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Topologically completely positive entropy and zero-dimensional topologically completely positive entropy

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Abstract. In a previous paper [Pavlov, A characterization of topologically completely positive entropy for shifts of finite type. *Ergod. Th. & Dynam. Sys.* **34** (2014), 2054–2065], the author gave a characterization for when a $\mathbb{Z}^d$-shift of finite type has no non-trivial subshift factors with zero entropy, a property which we here call zero-dimensional topologically completely positive entropy. In this work, we study the difference between this notion and the more classical topologically completely positive entropy of Blanchard. We show that there are one-dimensional subshifts and two-dimensional shifts of finite type which have zero-dimensional topologically completely positive entropy but not topologically completely positive entropy. In addition, we show that strengthening the hypotheses of the main result of Pavlov [A characterization of topologically completely positive entropy for shifts of finite type. *Ergod. Th. & Dynam. Sys.* **34** (2014), 2054–2065] yields a sufficient condition for a $\mathbb{Z}^d$-shift of finite type to have topologically completely positive entropy.

1. *Introduction*

This work is motivated by an unfortunate misuse of the term ‘topologically completely positive entropy’ (hereafter called TCPE) in some works written or co-written by the author (see [**4**, **9**]). Blanchard [**1**] originally defined TCPE to mean that a topological dynamical system has no non-trivial (i.e., containing more than one point) factors with zero entropy. However, in [**4**] and [**9**], the proofs furnished were for $\mathbb{Z}^d$ subshifts and only proved that all non-trivial subshift factors have positive entropy. It is quite simple to show that any system has a non-trivial subshift factor with zero entropy if and only if it has a non-trivial zero-dimensional factor with zero entropy (see Theorem 3.1), and so we say that a system has zero-dimensional TCPE (or ZTCPE) if all non-trivial zero-dimensional factors have positive entropy. It is obvious that TCPE implies ZTCPE, and just as obvious that the converse is false in general: any topological dynamical system on a connected space trivially has ZTCPE since it has no non-trivial zero-dimensional factors.

However, it is natural to wonder whether or not the two notions coincide if (X, T) is itself assumed to be zero-dimensional, a subshift, or even a shift of finite type (SFT). This is not the case; we will construct several examples of systems in these classes with ZTCPE but not TCPE. We prove the following results in the one-dimensional case.

THEOREM 1.1. *There exists a $\mathbb{Z}$-subshift which has ZTCPE but not TCPE.*

THEOREM 1.2. *Any $\mathbb{Z}$-SFT with ZTCPE also has TCPE.*

In the two-dimensional case, the picture is even more interesting. The following theorem was proved by the author in [**9**]; it erroneously purported to give conditions equivalent to TCPE for multidimensional SFTs (and had an unfortunate typo which replaced 'SFT' with 'subshift'), but we have given the corrected version below.

THEOREM 1.3. [**9**, Theorem 1.1] *A $\mathbb{Z}^d$-SFT has ZTCPE if and only if it has the following two properties: every $w \in L(X)$ has positive measure for some $\mu \in \mathcal{M}(X)$ and, for every $S \subset \mathbb{Z}^d$ and $w, w' \in L_S(X)$, there exist patterns $w = w_1, w_2, \ldots, w_n = w'$ so that, for $1 \leq i < n$, there exist homoclinic points $x, x' \in X$ with $x(S) = w_i$ and $x'(S) = w_{i+1}$.*

The second property in Theorem 1.3 was called 'chain exchangeability' of w and w' in [**9**]. Note that even if all pairs of patterns with the same shape are chain exchangeable, it is theoretically possible that the number n of required 'exchanges' could increase with the size of the patterns. In fact, this is related to TCPE as well, as shown by the following theorem.

THEOREM 1.4. *If a $\mathbb{Z}^d$-SFT satisfies the hypotheses of Theorem 1.3 with a uniform bound on the required n over all patterns w, w', then X has TCPE.*

Theorem 1.4 implies that the previously mentioned results of [**4**] and [**9**], in fact, do yield TCPE for the subshifts in question (see Corollaries 3.3 and 3.4), since all of those proofs included such a uniform bound on n. The remaining question of whether ZTCPE, in fact, implies TCPE for $\mathbb{Z}^d$-SFTs is answered negatively by the following.

THEOREM 1.5. *There exists a $\mathbb{Z}^2$-SFT X which has ZTCPE but not TCPE.*

We note that, by necessity, the X from Theorem 1.5 has the property that all pairs of patterns are chain exchangeable, but that larger and larger patterns may require more and more exchanges. We do not know whether this property is sufficient as well as necessary, i.e., whether the converse of Theorem 1.4 holds as well.

Question 1.6. Does every $\mathbb{Z}^d$ SFT with TCPE satisfy the hypotheses of Theorem 1.3 with a uniform bound on n over all w, w'?

2. *Definitions*

We begin with some definitions from topological/symbolic dynamics.

Definition 2.1. A $\mathbb{Z}^d$ *topological dynamical system* (X, T_v) is given by a compact metric space X and a $\mathbb{Z}^d$ action $\{T_v\}_{v\in\mathbb{Z}^d}$ by homeomorphisms on X. In the special case $d = 1$, it is standard to refer to the system as (X, T) rather than (X, T_n); the single homeomorphism T generates the entire action in this case, anyway.

Definition 2.2. For any finite set A (called an *alphabet*), the $\mathbb{Z}^d$*-shift action* on $A^{\mathbb{Z}^d}$, denoted by $\{\sigma_t\}_{t\in\mathbb{Z}^d}$, is defined by $(\sigma_t x)(s) = x(s + t)$ for $s, t \in \mathbb{Z}^d$.

We always endow $A^{\mathbb{Z}^d}$ with the product discrete topology, with respect to which it is obviously compact metric.

Definition 2.3. A $\mathbb{Z}^d$*-subshift* is a closed subset of $A^{\mathbb{Z}^d}$ which is invariant under the $\mathbb{Z}^d$-shift action. When the dimension is clear from context, we often just use the term *subshift*.

Any $\mathbb{Z}^d$-subshift inherits a topology from $A^{\mathbb{Z}^d}$, and is compact. Each σ_t is a homeomorphism on any $\mathbb{Z}^d$-subshift, and so any $\mathbb{Z}^d$-subshift, when paired with the $\mathbb{Z}^d$-shift action, is a topological dynamical system. Where it will not cause confusion, we suppress the action σ_v and just refer to a subshift by the space X.

Definition 2.4. A *pattern* over A is a member of A^S for some finite $S \subset \mathbb{Z}^d$, which is said to have *shape* S. When $d = 1$ and S is an interval of integers, we use the term *word* rather than pattern.

For any set $\mathcal{F}$ of patterns over A, one can define the set $X(\mathcal{F}) := \{x \in A^{\mathbb{Z}^d} : x(S) \notin \mathcal{F}\ \forall \text{ finite } S \subset \mathbb{Z}^d\}$. It is well known that any $X(\mathcal{F})$ is a $\mathbb{Z}^d$-subshift, and all $\mathbb{Z}^d$-subshifts are representable in this way. All subshifts are assumed to be non-empty in this paper.

For any patterns $v \in A^S$ and $w \in A^T$ with $S \cap T = \varnothing$, define vw to be the pattern in $A^{S\cup T}$ defined by $(vw)(S) = v$ and $(vw)(T) = w$.

Definition 2.5. A $\mathbb{Z}^d$*-shift of finite type (SFT)* is a $\mathbb{Z}^d$-subshift equal to $X(\mathcal{F})$ for some finite $\mathcal{F}$. The *type* of X is defined to be the minimum integer t so that $\mathcal{F}$ can be chosen with all patterns on shapes which are subsets of $[1, t]^d$.

Throughout this paper, for $a < b \in \mathbb{Z}$, $[a, b]$ will be used to denote $\{a, \ldots, b\}$, except for the special case $[0, 1]$, which will have its usual meaning as an interval of real numbers.

Definition 2.6. The *language* of a $\mathbb{Z}^d$-subshift X, denoted by $L(X)$, is the set of all patterns which appear in points of X. For any finite $S \subset \mathbb{Z}^d$, $L_S(X) := L(X) \cap A^S$, the set of patterns in the language of X with shape S.

The following definitions are from [**9**] and relate to the conditions given there characterizing ZTCPE.

Definition 2.7. For any $\mathbb{Z}^d$-subshift X and any finite $S \subseteq \mathbb{Z}^d$, patterns $w, w' \in L_S(X)$ are *exchangeable* in X if there exist homoclinic points $x, x' \in X$ such that $x(S) = w$ and $x'(S) = w'$.

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It should be reasonably clear that if X is an $\mathbb{Z}^d$-SFT with type t, then w, w' are exchangeable if and only if there exist N and $\delta \in L_{[-N,N]^d \setminus [-N+t,N-t]^d}(X)$ such that the shapes of w, w' lie in $[-N+t, N-t]^d$ and $\delta w, \delta w' \in L(X)$.

Definition 2.8. For any $\mathbb{Z}^d$-subshift X and any finite $S \subseteq \mathbb{Z}^d$, patterns $w, w' \in L_S(X)$ are *chain exchangeable* in X if there exists n and patterns $(w_i)_{i=1}^n$ in $L_S(X)$ such that $w_1 = w$, $w_n = w'$, and w_i and w_{i+1} are exchangeable in X for $i \in [1, n)$.

Alternately, the chain exchangeability relation is just the transitive closure of the exchangeability relation.

Definition 2.9. The *topological entropy* of a $\mathbb{Z}^d$ topological dynamical system (X, T_v) is given by

$$h(X, T_v) := \sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{n^d} N\bigg(\bigvee_{v\in[1,n]^d} T_v \mathcal{U}\bigg),$$

where $\mathcal{U}$ ranges over open covers of X and $N(\mathcal{U})$ is the minimal size of a subcollection of $\mathcal{U}$ which covers X.

We will not need any advanced properties of topological entropy in this paper. (For a detailed treatment of topological entropy, see [**10**].) We do, however, note the following sufficient condition for positive topological entropy. If K and K' are disjoint non-empty closed sets in X, and if there exists a subset S of $\mathbb{Z}^d$ with positive density so that, for any $y \in \{0, 1\}^S$, there exists $x \in X$ with $T_s x \in K$ when $y(s) = 0$ and $T_s x \in K'$ when $y(s) = 1$, then it follows that $h(X, T_v) > 0$; in particular, for $\mathcal{U} = \{K^c, K'^c\}$, the limit in the definition is at least log 2 times the density of S. For brevity, we refer to this property by saying that (X, T_v) contains points which 'independently visit K and K' in any predetermined way along a set of iterates of positive density.'

Definition 2.10. A (topological) *factor map* is any continuous shift-commuting map ϕ from a $\mathbb{Z}^d$ topological dynamical system (X, T_v) to a $\mathbb{Z}^d$ topological dynamical system (Y, S_v). Given such a factor map ϕ, the system $(\phi(X), S_v)$ is called a *factor of* (X, T_v).

It is well known that topological entropy does not increase under factor maps; again, see [**10**] for a proof.

Definition 2.11. A $\mathbb{Z}^d$ topological dynamical system (X, T_v) has *topologically completely positive entropy* (or *TCPE*) if, for every surjective factor map from (X, T_v) to a $\mathbb{Z}^d$ topological dynamical system (Y, S_v), either $h(Y, S_v) > 0$ or $|Y| = 1$.

Definition 2.12. A $\mathbb{Z}^d$ topological dynamical system (X, T_v) has *zero-dimensional topologically completely positive entropy* (or *ZTCPE*) if, for every surjective factor map from (X, T_v) to a $\mathbb{Z}^d$ zero-dimensional topological dynamical system (Y, S_v), either $h(Y, S_v) > 0$ or $|Y| = 1$.

Our final set of definitions relates to so-called balanced sequences. For any word w on $\{0, 1\}$, we use $\#(w, 1)$ to denote the number of 1 symbols in w.

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Definition 2.13. A sequence $x \in \{0, 1\}^{\mathbb{Z}}$ is *k-balanced* if every two subwords of x of the same length have numbers of 1 symbols within k: i.e., if, for every n, i, j, $|\#(w([i, i + n - 1]), 1) - \#(w([j, j + n - 1]), 1)| \leq k$. We use simply the term *balanced* to mean 1-balanced.

The following lemma and corollary are standard; see, for instance, [**7**, Ch. 2] for proofs in the 1-balanced case which trivially extend to arbitrary k.

LEMMA 2.14. *For every k-balanced sequence x, there is a uniform frequency of* 1*s: i.e., there exists* $\alpha \in [0, 1]$ *so that, for every* $\epsilon > 0$*, there exists N such that for* $n > N$*, every n-letter subword of x has proportion of* 1 *symbols between* $\alpha - \epsilon$ *and* $\alpha + \epsilon$.

COROLLARY 2.15. *For every k-balanced sequence x with frequency* α*, every* $n \in \mathbb{N}$ *and every* $i \in \mathbb{Z}$, $|n\alpha - \#(x([i, i + n - 1]), 1)| \leq k$.

For convenience, we refer to the uniform frequency of 1s in a k-balanced sequence as its *slope*. The following is immediate.

COROLLARY 2.16. *For any balanced sequence x with slope* α *and* $n \in \mathbb{N}$*, if* $n\alpha \notin \mathbb{Z}$*, then, for every* $i \in \mathbb{Z}$, $\#(x([i, i + n - 1]), 1)$ *is either* $\lfloor n\alpha \rfloor$ *or* $\lceil n\alpha \rceil$.

Here are two examples of simple algorithmically generated balanced sequences; see [**7**, Ch. 2] for a proof that they are, in fact, balanced with slope α.

Definition 2.17. For any $\alpha \in [0, 1]$, the *lower characteristic sequence* $\underline{x_\alpha}$ is defined by $\underline{x_\alpha}(n) = \lfloor (n + 1)\alpha \rfloor - \lfloor n\alpha \rfloor$ for all $n \in \mathbb{Z}$. The *upper characteristic sequence* $\overline{x_\alpha}$ is defined by $\overline{x_\alpha}(n) = \lceil (n + 1)\alpha \rceil - \lceil n\alpha \rceil$ for all $n \in \mathbb{Z}$.

The lower and upper characteristic sequences are not shifts of each other for irrational α, but for rational α we note that they are. If we write $\alpha = i/j$ in lowest terms, then there exists $k \in \mathbb{N}$ so that $k\alpha = m + 1/j$ for an integer m. Then

$$\begin{aligned}\sigma^k(\overline{x_\alpha}(n)) &= \overline{x_\alpha}(k + n) = \lceil (k + n + 1)\alpha \rceil - \lceil (k + n)\alpha \rceil \\ &= m + \left(\left\lceil (n + 1)\alpha + \frac{1}{j} \right\rceil\right) - m - \left(\left\lceil n\alpha + \frac{1}{j} \right\rceil\right) \\ &= \lfloor (n + 1)\alpha \rfloor - \lfloor n\alpha \rfloor = \underline{x_\alpha}(n).\end{aligned}$$

(We used here the easily checked fact that, for any rational x with denominator j, $\lceil x + 1/j \rceil = 1 + \lfloor x \rfloor$.)

Characteristic sequences also have useful convergence properties.

LEMMA 2.18. *If* α_n *approaches a limit* α *from above, then the lower characteristic sequences* $\underline{x_{\alpha_n}}$ *converge to the lower characteristic sequence* $\underline{x_\alpha}$. *Similarly, if* α_n *approaches* α *from below, then the upper characteristic sequences* $\overline{x_{\alpha_n}}$ *converge to the upper characteristic sequence* $\overline{x_\alpha}$.

Proof. This follows immediately from the continuity of the floor function from the right and the continuity of the ceiling function from the left. □

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For irrational α, the structure of balanced sequences is well known; the set of such balanced sequences is just the so-called Sturmian subshift with rotation number α. We omit a full treatment of Sturmian sequences here and instead refer the reader to [**7**, Ch. 2] for a detailed analysis. We will say that Sturmian sequences are defined similarly to upper and lower characteristic sequences, with the change that one is also allowed to add any constant to the terms inside floor or ceiling functions (e.g., x defined by $x(n) = \lceil \pi + (n + 1)\alpha \rceil - \lceil \pi + n\alpha \rceil$ is Sturmian).

For rational α, the structure of balanced sequences is more complicated. All balanced sequences with rational slope are eventually periodic [**7**, Proposition 2.1.11]. The periodic balanced sequences are easy to describe; the following is essentially [**7**, Lemma 2.1.15], combined with the above observation that upper and lower characteristic sequences are shifts of each other for rational α.

LEMMA 2.19. *Every balanced sequence with rational slope α which is periodic is a shift of $\underline{x_\alpha}$.*

The eventually periodic but not periodic balanced sequences are more complicated; they are described as 'skew sequences' in [**8**]. Luckily, we do not need a complete description of such sequences in this work, only one important property. Proposition 2.1.17 in [**7**] states that every finite balanced word is a subword of some Sturmian sequence (in fact, the reverse direction is also true), which clearly implies the following fact.

LEMMA 2.20. *Every balanced sequence can be written as the limit of balanced sequences with irrational slopes, i.e., Sturmian sequences.*

3. *Proofs*

We first establish the claim from the introduction that 'subshift TCPE' is in fact the same as ZTCPE.

THEOREM 3.1. *A $\mathbb{Z}^d$ topological dynamical system has a non-trivial $\mathbb{Z}^d$-subshift factor with zero entropy if and only if it has a non-trivial zero-dimensional factor with zero entropy.*

Proof. The forward direction is trivial, so we prove only the reverse. Suppose that (X, T_v) is a topological dynamical system with a factor (Y, S_v), where $|Y| > 1$, Y is zero-dimensional, and $h(Y, S_v) = 0$. Then, since $|Y| > 1$ and Y is zero-dimensional, there exists a non-trivial partition of Y into clopen sets A and B. Then, define the map $\phi : Y \to \{0, 1\}^{\mathbb{Z}^d}$ as follows: $(\phi(y))(v) = \chi_B(S_v(y))$, i.e., $(\phi(y))(v) = 0$ if $S_v(y) \in A$ and $(\phi(y))(v) = 1$ if $S_v(y) \in B$. Since A and B are closed, ϕ is a surjective factor map from (Y, S_v) to the subshift $(\phi(Y), \sigma_v)$. Moreover, if $a \in A$ and $b \in B$, then $(\phi(a))(0) = 0$ and $(\phi(b))(0) = 1$, meaning that $|\phi(Y)| > 1$. Therefore, $(\phi(Y), \sigma_v)$ is a non-trivial subshift factor of (X, T_v), and it has zero entropy since it is a factor of the zero entropy system (Y, S_v). □

Theorem 1.2 is a corollary of well known results, but, for completeness, we supply the simple proof here.

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Proof of Theorem 1.2. We assume basic knowledge of the structure of $\mathbb{Z}$-SFTs; for more information, see [**6**].

Consider a $\mathbb{Z}$-SFT X, which, without loss of generality, we may assume to be nearest neighbor. If X is not mixing, then it is either reducible or periodic. If X is reducible, then the factor map which carries each letter to its irreducible component has (zero-dimensional) image which is a non-trivial (there are at least two irreducible components) SFT given by a directed graph whose only cycles are self-loops, and which therefore has zero entropy. If X is irreducible and periodic, then the factor map which carries each letter to its period class has an image which is a non-trivial (and zero-dimensional) finite union of periodic orbits, and is thereby of zero entropy. We have shown that any $\mathbb{Z}$-SFT with ZTCPE is mixing.

Then it is well known that a $\mathbb{Z}$-SFT is mixing if and only if it has the specification property of Bowen. (A full description of the specification property is beyond the scope of this work, but for subshifts it is basically a stronger 'uniform' version of the topological mixing property. For more details on specification, see [**3**].) This implies TCPE since specification is preserved under factors, and every non-trivial dynamical system with specification has positive entropy. □

Proof of Theorem 1.4. We assume some familiarity with the proof of Theorem 1.3 from [**9**], and so only summarize the required changes. Suppose that X is a $\mathbb{Z}^d$-SFT of type t with the properties that every $w \in L(X)$ has positive measure for some $\mu \in \mathcal{M}(X)$ and that there exists N so that, for all $S \subseteq \mathbb{Z}^d$ and $w, w' \in L_S(X)$, there exist $w = w_1, w_2, \ldots, w_N = w'$ so that, for every $i \in [1, N)$, there are homoclinic points in $[w_i]$ and $[w_{i+1}]$. Now consider any surjective factor map $\phi : (X, \sigma_v) \to (Y, S_v)$ with $|Y| > 1$. Since $|Y| > 1$, there exist $y, y' \in Y$ with $d_Y(y, y') = \alpha > 0$. By uniform continuity of ϕ, there exists $\delta > 0$ so that $d_X(x, x') < \delta \Longrightarrow d_Y(y, y') < \alpha/N$. Choose n so that the cylinder set of any $w \in L_{[-n,n]^d}(X)$ has diameter less than δ.

Choose $x \in \phi^{-1}(y)$ and $x' \in \phi^{-1}(y')$, and define $w = x([-n, n]^d)$ and $w' = x'([-n, n]^d)$. Then, by assumption, there exist $w = w_1, w_2, \ldots, w_N = w'$ with the above described properties. Note that each $\phi([w_i])$ has diameter less than α/N, $y \in \phi([w_1])$, $y' \in \phi([w_N])$, and $d(y, y') = \alpha$. This implies that there exists i for which $\phi([w_i])$ and $\phi([w_{i+1}])$ are disjoint closed subsets of Y. From here, the proof proceeds essentially as in [**9**]; we again will only briefly summarize. Firstly, since there exist homoclinic points in $[w_i]$ and $[w_{i+1}]$, there exists a boundary pattern δ of thickness t which can be filled with either w_i or w_{i+1} at the center. By assumption, there exists $\mu \in \mathcal{M}(X)$ with $\mu([\delta]) > 0$ and therefore a point $x \in X$ with a positive frequency of occurrences of δ. Then, since X is an SFT with type t, each occurrence of δ in x can be independently filled with either w_i or w_{i+1} at the center. The ϕ-images of this family of points then visit the disjoint closed sets $\phi([w_i])$ and $\phi([w_{i+1}])$ under S_v in any predesignated way along a set of $v \in \mathbb{Z}^d$ of positive density, which is enough to imply positive entropy of (Y, S_v). □

In particular, several existing proofs in the literature which purported to prove TCPE while, in truth, only verifying ZTCPE can be shown to actually yield TCPE via Theorem 1.4.

COROLLARY 3.2. *The topologically mixing $\mathbb{Z}^2$-SFT defined in* [**5**, §6.3] *has TCPE.*

Proof. It was shown in [**4**] that any two patterns in the example in question are exchangeable, i.e., that the hypotheses of Theorem 1.3 are satisfied with $n = 2$. Therefore, Theorem 1.4 implies TCPE. □

COROLLARY 3.3. *Every non-trivial block gluing $\mathbb{Z}^d$-SFT has TCPE.*

Proof. Theorem 1.5 from [**9**] shows that any two patterns in a block gluing $\mathbb{Z}^d$-SFT are exchangeable, i.e., that the hypotheses of Theorem 1.3 are satisfied with $n = 2$. Therefore, Theorem 1.4 implies TCPE. □

COROLLARY 3.4. *The $\mathbb{Z}^2$-SFT from* [**9**, Examples 1.2 and 1.3] *has TCPE.*

Proof. It is shown in [**9**] that the example in question satisfies the hypotheses of Theorem 1.3 with $n = 3$. Therefore, Theorem 1.4 implies TCPE. □

Our main tool for constructing the examples of Theorems 1.1 and 1.5 is the following 'black box' which, given an input subshift X with some very basic transitivity properties, yields a subshift with TCPE. Although the technique should work in any dimension, for brevity, we here restrict ourselves to $d \leq 2$.

THEOREM 3.5. *For $d \leq 2$ and any alphabet A, there exists an alphabet B and a map f taking any orbit of a point in $A^{\mathbb{Z}^d}$ to a union of orbits of points in $B^{\mathbb{Z}^d}$ with the following properties.*

(1) $O(x) \neq O(x') \Longrightarrow f(O(x)) \cap f(O(x')) = \varnothing$.

(2) *For any $\mathbb{Z}^d$-subshift (respectively, SFT) X, $f(X)$ is a $\mathbb{Z}^d$-subshift (respectively, SFT).*

(3) *If X is a $\mathbb{Z}^d$-subshift with the two properties that:*

  (3a) *every $w \in L(X)$ has positive measure for some $\mu \in \mathcal{M}(X)$; and*

  (3b) *there exists N so that for every $w, w' \in L_{[-n,n]^d}(X)$, there exist patterns $w = w_1, w_2, \ldots, w_N = w'$ in $L_{[-n,n]^d}(X)$ so that for all $i \in [1, N)$, w_i and w_{i+1} coexist in some point of X,*

then $f(X)$ has TCPE.

Proof. Before beginning the proof, we note that, by the pointwise ergodic theorem, (3a) is clearly equivalent to the statement that, for every w, X contains a point with a positive frequency of occurrences of w. Where it is useful, we prove/use this equivalent version without further comment.

We first deal with $d = 1$, which is a significantly easier proof and shows the ideas required for the more difficult $d = 2$ case. Consider any alphabet A, take a symbol $0 \notin A$ and, to any orbit $O(x)$, define $f(O(x))$ to be the set of all points in $(A \cup 0)^{\mathbb{Z}}$ of the form $\ldots a_{-1}0^{n_{-1}}a_0 0^{n_0} a_1 0^{n_1} \ldots$, where $\ldots a_{-1}a_0a_1 \ldots \in O(x)$ and each n_i is 2, 3, or 4.

It is easily checked that $f(O(x))$ is shift-invariant, i.e., a union of orbits. Clearly, if some point $\ldots a_{-1}0^{n_{-1}}a_0 0^{n_0} a_1 0^{n_1} \ldots$ is in $f(O(x)) \cap f(O(x'))$, then $\ldots a_{-1}a_0a_1 \ldots \in O(x) \cap O(x')$, which verifies (1).

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If X is a $\mathbb{Z}$-subshift defined by a set $\mathcal{F}$ of forbidden words, then the reader may check that $f(X)$ is a $\mathbb{Z}$-subshift defined by the forbidden list

$$\mathcal{F}' = \{00000\} \cup \{ab : a, b \in A\} \cup \{a0b : a, b \in A\} \\ \cup \{a_1 0^{n_1} a_2 0^{n_2} \ldots 0^{n_{k-1}} a_k : n_i \in \{2, 3, 4\}, a_1 a_2 \ldots a_k \in \mathcal{F}\}.$$

If X is an SFT, then $\mathcal{F}$ can be chosen to be finite, in which case $\mathcal{F}'$ is also finite, showing that $f(X)$ is an SFT and verifying (2).

It remains only to show (3). We begin with some notation. For any point $y \in f(X)$, there exists x for which $y \in f(x)$ and, by (1), x is uniquely determined up to shifts. We say for any such x that y *extends* x. Similarly, for any finite word $w \in L(f(X))$, the non-zero letters of w (in the same order) form a word $v \in L(X)$, and we say that w *extends* v. Suppose that X satisfies (3a) and (3b) and consider any surjective factor map $\phi : (f(X), \sigma) \to (Y, S)$ with $|Y| > 1$. By considering the words in $L(X)$ 'inducing' arbitrary words $w, w' \in L(f(X))$, it is easily checked that (3a) and (3b) hold for $f(X)$ as well.

Since N does not depend on the words chosen in (3b), we can proceed as in the proof of Theorem 1.4 to find words $w_i, w_{i+1} \in L(f(X))$ of the same length L for which $\phi([w_i])$ and $\phi([w_{i+1}])$ are disjoint closed sets and w_i and w_{i+1} coexist in some word $u \in L(f(X))$. We can assume, without loss of generality, that u begins and ends with non-zero letters by extending it slightly on the left and right, and we define $v \in L(X)$ which induces u. We fix single occurrences of w_i and w_{i+1} within u, assume without loss of generality that w_i appears to the left of w_{i+1} and denote by k the horizontal distance between them.

Since $v \in L(X)$, we may choose $p, s \in L_k(X)$ so that $pvs \in L(X)$. Then we define the words

$$u' = p(1)0^3 p(2)0^3 \ldots 0^3 p(k)0^3 u 0^3 s(1)0^3 s(2)0^3 \ldots 0^3 s(k) \quad \text{and}$$
$$u'' = p(1)0^2 p(2)0^2 \ldots 0^2 p(k)0^2 u 0^4 s(1)0^4 s(2)0^4 \ldots 0^4 s(k).$$

Since both u' and u'' extend $pvs \in L(X)$, u' and u'' are both in $L(f(X))$. They also have the same length. In addition, since u'' is created by reducing the first k gaps of 0s in u' by one and increasing the last k gaps of 0s in u' by one, the occurrence of w_{i+1} in the central u of u' occurs at a location k units further to the left within u''. In other words, there exists j so that $u'([j, j + L - 1]) = w_i$ and $u''([j, j + L - 1]) = w_{i+1}$.

Since $u' \in L(f(X))$, by (3a) there exists $y \in f(X)$ containing a positive frequency of occurrences of u'. The rules defining $f(X)$ should make it clear that any subset of these occurrences of u' can be replaced by u'' to yield a collection of points of $f(X)$. Then, as before, the image under ϕ of this collection yields a collection of points of Y which independently visit the disjoint closed sets $\phi([w_i])$ and $\phi([w_{i+1}])$ under S in any predesignated way along a set of iterates of positive density, proving that $h(Y, S) > 0$ and completing the proof of (3).

Now, we must describe f and prove (1)–(3) for $d = 2$ as well. Many portions of the argument are quite similar, and so we will only comment extensively on the portions which require significantly more details. First, we describe auxiliary $\mathbb{Z}^2$ shifts of finite type X_H and X_V which will help with the definition of f. The alphabet for X_H is $\{0, H\}$, and the SFT rules are as follows.

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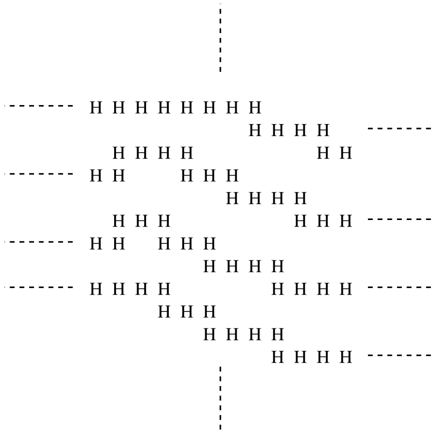

FIGURE 1. Part of a point of X_H.

- Each column consists of H symbols separated by gaps of 0 symbols with lengths 2, 3, or 4.
- Given any H symbol, exactly two of its neighbors (in cardinal directions) are H symbols.
- If two H symbols are diagonally adjacent, they must have exactly one H symbol as a common neighbor.
- Three H symbols may not comprise a vertical line segment.
- If three H symbols comprise a diagonal line segment, then the central of the three must have H symbols to its left and right. (For instance, $\begin{smallmatrix} H & & \\ H & H & H \\ & & H \end{smallmatrix}$ is legal, but $\begin{smallmatrix} H & H & \\ & H & H \\ & & H \end{smallmatrix}$ is not.)

The reader may check these rules make X_H a $\mathbb{Z}^2$-SFT of type 5. Points of X_H consist of biinfinite meandering ribbons of H symbols, which either move up one unit, down one unit, or stay at the same height for each unit moved to the right or left, and which may not 'meander' twice consecutively in the same direction. Informally, this means that any horizontal ribbon has 'slope' with absolute value less than or equal to $1/2$. Every point of X_H contains infinitely many such ribbons, and any pair of closest ribbons may not touch diagonally and are always separated by a vertical gap of 0 symbols of length either 2, 3, or 4 (see Figure 1).

We also define the $\mathbb{Z}^2$-SFT X_V with alphabet $\{0, V\}$ with vertical rather than horizontal ribbons, where legal points are just legal points of X_H, rotated by ninety degrees, with H symbols replaced by V symbols. In particular, vertical ribbons have 'slope' with absolute value at least 2. We will require the following fact about X_H.

CLAIM A1. *Every pattern $w \in L(X_H)$ appears in a point $x \in X_H$ homoclinic to the point $x_0 \in X_H$ consisting of flat horizontal ribbons, equispaced by three units, one of which passes through the origin.*

Proof. Our proof is quite similar to a proof given in [**9**] for a slightly different system, and so for brevity we do not include every technical detail here. It clearly suffices to only treat $w \in L_{[-n,n]^d}(X_H)$ for some n. We proceed in three steps.

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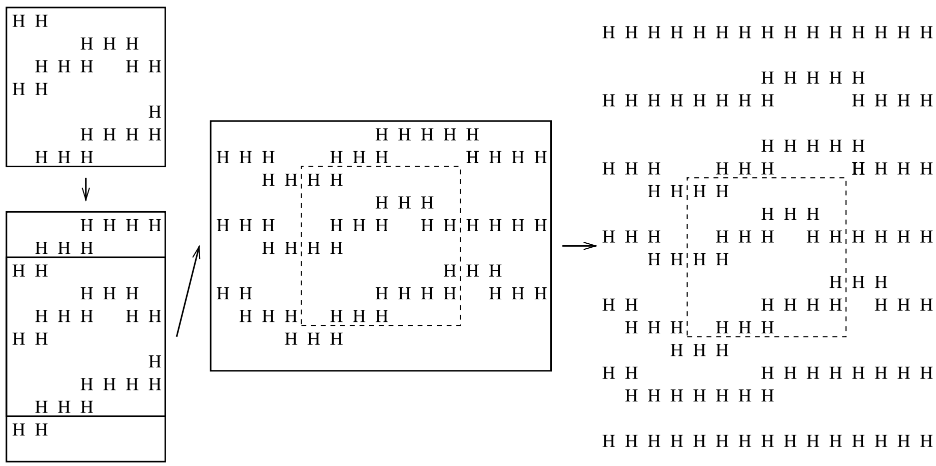

FIGURE 2. The steps of embedding w in a point $x \in X_H$ homoclinic to x_0.

Step 1: Complete each horizontal ribbon segment in w to create w' in which each horizontal ribbon segment touches the infinite vertical lines given by the left and right edges of w.

Step 2: Allow the ribbons in w' to meander on the left and right until they are equispaced with distance 3, each ribbon has the same height at the left and right edge, and those heights are the same as those of ribbons in x_0, i.e., multiples of four.

Step 3: Place additional ribbons above, one at a time, with left and right edges equispaced with distance 3, each of which 'unravels' the leftmost meandering in the ribbon below, until arriving at a completely horizontal ribbon; then continue with infinitely many more completely horizontal ribbons equispaced with distance 3. Perform a similar procedure below. (See Figure 2 for an illustration.)

The resulting point $x \in X_H$ is clearly homoclinic with x_0 and contains w, which completes the proof. □

Every point of X_H must contain infinitely many ribbons; in a point of X_H, we index these by $\mathbb{Z}$, beginning with the zeroth as the first encountered when beginning from the origin and moving straight up, and then proceeding with the positively-indexed ribbons above it and negatively indexed ribbons below it. Similarly, the vertical ribbons of a point of X_V are indexed by starting with the zeroth ribbon as the first encountered by beginning from the origin and moving to the right, with positively-indexed ribbons to the right and negatively indexed ribbons to the left. By the earlier noted restrictions on slopes, any horizontal ribbon and any vertical ribbon must intersect at either a single site or a pair or triple of adjacent (including diagonals) sites (see Figure 3), and so, for any pair of points $x \in X_H$ and $y \in X_V$, we can assign an injection $f_{x,y}$ from $\mathbb{Z}^2$ to itself by defining $f_{x,y}(i, j)$ to be the lexicographically least site within the intersection of the ith horizontal ribbon and jth vertical ribbon.

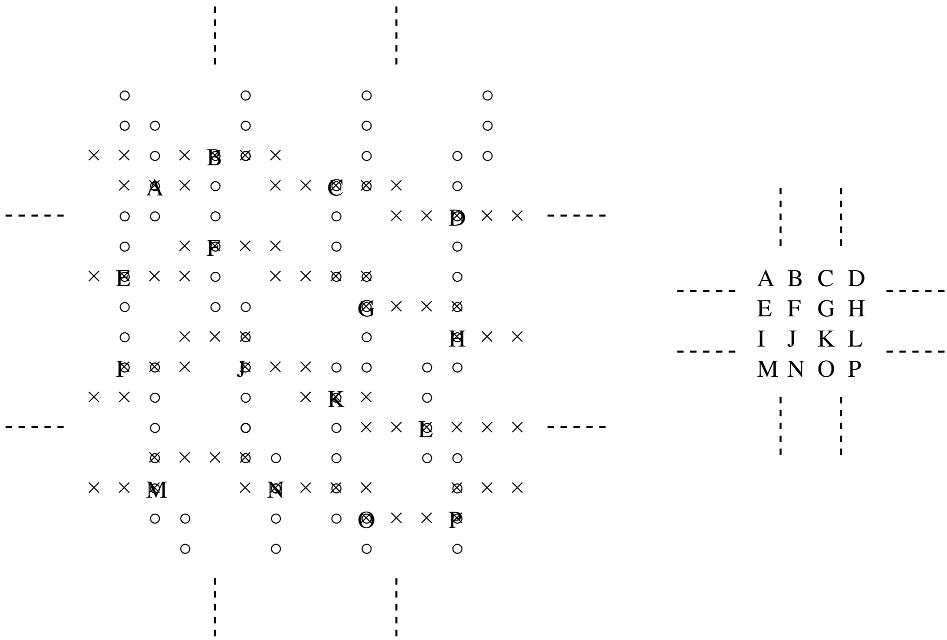

FIGURE 3. A point of $f(X)$ and the pattern in $L(X)$ given by the letters of A at (lexicographically minimal sites within) its ribbon intersections; $\times$ and $\circ$ here represent H and V, respectively.

We are now ready to define f. For any alphabet A and orbit $O(x)$ in $A^{\mathbb{Z}^2}$, choose a symbol $0 \notin A$ and define $f(x)$ to be the collection of all points z on the alphabet $B = \{(0, 0, 0), (0, V, 0), (H, 0, 0)\} \cup (\{(H, V)\} \times (A \sqcup \{0\}))$ with the following properties.

- The first coordinate of z is a point x of X_H.
- The second coordinate of z is a point y of X_V.
- Letters of A may only appear in the third coordinate at the lexicographically least sites within intersections of ribbons from x and y. If we define $t \in A^{\mathbb{Z}^2}$ by taking $t(i, j)$ to be the letter of A in the third layer at $z(f_{x,y}(i, j))$, then $t \in O(x)$.

We claim now that this f has the desired properties. It is easily checked that $f(x)$ is shift-invariant, i.e., a union of orbits. Clearly, if $f(O(x))$ and $f(O(x'))$ share a point z, then t defined as in the third bullet point above must be in $O(x) \cap O(x')$, verifying (1).

An explicit description of a forbidden list inducing $f(X)$ for a $\mathbb{Z}^2$-subshift X would be needlessly long and complicated; instead, we just informally describe the restrictions. Firstly, a finite forbidden list can be used to force the first and second coordinates of any point in $f(X)$ to be in X_H and X_V, respectively. Secondly, since intersections of ribbons are finite sets of adjacent or diagonally adjacent sites which cannot be adjacent or diagonally adjacent to each other, a finite forbidden list can force letters of A to occur on the third coordinate precisely at lexicographically minimal sites within ribbon crossings. Finally, one must only choose a forbidden list $\mathcal{F}$ which induces X and forbid all finite patterns whose first and second coordinates form legal patterns in X_H and X_V, but whose third coordinate contains a pattern from $\mathcal{F}$ on the lexicographically least sites within intersections of ribbons from the first two. Then $f(X)$ is a subshift, and again it should be clear that if X is an SFT, then $\mathcal{F}$ can be chosen to be finite, yielding a finite forbidden list for $f(X)$, implying that $f(X)$ is an SFT and completing the proof of (2).

It remains to prove (3). Choose any X satisfying (3a) and (3b), and again we begin with notation: any point $y \in f(X)$ *extends* $x \in X$ if $y \in f(x)$ and, by (1), x is uniquely determined up to shifts. For finite patterns, the geometry of the ribbons makes a similar definition trickier. For any $w \in L_S(f(X))$, choose $y \in f(X)$ with $w = y(S)$, define a, b to be the first and second coordinates of y, define $T = f_{a,b}^{-1}(S)$ and define $v \in L_T(X)$ by taking $v(i, j)$ to be the third coordinate of $y(f_{a,b}(i, j)) = w(f_{a,b}(i, j))$; we say that w *extends* v. We begin with the following auxiliary claim.

CLAIM A2. *$f(X)$ satisfies (3a) and (3b).*

Proof. Choose any $v \in L_S(f(X))$, which extends $v' \in L_T(X)$. Since X satisfies (3a), there is a point $x \in X$ with positive frequency of occurrences of v'; say, $x(i + T) = v'$ for all $i \in I$ a subset of $\mathbb{Z}^2$ with positive density. Define $v'' \in L_S(X_H)$ and $v''' \in L_S(X_V)$ to be the restrictions of v to its first and second coordinates, respectively. By Claim A1 above, v'' appears within a point $x'' \in X_H$ homoclinic to the point x_0 of equispaced flat horizontal ribbons, and v''' appears within a point of $x''' \in X_V$ homoclinic to the point y_0 of equispaced flat vertical ribbons. Then the pair (v'', v''') appears within a point of $X_H \times X_V$ homoclinic to the doubly periodic point (x_0, y_0) of equispaced horizontal and vertical ribbons. Finally, since $X_H \times X_V$ is an SFT, this means that (v'', v''') appears within some periodic point $(y_1, y_2) \in X_H \times X_V$, which then contains (v'', v''') at a set of sites forming a coset G of $\mathbb{Z}^2$ of finite index. Denote by H the set $f_{y_1,y_2}^{-1}(G)$, which is also a finite index coset of $\mathbb{Z}^2$. Since I has positive density, there exists $t \in \mathbb{Z}^2$ so that $H \cap (t + I)$ also has positive density. Construct a point $z \in f(X)$ by 'superimposing' $\sigma_t x$ in the third coordinate at (lexicographically minimal sites within) intersections of ribbons in (y_1, y_2). Then v appears with positive frequency in z, which proves (3a) for $f(X)$.

Now, consider any two patterns $v \neq w \in L_{[-n,n]^2}(f(X))$. As above, they extend $v' \in L_T(X)$ and $w' \in L_{T'}(X)$. respectively. We may extend v' and w' to patterns $t', u' \in L_{[-n,n]^2}(X)$ which induce $t, u \in L(f(X))$ containing v, w. respectively. Then, by assumption, there exist $t' = w'_1, w'_2, \ldots, w'_N = u'$, all in $L_{[-n,n]^2}(X)$, so that, for every i, w'_i and w'_{i+1} coexist in a point of X. We may, in fact, assume that both occur infinitely many times in the same point of X since, by (3a), any pattern containing both occurs with positive frequency in some point of X. In particular, w'_i and w'_{i+1} appear with arbitrarily large separation in some point of X. Choose any patterns $t = w_1, w_2, \ldots, w_N = u$, all in $L(f(X))$, where each w_i extends w'_i. Each w_i then has shape containing $[-n, n]^2$ since the letters of w'_i are 'stretched out' to be placed within intersections of ribbons. For each i, define w''_i to be the pattern given by the first two coordinates of w_i. By Claim A1, for $i \in [1, N)$, for any large enough $v \in (4\mathbb{Z})^2$, we may place w''_i and w''_{i+1}, separated by v, in some point of $X_H \times X_V$ homoclinic to the doubly periodic point (x_0, y_0).

We can then create a point of X containing w'_i and w'_{i+1} with large enough separation that they may be superimposed over w''_i and w''_{i+1} in such a point of $X_H \times X_V$ as to yield a point of $f(X)$ containing w_i and w_{i+1}. Although the patterns w'_i do not have the proper shape $[-n, n]^2$, each has shape containing $[-n, n]^2$, and we can pass to subpatterns with that shape (yielding v from w_1 and w from w_N, in particular) which still have the desired properties. We have then shown that $f(X)$ satisfies (3b). □

Now, consider any surjective factor map $\phi : (f(X), \sigma_v) \to (Y, S_v)$ with $|Y| > 1$. Again, as was done in the proof of Theorem 1.4, we can find patterns $w_i, w_{i+1} \in L_{[-n,n]^2}(f(X))$ for which $\phi([w_i])$ and $\phi([w_{i+1}])$ are disjoint closed sets and w_i and w_{i+1} coexist in some pattern $u \in L(f(X))$.

We fix single occurrences of w_i and w_{i+1} within u, denote by t the vector pointing from w_i to w_{i+1} in u, and denote by k the ℓ_1-norm $|t_1| + |t_2|$ of t. Our goal is now to extend u to a larger pair of patterns in $L(f(X))$ which contain occurrences of w_i and w_{i+1}, respectively, at the same location. We begin by defining a pair u_1 and u'_1 which contain occurrences of w_i and w_{i+1}, respectively, separated by a vector t_1 with ℓ_1-norm smaller than k.

First, by Claim A1, we can extend u to an entire point $y_1 \in f(X)$ whose first two coordinates (i.e., 'ribbon structure') are homoclinic to $x_0 \times y_0$, the point with equispaced horizontal and vertical ribbons. The (lexicographically minimal sites within) ribbon crossings of y_1 are filled as in some point $x \in X$ extending u.

Then we perturb y_1 to create a new point y'_1 in a way controlled by t. If the first coordinate of t is non-zero, then we choose two vertical ribbons in y_1 identical to corresponding ribbons in y_0, one to the left of u and one to the right, such that between them lie all vertical ribbons in y_1 which are not identical to corresponding ribbons in y_0. These ribbons, and all vertical ribbons between them, are forced to meander a single unit somewhere in the equispaced part of y_1 homoclinic to $x_0 \times y_0$. This has the effect of moving the occurrence of u within y_1 either left or right depending on whether the first coordinate of t is positive or negative, respectively. The resulting point, which we call y'_1, is still in $f(X)$ since we changed no horizontal ribbons, did not change the A letters at crossing points of ribbons and the horizontal separation between vertical ribbons could only have been changed from 3 to 2 or 4, both legal in $f(X)$. If it was the second coordinate of v that was non-zero, then we force horizontal ribbons to meander to move the occurrence of u within y_1, either down or up depending on whether the second coordinate of v is positive or negative, respectively (see Figure 4).

Since we only changed finitely many ribbons of y_1 at finitely many locations to create y'_1, y_1 and y'_1 are homoclinic. Clearly u is still a subpattern of both y_1 and y'_1, and so we may restrict y_1 and y'_1 to some finite box to create patterns u_1 and u'_1 which are equal on their boundaries of thickness 5. By the movement of the copy of u within u_1 to create u'_1, there exists a vector t_1 with ℓ_1 norm less than k and a location i_1 so that $u_1(i_1 + S) = w_i$ and $u'_1(i_1 + t_1 + S) = w_{i+1}$.

We now simply repeat this procedure a finite number of times until arriving at u_k and u'_k in $L(f(X))$ which agree on their boundaries of thickness t and for which there exists i_k with $u_k(i_k + S) = w_i$ and $u'_k(i_k + S) = w_{i+1}$. Then, since $u_k \in L(f(X))$, by (3a) there is a point $y \in f(X)$ containing a positive frequency of occurrences of u_k. Since u_k and u'_k carry the same letters of A at (lexicographically minimal sites within) ribbon intersections and have the same boundaries of thickness 5 (the type of $X_H \times X_V$), the rules of $f(X)$ should make it clear that any subset of these occurrences of u_n in y can be replaced by

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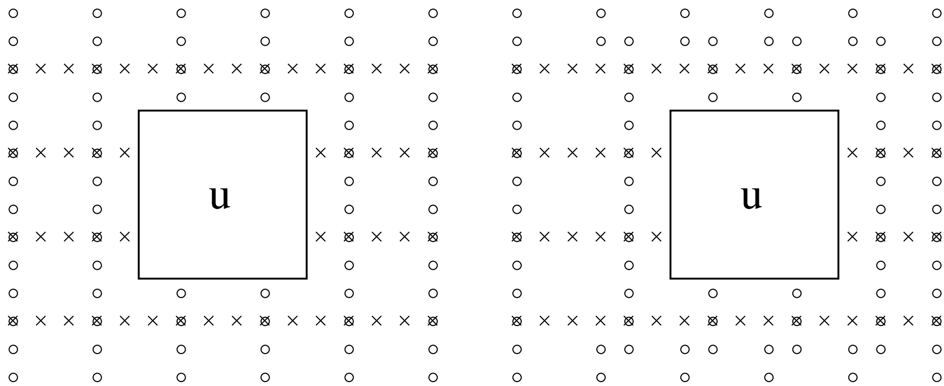

FIGURE 4. Changing y_1 to y_1' when the first coordinate of t is negative; again, $\times$ and $\circ$ represent H and V, respectively.

u_n' to yield a collection of points of $f(X)$. Then, as before, the image under ϕ of this collection yields a collection of points of Y which independently visit the disjoint closed sets $\phi([w_i])$ and $\phi([w_{i+1}])$ under S_v in any predetermined way along a set of positive density, proving that $h(Y, S_v) > 0$ and completing the proof of (3). □

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. For every $\alpha \in [0, 1]$, denote by B_α the $\mathbb{Z}$-subshift consisting of all balanced sequences on $\{0, 1\}$ with slope α. (Recall that $[0, 1]$ denotes the usual interval of real numbers, not the set $\{0, 1\}$.) Then define

$$B = \bigsqcup_{\alpha \in [0,1]} B_\alpha,$$

the $\mathbb{Z}$-subshift consisting of all balanced sequences on $\{0, 1\}$. (This subshift is called the grand Sturmian subshift in [**2**].) Then, by (2) of Theorem 3.5, $f(B)$ is a $\mathbb{Z}$-subshift and, by (1) of Theorem 3.5, it can be written as

$$f(B) = \bigsqcup_{\alpha \in [0,1]} f(B_\alpha).$$

CLAIM B1. *$(f(B), \sigma)$ does not have TCPE.*

Proof. We will show that there is a surjective factor map from $(f(B), \sigma)$ to the non-trivial zero entropy system $([0, 1], \mathrm{id})$. The map π is defined as follows. For every $c \in f(B)$, $\pi(c)$ is defined to be the unique α so that $c \in f(B_\alpha)$, i.e., the slope of any point $b \in B$ inducing c. Since each $f(B_\alpha)$ is shift-invariant, clearly, $\pi(\sigma(c)) = \pi(c) = \mathrm{id}(\pi(c))$ for every $c \in f(B)$. Also, π is clearly surjective. It remains only to show that π is continuous.

Consider any sequence $(c_n) \in f(B)$ for which $c_n \to c$. Define $\alpha_n = \pi(c_n)$ and $\alpha = \pi(c)$, so that $c_n \in f(B_{\alpha_n})$ and $c \in f(B_\alpha)$. It remains to prove that $\alpha_n \to \alpha$. Since $c_n \in f(B_{\alpha_n})$, there exists $b_n \in B_{\alpha_n}$ inducing c_n and, similarly, there exists $b \in B_\alpha$ inducing c. Clearly, since $c_n \to c$, it must be the case that $b_n \to b$ as well.

By Corollary 2.15, the slope of any balanced sequence containing the word $b([1, k])$ is trapped between $(\#(b([1, k]), 1) - 1)/k$ and $(\#(b([1, k]), 1) + 1)/k$. Since, for every

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k, $b_n([1, k])$ eventually agrees with $b([1, k])$, $\alpha_n \to \alpha$, which completes the proof that B does not have TCPE. □

CLAIM B2. *$(f(B), \sigma)$ has ZTCPE.*

Proof. Consider any surjective factor map $\psi : (f(B), \sigma) \to (Y, S)$, where $h(Y, S) = 0$ and Y is a zero-dimensional topological space. We must show that $|Y| = 1$. We first note that all Sturmian shifts are minimal (see [**7**, Ch. 2]) and so satisfy (3a) and (3b) in Theorem 3.5. Therefore, for every irrational α, $f(B_\alpha)$ has TCPE by Theorem 3.5. For every α, since $\psi(f(B_\alpha)) \subset Y$, clearly, $(\psi(f(B_\alpha)), S)$ has zero entropy as well. Therefore, for every $\alpha \notin \mathbb{Q}$, $\psi(f(B_\alpha))$ consists of a single point; call it $g(\alpha)$.

Now we consider the more complicated case of rational α. We first define $B_{\alpha,0} \subset B_\alpha$ to be the orbit $O(\underline{x_\alpha})$ of the lower characteristic sequence for α defined in §2. Then $B_{\alpha,0}$ is a single periodic orbit and so satisfies (3a) and (3b) in Theorem 3.5. Therefore $f(B_{\alpha,0})$ has TCPE and, as above, $\psi(f(B_{\alpha,0}))$ consists of a single point, which we again denote by $g(\alpha)$.

We have now defined g on all of [0, 1], and we claim that it is continuous. First, by Lemma 2.18, for any sequence $\alpha_n \in [0, 1]$ converging to a limit α from above, the corresponding lower characteristic sequences $\underline{x_{\alpha_n}} \in B_{\alpha_n}$ converge to $\underline{x_\alpha} \in B_{\alpha,0}$. Then we can define $\underline{y_{\alpha_n}} \in f(B_{\alpha_n})$ extending $\underline{x_{\alpha_n}}$ and $\underline{y_\alpha} \in f(B_\alpha)$ extending $\underline{x_\alpha}$ so that $\underline{y_{\alpha_n}} \to \underline{y_\alpha}$; just give them all the same pattern of 0 and non-0 symbols (say, by making all gaps of 0s have length 3). Continuity of ψ then means that $\psi(\underline{y_{\alpha_n}}) = g(\alpha_n)$ approaches $\psi(\underline{y_\alpha}) = g(\alpha)$, proving that g is continuous from the right. A similar argument using upper characteristic sequences proves that g is also continuous from the left, and is therefore continuous.

Now choose any $\alpha \in \mathbb{Q}$ and $c_\alpha \in f(B_\alpha)$. Then c_α extends some $b_\alpha \in B_\alpha$ and, by Lemma 2.20, b_α is the limit of a sequence $b_{\alpha_n} \in B_{\alpha_n}$ for some sequence of irrational α_n converging to α. Then we can define $c_{\alpha_n} \in f(B_{\alpha_n})$ so that $c_{\alpha_n} \to c_\alpha$ by using the same structure of 0 and non-0 symbols as c_α for all c_{α_n}. Then, by continuity, $\psi(c_{\alpha_n}) = g(\alpha_n)$ converges to $\psi(c_\alpha)$, which implies that $\psi(c_\alpha) = g(\alpha)$ by continuity of g. We have then shown that ψ collapses every $f(B_\alpha)$ to a single point $g(\alpha)$ for a continuous function g on [0, 1]. Then $g([0, 1]) = Y$ must be connected (as the continuous image of a connected set), and the only connected subsets of Y are singletons. We have therefore shown that g is constant, and so $|Y| = 1$. Since ψ was arbitrary, this shows that $(f(B), \sigma)$ has ZTCPE. □

We have shown that $(f(B), \sigma)$ has ZTCPE but not TCPE, which completes the proof of Theorem 1.1. □

We are finally ready to present the proof of Theorem 1.5. It is quite similar to that of Theorem 1.1, but requires a somewhat technical description of a $\mathbb{Z}^2$-SFT which will play the role of B from the former proof.

Proof of Theorem 1.5. We begin with the description of a $\mathbb{Z}^2$-SFT X in which all rows of points in X have a property similar to being balanced. The alphabet is $A = \{0, 1\}^3$, and the rules are as follows.

- The first coordinate is constant in the vertical direction: i.e., for any $x \in X$ and $(i, j) \in \mathbb{Z}^2$, $(x(i, j))(1) = (x(i, j + 1))(1)$.

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- The second coordinate is constant along the line with slope 1: i.e., for any $x \in X$ and $(i, j) \in \mathbb{Z}^2$, $(x(i, j))(2) = (x(i+1, j+1))(2)$.
- The third coordinate is a running total of the differences between the first two coordinates: i.e., for any $x \in X$ and $(i, j) \in \mathbb{Z}^2$, $(x(i, j))(3) = (x(i-1, j))(3) + (x(i, j))(2) - (x(i, j))(1)$.

It should be clear that X is an SFT. For any $x \in X$, define $a(x), b(x) \in \{0, 1\}^{\mathbb{Z}}$ by $(a(x))(n) = (x(n, 0))(1)$ and $(b(x))(n) = (x(n, 0))(2)$. Note that $a(x)$ and $b(x)$ completely determine the first and second coordinates of x due to the constancy of the first and second coordinates in their respective directions. Also note that given the first and second coordinates in a row of a point of x, the third coordinate along that row is determined up to an additive constant. For any row of a point of x in which the first and second coordinates along a row are not equal sequences, the third coordinate contains a 0 and 1 and therefore is completely forced by the first two coordinates (since no constant can be added to keep the third coordinate using only 0 and 1). This means that $a(x)$ and $b(x)$ uniquely determine x as long as $a(x)$ and $b(x)$ are not shifts of each other. If $a(x) = \sigma^n(b(x))$ for some n, then the nth row of x has first and second coordinates both equal to $a(x)$, meaning that the third coordinate may either be all 0 or all 1 along that row. Similar facts are true even for finite patterns; in any rectangular pattern, the first and second coordinates along a row force the third unless the first and second coordinates are equal words along that row, in which case it is locally allowed for the third coordinate to either be all 0 or all 1. We note that this does not necessarily mean that both choices are globally admissible; it may be the case that a rectangular pattern with equal first and second coordinates along a row can only be extended in such a way that the first and second coordinates along that row are eventually unequal, forcing the entire row.

Since $a(x)$ and $b(x)$ determine x up to some possible constant third coordinates of rows, we wish to understand the structure of which pairs $a(x), b(x)$ may appear for $x \in X$, for which we need a definition.

Definition 3.6. Two sequences $a, b \in \{0, 1\}^{\mathbb{Z}}$ are *jointly balanced* if for every n and every pair of subwords w, w' of a, b of length n, the numbers of 1s in w and w' differ by at most 1: i.e., $|\#(w, 1) - \#(w', 1)| \leq 1$.

CLAIM C1. *There exists $x \in X$ with $a(x) = a$ and $b(x) = b$ if and only if a and b are jointly balanced.*

Proof. $\Longrightarrow$: Consider any $x \in X$ and arbitrary n-letter subwords $w = (a(x))([i, i+n-1])$ of $a(x)$ and $w' = (b(x))([j, j+n-1])$ of $b(x)$. The $(i-j)$th row of x contains $a(x)$ and $\sigma^{i-j}b(x)$ as its first two coordinates and, by the third rule defining X,

$$\begin{aligned}
&(x(i+n-1, i-j))(3) - (x(i-1, i-j))(3)\\
&\quad = \sum_{k=i}^{i+n-1} (x(k, i-j))(2) - (x(k, i-j))(1)\\
&\quad = \sum_{k=i}^{i+n-1} (x(k, i-j))(2) - \sum_{k=i}^{i+n-1} (x(k, i-j))(1)
\end{aligned}$$

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$$= \sum_{\ell=j}^{j+n-1} (b(x))(\ell) - \sum_{k=i}^{i+n-1} (a(x))(k)$$
$$= \#(w', 1) - \#(w, 1).$$

Since $(x(i+n-1, i-j))(3)$ and $(x(i-1, i-j))(3)$ are either 0 or 1, their difference is -1, 0 or 1, and so $a(x)$ and $b(x)$ are jointly balanced.

$\Longleftarrow$: Suppose that a and b are jointly balanced. Then define $x \in (\{0, 1\}^2 \times \{-1, 0, 1\})^{\mathbb{Z}^2}$ as follows. The first two coordinates are given by $(x(i, j))(1) = a(i)$ and $(x(i, j))(2) = b(j-i)$ for every $(i, j) \in \mathbb{Z}^2$. The third coordinate is defined piecewise. Firstly, $(x(0, j))(3) = 0$ for all $j \in \mathbb{Z}$. Secondly, for $i > 0$, $(x(i, j))(3) = \#(b([1-j, i-j]), 1) - \#(a([1, i]), 1)$. Finally, for $i < 0$, $(x(i, j))(3) = -\#(b([i-j, -j]), 1) + \#(a([i, 0]), 1)$. By joint balancedness, the third coordinate clearly takes only the values -1, 0 and 1. The reader may check that x satisfies the rules (the three from the bulleted list given at the beginning of the proof) defining X; but it may not be a point of X since its third coordinate may take the value -1. However, it is not possible for the third coordinate of any row of x to contain 1 and -1; if $(x(i, j))(3) = 1$, $(x(i', j))(3) = -1$ and $i > i'$, then $(x(i, j))(3) - (x(i', j))(3) = \#(b([i'-j, i-j]), 1) - \#(a([i', i]), 1) = 2$, which contradicts joint balancedness of a and b. (The case $i < i'$ is trivially similar.) Therefore, for any j at which the jth row of x contains -1, that row can only contain -1 and 0, and so we simply add 1 to the third coordinate of that entire row. This new point, call it $x' \in (\{0, 1\}^3)^{\mathbb{Z}^2}$, is a point of X with $a(x') = a$ and $b(x') = b$, which completes the proof. □

We will now classify the jointly balanced pairs (a, b).

CLAIM C2. *All jointly balanced pairs (a, b) fall into at least one of the following four categories.*

(1) $\alpha \notin \mathbb{Q}$ *and a and b are* 1*-balanced sequences with slope* α.
(2) $\alpha \in \mathbb{Q}$ *and a and b are* 1*-balanced sequences with slope* α.
(3) $\alpha \in \mathbb{Q}$, *a is* 2*-balanced and jointly balanced with* $\underline{x_\alpha}$, *and* $b \in O(\underline{x_\alpha})$.
(4) $\alpha \in \mathbb{Q}$, *b is* 2*-balanced and jointly balanced with* $\underline{x_\alpha}$, *and* $a \in O(\underline{x_\alpha})$.

Proof. Firstly, if (a, b) are jointly balanced, then clearly both a and b are 2-balanced; any two n-letter subwords of a have number of 1s within 1 of some n-letter subword of b, and so their numbers of 1s may differ by at most 2. This implies, by Lemma 2.14, that a and b both have some uniform frequency of 1s (or slope), which must be the same since a, b are jointly balanced.

Next, suppose that a is 2-balanced but not 1-balanced. Then there exist n and two n-letter subwords v, v' of a with $\#(v, 1)$ and $\#(v', 1)$ differing by 2, say, that they are k and $k+2$, respectively. But then, since a and b are jointly balanced, every n-letter subword of b must have exactly $k+1$ 1s. This implies that b is periodic with period n (this is not necessarily the least period of b though). We claim that b must be 1-balanced as well.

Assume, for a contradiction, that a, b are both 2-balanced but not 1-balanced. Then there are m, n (which we take to be minimal), two n-letter subwords v, v' of a with

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$|\#(v, 1) - \#(v', 1)| = 2$ and two m-letter subwords w, w' of b with $|\#(w, 1) - \#(w', 1)| = 2$. Without loss of generality, assume that $n \leq m$. As above, every n-letter subword of b has the same number of 1s, and so we can remove the first n letters of w, w' to yield shorter words with the same property. However, this contradicts minimality of m. Therefore, our assumption was wrong, and if a is 2-balanced but not 1-balanced, then b is 1-balanced. Then, by Lemma 2.19, since b is periodic, it must be in the orbit $O(\underline{x_\alpha})$ of the lower characteristic sequence $\underline{x_\alpha}$.

We conclude several things from this: if (a, b) is jointly balanced and a is not 1-balanced, then a is 2-balanced, b is periodic, both have the same rational slope α and $b \in O(\underline{x_\alpha})$. Similarly, if b is not 1-balanced, then b is 2-balanced, a is periodic, both have the same rational slope α and $a \in O(\underline{x_\alpha})$. These correspond to categories (3) and (4).

This means that if a and b have irrational slope α, then they must both be 1-balanced; this corresponds to category (1). The only remaining case is that a and b have rational slope α and both are 1-balanced; this corresponds to category (2) and completes the proof. □

We briefly note that, by Corollary 2.16, any two 1-balanced sequences with irrational slope α are jointly balanced, and so all (a, b) in category (1) are jointly balanced. By description, all (a, b) in categories (3) and (4) are clearly jointly balanced, but (a, b) in category (2) need not be jointly balanced; for instance, $a = \ldots 0101001010 \ldots$ and $b = \ldots 1010110101 \ldots$ are both 1-balanced sequences with slope $\alpha = 1/2$, but a contains 00 and b contains 11, and so a and b are not jointly balanced.

For any $\alpha \in [0, 1]$, we write $X_\alpha = \{x \in X \ : \ a(x), b(x) \text{ have slope } \alpha\}$; clearly, $X = \bigsqcup X_\alpha$ and so, by (1) of Theorem 3.5, $f(X) = \bigsqcup f(X_\alpha)$. By (2) of Theorem 3.5, $f(X)$ is a $\mathbb{Z}^2$-SFT. We will show that $f(X)$ has ZTCPE but not TCPE, for which we need to prove several properties about the subshifts $f(X_\alpha)$.

Claim C3. *For every $\alpha \notin \mathbb{Q}$, $f(X_\alpha)$ has TCPE.*

Proof. Choose any $\alpha \notin \mathbb{Q}$. We will show that X_α satisfies (3a) and (3b) from Theorem 3.5, which will imply that $f(X_\alpha)$ has TCPE. Choose any pattern $w \in L_{[-n,n]^2}(X_\alpha)$, and define $x \in X_\alpha$ for which $x([-n, n]^2) = w$. Then $a(x)$ and $b(x)$ are balanced sequences with irrational slope α, and are therefore Sturmian with slope α. We break the proof into two cases.

If $a(x) \neq \sigma^i b(x)$ for all $i \in [-n, n]$, then there exists $N > n$ so that all rows of $w' := x([-N, N] \times [-n, n])$ have unequal first and second coordinates, and therefore the first two coordinates of w' force the third. Then we define the words $u = (a(x))([-N - n, N + n])$ and $v = (b(x))([-N - n, N + n])$; by the rules defining X, for $y \in X$, if the first two coordinates of $y([-N - n, N + n] \times \{0\})$ are u and v, then the first two coordinates of $y([-N, N] \times [-n, n])$ match those of $w' = x([-N, N] \times [-n, n])$ and, since the first and second coordinates of w' force the third, $y([-N, N] \times [-n, n]) = w'$ and $y([-n, n]^2) = w$. We say that u and v force an occurrence of w in any point of X.

If $a(x) = \sigma^i b(x)$ for some $i \in [-n, n]$, then we wish to slightly change one of $a(x)$ and $b(x)$ to another Sturmian sequence with the same slope α so that they are no longer shifts of one another, but without changing $x([-n, n])^2$. Since $a(x)$ and $b(x)$ are not periodic,

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for all $j \neq i$, $a(x) \neq \sigma^j b(x)$. Then we can extend w to $w' := x([-N, N] \times [-n, n])$ for which every row has unequal first and second coordinates, except for the ith row, which must have equal first and second coordinates since $a(x) = \sigma^i b(x)$. Recall that $a(x) = \sigma^i b(x)$ is Sturmian, and so can be written as a version of a lower/upper characteristic sequence with a constant added inside the floor/ceiling function. Choose another Sturmian sequence a' with slope α which is not equal to $a(x)$, but with $a'([-N-n, N+n]) = a([-N-n, N+n])$; this can be accomplished by adding a tiny constant inside the floor/ceiling function defining $a(x)$. Then define k to be the minimal positive integer for which $(a(x))(k) \neq a'(k)$ and j to be the maximal such negative integer; note that $j, k \notin [-N-n, N+n]$. Then it cannot be the case that $(a(x))(j) = (a(x))(k) = 0$; if so, then $a'(j) = a'(k) = 1$ and $|\#((a(x))([j, k]), 1) - \#(a'([j, k]), 1)| = 2$, which would contradict the fact that any two Sturmian sequences with slope α are jointly balanced. Similarly, $(a(x))(j) = (a(x))(k) = 1$ is impossible. Therefore, either $(a(x))(j) = 0$, $(a(x))(k) = 1$, $a'(j) = 1$ and $a'(k) = 0$, or all of these values are the opposite. We assume the former, as the proof of the latter is almost exactly the same. We use Claim C1 to define $x' \in X_\alpha$ with $a(x') = a'$ and $b(x') = b(x)$. Then, since $a'([-N-n, N+n]) = a([-N-n, N+n])$, the first two coordinates of $x'([-N, N] \times [-n, n])$ and $w' = x([-N, N] \times [-n, n])$ are equal, and therefore they have the same third coordinates as well, except possibly in the ith row. In the ith row of x', the first and second coordinates are $a(x') = a'$ and $\sigma^i b(x') = \sigma^i b(x) = a(x)$, respectively. Then $x'(i, j)$ has first and second coordinates $a'(j) = 1$ and $(a(x))(j) = 0$, respectively, implying that the third coordinate of $x'(i, j)$ is 0. Since a' and $a(x)$ agree on (j, k), the third coordinate of the ith row of x' is 0 on that entire interval, including $[-N, N]$. If the third coordinate of w' is 0 on the entire ith row, then we have constructed x' with $x'([-N, N] \times [n, n]) = w'$, and so $x'([-n, n]^2) = w$. If the third coordinate of w' was instead 1 on the ith row, then the reader may check that if we define x' instead by $a(x') = a(x)$ and $b(x') = \sigma^{-i} a'$, then x' would have third coordinate 1 on the ith row and again $x'([-n, n]^2) = w$. In either case, $x'([-n, n]^2) = w$ and $a(x') \neq \sigma^i b(x')$ for all $i \in [-n, n]$, and so the argument of the last paragraph yields a pair of finite words u and v which force an occurrence of w in any point of X.

Now, since Sturmian subshifts satisfy (3a) from Theorem 3.5, there exist points a_u and b_v which contain occurrences of u and v, respectively, beginning at sets of indices A, B with positive density in $\mathbb{Z}$ (in fact every point has this property). By Claim C1, define x'' with $a(x'') = a_u$ and $b(x'') = b_v$. Then, for every pair (i, j) with $i \in A$ and $i + j \in B$, $x''(i, j)$ begins occurrences of u and v on its first two coordinates, yielding an occurrence of w. The set of such (i, j) has positive density in $\mathbb{Z}^2$ since A, B had positive density in $\mathbb{Z}$, and so we have proved that X_α satisfies (3a) from the hypotheses of Theorem 3.5.

Now consider any two patterns $w, w' \in L_{[-n,n]^2}(X_\alpha)$. As above, we can find words u, v, u', v' on $\{0, 1\}^2$ so that a location containing u, v on its first two coordinates forces an occurrence of w, and u', v' similarly force w'. Since Sturmian subshifts are minimal, there exists Sturmian a with slope α containing both u and u'; say, $a(i)$ begins an occurrence of u and $a(j)$ begins an occurrence of u'. Similarly, there exists Sturmian b with slope α containing both v and v'; say, $b(k)$ begins an occurrence of v and $b(\ell)$ begins an occurrence of v'. Since any pair of Sturmian sequences of the same slope are jointly balanced, by Claim C1 we may define $x \in X_\alpha$ with $a(x) = a$ and $b(x) = b$. Then

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$x(i, i-k)$ begins occurrences of u and v in the first two coordinates (forcing an occurrence of w), and $x(j, j-\ell)$ begins occurrences of u' and v' in the first two coordinates (forcing an occurrence of w'). Therefore, x contains both w and w', verifying (3b) from the hypotheses of Theorem 3.5. We then know that $f(X_\alpha)$ has TCPE. □

We now move to the more difficult case of rational α. By Claim C2, we know that X_α can be written as the union of three subshifts, defined as follows.

- $X_{\alpha,1}$ consists of $x \in X_\alpha$ for which $a(x)$ and $b(x)$ are both 1-balanced with slope α.
- $X_{\alpha,2}$ consists of $x \in X_\alpha$ for which $a(x)$ is 2-balanced and jointly balanced with $\underline{x_\alpha}$, and $b(x) \in O(\underline{x_\alpha})$.
- $X_{\alpha,3}$ consists of $x \in X_\alpha$ for which $b(x)$ is 2-balanced and jointly balanced with $\underline{x_\alpha}$, and $a(x) \in O(\underline{x_\alpha})$.

We also define a useful subshift of $X_{\alpha,1}$:

- $X_{\alpha,0} \subset X_{\alpha,1}$ consists of $x \in X_\alpha$ for which $a(x), b(x) \in O(\underline{x_\alpha})$.

Every point $x \in X_{\alpha,0}$ has a strange property: since $a(x)$ and $b(x)$ are in the same periodic orbit, there are infinitely many rows for which the first and second coordinates are the same, and so in each of those rows the third coordinate can be chosen to be all 0 or all 1, each independent of every other such row. This means that many points in $X_{\alpha,0}$ are, in fact not limits of points of X_{α_n} for irrational α_n, in contrast to the proof of Theorem 1.1. We instead use the following fact.

CLAIM C4. *For every $\alpha \in \mathbb{Q}$, $f(X_{\alpha,0})$ has TCPE.*

Proof. Choose any $\alpha \in \mathbb{Q}$ and any pattern w in $L_{[-n,n]^2}(X_{\alpha,0})$. For each row, we extend w on the left and right to make the first and second coordinates different, if possible, arriving at a new pattern $w' \in L_{[-N,N]\times[-n,n]}(X_{\alpha_0})$ with the following property. For each row where the first and second coordinates of w' match, any point $x \in X_{\alpha,0}$ with $x([-N, N] \times [-n, n]) = w'$ must have equal first and second coordinates in that entire row. Define by w'' the pattern given by the first two coordinates of w'. Choose an arbitrary point $x \in X_{\alpha,0}$; w'' clearly appears with positive frequency (in fact, along a subgroup of finite index of $\mathbb{Z}^2$) in x since the first and second coordinates of x come from the single periodic orbit $O(\underline{x_\alpha})$. At each occurrence of w'' in x, the third coordinate is forced to match that of w' except, possibly, in some rows where the first and second coordinates are equal. Recall, though, that this forces that entire row of x to have equal first and second coordinates, and so the third coordinates of any such rows in x can be judiciously changed to create a point $x' \in X_{\alpha,0}$ in which w' itself (and therefore w as well) appears with positive frequency, which proves (3a) from Theorem 3.5.

Now choose any $v, w \in L_{[-n,n]^2}(X_{\alpha,0})$. As above, v and w can be extended to $v', w' \in L_{[-N,N]\times[-n,n]}(X_{\alpha,0})$ such that any rows with equal first and second coordinates within v' (or w') force equal first and second coordinates throughout the corresponding entire biinfinite row of any point of $X_{\alpha,0}$ containing v' (or w'). Then, if we denote by v'' and w'' the patterns given by the first two coordinates of v' and w', respectively, and choose any $x \in X_{\alpha,0}$, then again v'' and w'' appear with positive frequency in x. Therefore, it is possible to choose occurrences of v'' and w'' within x which share no row. Then, as above, for any rows in which v'' and w'' have equal first and second coordinates, the corresponding

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rows of x have equal first and second coordinates. The third coordinate on those rows can then be changed (if necessary) to create a new point x' in which v' and w' (and therefore v and w) both appear, verifying (3b) and implying that $f(X_{\alpha,0})$ has TCPE via Theorem 3.5. □

Now, similarly to the proof of Theorem 1.1, we wish to deal with points of $f(X_{\alpha,1}) \setminus f(X_{\alpha,0})$ by representing them as limits of points from the simpler irrational case.

CLAIM C5. *Every $c \in f(X_{\alpha,1}) \setminus f(X_{\alpha,0})$ can be written as the limit of a sequence $c_n \in f(X_{\alpha_n})$ for some sequence of irrational α_n converging to α.*

Proof. Choose any $c \in f(X_{\alpha,1}) \setminus f(X_{\alpha,0})$, which extends some $x \in X_{\alpha,1} \setminus X_{\alpha,0}$. We will show that x can be written as a limit of $x_n \in X_{\alpha_n}$, as claimed; then clearly we can create $c_n \in f(X_{\alpha_n})$ converging to c by simply copying the 'ribbon structure' of c.

Since $x \in X_{\alpha,1} \setminus X_{\alpha,0}$, at least one of $a(x)$ and $b(x)$ is not periodic (although, in fact, both must be eventually periodic). We break the proof into two cases depending on whether or not $a(x)$ and $b(x)$ are shifts of each other.

Case 1. Suppose that $a(x)$ and $b(x)$ are not shifts of each other. Then for every n, there exists N so that all rows of $x([-N, N] \times [-n, n])$ have unequal first and second coordinates, and thereby the third coordinate of $x([-N, N] \times [-n, n])$ is forced by the first two coordinates on that pattern. Write $u_n := (a(x))([-N-n, N+n])$ and $v_n := (b(x))([-N-n, N+n])$. Since the first and second coordinates of all rows of x are balanced sequences, u_n and v_n are balanced words and so, by Lemma 2.20, there exist Sturmian sequences a_n and b_n for which $a_n([-N-n, N+n]) = u_n$ and $b_n([-N-n, N+n]) = v_n$.

We wish to choose a_n and b_n with the same slope, which requires a more detailed examination of the proof of [**7**, Lemma 2.20]. In that proof, it is shown that, in fact, a balanced word w is a subword of any Sturmian sequence with irrational slope strictly between

$$\alpha'(w) := \max_v \left(\frac{\#(v, 1) - 1}{|v|}\right) \quad \text{and} \quad \alpha''(w) := \min_v \left(\frac{\#(v, 1) + 1}{|v|}\right),$$

where v ranges over all subwords of w. We then need to show that $(\alpha'(u_n), \alpha''(u_n)) \cap (\alpha'(v_n), \alpha''(v_n)) \neq \varnothing$. First, note that since u_n and v_n are subwords of balanced sequences with slope α, by Corollary 2.15 $\alpha'(u_n), \alpha'(v_n) \leq \alpha \leq \alpha''(u_n), \alpha''(v_n)$. The only case in which we are not finished is if either $\alpha'(u_n) = \alpha = \alpha''(v_n)$ or $\alpha'(v_n) = \alpha = \alpha''(u_n)$. For a contradiction, we assume the former; the other case is trivially similar. Since $\alpha'(u_n) = \alpha$, u_n has a subword s with $\#(s, 1) = |s|\alpha + 1$. If we write $\alpha = i/j$ in lowest terms, then $|s|$ must be a multiple of j since $\#(s, 1)$ is an integer. But then we may partition s into j-letter subwords, and one of them, call it s', must have $\#(s', 1) \geq i + 1$. However, a similar argument shows that, since $\alpha''(v_n) = \alpha$, v_n contains a j-letter subword t' with $\#(t', 1) \leq i - 1$, which violates the fact that u_n and v_n are jointly balanced. Therefore, $(\alpha'(u_n), \alpha''(u_n)) \cap (\alpha'(v_n), \alpha''(v_n)) \neq \varnothing$, and so we may choose a_n and b_n to have the same irrational slope α_n.

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Then a_n and b_n are jointly balanced, so, by Claim C1, we may define $x_n \in X_{\alpha_n}$ with $a(x_n) = a_n$ and $b(x_n) = b_n$. Then $x_n([-N, N] \times [-n, n])$ has first two coordinates agreeing with those of $x([-N, N] \times [-n, n])$, and we argued above that their third coordinates must agree as well, meaning that $x_n([-N, N] \times [-n, n]) = x([-N, N] \times [-n, n])$. Therefore, x is the limit of the sequence $x_n \in X_{\alpha_n}$ for a sequence of irrational α_n converging to α.

Case 2. Suppose that $a(x)$ and $b(x)$ are shifts of each other, say, $a(x) = \sigma^k b(x)$. Then both $a(x)$ and $b(x)$ are not periodic (since $x \notin X_{\alpha,0}$), and so, for $m \neq k$, $a(x) \neq \sigma^m b(x)$. Therefore, for every $n > |k|$, we can choose N so that all rows of $x([-N, N] \times [-n, n])$ except the kth have unequal first and second coordinates, and thereby the third coordinate of $x([-n, n] \times [-N, N])$ is forced by the first two coordinates on each row except the kth. Then, again by Lemma 2.20, there exists a Sturmian sequence b_n, with irrational slope α_n, for which $b_n([-N - n, N + n]) = (b(x))([-N - n, N + n])$. By Claim C1, we define $x_n \in X_{\alpha_n}$ with $a(x_n) = \sigma^k b_n$ and $b(x_n) = b_n$. Then $x_n([-N, N] \times [-n, n])$ has first two coordinates agreeing with those of $x([-N, N] \times [-n, n])$ and, by the above argument, the third coordinates are forced to agree as well, except possibly on the kth row. However, the kth row of x_n has equal first and second coordinates, and so the third coordinate can be chosen to be either all 0s or all 1s, whichever matches the third coordinate of the kth row of $x([-N, N] \times [-n, n])$. Then $x_n([-N, N] \times [-n, n]) = x([-N, N] \times [-n, n])$, and so again x is the limit of the sequence $x_n \in X_{\alpha_n}$ for a sequence of irrational α_n converging to α.

Then, the sequence x_n induces a sequence $c_n \in f(X_{\alpha_n})$ with the same 'ribbon structure' as that of c, and clearly $c_n \to c$, which completes the proof. □

Finally we must treat the subshifts $X_{\alpha,2}$ and $X_{\alpha,3}$. We first treat the special cases $\alpha = 0$ and $\alpha = 1$.

CLAIM C6. *For $\alpha \in \{0, 1\}$, $X_{\alpha,2} \cup X_{\alpha,3} \subseteq X_{\alpha,1}$.*

Proof. We treat only $\alpha = 0$, as $\alpha = 1$ is trivially similar. Note that $\underline{x_\alpha} = 0^\infty = \ldots 000 \ldots$, and the only sequences jointly balanced with $\underline{x_\alpha}$ are $\underline{x_\alpha}$ itself and the orbit of $0^\infty 1 0^\infty = \ldots 0001000 \ldots$. All of these sequences are, however, also 1-balanced. Therefore, $X_{0,2} \cup X_{0,3} \subseteq X_{0,1}$ and, similarly, $X_{1,2} \cup X_{1,3} \subseteq X_{1,1}$. □

A key technique used in the proof of Theorem 1.1 was to show that any point of B_α could be written as the limit of a sequence of points from B_{α_n} for some irrational $\alpha_n \to \alpha$. However, the analogous fact here is not true; the set of 1-balanced sequences is closed, so no point in $X_{\alpha,2} \cup X_{\alpha,3}$ (where one of $a(x)$ or $b(x)$ is not 1-balanced) can be written as the limit of a sequence from X_{α_n} for irrational α_n (for which both $a(x)$ and $b(x)$ are 1-balanced). Instead, we will prove the following claim.

CLAIM C7. *For $\alpha \in \mathbb{Q} \setminus \{0, 1\}$, $f(X_{\alpha,2} \cup X_{\alpha,3})$ has TCPE.*

Proof. We need only show that $X_{\alpha,2} \cup X_{\alpha,3}$ satisfies (3a) and (3b) from the hypotheses of Theorem 3.5. For the first part, consider any $w \in L_{[-n,n]^2}(X_{\alpha,2})$. As in the proof of Claim C4, we may extend w to $w' \in L_{[-N,N]\times[-n,n]}(X_{\alpha,2})$ with the following property.

For each row of w' with equal first and second coordinates, the first and second coordinates of the entire corresponding row are forced to agree for any $x \in X_{\alpha,2}$ with $x([-N, N] \times [-n, n]) = w'$. Choose such an $x \in X_{\alpha,2}$. By definition, $b(x) \in O(\underline{x_\alpha})$ and $a(x)$ is jointly balanced with $\underline{x_\alpha}$. Define $u = (a(x))([-N - n, N + n])$ and $v = (b(x))([-N - n, N + n])$; by the rules defining X, any $x' \in X_{\alpha,2}$ with $(a(x'))([-N - n, N + n]) = u$ and $(b(x'))([-N - n, N + n]) = v$ must have first and second coordinates on $[-N, N] \times [-n, n]$ matching those of w'. We now wish to show that there exists $x' \in X_{\alpha,2}$ for which the pair $a(x'), b(x')$ sees the pair u, v with positive frequency. We begin by proving that we can find a periodic sequence a which is jointly balanced with $\underline{x_\alpha}$ and for which $a(-N - n, N + n) = u$. Recall that u is contained in $a(x)$, which is jointly balanced with $\underline{x_\alpha}$. Let us write $\alpha = i/j$ in lowest terms; then, by definition, $\underline{x_\alpha}$ is periodic with period j and every j-letter subword of $\underline{x_\alpha}$ contains exactly i 1s. Let us also fix any $m \in \mathbb{N}$ so that $mj > |u| + j$.

We break the proof into two cases. Firstly, assume that there exists some subword t of $a(x)$ which contains u, has length mj, and contains exactly mi 1s. Secondly, we claim that t^∞ is jointly balanced with $\underline{x_\alpha}$. To see this, consider any subword y of t^∞; clearly, y can be written as $st^k p$ for some p a prefix of t and s a suffix of t. If $|p| + |s| \leq |t|$, then t can be written as pzs, and then $\#(y, 1) = \#(st^k p, 1) = (k + 1)mi - \#(z, 1)$. Then z is a subword of $a(x)$, and we recall that $a(x)$ is jointly balanced with $\underline{x_\alpha}$. If $|z|$ is a multiple of j, then every $|z|$-letter subword z' of $\underline{x_\alpha}$ has $\#(z', 1) = |z|\alpha$. If $|z|$ is not a multiple of j, then, by Corollary 2.16, there exist $|z|$-letter subwords z', z'' of $\underline{x_\alpha}$ with $\#(z', 1) = \lfloor |z|\alpha \rfloor$ and $\#(z'', 1) = \lceil |z|\alpha \rceil$. Either way, we see that $\left|\#(z, 1) - |z|\alpha\right| \leq 1$. Therefore, $\#(y, 1)$ is within 1 of $(k + 1)mi - |z|\alpha = ((k + 1)mj - |z|)\alpha = |y|\alpha$. If instead $|p| + |s| > |t|$, then we can write $s = zs'$ and $t = ps'$. Then $\#(y, 1) = \#(st^k p, 1) = (k + 1)mi + \#(z, 1)$. Again $\left|\#(z, 1) - |z|\alpha\right| \leq 1$, so $\#(y, 1)$ is within 1 of $(k + 1)mi + |z|\alpha = ((k + 1)mj + |z|)\alpha = |y|\alpha$. Either way, we have shown that every subword of t^∞ has number of 1s within 1 of α times its length, and so $a := t^\infty$ is jointly balanced with $\underline{x_\alpha}$. Also, since $a(-N - n, N + n)$ was unchanged from $a(x)$, it is equal to u.

The remaining case is that every subword t of $a(x)$ with length mj which contains u does not have mi 1s. By the fact that $a(x)$ is jointly balanced with $\underline{x_\alpha}$, the only possibilities are that such subwords have either $mi - 1$ or $mi + 1$ 1s. If both numbers occurred, then there would have to be an intermediate subword with length mj containing u with exactly mi 1s, which is a contradiction. Therefore, either $\#(t, 1) = mi - 1$ for every mj-letter subword t of $a(x)$ containing u or $\#(t, 1) = mi + 1$ for all such t; we treat only the former case, as the latter is trivially similar. Consider an mj-letter subword t of $a(x)$ which contains u and ends with a 0 (such a word must exist; $\alpha \leq (j - 1)/j$, and so, since $a(x)$ is jointly balanced with $\underline{x_\alpha}$, the $j + 1$ letters immediately following u in $a(x)$ contain at least one 0). Then t is jointly balanced with $\underline{x_\alpha}$, and we claim that if we change the final letter of t to a 1, yielding a new word t', then t' is jointly balanced with $\underline{x_\alpha}$ as well. To see this, we need only show that every subword of t' has number of 1s within 1 of α times its length. Since we changed only the last letter of t, it suffices to show this for suffixes of t'. For this purpose, choose any suffix s' of t', and denote by s the suffix of t of the same length. Take y to be the subword of $a(x)$ ending with t with length mj. Then, by assumption, $\#(y, 1) = mi - 1$. We write $y = zs$ and, as before, since z is a subword of $a(x)$, which is

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jointly balanced with $\underline{x_\alpha}$, $\#(z, 1) \geq |z|\alpha - 1$. So $\#(s, 1) \leq (mi - 1) - (|z|\alpha - 1) = (mj - |z|)\alpha = |s|\alpha$. Then, again using Corollary 2.15, $\#(s, 1) \in [|s|\alpha - 1, |s|\alpha]$, which implies that $\#(s', 1) \in [|s'|\alpha, |s'|\alpha + 1]$, since it is exactly one greater. But then we have shown that t' is jointly balanced with $\underline{x_\alpha}$, and it is a word with length mj which contains u and has exactly mi 1s, and so, by the previous paragraph, $a := (t')^\infty$ is jointly balanced with $\underline{x_\alpha}$ and has $a(-N - n, N + n) = u$.

In both cases, we have found a which is periodic and jointly balanced with $\underline{x_\alpha}$, with $a(-N - n, N + n) = u$. By Claim C1, define $x' \in X_{\alpha,2}$ with $a(x') = a$ and $b(x') = b(x)$. Then $b(x')$ is periodic and $(b(x')([-N - n, N + n]) = v$, meaning that the pair u, v appears along $a(x'), b(x')$ periodically with period the product of those of $a(x'), b(x')$. Since $b(x')$ is periodic with period j and the first coordinate of x' is constant vertically, every jth row of x', in fact, also contains u, v in its first two coordinates with positive frequency. As explained above, each occurrence of u, v forces a pattern with shape $[-N, N] \times [-n, n]$ which has the same first two coordinates as w' and, for all rows where those coordinates are unequal, the third coordinate is forced and must match that of w' as well. If any rows have equal first and second coordinates, then, as argued above, the entire associated biinfinite rows of x' must have equal first and second coordinates as well, and then the third coordinate can be changed (if necessary) in each row to match that of w' in the relevant row. This yields a point $x'' \in X$ which contains a positive frequency of occurrences of w', and therefore of w. The proof for patterns in $L(X_{\alpha,3})$ is trivially similar, and so we have shown (3a) from Theorem 3.5 for $X_{\alpha,2} \cup X_{\alpha,3}$.

Now we must prove (3b). Again, choose any $w \in L_{[-n,n]^2}(X_{\alpha,2})$, extend to $w' \in L_{[-N,N]\times[-n,n]}(X_{\alpha,2})$, as above, choose $x \in X_{\alpha,2}$ containing w' and define u and v subwords of $a(x)$ and $b(x)$ with the same properties as in the proof of (3a). Specifically, the first and second coordinates of x are equal in any row on which w' has equal first and second coordinates, $b(x) \in O(\underline{x_\alpha})$, $a(x)$ is jointly balanced with $\underline{x_\alpha}$ and any $x' \in X_{\alpha,2}$ with $(a(x'))([-N - n, N + n]) = u$ and $(b(x'))([-N - n, N + n]) = v$ has first and second coordinates on $[-N, N] \times [-n, n]$ matching those of w'. Consider the sequence $a(x)$. It must contain a subword of length j with exactly i 1s somewhere to the right of u; if not, then, as above, every such word would have to have exactly $i - 1$ 1s or every such word would have exactly $i + 1$ 1s, each of which contradicts the fact that $a(x)$ has slope $\alpha = i/j$. Similar reasoning shows that $a(x)$ also contains a subword of length j with exactly i 1s somewhere to the left of u. We then can write the subword of $a(x)$ between these two j-letter words (inclusive) as $ptuvq$, where p and q are length j and $\#(p, 1) = \#(q, 1) = i$. Since $ptuvq$ was a subword of $a(x)$, it is jointly balanced with $\underline{x_\alpha}$. We now claim that $a := p^\infty tuvq^\infty$ is also jointly balanced with $\underline{x_\alpha}$. To see this, choose any subword s of a. We need to show that $\big|\#(s, 1) - |s|\alpha\big| \leq 1$. We can clearly write s as $s = p^k z q^\ell$ for some $k, \ell \geq 0$, where z is a subword of $ptuvq$. As before, since z is a subword of $a(x)$, which is jointly balanced with $\underline{x_\alpha}$, $\big|\#(z, 1) - |z|\alpha\big| \leq 1$. Then $\#(s, 1) = ik + i\ell + \#(z, 1)$, and therefore is within 1 of $ik + i\ell + |z|\alpha = (jk + j\ell + |z|)\alpha = |s|\alpha$. We have then shown that $a = p^\infty tuvq^\infty$ is jointly balanced with $\underline{x_\alpha}$. We note that, since a is jointly balanced with $\underline{x_\alpha}$, the biinfinite sequences p^∞ and q^∞ must be as well. We then claim that these sequences are, in fact, (1-)balanced. For any length m which is not a multiple of j, Corollary 2.16 implies that every m-letter subword of p^∞ or q^∞ has either $\lfloor m\alpha \rfloor$ or

$\lceil m\alpha \rceil$ 1s. For any m a multiple of j, since p^∞ is periodic with period j, every m-letter subword of p^∞ or q^∞ has the same number of 1s (namely, $m\alpha$). Therefore p^∞ and q^∞ are, in fact, balanced and, by Lemma 2.19, must be in $O(\underline{x_\alpha})$ themselves.

By Claim C1, define $x' \in X_{\alpha,2}$ with $a(x') = a$ and $b(x') = b(x)$. Clearly, $(a(x'))([-N-n, N+n])$ is unchanged from $a(x)$ and so equals u, and $(b(x'))([-N-n, N+n]) = v$. As argued before, these occurrences of u, v force the first two coordinates of $x'([-N, N] \times [-n, n])$ to match those of w', and the third coordinate on any rows of x' with equal first and second coordinates can be changed to yield x'' containing w' (and thereby w). Since $a(x'')$ and $b(x'')$ both terminate with a shift of $\underline{x_\alpha}$, due to the periodicity of the first and second coordinates of x'', there must be a subpattern of x'' of shape $[-n, n]^2$, call it w'', whose first and second coordinates on every row are just subwords of $\underline{x_\alpha}$. Call the set of such patterns S_n. We have then shown that any pattern in $L_{[-n,n]^2}(X_{\alpha,2})$ coexists in a point of $X_{\alpha,2}$ with a pattern from the set S_n and, similarly, one can prove that any pattern in $L_{[-n,n]^2}(X_{\alpha,3})$ coexists in a point of $X_{\alpha,3}$ with a pattern from S_n.

Finally, we claim that any two patterns $s, t \in S_n$ coexist in some point of $X_{\alpha,0} \subset X_{\alpha,2} \cup X_{\alpha,3}$. We may, without loss of generality, assume that $n > j$. Define s' and t' to be the patterns given by the first two coordinates of s and t, respectively, and use Claim C1 to define $x \in X_{\alpha,0}$ with $a(x) = b(x) = \underline{x_\alpha}$. Then all possible 'phase shifts' of the first and second coordinates appear in infinitely many rows, and so s' and t' appear infinitely many times in x; in particular, there are occurrences of them which share no row. Since $n > j$, in any rows where the first and second coordinates of those occurrences of s' or t' agree, the corresponding entire rows of x have equal first and second coordinates. Then, in any such rows, the third coordinate of x can be changed (if necessary) to yield $x' \in X_{\alpha,0}$ containing s and t.

We have then proved (3b) from Theorem 3.5 (with $N = 3$) for $X_{\alpha,2} \cup X_{\alpha,3}$; for any two patterns w, w' in $L_{[-n,n]^2}(X_{\alpha,2} \cup X_{\alpha,3})$, each coexists with a pattern from S_n in some point of $X_{\alpha,2} \cup X_{\alpha,3}$, and then the two patterns from S_n coexist in some point of $X_{\alpha,0} \subset X_{\alpha,2} \cup X_{\alpha,3}$. Finally, we apply Theorem 3.5 to see that $f(X_{\alpha,2} \cup X_{\alpha,3})$ has TCPE. □

We are now prepared to prove that $(f(X), \sigma_v)$ has ZTCPE but not TCPE.

CLAIM C8. *$(f(X), \sigma_v)$ does not have TCPE.*

Proof. As in the corresponding proof from Theorem 1.1, we define a surjective factor map from $(f(X), \sigma_v)$ to the non-trivial zero entropy system ([0, 1], id). The map π is defined as follows. For every $c \in f(X)$, $\pi(c)$ is defined to be the unique α so that $c \in f(X_\alpha)$. The arguments that π is shift-invariant and surjective are the same as before. It remains only to show that π is continuous, but this is simple: if $c_n \in f(X)$ approaches c, then each c_n extends some x_n where the sequence $x_n$approaches a limit x, and then $a(x_n)$ approaches $a(x)$. But then $a(x_n)$ and $a(x)$ are 2-balanced sequences, and the proof that the slopes of $a(x_n)$ approach the slope of $a(x)$ is the same as in the one from Theorem 1.1. This implies that $\pi(c_n) \to \pi(c)$, and that π is continuous, meaning that $(f(X), \sigma_v)$ does not have TCPE. □

CLAIM C9. *$(f(X), \sigma_v)$ has ZTCPE.*

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Proof. Again we proceed by showing that every factor map on $(f(X), \sigma_v)$ with image having zero entropy factors through π. Consider any surjective factor map $\psi : (f(X), \sigma) \to (Y, S_v)$, where $h(Y, S_v) = 0$ and Y is a zero-dimensional topological space. We must show that $|Y| = 1$.

For every α, $\psi(f(X_\alpha)) \subset Y$, and so, clearly, $h(\psi(f(X_\alpha)), S_v) = 0$. Therefore, by Claim C3 above, for $\alpha \notin \mathbb{Q}$, $\psi(f(X_\alpha))$ is a single point, which we denote by $g(\alpha)$.

Similarly, for any $\alpha \in \mathbb{Q}$, by Claim C4, $\psi(f(X_{\alpha,0}))$ consists of a single point, which we denote by $g(\alpha)$. For $\alpha \in \mathbb{Q} \setminus \{0, 1\}$, Claim C7 implies that $\psi(f(X_{\alpha,2}) \cup f(X_{\alpha,3}))$ consists of a single point. Since $f(X_{\alpha,0}) \subset f(X_{\alpha,2}) \cup f(X_{\alpha,3})$, this point must also be $g(\alpha)$.

We have now defined g on all of $[0, 1]$, and we claim that it is continuous. This is similar to the corresponding proof from Theorem 1.1, but the third coordinate causes some technical difficulties. Consider any sequence α_n which approaches a limit α from above. Then, by Claim C1, define $x_n \in X_{\alpha_n}$ by taking $a(x_n)$ and $b(x_n)$ to both be the lower characteristic sequence $\underline{x_{\alpha_n}}$; note that if $\alpha_n \in \mathbb{Q}$, then, in addition, $x_n \in X_{\alpha_n,0}$. For any row where the third coordinate is not forced by the first two (including the 0th row), label the third coordinate by all 1s if it is a non-negatively indexed row, and by all 0s if it is a negatively indexed row. From Lemma 2.18, the first two coordinates of x_n clearly approach a limit, and any point x with those first two coordinates would have $a(x) = b(x) = \underline{x_\alpha}$. It remains to show that the third coordinates of x_n actually converge. To see this, choose any row, say the kth, and examine what happens to the third coordinates of x_n along that row as n increases. If $\sigma^k \underline{x_\alpha} \neq \underline{x_\alpha}$, then there is a place in the kth row where the first and second coordinates are unequal for large enough n, meaning that the third coordinate is forced by the first two for large n and therefore must approach a limit since the first two do. If $\sigma^k \underline{x_\alpha} = \underline{x_\alpha}$, then $k\alpha \in \mathbb{Z}$ and $\alpha \in \mathbb{Q}$. Since $\alpha_n > \alpha$, if we denote by i_n, j_n the negative and positive indices at which $\underline{x_{\alpha_n}}$ and $\underline{x_\alpha}$ first differ, then $\underline{x_{\alpha_n}}(i_n) = \underline{x_{\alpha_n}}(j_n) = 1$ and $\underline{x_\alpha}(i_n) = \underline{x_\alpha}(j_n) = 0$. Choose n large enough that $|i_n|, |j_n| > k$. If $k > 0$, then the first and second coordinates of the kth row of x_n agree from $i_n + k + 1$ to $j_n - 1$, and at j_n the first coordinate has a 1 and the second has a 0. This forces the third coordinate at $j_n - 1$ to be a 1 and, since the first and second coordinates agree from $i_n + k + 1$ to $j_n - 1$, the third coordinate is 1 throughout that range. As $n \to \infty$, $|i_n|, |j_n| \to \infty$, and so the third coordinate on the kth row approaches all 1s. Similarly, for $k < 0$, the third coordinate will approach all 0s. Therefore, x_n does, in fact, approach a limit x, where all non-negatively indexed rows with non-forced third coordinate have that coordinate labeled with all 1s, and all similar negatively indexed rows have third coordinate all 0s. This point x is in X_α by definition, and in $X_{\alpha,0}$ if $\alpha \in \mathbb{Q}$. We may create $c_n \in f(X_{\alpha_n})$ extending x_n and $c \in f(X_\alpha)$ (or $f(X_{\alpha,0})$) extending x for which $c_n \to c$; just use the same 'ribbon structure' for all of the points. Then, by continuity of ψ, $\psi(c_n) \to \psi(c)$. However, $\psi(c_n) = g(\alpha_n)$ and $\psi(c) = g(\alpha)$, and so we have shown that $g(\alpha_n) \to g(\alpha)$ and therefore that g is continuous from the right. A similar argument using upper characteristic sequences and third coordinate 0 in the upper half-plane and 1 in the lower half-plane shows that g is continuous from the left, and is therefore continuous.

The only points of $f(X)$ which have not yet been considered are those in $f(X_{\alpha,1}) \setminus f(X_{\alpha,0})$. (It may look as if we have ignored $f(X_{\alpha,2}) \cup f(X_{\alpha,3})$ for $\alpha \in \{0, 1\}$, but by Claim C6 such points are already contained in $X_{\alpha,1}$.) By Claim C5 above, every point

$y \in f(X_{\alpha,1}) \setminus f(X_{\alpha,0})$ can be written as a limit from points of $f(X_{\alpha_n})$ for some sequence of irrationals $\alpha_n \to \alpha$. But then $\psi(y)$ is the limit of $g(\alpha_n)$ and, by continuity of g, this implies that $\psi(y) = g(\alpha)$. We have then shown that, for every α, $\psi(f(X_\alpha)) = g(\alpha)$.

Since g is continuous on [0, 1], $g([0, 1]) = Y$ must be connected (as the continuous image of a connected set), and the only connected subsets of Y are singletons. We have therefore shown that g is constant, and so $|Y| = 1$. Since ψ was arbitrary, (X, σ_v) has ZTCPE. □

We have shown that the $\mathbb{Z}^2$-SFT $(f(X), \sigma_v)$ has ZTCPE but not TCPE, which completes the proof of Theorem 1.5. □

We end by briefly remarking on a comment made in the introduction. By Theorems 1.3 and 1.4, for X as in Theorem 1.5, any two patterns in $L_{[-n,n]^2}(f(X))$ must be chain exchangeable, but the maximum number of required exchanges between two such patterns must increase as $n \to \infty$. This can be seen informally without reference to TCPE or ZTCPE, as follows. As shown in the proof of Theorem 3.5, two patterns $v, w \in L_{[-n,n]^2}(f(X))$ extending $v', w' \in L(X)$ are exchangeable only if v' and w' appear in the same point of X. Such v' and w' are essentially determined by pairs of jointly balanced words. A balanced word of length n generally determines the slope of a balanced sequence containing it within a tolerance which approaches 0 as $n \to \infty$. So two pairs of jointly balanced words of length n may appear in the same pair of jointly balanced sequences only if their frequencies of 1s are close enough. Therefore, if v', w' have frequencies of 1s quite far apart, then the number of exchanges required to get from v to w will increase with n.

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