Teaching Notes

Rewriting polynomials: a tool for teaching secondary mathematics

The rewriting of polynomials in powers of $(x - a)$ is a way of anticipating the Taylor polynomials of a function. Our goal is to show that this rewrite can also be used as a didactic tool in secondary mathematics courses. This allows us to make easy explorations and to find nice and visual results that help us to understand the relationship between the tangent lines to graphs of polynomials of degree 2, 3 and 4. Also, we obtain results about area calculations, with visual characteristics that can encourage the students.

Initial exploration

First, we analyse how to rewrite a polynomial in powers of $(x - a)$. We start with a cubic polynomial $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3$. We want to rewrite it in the form,

$$f(x) = A(x - a)^3 + B(x - a)^2 + C(x - a) + D.$$

By replacing x by a we have $D = f(a)$. To find C, we make the factor $(x - a)$ disappear from its side using the derivative $C = f'(a)$. Repeating the process we obtain $B = \frac{f''(a)}{2}$ and $A = \frac{f'''(a)}{6}$. Applying the above to a polynomial $f(x)$ of degree n, we have

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)(x - a)^2}{2!} + \frac{f'''(a)(x - a)^3}{3!} + \ldots + \frac{f^{(n)}(a)(x - a)^n}{n!}.$$

We observe that the first two terms correspond to the tangent line at $x = a$, $T_f(x) = f(a) + f'(a)(x - a)$. Therefore

$$f(x) - T_f(x) = (x - a)^2\left[\frac{f''(a)}{2!} + \frac{f'''(a)(x - a)}{3!} + \ldots + \frac{f^{(n)}(a)(x - a)^{n-2}}{n!}\right]. \quad (1)$$

Thus, $f(x) - T_f(x)$ contains the factor $(x - a)^2$.

If $f(x)$ has an inflection point at $x = a$, we will have $f''(a) = 0$. Then,

$$f(x) - T_f(x) = (x - a)^3\left[\frac{f'''(a)}{3!} + \frac{f''''(a)(x - a)}{4!} + \ldots + \frac{f^{(n)}(a)(x - a)^{n-3}}{n!}\right]. \quad (2)$$

Thus, $f(x) - T_f(x)$ contains the factor $(x - a)^3$.

Polynomials of degree 2

Let $f(x) = a_0x^2 + a_1x + a_2$, with $a_0 \neq 0$. The expressions of $f(x)$ in terms of the powers of $(x - a)$ and $(x - b)$ are

$$f(x) = f(a) + f'(a)(x - a) + \frac{f''(a)(x - a)^2}{2!},$$

$$f(x) = f(b) + f'(b)(x - b) + \frac{f''(b)(x - b)^2}{2!}.$$

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We are going to explore the tangent lines $T_1(x)$, $T_2(x)$ at $x = a$ and $x = b$. Considering that $f''(a) = f''(b) = 2a_0$, we have,

$$T_1(x) - T_2(x) = \big(f(x) - T_2(x)\big) - \big(f(x) - T_1(x)\big)$$
$$= a_0(x - b)^2 - a_0(x - a)^2.$$

Thus, $T_1(x)$ and $T_2(x)$ intersect at $x = \frac{1}{2}(a + b)$.

Applying this result to the area enclosed between the graphs of $f(x)$, $T_1(x)$ and $T_2(x)$ we obtain

$$\int_a^{\frac{a+b}{2}} \big(f(x) - T_1(x)\big)\,dx + \int_{\frac{a+b}{2}}^b \big(f(x) - T_2(x)\big)\,dx$$
$$= \int_a^{\frac{a+b}{2}} a_0(x - a)^2\,dx + \int_{\frac{a+b}{2}}^b a_0(x - b)^2\,dx = 2\int_a^{\frac{a+b}{2}} a_0(x - a)^2\,dx.$$

Figure 1 shows a visualisation of this result for $a_0 > 0$.

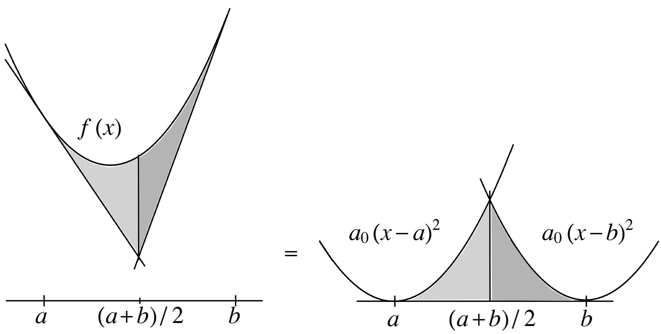

FIGURE 1: for $a_0 > 0$

Thus, the above area is $\left|2\int_a^{\frac{a+b}{2}} a_0(x - a)^2\,dx\right| = \dfrac{1}{12}\left|a_0\right|(b - a)^3.$

Polynomials of degree 3

Let $f(x) = a_0x^3 + a_1x^2 + a_2x + a_3$ with $a_0 \neq 0$. We see that $f''(a) = 6a_0a + 2a_1$ and $f'''(a) = 6a_0$.

(a) If $f(x)$ does not have an inflection point at $x = a$, then $f''(a) \neq 0$, and applying (1), we will have

$$f(x) - T_f(x) = (x - a)^2\left[\frac{6a_0a + 2a_1}{2} + \frac{6a_0(x - a)}{6}\right]$$
$$= a_0(x - a)^2(x - b) \qquad (3)$$

for $b = -2a - \dfrac{a_1}{a_0}$. (We note that $b \neq a$ because $f''(a) \neq 0$.)

Thus, $f(x)$ and $T_f(x)$ intersect at $x = a$ and $x = b$.

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Let us take $g(x) = a_0 x^3 + a_1 x^2$ (or any other cubic polynomial, with the same coefficients a_0 and a_1) and $T_g(x)$ the tangent line to graph of $g(x)$ at $x = a$. It is clear by (1) that $g(x) - T_g(x) = f(x) - T_f(x)$, and therefore, by (3), $g(x)$ and $T_g(x)$ intersect at $x = a$ and $x = b$ as well.

Thus, the area between the graphs of $f(x)$ and $T_f(x)$ is the same as the area between the graphs of $g(x)$ and $T_g(x)$ (Figure 2).

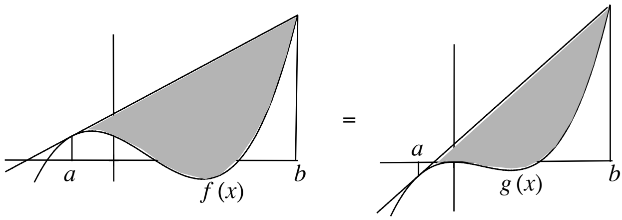

FIGURE 2: for $a_0 > 0, a < b$

This area can be calculated using (3),

$$\left|\int_a^b a_0 (x - a)^2 (x - b)\, dx\right| = \frac{1}{12}|a_0|(b - a)^4 = \frac{1}{12}|a_0|\left(3a + \frac{a_1}{a_0}\right)^4.$$

Finally, we observe that this value is the same as the area enclosed between the graph of $h(x) = a_0 (x - a)^2 (x - b)$ and the x-axis (Figure 3).

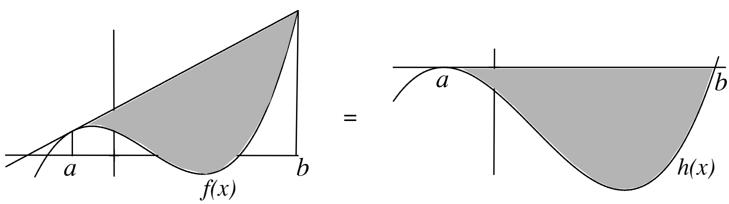

FIGURE 3: for $a_0 > 0, a < b$

(b) If the only inflection point of $f(x)$ is $x = a$, then $f''(a) = 0$ and by (2) we have

$$f(x) - T_f(x) = a_0 (x - a)^3.$$

As a consequence, the area between the graphs of $f(x)$ and $T_f(x)$ at $x = a$, over the interval $[a, b]$ is

$$\left|\int_a^b a_0 (x - a)^3\, dx\right| = \tfrac{1}{4}|a_0|(b - a)^4.$$

This value is the same as the area between the graph of $g(x) = a_0 (x - a)^3$ and the x-axis over $[a, b]$ (Figure 4).

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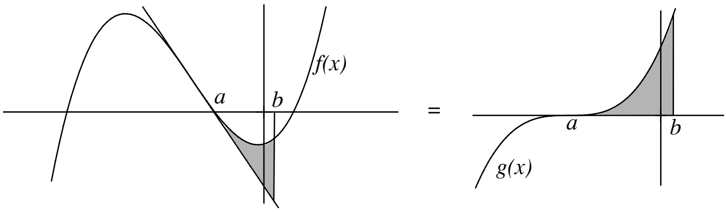

FIGURE 4: for $a_0 > 0, a < b$

Polynomials of degree 4

Let $f(x) = a_0x^4 + a_1x^3 + a_2x^2 + a_3x + a_4$, with $a_0 \neq 0$, and $T_f(x)$ the tangent line to the graph of $f(x)$ at an inflection point $x = a$. Then $f''(a) = 0, f'''(a) = 6a_1 + 24aa_0 \neq 0$ and $f^{(4)}(a) = 24a_0$. Therefore, by (2) we have,

$$f(x) - T_f(x) = (x-a)^3\left(\frac{6a_1 + 24aa_0}{6} + \frac{24a_0}{24}(x-a)\right) = a_0(x-a)^3(x-b)$$

for $b = -3a - \dfrac{a_1}{a_0}$. (We note that $b \neq a$ because $f'''(a) \neq 0$.)

Consequently the area between the graphs of $f(x)$ and $T_f(x)$ is

$$\left|\int_a^b a_0(x-a)^3(x-b)\,dx\right| = \left|\frac{1}{20}a_0(b-a)^5\right| = \left|\frac{1}{20}a_0\left(4a + \frac{a_1}{a_0}\right)^5\right|.$$

This value is the same as the area between the graph of $g(x) = a_0(x-a)^3(x-b)$ and the x-axis (Figure 5).

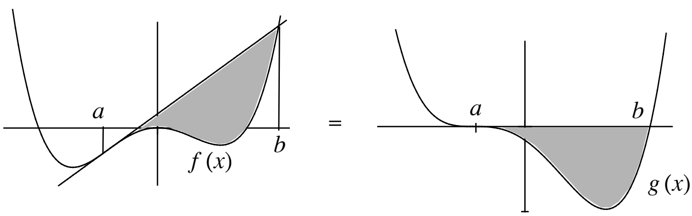

FIGURE 5: for $a_0 > 0, a < b$

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FÉLIX MARTÍNEZ de la ROSA
Department of Mathematics,
University of Cádiz, Spain
e-mail: *felix.martinez@uca.es*

The full story of invariant lines

The topics of invariant points and lines (for 2 × 2 matrix transformations) and eigenvalues and eigenvectors appear on some of the current AS/A level Further Mathematics specifications. These topics are treated separately in the textbooks. The aim of this short Note is to describe how the latter enables a full and succinct treatment of the former which also explains the rather limited range of examples.