldsymbol{F}, Q)$- and $(\boldsymbol{G}, Q)$-characteristics of X are given by (2.5);*

(ii) *for each $Q \in \mathcal{M}^{\mathrm{sp}}$, there exist a pair $(\beta, U) \in \mathcal{Y}$ such that X has $(\boldsymbol{F}, Q)$- and $(\boldsymbol{G}, Q)$-characteristics given by (2.5).*

We stress that the integrability assumptions in the definition of $\mathcal{Y}$ have an almost sure character in contrast to classical exponential moment conditions of Novikov-type, which are typically imposed to guarantee the existence of an equivalent martingale measure.

We provide a short outline of the proof of Theorem 2.1. For given Girsanov quantities (β, U), we may define the candidate density process

$$Z := \mathcal{E}\bigg(\beta \cdot X^c + \bigg\{U - 1 + \frac{a' - a}{1 - a}\mathbf{1}_{\{a<1\}}\bigg\} * (\mu^X - \nu^X)\bigg), \tag{2.6}$$

where $\mathcal{E}$ denotes the stochastic exponential. We show in Lemma 4.2 below that (i), (ii), and (iv) of Definition 2.3 imply that Z is a positive martingale. The proof is based on the observation that $\log Z$ is a $\mathcal{H}$-SII and that exponential $\mathcal{H}$-SIIs are martingales if they are local martingales. Now we define a candidate measure Q for $\mathcal{M}^{\mathrm{sp}}$ by $Q(G) = \mathbb{E}_{\mathbb{P}}[Z_T \mathbf{1}_G]$ for $G \in \mathcal{F}$.

On the infinite time horizon, Z may not be a uniformly integrable martingale. This is the only point where the proof of sufficiency depends on the finite time horizon. However, when we consider an infinite time horizon, we can define the consistent family $(\mathcal{F}_t, Q_t)_{t\geq 0}$ by $Q_t(G) = \mathbb{E}_{\mathbb{P}}[Z_t \mathbf{1}_G]$ for $G \in \mathcal{F}_t$. Now, if the filtered space $(\Omega, \mathcal{F}, \boldsymbol{F})$ allows extensions of consistent families, there exists a probability measure Q on $(\Omega, \mathcal{F})$ such that $Q = Q_t$ on $\mathcal{F}_t$ for all $t \in [0, \infty)$. Since Z is positive, this implies that Q and $\mathbb{P}$ are locally equivalent. These considerations lead to a *local* version of Theorem 2.1. We stress that classical path spaces allow the extension of consistent families; see [3, Proposition 3.9.17].

Checking that Q satisfies (ii) and (iii) of Definition 2.2 is identical for the finite and the infinite time horizon.

Let us also comment on the converse direction. If $Q \in \mathcal{M}^{\mathrm{sp}}$ then Girsanov's theorem yields the existence of candidate Girsanov quantities (β, U). The integrability conditions follow from general results on absolute continuity of laws of semimartingales, and the equivalence of $\mathbb{P}$ and Q allows a modification of (β, U) such that Definition 2.3(i) is satisfied. These results can be applied irrespective of a finite or an infinite time horizon.

2.3. $\mathcal{H}$-SII models with a continuous local martingale part

We now discuss a simplified situation in which X has a nontrivial continuous local martingale part.

It is well known that the $(\boldsymbol{F}, \mathbb{P})$-characteristics of X have a decomposition

$$B^X = b^X \cdot A^X, \qquad C^X = c^X \cdot A^X, \qquad \nu^X = F^X \otimes A^X;$$

see [11, Proposition II.2.9]. Here, A^X is an $\boldsymbol{F}$-predictable process in $\mathcal{A}^+_{\mathrm{loc}}(\boldsymbol{F}, \mathbb{P})$, b^X is an $\boldsymbol{F}$-predictable process, c^X is an $\boldsymbol{F}$-predictable nonnegative process, and $F^X_{\omega,t}(\mathrm{d}x)$ is a transition kernel from $(\Omega \times [0, T], \mathcal{P}(\boldsymbol{F}))$ to $(\mathbb{R}, \mathcal{B})$. We call $(b^X, c^X, F^X; A^X)$ *local* $(\boldsymbol{F}, \mathbb{P})$-*characteristics of* X. Here, $\mathcal{A}^+_{\mathrm{loc}}(\boldsymbol{F}, \mathbb{P})$ is defined as in [11, p. 29]. Thanks to Assumption 2.2, $(b^X, c^X, F^X; A^X)$ are also local $(\boldsymbol{G}, \mathbb{P})$-characteristics of X.

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Corollary 2.1. *Suppose that* $\nu^X(\{t\}\times\mathbb{R}) = 0$ *for all* $t \in [0, T]$*,* $c^X \neq 0$*, and there exists an* $\mathcal{H}\otimes\mathcal{B}([0,T])\otimes\mathcal{B}$*- and* $\mathcal{P}(\boldsymbol{F})\otimes\mathcal{B}$*-measurable positive function* U *such that* $\mathbb{P}$*-a.s.*

$$|h(x)(U-1)| * \nu^X_T + |\mathrm{e}^x - 1| U \mathbf{1}_{\{x>1\}} * \nu^X_T + \beta^2 \cdot C^X_T + (1-\sqrt{U})^2 * \nu^X_T < \infty,$$

where

$$\beta := -\frac{1}{c^X}\bigg(\frac{1}{2}c^X + b^X + \int_{\mathbb{R}} ((\mathrm{e}^x - 1)U(\cdot, x) - h(x)) F^X(\mathrm{d}x)\bigg),$$

then $\mathcal{M}^{\mathrm{sp}} \neq \varnothing$*. Moreover, there exists a* $Q \in \mathcal{M}^{\mathrm{sp}}$ *such that* X *has local* $(\boldsymbol{F}, Q)$*- and local* $(\boldsymbol{G}, Q)$*-characteristics* $(b^{Q,X}, c^X, F^{Q,X}; A^X)$*, where*

$$b^{Q,X} = -\bigg(\frac{1}{2}c^X + \int_{\mathbb{R}} (\mathrm{e}^x - 1 - h(x)) U(\cdot, x) F^X(\mathrm{d}x)\bigg)$$

and $F^{Q,X}(\mathrm{d}x) = U(\cdot, x)F^X(\mathrm{d}x)$*.*

Proof. If $(\beta, U) \in \mathcal{Y}$ then Theorem 2.1 implies the stated claims. It is assumed that (β, U) satisfies Definition 2.3(i)–(iii). Moreover, it follows along the lines of the proof of Lemma A.2 in Appendix A that (β, U) also satisfies Definition 2.3(iv). □

If we choose $U = 1$, there exists a measure $Q \in \mathcal{M}^{\mathrm{sp}}$ which does not change the jump structure of X.

3. Examples

In this section we discuss two examples. First, we investigate a generalization of the Nobel Prize-winning model of Black and Scholes [4], introducing an additional independent stochastic factor, which, for instance, may be a fractional Brownian motion. Second, we consider a time-changed Lévy Carr–Geman–Madan–Yor (CGMY) model as introduced by Carr *et al.* [5].

3.1. A generalized Black–Scholes model with independent factor

Let $\boldsymbol{F}, \mathcal{H}, Y$, and $V := (I, W)$ be as in Example B.2. Here, we denote $I_t = t$. Then Assumption 2.1 holds. We assume that W is a Brownian motion which is $\mathbb{P}$-independent of Y. Moreover, let $\gamma\colon \mathbb{D}^m \times [0,T] \to \mathbb{R}$ and $\sigma\colon \mathbb{D}^m \times [0,T] \to (0,\infty)$ be such that $\gamma(Y), \sigma(Y)$ are $\boldsymbol{F}^Y$-predictable and

$$\mathbb{P}\bigg(\int_0^T |\gamma(Y,s)|\,\mathrm{d}s + \int_0^T \sigma(Y,s)^2\,\mathrm{d}s < \infty\bigg) = 1. \tag{3.1}$$

We now set

$$X := \int_0^{\cdot} \gamma(Y,s)\,\mathrm{d}s + \sigma(Y)\cdot W.$$

Assumption 2.2 holds thanks to Corollary B.1. We obtain very mild sufficient and necessary conditions for $\mathcal{M}^{\mathrm{sp}} \neq \varnothing$.

Corollary 3.1. *We have* $\mathcal{M}^{\mathrm{sp}} \neq \varnothing$ *if and only if*

$$\mathbb{P}\bigg(\int_0^T \bigg(\frac{\gamma(Y,s)}{\sigma(Y,s)}\bigg)^2 \mathrm{d}s < \infty\bigg) = 1. \tag{3.2}$$

Proof. The implication '$\Longleftarrow$' follows from Corollary 2.1 and (3.1). If $\mathcal{M}^{\mathrm{sp}} \neq \varnothing$ then Theorem 2.1 yields that the MPRE (2.4) has a solution β such that $\mathbb{P}$-a.s. $\beta^2 \cdot C^X_T < \infty$. We obtain $\beta\sigma^2(Y) = -\gamma(Y) - \sigma^2(Y)/2$ up to a $\mathbb{P}\otimes\mathrm{d}t$-null set. Hence, we deduce (3.2) from (3.1) together with $\mathbb{P}$-a.s. $\beta^2\cdot C^X_T < \infty$. □

3.2. The CGMY model with independent stochastic volatility

We proceed in the setting introduced in Example B.3. Let Y be an Ornstein–Uhlenbeck process driven by a Lévy subordinator L with constant initial value $Y_0 > 0$ and parameter $\lambda > 0$. More precisely, we assume that

$$Y_t := Y_0 \mathrm{e}^{-\lambda t} + \mathrm{e}^{-\lambda(t-s)} \cdot L_t, \qquad t \in [0, T].$$

From this definition, we immediately deduce that, for all $t \in [0, T]$,

$$Y_t \geq Y_0 \mathrm{e}^{-\lambda t} \geq Y_0 \mathrm{e}^{-\lambda T} > 0. \tag{3.3}$$

Let V be a one-dimensional Lévy process with Lévy–Khinchine triplet (b^V, c^V, F^V) and $h(x) = x\mathbf{1}_{\{|x|\leq 1\}}$. Then we set

$$X_t := \mu t + V_{\int_0^t Y_{s-}\,\mathrm{d}s}, \qquad t \in [0, T].$$

Recall that Y and V are assumed to be $\mathbb{P}$-independent and note that both Assumptions 2.1 and 2.2 hold.

Proposition 3.1. *We have $\mathcal{M}^{\mathrm{sp}} \neq \varnothing$ if at least one of the following conditions hold:*

(i) $c^V \neq 0$*;*

(ii) $F^V((-\infty, -1)) > 0$ *and* $F^V((1, \infty)) > 0$.

Proof. (i) The local characteristics of X are given as in Lemma B.2. We choose

$$U(\omega, t, x) := \frac{1}{1-\mathrm{e}^x}\mathbf{1}_{\{x<-1\}} + \frac{x}{\mathrm{e}^x - 1}\mathbf{1}_{\{|x|\leq 1\}\setminus\{0\}} + \mathbf{1}_{\{0\}} + \frac{1}{\mathrm{e}^x - 1}\mathbf{1}_{\{x>1\}},$$
$$\beta_t := \frac{-\mu}{c^V Y_{t-}} - \frac{1}{c^V}\bigg(b^V + \int_{\mathbb{R}} (\mathbf{1}_{\{x>1\}} - \mathbf{1}_{\{x<-1\}})F^V(\mathrm{d}x)\bigg) - \frac{1}{2}.$$

Obviously, U is positive and $\mathcal{H}\otimes\mathcal{B}([0, T|)\otimes\mathcal{B}$- and $\mathcal{P}(\boldsymbol{F})\otimes\mathcal{B}$-measurable. Taylor's theorem yields the existence of a nonnegative constant K such that

$$|h(x)(U(x) - 1)| + (1 - \sqrt{U(x)})^2 + (\mathrm{e}^x - 1)U\mathbf{1}_{\{x>1\}} \leq K(1 \wedge |x|^2).$$

Together with the bound (3.3), we conclude that the integrability condition of Corollary 2.1, and, hence, the claim, holds.

(ii) Without loss of generality, we may assume that $c^V = 0$. We set $\beta := 0$ and

$$\begin{aligned} U(t, x) := {} & \bigg(\frac{b^V\mathbf{1}_{\{b^V\geq 0\}} + Y_{t-}^{-1}\mu\mathbf{1}_{\{\mu\geq 0\}}}{(1-\mathrm{e}^x)F^V((-\infty, -1))} + \frac{F^V((1,\infty))}{1-\mathrm{e}^x}\bigg)\mathbf{1}_{\{x<-1\}} \\ & + \bigg(\frac{-b^V\mathbf{1}_{\{b^V<0\}} - Y_{t-}^{-1}\mu\mathbf{1}_{\{\mu<0\}}}{(\mathrm{e}^x - 1)F^V((1,\infty))} + \frac{F^V((-\infty,-1))}{\mathrm{e}^x - 1}\bigg)\mathbf{1}_{\{x>1\}} \\ & + \frac{x}{\mathrm{e}^x - 1}\mathbf{1}_{\{|x|\leq 1\}\setminus\{0\}} + \mathbf{1}_{\{0\}}. \end{aligned}$$

Definition 2.3(i) holds trivially and it is routine to check that Definition 2.3(iii) is satisfied. Definition 2.3(ii) follows from Taylor's theorem as above. Since U is $\mathcal{H}\otimes\mathcal{B}([0, T])\otimes\mathcal{B}$-measurable, similar arguments as used in the proof of Lemma A.2 yield Definition 2.3(iv). Thus, $(0, U) \in \mathcal{Y}$ and the claim follows from Theorem 2.1. □

4. Proof of Theorem 2.1

4.1. Martingale property of exponential $\mathscr{H}$-SII processes

The following lemma generalizes [16, Lemma A.1] to arbitrary $\mathscr{H}$-SIIs.

Lemma 4.1. *Let Y be a $(\mathscr{H}, \boldsymbol{F}, \mathbb{P})$-SII with $(\boldsymbol{G}, \mathbb{P})$-characteristics (B^Y, C^Y, ν^Y).*

(i) *The following are equivalent:*

(I) e^Y *is a $(\boldsymbol{G}, \mathbb{P})$-martingale;*

(II) e^Y *is a local $(\boldsymbol{G}, \mathbb{P})$-martingale;*

(III) e^Y *is a sigma $(\boldsymbol{G}, \mathbb{P})$-martingale;*

(IV) *we have $\mathrm{e}^x \mathbf{1}_{\{x>1\}} * \nu^Y \in \mathcal{V}(\boldsymbol{G}, \mathbb{P})$ and $\mathbb{P}$-a.s.*

$$B^Y + \tfrac{1}{2} C^Y + (\mathrm{e}^x - 1 - h(x)) * \nu^Y + \sum_{s \in [0,\cdot]} (\log(1 + \widehat{Y}_s) - \widehat{Y}_s) = 0,$$

where $\widehat{Y}_t := \int_{\mathbb{R}} (\mathrm{e}^x - 1) \nu^Y(\{t\} \times \mathrm{d}x)$.

(ii) *In addition, if Y is an $(\boldsymbol{F}, \mathbb{P})$-semimartingale and its $(\boldsymbol{F}, \mathbb{P})$-characteristics coincide with (B^Y, C^Y, ν^Y), then (I) $\Longleftrightarrow$ (II) $\Longleftrightarrow$ (III) $\Longleftrightarrow$ (IV), where (I)–(IV) are given as in (i) with $\boldsymbol{G}$ replaced by $\boldsymbol{F}$.*

Proof. (i) The implication (I) $\Longrightarrow$ (II) is trivial and the implication (II) $\Longrightarrow$ (III) holds due to [11, Proposition III.6.34]. The implication (III) $\Longrightarrow$ (II) follows from the fact that nonnegative sigma-martingales are local martingales; see [11, p. 216]. An exponential semimartingale is a local martingale if and only if it is exponentially special and its exponential compensator vanishes; see [17, Lemma 2.15]. Hence, the equivalence (II) $\Longleftrightarrow$ (IV) follows from [17, Lemma 2.13, Theorem 2.18, Theorem 2.19]. It remains to prove the implication (IV) $\Longrightarrow$ (I). Thanks to the equivalence (III) $\Longleftrightarrow$ (IV) and [14, Proposition 3.1], the process e^Y is a nonnegative $(\boldsymbol{G}, \mathbb{P})$-supermartingale. Thus, we have to show that $\mathbb{E}_{\mathbb{P}}[\mathrm{e}^{Y_T}] = 1$. Thanks to Remark 2.1(ii), Lemma 2.2, and [17, Lemma 2.13, Theorem 2.18, Theorem 2.19] $\mathbb{P}$-a.s., the process Y is a $(\{\Omega, \varnothing\}, \boldsymbol{G}, \mathbb{P}(\cdot \mid \mathscr{H}))$-SII and e^Y is $\mathbb{P}$-a.s. a local $(\boldsymbol{G}, \mathbb{P}(\cdot \mid \mathscr{H}))$-martingale. Hence, using [15, Proposition 3.12], the process e^Y is $\mathbb{P}$-a.s. a $(\boldsymbol{G}, \mathbb{P}(\cdot \mid \mathscr{H}))$-martingale. This implies that $\mathbb{P}$-a.s. $\int_\Omega \mathrm{e}^{Y_T(\omega')} \mathbb{P}(\mathrm{d}\omega' \mid \mathscr{H}) = 1$. Taking the $\mathbb{P}$-expectation completes the proof.

(ii) (I) $\Longrightarrow$ (II) $\Longleftrightarrow$ (III) $\Longleftrightarrow$ (IV) follow as in (i). Thanks to (i), (IV) implies $\mathbb{E}_{\mathbb{P}}[\mathrm{e}^{Y_T}] = 1$. Hence, we can also conclude that (IV) $\Longrightarrow$ (I). □

4.2. A candidate density process

Assumption 2.2 and [11, Theorem II.2.34] imply that the continuous local $(\boldsymbol{F}, \mathbb{P})$- and $(\boldsymbol{G}, \mathbb{P})$-martingale parts of X coincide. We denote them by X^c.

Lemma 4.2. *Let $(\beta, U) \in \mathscr{Y}$, then the process Z, as given by (2.6), is a positive $(\boldsymbol{F}, \mathbb{P})$- and $(\boldsymbol{G}, \mathbb{P})$-martingale.*

Proof. Let us start by showing that Z is a positive local $(\boldsymbol{F}, \mathbb{P})$- and $(\boldsymbol{G}, \mathbb{P})$-martingale. We have $\boldsymbol{F} \subseteq \boldsymbol{G}$, i.e. $\mathscr{F}_t \subseteq \mathscr{G}_t$ for all $t \in [0, T]$. Since β is $\boldsymbol{F}$-predictable, $\boldsymbol{F} \subseteq \boldsymbol{G}$ implies that β is $\boldsymbol{G}$-predictable. Now, $\mathbb{P}$-a.s. $\beta^2 \cdot C_T^X < \infty$ yields that $\beta \cdot X^c$ is a local $(\boldsymbol{F}, \mathbb{P})$- and

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$(\boldsymbol{G}, \mathbb{P})$-martingale. We denote

$$V(t,x) := U(t,x) - 1 + \frac{a'_t - a_t}{1-a_t}\mathbf{1}_{\{a_t<1\}},$$
$$\widehat{V}(t,x) := \int_{\mathbb{R}} V(t,x)\nu^X(\{t\}\times \mathrm{d}x) = \frac{a'_t - a_t}{1-a_t}\mathbf{1}_{\{a_t<1\}},$$

where we use the assumption that $\{a=1\} = \{a'=1\}$. Recalling $\boldsymbol{F} \subseteq \boldsymbol{G}$, it follows that V is $\mathcal{P}(\boldsymbol{F}) \otimes \mathcal{B}$- and $\mathcal{P}(\boldsymbol{G}) \otimes \mathcal{B}$-measurable. Moreover, we have

$$\widetilde{V}_t := V(t, \Delta X_t)\mathbf{1}_{\{\Delta X_t \neq 0\}} - \widehat{V}_t = \begin{cases} U(t,\Delta X_t) - 1 & \text{on } \{\Delta X_t \neq 0\}, \\ -\dfrac{a'_t - a_t}{1-a_t}\mathbf{1}_{\{a_t<1\}} & \text{on } \{\Delta X_t = 0\}. \end{cases} \tag{4.1}$$

Since $\{U>0\} = \Omega \times [0,T]$ and $\{a'<1\} = \{a<1\}$, we have $\{\widetilde{V} > -1\} = \Omega\times[0,T]$. Now, [11, Theorem II.1.33(d)] yields that $V * (\mu^X - \nu^X)$ is a local $(\boldsymbol{F},\mathbb{P})$- and $(\boldsymbol{G},\mathbb{P})$-martingale if $\mathbb{P}$-a.s.

$$K_T := \Big(1 + \sqrt{1 + V - \widehat{V}}\Big)^2 * \nu^X_T + \sum_{s\in[0,T]} (1-a_s)\Big(1 - \sqrt{1-\widehat{V}_s^2}\Big)^2 < \infty.$$

This holds since $K_T \le H_T$ and $\mathbb{P}$-a.s. $H_T < \infty$; see (2.3) for the definition of H_T. Therefore, Z is a local $(\boldsymbol{F},\mathbb{P})$- and $(\boldsymbol{G},\mathbb{P})$-martingale, which is positive due to the fact that $\{\widetilde{V} > -1\} = \Omega \times [0,T]$ together with [11, Theorem I.4.61(c)].

Using Lemma 4.1, Z is a $(\boldsymbol{G},\mathbb{P})$-martingale if $\log Z$ is a $(\mathcal{H},\boldsymbol{F},\mathbb{P})$-SII. Since Z is an $(\boldsymbol{F},\mathbb{P})$-supermartingale, this also yields that Z is an $(\boldsymbol{F},\mathbb{P})$-martingale. We proceed in two steps. First, we compute the $(\boldsymbol{G},\mathbb{P})$-characteristics of $\log Z$. Second, we show that they have $\mathcal{H}$-measurable $\mathbb{P}$-versions. We define the local $(\boldsymbol{G},\mathbb{P})$-martingale $N := \beta\cdot X^c + V*(\mu^X - \nu^X)$ and denote its $(\boldsymbol{G},\mathbb{P})$-characteristics by (B^N, C^N, ν^N). The continuous local $(\boldsymbol{G},\mathbb{P})$-martingale part of N is given by $\beta\cdot X^c$ and, hence, $C^N = \beta^2\cdot C^X$. Similarly as in [13], it follows that the $(\boldsymbol{G},\mathbb{P})$-compensator ν^N of μ^N is given by

$$\mathbf{1}_G * \nu^N = \mathbf{1}_G(U-1)*\nu^X + \sum_{t\in[0,\cdot]} \mathbf{1}_{\{a_t>0\}}\mathbf{1}_G\Big(-\frac{a'_t - a_t}{1-a_t}\Big)(1-a_t) \quad \text{for } G\in\mathcal{B},\ 0\notin G. \tag{4.2}$$

Since N is a local $(\boldsymbol{G},\mathbb{P})$-martingale, [11, Proposition II.2.29] yields that $B^N(h') = -(x - h'(x)) * \nu^N$. Since identically $\Delta N = \widetilde{V} > -1$, [11, Theorem II.8.10] yields that $\log Z$ has $(\boldsymbol{G},\mathbb{P})$-characteristics given by

$$B^{\log Z} = B^N - \tfrac{1}{2}C^N + (h(\log(1+x)) - h(x)) * \nu^N,$$
$$C^{\log Z} = C^N, \qquad \mathbf{1}_A * \nu^{\log Z} = \mathbf{1}_A(\log(1+x)) * \nu^N, \qquad A\in\mathcal{B},\ 0\notin A.$$

Since ν^X and $U\cdot\nu^X$ have $\mathcal{H}$-measurable $\mathbb{P}$-versions and $\mathbb{P}$-a.s. $|h(x)(U-1)| * \nu^X_T < \infty$, Lemma A.2(i) yields that $h(x)(U-1)*\nu^X$ has a $\mathcal{H}$-measurable $\mathbb{P}$-version. Hence, since B^X and $B^X + \beta\cdot C^X + h(x)(U-1)*\nu^X$ have $\mathcal{H}$-measurable $\mathbb{P}$-versions, so does $\beta\cdot C^X$. Now Lemma A.3 implies that C^N has a $\mathcal{H}$-measurable $\mathbb{P}$-version. Recalling (4.2), Lemma A.2 yields that ν^N has a $\mathcal{H}$-measurable $\mathbb{P}$-version. For all $x > -1$ and $g\in\mathcal{I}$, we have $|g(\log(1+x))| \le 3(1\wedge |x|^2)$. Moreover, $\mathbb{P}$-a.s. $(|x - h'(x)| + |h(\log(1+x)) - h(x)|) * \nu^N_T < \infty$ follows from $\mathbb{P}$-a.s.

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$(|x| \wedge |x|^2) * \nu^N_T < \infty$; see [11, Proposition II.2.29]. Therefore, since ν^N has a $\mathscr{H}$-measurable $\mathbb{P}$-version, Lemma A.2(i) implies that $B^{\log Z}(h)$ and $\nu^{\log Z}$ also have $\mathscr{H}$-measurable $\mathbb{P}$-versions. This concludes the proof. □

Remark 4.1. The statement of Lemma 4.2 holds if the pair (β, U) only satisfies (i), (ii), and (iv) of Definition 2.3.

4.3. Proof of Theorem 2.1

Let $(\beta, U) \in \mathscr{Y}$ and Z be as in (2.6). Thanks to Lemma 4.2, Z is a positive $(\boldsymbol{F}, \mathbb{P})$- and $(\boldsymbol{G}, \mathbb{P})$-martingale. Define Q by $Q(A) = \mathbb{E}_{\mathbb{P}}[Z_T \mathbf{1}_A]$ for $A \in \mathscr{F}$. Since $\mathbb{P}$-a.s. $Z_T > 0$, it holds that $Q \sim \mathbb{P}$. Thanks to the martingale property of Z, $Q(A) = \mathbb{E}_{\mathbb{P}}[Z_t \mathbf{1}_A]$ for $A \in \mathscr{G}_t$ and $t \in [0, T]$. From (4.1), it follows that $\mathbb{P}$-a.s. $\langle Z^c, X^c\rangle^{\mathbb{P},\boldsymbol{K}} = Z_-\beta \cdot C^X$ and $M^{\mathbb{P}}_{\mu^X}(Z \mid \mathscr{P}(\boldsymbol{K}) \otimes \mathscr{B}) = Z_- U$ for $\boldsymbol{K} \in \{\boldsymbol{F}, \boldsymbol{G}\}$. For the notation, we refer to [11, Section III.3.c]. Now, using Girsanov's theorem [11, Theorem III.3.24], X is an $(\boldsymbol{F}, Q)$- and $(\boldsymbol{G}, Q)$-semimartingale with $(\boldsymbol{F}, Q)$- and $(\boldsymbol{G}, Q)$-characteristics given by (2.5). Since $(\beta, U) \in \mathscr{Y}$ and $Q \sim \mathbb{P}$, these characteristics have an $\mathscr{H}$-measurable Q-version, i.e. X is a $(\mathscr{H}, \boldsymbol{F}, Q)$-SII. Moreover, since the MPRE (2.4) holds, S is an $(\boldsymbol{F}, Q)$-martingale by Lemma 4.1. Therefore, we have shown that (i) holds and $\mathscr{Y} \neq \varnothing$ implies that $\mathscr{M}^{\text{sp}} \neq \varnothing$.

Next, we prove (ii) and $\mathscr{M}^{\text{sp}} \neq \varnothing \Longrightarrow \mathscr{Y} \neq \varnothing$. Take $Q \in \mathscr{M}^{\text{sp}}$ and denote the $\boldsymbol{F}$-density process of Q with respect to $\mathbb{P}$ by Z^*. It follows from $Q \sim \mathbb{P}$ and [11, Proposition III.3.5] that $\mathbb{P}$-a.s. $Z^*_t > 0$ and $Z^*_{t-} > 0$ for all $t \in [0, T]$. Denote $\Lambda := \{M^{\mathbb{P}}_{\mu^X}(Z^* \mid \mathscr{P}(\boldsymbol{F}) \otimes \mathscr{B}) > 0\} \cap \{Z^*_- > 0\} \times \mathbb{R}$ and

$$U^*(\omega, t, x) := \begin{cases} \dfrac{1}{Z^*_{t-}(\omega)} M^{\mathbb{P}}_{\mu^X}(Z^* \mid \mathscr{P}(\boldsymbol{F}) \otimes \mathscr{B})(\omega, t, x) & \text{on } \Lambda, \\ 1 & \text{otherwise.} \end{cases}$$

Girsanov's theorem [11, Theorem III.3.24] yields the existence of an $\boldsymbol{F}$-predictable process β such that the $(\boldsymbol{F}, Q)$-characteristics of X are given by (2.5) with U replaced by U^*. Moreover, since X is an $(\mathscr{H}, \boldsymbol{F}, Q)$-SII and its $(\boldsymbol{F}, Q)$-characteristics coincide with its $(\boldsymbol{G}, Q)$-characteristics, there exists an $\mathscr{H}$-measurable Q-version of these characteristics. Since $\mathbb{P} \sim Q$, there also exists an $\mathscr{H}$-measurable $\mathbb{P}$-version. Using again Girsanov's theorem and $Q \sim \mathbb{P}$, we obtain $\mathbb{P}$-a.s. $|h(x)(U^* - 1)| * \nu^X_T < \infty$. Since e^X is an $(\boldsymbol{F}, Q)$-martingale and $Q \sim \mathbb{P}$, [17, Lemma 2.13, Theorem 2.19] implies that $\mathbb{P}$-a.s. $(\mathrm{e}^x - 1)\mathbf{1}_{\{x>1\}} U^* * \nu^X_T < \infty$ and that the MPRE (2.4) holds $\mathbb{P}$-a.s. for all $t \in [0, T]$ with U replaced by U^*. Denote by H^* the process H with U replaced by U^*. Since $Q \sim \mathbb{P}$, we deduce from [12, Theorem 1**] that $\mathbb{P}$-a.s. $H^*_T < \infty$. We now show that there exists a $\mathscr{P}(\boldsymbol{F}) \otimes \mathscr{B}$-measurable function U and a $\mathbb{P}$-evanescence set Λ' such that $U = U^*$ on $\Lambda'^c \times \mathbb{R}$, $\{U > 0\} = \{a' \leq 1\} = \Omega \times [0, T]$, and $\{a = 1\} = \{a' = 1\}$. The properties of U^* then readily extend to U and $(\beta, U) \in \mathscr{Y}$ follows. Denote $a^*_t := (U^* \cdot \nu^X)(\{t\} \times \mathbb{R})$. Due to the fact that progressive subsets of thin sets are themselves thin (see [8, Theorem 3.19]), the set $\{a = 1\} \subseteq \{a > 0\}$ is thin. Hence, [11, Lemma I.2.23] yields the existence of a sequence of $\boldsymbol{F}$-predictable times $(\tau_n)_{n\in\mathbb{N}}$ such that $\{a = 1\} = \bigcup_{n\in\mathbb{N}} [\![\tau_n]\!]$ up to $\mathbb{P}$-evanescence. Using [11, Proposition II.1.17] similarly as in the proof of [11, Theorem III.3.17], we obtain $Q(a^*_{\tau_n} = 1, \tau_n < \infty) = 1$. By the equivalence $Q \sim \mathbb{P}$, it also holds that $\mathbb{P}(a^*_{\tau_n} = 1, \tau_n < \infty) = 1$. Hence, $\{a = 1\} \subseteq \{a^* = 1\}$ up to $\mathbb{P}$-evanescence. For the converse direction, we slightly modify the argument. Since also $\{a^* = 1\} \subseteq \{a > 0\}$, there exists a sequence of $\boldsymbol{F}$-predictable times $(\rho_n)_{n\in\mathbb{N}}$ such that

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$\{a^* = 1\} = \bigcup_{n\in\mathbb{N}} [\![\rho_n]\!]$ up to $\mathbb{P}$-evanescence. Set $D := \{\Delta X \neq 0\}$. Now [11, Proposition II.1.17] yields that $Q(\rho_n \in D \mid \mathcal{F}_{\rho_n-}) = a^*_{\rho_n}$ on $\{\rho_n < \infty\}$ for each $n \in \mathbb{N}$. Hence, we deduce from [11, Theorem III.3.4] that $Q(\rho_n \notin D,\ \rho_n < \infty) = \mathbb{E}_{\mathbb{P}}[Z_{\rho_n}(1 - a^*_{\rho_n})\mathbf{1}_{\{\rho_n<\infty\}}] = 0$, which implies that $\mathbb{P}(\rho_n \notin D,\ \rho_n < \infty) = 0$ since $Q \sim \mathbb{P}$. Using [11, Proposition II.1.17] yields that $\mathbb{P}(a_{\rho_n} = 1,\ \rho_n < \infty) = 1$ for each $n \in \mathbb{N}$. This proves that $\{a^* = 1\} \subseteq \{a = 1\}$ up to $\mathbb{P}$-evanescence. It follows, as in the proof of [19, Lemma 3.3.1], that $\{a^* > 1\}$ is a Q-evanescence set. Again, since $Q \sim \mathbb{P}$, $\{a^* > 1\}$ is also a $\mathbb{P}$-evanescence set. Define $\Lambda' := \{(\omega, t) \in \Omega \times [0, T]\colon (a_t(\omega) = 1 \text{ and } a^*_t(\omega) \neq 1) \text{ or } (a_t(\omega) \neq 1 \text{ and } a^*_t(\omega) = 1) \text{ or } a^*_t(\omega) > 1\}$ and

$$U(\omega, t, x) := \begin{cases} 1 & \text{on } \Lambda' \times \mathbb{R}, \\ U^*(\omega, t, x) & \text{otherwise}, \end{cases}$$

which is a $\mathcal{P}(\boldsymbol{F}) \otimes \mathcal{B}$-measurable function. Recalling (2.2), we obtain $\{U > 0\} = \{a' \le 1\} = \Omega \times [0, T]$ and $\{a = 1\} = \{a' = 1\}$. This completes the proof. □

Appendix A. Measurability lemmata

In this appendix we collect some measurability results which are used in the proof of Lemma 4.2. We start with an elementary observation.

Lemma A.1. *A nonnegative random variable Y has a $\mathcal{H}$-measurable $\mathbb{P}$-version if and only if there exists a $\mathbb{P}$-null set $N \in \mathcal{F}$ such that, for all $\omega \in N^c$, we have $\mathbb{P}(Y = Y(\omega) \mid \mathcal{H})(\omega) = 1$.*

Proof. First, we show the implication '$\Longleftarrow$'. Thanks to Remark 2.1(i), for all $A \in \mathcal{F}$, we have $\mathbb{E}[\mathbf{1}_A \mathbb{E}[Y \mid \mathcal{H}]] = \mathbb{E}[\mathbf{1}_A \int Y(\omega')\mathbb{P}(\mathrm{d}\omega' \mid \mathcal{H})] = \mathbb{E}[\mathbf{1}_A Y]$. Hence, $\mathbb{E}[Y \mid \mathcal{H}]$ is a $\mathcal{H}$-measurable $\mathbb{P}$-version of Y.

Second, assume that Y has a $\mathcal{H}$-measurable $\mathbb{P}$-version K. In view of Remark 2.1(ii) and (2.1), there exists a $\mathbb{P}$-null set $N \in \mathcal{F}$ such that, for all $\omega \in N^c$, it holds that $\mathbb{P}(Y = Y(\omega) \mid \mathcal{H})(\omega) = \mathbb{P}(K = K(\omega) \mid \mathcal{H})(\omega) = \mathbf{1}_{\{K(\omega)=K(\omega)\}} = 1$. □

Next, we study measurability of integrals with respect to random measures.

Lemma A.2. *Assume that ν is a $(\boldsymbol{G}, \mathbb{P})$-compensator of a random measure of jumps with a $\mathcal{H}$-measurable $\mathbb{P}$-version. Let $U\colon \Omega \times [0, T] \times \mathbb{R} \to \mathbb{R}^+$ be $\mathcal{P}(\boldsymbol{G}) \otimes \mathcal{B}$-measurable such that $\mathbb{P}$-a.s. $(1 \wedge |x|^2)U * \nu_T < \infty$ and $U \cdot \nu$ has a $\mathcal{H}$-measurable $\mathbb{P}$-version.*

(i) *Let $g\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Borel function such that $\mathbb{P}$-a.s. $|g(x, U)| * \nu_T < \infty$, then $g(x, U) * \nu$ has a $\mathcal{H}$-measurable $\mathbb{P}$-version.*

(ii) *Let $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Borel function such that $f(0, y) = 0$ for all $y \in \mathbb{R}$ and denote $a_t := \nu(\{t\} \times \mathbb{R})$ and $a'_t := \int_{\mathbb{R}} U(t, x)\nu(\{t\} \times \mathrm{d}x)$. Suppose that $a'_t \le 1$ for all $t \in [0, T]$ and $\mathbb{P}$-a.s. $\sum_{s\in[0,T]} |f(a_s, a'_s)| < \infty$. The process $\sum_{s\in[0,\cdot]} f(a_s, a'_s)$ has a $\mathcal{H}$-measurable $\mathbb{P}$-version.*

Proof. Denote $B_n := \{x \in \mathbb{R}\colon |x| < 1/n\}$ and take $0 \le r \le s \le T$ and $G \in \mathcal{B}$. There exists a constant K such that $\mathbf{1}_{B_n^c}(x) \le K(1 \wedge |x|^2)$. Hence, since ν and $U \cdot \nu$ have $\mathcal{H}$-measurable $\mathbb{P}$-versions, the random variables $\nu((r, s] \times G \cap B_n^c)$ and $(U \cdot \nu)((r, s] \times G \cap B_n^c)$ also have $\mathcal{H}$-measurable $\mathbb{P}$-versions. By Remark 2.1(ii) and Lemma A.1, there exists a $\mathbb{P}$-null set $N \in \mathcal{F}$ such that, for all $\omega \in N^c$, there is a $\mathbb{P}(\cdot \mid \mathcal{H})(\omega)$-null set $N_\omega \in \mathcal{F}$ such that, for all $\omega^* \in N_\omega^c$,

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we have

$$
\begin{aligned}
&(\mathbf{1}_{B_n^c} U * \nu_T)(\omega) + (\mathbf{1}_{B_n^c} U * \nu_T)(\omega^*) + (|g(x, U)| * \nu_T)(\omega) + (|g(x, U)| * \nu_T)(\omega^*) \\
&\quad + \sum_{s \in [0,T]} |f(a_s(\omega), a'_s(\omega))| + \sum_{s \in [0,T]} |f(a_s(\omega^*), a'_s(\omega^*))| \\
&\quad < \infty
\end{aligned}
$$

and

$$
\begin{aligned}
(\mathbf{1}_G \mathbf{1}_{B_n^c} * \nu_T)(\omega^*) &= \int_0^T \int_{\mathbb{R}} \mathbf{1}_G(\omega^*, s, x) \mathbf{1}_{B_n^c}(x) \nu(\omega, \, \mathrm{d}s \times \mathrm{d}x), \\
(\mathbf{1}_G \mathbf{1}_{B_n^c} * (U \cdot \nu)_T)(\omega^*) &= \int_0^T \int_{\mathbb{R}} \mathbf{1}_G(\omega^*, s, x) \mathbf{1}_{B_n^c}(x) U(\omega, s, x) \nu(\omega, \, \mathrm{d}s \times \mathrm{d}x)
\end{aligned}
\tag{A.1}
$$

for all $n \in \mathbb{N}$, $G = \Omega \times [0, T] \times \mathbb{R}$, and $G = A \times (r, s] \times (c, d]$ with $A \in \mathcal{F}$, $r, s, c, d \in \mathbb{Q}$, $0 \le r \le s \le T$ and $c \le d$. By a monotone class argument, (A.1) holds for all $n \in \mathbb{N}$ and $G \in \mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}$. Letting $n \to \infty$ and using the monotone convergence theorem yields

$$
\begin{aligned}
(\mathbf{1}_G * \nu_T)(\omega^*) &= \int_0^T \int_{\mathbb{R}} \mathbf{1}_G(\omega^*, s, x) \nu(\omega, \, \mathrm{d}s \times \mathrm{d}x), \\
(\mathbf{1}_G * (U \cdot \nu)_T)(\omega^*) &= \int_0^T \int_{\mathbb{R}} \mathbf{1}_G(\omega^*, s, x) U(\omega, s, x) \nu(\omega, \, \mathrm{d}s \times \mathrm{d}x)
\end{aligned}
$$

for all $G \in \mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}$.

Therefore, for all $t \in [0, T]$, we have $a_t(\omega^*) = a_t(\omega)$ and $a'_t(\omega^*) = a'_t(\omega)$, which implies that

$$
\sum_{s \in [0,t]} f(a_s(\omega^*), a'_s(\omega^*)) = \sum_{s \in [0,t]} f(a_s(\omega), a'_s(\omega)).
$$

By Lemma A.1, this proves the claim of Lemma A.2(ii).

Since each nonnegative $\mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}$-measurable function can be approximated from below by simple nonnegative $\mathcal{F} \otimes \mathcal{B}([0, T]) \otimes \mathcal{B}$-measurable functions, we have

$$
(g(x, U) * \nu_t)(\omega^*) = \int_0^t \int_{\mathbb{R}} g(x, U(\omega^*, s, x)) \nu(\omega, \, \mathrm{d}s \times \mathrm{d}x)
$$

and

$$
\int_0^T \int_{\mathbb{R}} \mathbf{1}_G(s, x) U(\omega^*, s, x) \nu(\omega, \, \mathrm{d}s \times \mathrm{d}x) = \int_0^T \int_{\mathbb{R}} \mathbf{1}_G(s, x) U(\omega, s, x) \nu(\omega, \, \mathrm{d}s \times \mathrm{d}x)
$$

for all $t \in [0, T]$ and $G \in \mathcal{B}([0, T]) \otimes \mathcal{B}$. Thus, $\nu(\omega, \mathrm{d}s \times \mathrm{d}x)$-almost everywhere $U(\omega^*, \cdot, \cdot) = U(\omega, \cdot, \cdot)$. We conclude that, for all $t \in [0, T]$,

$$
(g(x, U) * \nu_t)(\omega^*) = \int_0^t \int_{\mathbb{R}} g(x, U(\omega^*, s, x)) \nu(\omega, \, \mathrm{d}s \times \mathrm{d}x) = (g(x, U) * \nu_t)(\omega).
$$

Now, Lemma A.2(i) follows again from Lemma A.1. □

The same arguments as in the proof of Lemma A.2 yield the following lemma.

Lemma A.3. *Let* $k \colon \mathbb{R} \to \mathbb{R}^+$ *be a Borel function,* $\gamma \colon \Omega \times [0, T] \to \mathbb{R}$ *be* $\mathcal{F} \otimes \mathcal{B}([0, T])$*-measurable,* $C \in \mathcal{V}^+$*, and assume that* $\mathbb{P}$*-a.s.* $|\gamma| \cdot C_T < \infty$*. If* C *and* $\gamma \cdot C$ *have* $\mathcal{H}$*-measurable* $\mathbb{P}$*-versions, then so does* $k(\gamma) \cdot C$*.*

Appendix B. The scope of Assumption 2.2

We provide examples for situations where an $(\boldsymbol{F}, \mathbb{P})$-semimartingale is also a $(\boldsymbol{G}, \mathbb{P})$-semimartingale.

Example B.1. (*SIIs.*) If $\mathcal{H} := \{\Omega, \varnothing\}$ then $\boldsymbol{F} = \boldsymbol{G}$ and the $(\boldsymbol{G}, \mathbb{P})$- and $(\boldsymbol{F}, \mathbb{P})$-characteristics of X coincide.

Example B.2. (*Independent integrands.*) Let Y be an $\mathbb{R}^m$-valued càdlàg process and V be an $\mathbb{R}^n$-valued càdlàg process which are $\mathbb{P}$-independent. Define $\mathcal{F}_t^o := \sigma(Y_s, V_s, s \in [0, t])$, $\mathcal{F}_t^V := \sigma(V_s, s \in [0, t])$, $\boldsymbol{F}^V := (\mathcal{F}_{t+}^V)_{t\in[0,T]}$, $\mathbf{F}^Y$ analogously, and $\mathcal{H} := \sigma(Y_s, s \in [0, T])$. Lemma 2.1 yields that the σ-fields $\mathcal{F}_t^o$ and $\mathcal{H}$ are countably generated. Assume that V is an $(\boldsymbol{F}^V, \mathbb{P})$-semimartingale whose $(\boldsymbol{F}^V, \mathbb{P})$-characteristics are denoted by $(B^V(h), C^V, \nu^V)$. Let $\mu\colon \mathbb{D}^m \times \mathbb{R}^+ \to \mathbb{R}^d \otimes \mathbb{R}^n$ be such that $\mu(\cdot, Y) := \mu(Y) \in L(V, \boldsymbol{F}^Y, \mathbb{P})$. Next, we generalize [16, Lemma 2.3] to processes without absolutely continuous characteristics.

Lemma B.1. *The process $X := \mu(Y) \cdot V$ is an $\mathbb{R}^d$-valued $(\boldsymbol{G}, \mathbb{P})$- and $(\boldsymbol{F}, \mathbb{P})$-semimartingale and its $(\boldsymbol{G}, \mathbb{P})$- and $(\boldsymbol{F}, \mathbb{P})$-characteristics $(B(\tilde{h}), C, \nu)$ associated to a truncation function $\tilde{h}$ are given by*

$$\begin{aligned} B(\tilde{h})^i &= \sum_{k\le n} \mu(Y)^{i,k} \cdot B^V(h)^k + \Big(\tilde{h}^i(\mu(Y)x) - \sum_{k\le n} \mu(Y)^{i,k} h^k(x)\Big) * \nu^V, \\ C^{i,j} &= \sum_{k,l\le n} (\mu(Y)^{i,k}\mu(Y)^{j,l}) \cdot C^{V,k,l}, \qquad\qquad (B.1) \\ \nu(\mathrm{d}t \times G) &= \int_{\mathbb{R}^n} \mathbf{1}_G(\mu(Y)x)\nu^V(\mathrm{d}t \times \mathrm{d}x), \quad \textit{for } i, j \le d,\ G \in \mathcal{B}^d,\ 0 \notin G. \end{aligned}$$

Proof. Thanks to the inclusions $\boldsymbol{F}^V \subseteq \boldsymbol{F} \subseteq \boldsymbol{G}$, we deduce, from [11, Theorem II.2.42], [2, Theorem 15.5], and the tower rule, that V is an $(\boldsymbol{F}, \mathbb{P})$- and $(\boldsymbol{G}, \mathbb{P})$-semimartingale with $(\boldsymbol{F}, \mathbb{P})$- and $(\boldsymbol{G}, \mathbb{P})$-characteristics given by $(B^V(h), C^V, \nu^V)$. It follows from the inclusions $\boldsymbol{F}^Y \subseteq \boldsymbol{F} \subseteq \boldsymbol{G}$ and [11, Theorem III.6.30] that $\mu(Y) \in L(V, \boldsymbol{F}, \mathbb{P}) \cap L(V, \boldsymbol{G}, \mathbb{P})$. Hence, $\mu(Y) \cdot V$ is an $(\boldsymbol{F}, \mathbb{P})$- and $(\boldsymbol{G}, \mathbb{P})$-semimartingale, with $(\boldsymbol{F}, \mathbb{P})$- and $(\boldsymbol{G}, \mathbb{P})$-characteristics given by (B.1); see [11, Proposition IX.5.3]. □

Recalling that $\mu(Y)$ is $\boldsymbol{F}^Y$-predictable, we obtain the following corollary.

Corollary B.1. *If $(B^V(h), C^V, \nu^V)$ are deterministic then $X = \mu(Y) \cdot V$ is a $(\mathcal{H}, \boldsymbol{F}, \mathbb{P})$-SII and an $(\boldsymbol{F}, \mathbb{P})$-semimartingale whose $(\boldsymbol{F}, \mathbb{P})$- and $(\boldsymbol{G}, \mathbb{P})$-characteristics coincide and are given by (B.1).*

Corollary B.1 implies that the financial models suggested by [1], [9], and [22] are exponential $\mathcal{H}$-SII models as defined in Section 2.2.

Example B.3. (*Time-changed Lévy models.*) We assume that V, Y, and $\mathcal{H}$ are given as in Example B.2 and that Y is $\mathbb{R}^+$-valued. Let $\mu\colon \mathbb{R} \to \mathbb{R}^d$ be a Borel function such that $\mathbb{P}$-a.s. $|\mu(Y)| \cdot I_T < \infty$, where $\mu(Y)_t := \mu(Y_t)$. Then we set $X := \mu(Y_-) \cdot I + V_{Y_-\cdot I}$ and $\mathcal{F}_t^o := \sigma(X_s, Y_s, s \in [0, t])$. Lemma 2.1 yields that $\mathcal{F}_t^o$ is countably generated. Let V be an $(\boldsymbol{F}, \mathbb{P})$-Lévy process with $(\boldsymbol{F}, \mathbb{P})$-Lévy–Khinchine triplet (b^V, c^V, F^V). The following lemma is a restatement of [15, Lemma 2.4]. It shows that the time-changed Lévy model proposed by Carr *et al.* [5] is an exponential $\mathcal{H}$-SII model as defined in Section 2.2.

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Lemma B.2. *The process X is a $(\mathcal{H}, \boldsymbol{F}, \mathbb{P})$-SII and an $(\boldsymbol{F}, \mathbb{P})$-semimartingale whose $(\boldsymbol{G}, \mathbb{P})$- and $(\boldsymbol{F}, \mathbb{P})$-characteristics coincide and are given by*

$$B = (\mu(Y_-) + b^V Y_-) \cdot I, \qquad C = c^V Y_- \cdot I, \qquad \nu(\mathrm{d}t \times \mathrm{d}x) = Y_{t-}\, \mathrm{d}t F^V(\mathrm{d}x).$$

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