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GAPS BETWEEN DIVISIBLE TERMS IN $a^2(a^2+1)$

TSZ HO CHAN

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Abstract

Suppose $a^2(a^2+1)$ divides $b^2(b^2+1)$ with $b > a$. We improve a previous result and prove a gap principle, without any additional assumptions, namely $b \gg a(\log a)^{1/8}/(\log\log a)^{12}$. We also obtain $b \gg_\epsilon a^{15/14-\epsilon}$ under the abc conjecture.

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1. Introduction and main results

We are interested in increasing sequences of positive integers $(a_n)_{n\geq 0}$ with each term dividing the next one (that is, $a_n \mid a_{n+1}$). A simple example is $(2^n)_{n\geq 0}$. Another example is $(n!)_{n\geq 1}$. These are simple, recursively defined sequences. It is more interesting and challenging to require each term of the sequence to have a special form. For example, $(3^n)_{n\geq 0}$ has all terms odd. For another special example, consider the Fibonacci numbers

$$F_1 = 1, \quad F_2 = 1, \quad F_{n+1} = F_n + F_{n-1} \quad \text{for } n \geq 2.$$

By the well-known fact that $F_m \mid F_n$ if and only if $m \mid n$, the sequence $(F_{2^n})_{n\geq 0}$ has all terms of Fibonacci-type and each term dividing the next one. One may restrict the sequence to numbers of the form n^2+1 or other polynomials, and we are interested in the growth of such sequences. We shall focus on numbers of the form $n^2(n^2+1)$ and study the following question.

Question 1.1. Suppose $a^2(a^2+1)$ divides $b^2(b^2+1)$. Must it be true that there is some gap between a and b? More precisely, is it true that $b > a^{1+\epsilon}$ for some small $\epsilon > 0$?

In [1] the author studied this question with some additional restrictions on a and b. In this paper we remove all these restrictions and prove the following result.

Theorem 1.2. *Let a and b be positive integers with $3 \leq a < b$. Suppose $a^2(a^2+1)$ divides $b^2(b^2+1)$. Then*

$$b \gg \frac{a(\log a)^{1/8}}{(\log\log a)^{12}}.$$

Recently, Choi, Lam and the author [2] defined the gap principle of order n for polynomials with integer coefficients as follows.

Definition 1.3. Let n be a positive integer and $f(x)$ be a polynomial with integer coefficients. Consider the set of all positive integers $a_0 < a_1 < a_2 < \cdots < a_n$ such that $f(a_i)$ divides $f(a_{i+1})$ for $0 \le i \le n-1$. We say that $f(x)$ satisfies the gap principle of order n if $\lim a_n/a_0 = \infty$ as $a_0 \to \infty$.

Hence, Theorem 1.2 implies that the polynomial $f(x) = x^2(x^2 + 1)$ satisfies the gap principle of order 1.

Assuming the abc conjecture, we can obtain a better gap result than that in [1], without any extra assumptions.

Theorem 1.4. *Let a and b be positive integers with $a < b$. Suppose $a^2(a^2 + 1)$ divides $b^2(b^2 + 1)$. Then, under the abc conjecture with any small $\epsilon > 0$,*

$$b \gg_\epsilon a^{15/14-\epsilon}.$$

This answers Question 1.1 in the affirmative under the abc conjecture. We hope this article will inspire readers to study similar questions.

Notation. The symbol $a \mid b$ means that a divides b. The expressions $f(x) \ll g(x)$, $g(x) \gg f(x)$ and $f(x) = O(g(x))$ are all equivalent to $|f(x)| \le Cg(x)$ for some constant $C > 0$. Finally, $f(x) = O_\lambda(g(x))$, $f(x) \ll_\lambda g(x)$ and $g(x) \gg_\lambda f(x)$ mean that the implicit constant C may depend on λ.

2. Proof of Theorem 1.2

Since $a^2(a^2 + 1)$ divides $b^2(b^2 + 1)$, write

$$ta^2(a^2 + 1) = b^2(b^2 + 1)$$

for some integer $t > 1$. We may assume $t \le \log a$, for otherwise the theorem is true automatically. Let D be the greatest common divisor of a and b. Suppose $a = Dx$ and $b = Dy$ with $(x, y) = 1$, and let $T = (D^2x^2 + 1, D^2y^2 + 1)$. Then

$$tx^2 \frac{D^2x^2 + 1}{T} = y^2 \frac{D^2y^2 + 1}{T}. \tag{2.1}$$

Since $(x, y) = 1$, x^2 must divide $(D^2y^2 + 1)/T$. Write $(D^2y^2 + 1)/T = mx^2$ for some integer m. Then $t(D^2x^2 + 1)/T = my^2$ and

$$t(D^2x^2 + 1) = mTy^2 \tag{2.2}$$

and

$$D^2y^2 + 1 = mTx^2. \tag{2.3}$$

Multiplying (2.2) by D^2, (2.3) by mT and combining,

$$[(mT)^2 - tD^4]x^2 = tD^2 + mT. \tag{2.4}$$

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Similarly, multiplying (2.2) by mT, (2.3) by tD^2 and combining,

$$[(mT)^2 - tD^4]y^2 = tD^2 + tmT. \tag{2.5}$$

Subtracting (2.4) from (2.5),

$$s(y^2 - x^2) = (t-1)mT \tag{2.6}$$

with

$$s = (mT)^2 - tD^4. \tag{2.7}$$

From (2.3), we have $(mT, D) = 1$. Hence,

$$(s, mT) = ((mT)^2 - tD^4, mT) = (tD^4, mT) = (t, mT).$$

Combining this with (2.6) shows $s \mid (t-1)t$ and (2.7) gives a hyperelliptic curve

$$Y^2 = tX^4 + s$$

by setting $Y = mT$ and $X = D$. Let $\lambda := (t-1)t$. Voutier [3] studied such integral solutions on hyperelliptic curves using transcendental number theory. From [3, Theorem 1],

$$\max(X, Y) < e^{C_1\lambda(\max(1,\log\lambda))^{96}}$$

for some constant $C_1 \geq 1$. Suppose $\lambda \leq C_1^{-1}\log D/(\log\max(e, \log D))^{96}$. Then

$$D = X < \exp\Big(C_1 \frac{1}{C_1} \frac{\log D}{(\log\max(e, \log D))^{96}} (\log\max(e, \log D))^{96}\Big) = D,$$

which is a contradiction. Therefore, $(t-1)t > C_1^{-1}\log D/(\log\max(e, \log D))^{96}$ and

$$t \gg \frac{\sqrt{\log D}}{(\log\max(e, \log D))^{48}}. \tag{2.8}$$

From (2.2), $2tD^2x^2 \geq mTy^2$ and $tD^2 \gg mT$. This together with (2.4) gives $tD^2 \gg x^2$. Hence, as $t \leq \log a$ and $a = Dx$, we have $\log D \gg \log a$. Therefore, (2.8) gives

$$t \gg \frac{\sqrt{\log a}}{(\log\log a)^{48}}$$

and we have Theorem 1.2 as $b^4/a^4 \gg t$.

3. Proof of Theorem 1.4

Firstly, for any integer n, let $R(n) := \prod_{p|n} p$ be the radical or kernel of an integer n. We state the abc conjecture.

Conjecture 3.1. For every $\epsilon > 0$, there exists a constant C_ϵ such that for all triples (a, b, c) of coprime positive integers with $a + b = c$, we have

$$c < C_\epsilon R(abc)^{1+\epsilon}.$$

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Secondly, we state a lemma which follows easily from the unique factorisation of integers.

LEMMA 3.2. *Suppose* $a \mid A^2$ *and* $a = a_1 a_2^2$ *with* a_1 *squarefree. Then* $a_1 a_2 \mid A$.

PROOF OF THEOREM 1.4. Basically, we follow the proof of Theorem 1.2. Using the same notation as in Section 2,

$$a = Dx, \quad b = Dy \quad \text{and} \quad s + tD^4 = (mT)^2$$

with $(mT, D) = 1$. Let

$$(t, (mT)^2) = d \quad \text{and} \quad d = d_1 d_2^2 \quad \text{with } d_1 \text{ squarefree.}$$

By Lemma 3.2, $mT = d_1 d_2 S$. Then

$$\frac{s}{d} + \frac{t}{d} D^4 = d_1 S^2. \tag{3.1}$$

Here, the three terms are integers and pairwise relatively prime. We can apply the abc conjecture and obtain

$$\frac{t}{d} D^4 < C_\epsilon \Big(\frac{(t-1)t}{d} D d_1 S \Big)^{1+\epsilon} \tag{3.2}$$

since $s \mid (t-1)t$. Suppose $t \le 10D^4$. As $d_1 \le d$, (3.1) gives

$$(d_1 S)^2 \ll tD^4 \quad \text{or} \quad d_1 S \ll t^{1/2} D^2.$$

Putting this into (3.2),

$$\frac{t}{d} D^4 \ll_\epsilon \Big(\frac{t^{5/2}}{d} D^3 \Big)^{1+\epsilon}.$$

Hence,

$$D \ll_\epsilon t^{3/2+8\epsilon}. \tag{3.3}$$

As $s + tD^4 = (mT)^2$, $s \mid (t-1)t$ and $t \le D^4$, we have $mT \ll t^{1/2} D^2$. This together with (2.4) gives

$$x^2 \ll \frac{tD^2}{s} \quad \text{or} \quad x \ll t^{1/2} D.$$

Therefore, by (3.3),

$$a = Dx \ll t^{1/2} D^2 \ll_\epsilon t^{7/2+16\epsilon} \quad \text{or} \quad t \gg_\epsilon a^{2/7-2\epsilon}$$

which gives Theorem 1.4 as $b^4/a^4 \gg t$.

We are left with the case $t > 10D^4$. Suppose $y/x = b/a \le a^{1/5} = (Dx)^{1/5}$. From (2.1), $t^{1/4} \le \sqrt[4]{2}\, y/x$. Hence,

$$10D^4 < t \le 2(Dx)^{4/5}. \tag{3.4}$$

This gives

$$x > \frac{10}{2^{5/4}} D^4. \tag{3.5}$$

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On the other hand, (2.3) and (2.1) give

$$mT \le 2D^2y^2/x^2 \le 2\sqrt{2}t^{1/2}D^2.$$

Putting this into (2.4),

$$x^2 \le (1+2\sqrt{2})tD^2 \quad \text{or} \quad \frac{1}{1+2\sqrt{2}}\Big(\frac{x}{D}\Big)^2 \le t. \tag{3.6}$$

Combining (3.4) and (3.6),

$$\frac{1}{1+2\sqrt{2}}\Big(\frac{x}{D}\Big)^2 \le 2(Dx)^{4/5} \quad \text{or} \quad x \le [2(1+\sqrt{2})]^{5/6}D^{7/3}$$

which contradicts (3.5). Therefore, $b/a > a^{1/5}$, which also gives Theorem 1.4. □

References

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TSZ HO CHAN, Department of Mathematical Sciences,
University of Memphis, Memphis, TN 38152, USA
e-mail: thchan6174@gmail.com

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