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Minimal homeomorphisms on low-dimensional tori

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Abstract. We study minimal homeomorphisms (all orbits are dense) of the tori T^n, $n \leq 4$. The linear part of a homeomorphism φ of T^n is the linear mapping L induced by φ on the first homology group of T^n. It follows from the Lefschetz fixed point theorem that 1 is an eigenvalue of L if φ minimal. We show that if φ is minimal and $n \leq 4$, then L is quasi-unipontent, that is, all of the eigenvalues of L are roots of unity and conversely if $L \in GL(n, \mathbb{Z})$ is quasi-unipotent and 1 is an eigenvalue of L, then there exists a C^∞ minimal skew-product diffeomorphism φ of T^n whose linear part is precisely L. We do not know whether these results are true for $n \geq 5$. We give a sufficient condition for a smooth skew-product diffeomorphism of a torus of arbitrary dimension to be smoothly conjugate to an affine transformation.

1. *Introduction*

We first prove the following.

PROPOSITION 1.1. *Let φ be a minimal homeomorphism of a torus T^n and L be the induced mapping on $H_1(T^n, \mathbb{Z})$. Then the minimal polynomial $p(x)$ of L cannot be decomposed over $\mathbb{Q}[x]$, as $p(x) = q(x)r(x)$ where all of the roots of $q(x)$ are roots of unity and $r(x)$ is not constant with no roots in the unit circle.*

Proof. Assume that $p(x)$ has such a decomposition. Then by the Primary Decomposition theorem we have an invariant direct sum decomposition over $\mathbb{Q}$

$$\mathbb{R}^n = E \oplus V, \tag{1}$$

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where the restriction B of L to V is hyperbolic. Now $\Gamma = V \cap \mathbb{Z}^n$ is a discrete cocompact subgroup of V and $M = V/\Gamma$ is homeomorphic to a torus T^k, $k < n$.

Let b be the hyperbolic diffeomorphism of M induced by B and φ be given on the covering by $L + F$, where $F : \mathbb{R}^n \to \mathbb{R}^n$ is continuous and $F(x + \ell) = F(x)$ for all $x \in \mathbb{R}^n$ and $\ell \in \mathbb{Z}^n$. We claim that b is a factor of φ. Consider the continuous surjective mapping $h : T^n \to M$ given on the covering $\mathbb{R}^n$ by

$$h(x) = P(x) + H(x), \tag{2}$$

where $P : \mathbb{R}^n \to V$ is the projection associated to the decomposition (1) and $H : T^n \to V$ is a continuous solution of the cohomological equation

$$BH(x) - H(\varphi(x)) = P(F(x)). \tag{3}$$

Now since B is hyperbolic, a continuous solution of (3) exists see [**1**, Theorem 2.9.2] and since $P \circ L = B \circ P$, then $h \circ \varphi = b \circ h$. Observing that $h \circ \varphi^\ell = b^\ell \circ h$ for all $\ell \in \mathbb{Z}$ and since h is surjective we see that φ cannot be minimal because the lift of a periodic orbit of b defines a closed invariant set for φ which contradictions minimality. □

THEOREM 1. *Any minimal homeomorphism φ of a torus T^n, $n \leq 4$, is quasi-unipotent on the homology and* 1 *is an eigenvalue of its linear part.*

Proof. Minimality of φ and the Lefschetz fixed point theorem shows that 1 is a root of the minimal polynomial $p(x)$ of the linear part L of φ. Thus, $p(x) = (x - 1)s(x)$ where $\deg s(x) \leq 3$, since $\deg p \leq 4$.

If $\deg s(x) = 3$, then $s(x)$ factors over $\mathbb{Z}[x]$ as $(x \pm 1)q(x)$ and by Proposition 1.1 all of the roots of q are roots of unity. If $\deg s(x) \leq 2$ again by Proposition 1.1 all of the roots of $s(x)$ are roots of unity. □

We do not know whether the above theorem is true if $n \geq 5$. There are irreducible polynomials in $\mathbb{Q}[x]$ with roots of absolute value 1 and roots of absolute value different from 1.

Example 1.2. Eisenstein's criterion shows that the polynomial $p(x) = x^4 + 4x^3 - 6x^2 + 4x + 1$ is irreducible over $\mathbb{Q}[x]$ and as

$$p(x) = (x^2 + 2(1 - \sqrt{3})x + 1)(x^2 + 2(1 + \sqrt{3})x + 1)$$

we can see that

$$(\sqrt{3} - 1) \pm i\lambda, \quad \lambda = \sqrt{(1 - (\sqrt{3} - 1)^2)}$$

are roots of absolute value 1 and they are not roots of unity and the other two roots of $p(x)$ have absolute value different from 1.

2. *Minimal skew-product transformations of the torus*

In this section we show that every quasi-unipotent matrix $L \in GL(n, \mathbb{Z})$, $n \leq 4$ with 1 as eigenvalue is the linear part of a smooth minimal skew-product transformation of the torus T^n. Actually the skew-products are of the particular type given in (5). Note that T^{n-p} acts freely on $T^p \times T^{n-p}$ by translation on the second factor. Thus, if a homeomorphism

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ψ of T^n commutes with this action it induces a homeomorphism ψ_0 of the orbit space T^n/T^{n-p} which is homeomorphic to the torus T^p and we say that (T^n, ψ) is a free T^{n-p}-extension of (T^p, ψ_0) (see [**5**]).

THEOREM 2. *Let $L \in GL(n, \mathbb{Z})$, $n \le 4$, be quasi-unipotent having* 1 *as an eigenvalue. Then there exists a minimal smooth skew-product diffeomorphism φ of the torus T^n whose linear part is L.*

Proof. We may assume that using the work of Newman [**3**]

$$L = \begin{pmatrix} A & \mathbf{0} \\ C & B \end{pmatrix}, \tag{4}$$

where $A \in GL(p, \mathbb{Z})$, $A = I + N_1$ and $B \in GL(n - p, \mathbb{Z})$ such that $B^m = I + N_2$, $m \in \mathbb{Z}^+$, where N_1 and N_2 are nilpotent.

Let φ be diffeomorphism of T^n given on the covering $\mathbb{R}^n$ by

$$\varphi(X, Y) = (AX + \alpha, CX + BY + F(X)), \tag{5}$$

where $a(X) = AX + \alpha$ gives an affine minimal transformation of T^p (see [**2**]) and $F : \mathbb{R}^p \to \mathbb{R}^{n-p}$ is a smooth $\mathbb{Z}^p$-periodic function, that is, $F(X + \ell) = F(X)$ for all $\ell \in \mathbb{Z}^p$.

The iterates of φ are given by

$$\varphi^m(X, Y) = (a^m(X), C_m X + B^m Y + \alpha(m) + F_m(X)), \tag{6}$$

where

$$\begin{aligned} C_l X &= \sum_{j=1}^{l} B^{k-j} C A^{j-1}, \\ \alpha(m) &= \bigg(\sum_{j=1}^{m} C_{m-j} \bigg) \alpha, \\ F_m(X) &= \sum_{j=1}^{m} B^{m-j} F(a^{j-1}(X)). \end{aligned} \tag{7}$$

Recall that an action is *simple* if for each character $\gamma \in \hat{T}^{n-p}$ there exist a continuous function $f_\gamma : T^n \to S^1$ and $g : T^p \to S^1$ such that

$$f_\gamma(z, w) = g(z)\gamma(w) \tag{8}$$

for every $z \in T^p$ and $w \in T^{n-p}$.

Let ν be the endomorphism of T^{n-p} given on the covering $\mathbb{R}^{n-p}$ by the nilpotent matrix N_2 and consider the subgroup $H = \ker \nu$ of T^{n-p}. Note that $\psi = \varphi^m$ is invariant under the restriction of the action of T^{n-p} to the subgroup H, that is,

$$\psi(z, w + h) = \psi(z, w) + (0, h) \tag{9}$$

for all $h \in H$.

From now on we assume that the diffeomorphism ψ_0 of T^n/H induced by ψ is either minimal or uniquely ergodic.

Let $\pi : T^n \to T^n/H$ be the projection. By [**5**, Theorem 1] ψ is minimal(uniquely ergodic) if only if the equation

$$\frac{f(\psi_0(\pi(z, w)))}{f(z, w)} = \frac{f_\gamma(\psi(z, w))}{f_\gamma(z, w)} \tag{10}$$

has no continuous (measurable) solution for each character $\gamma \in \hat{H}$, $\gamma \neq 1$. Observe that the condition in (10) does not depend on the choice of the particular function f_γ. Note that the functions $f : T^n/H \to T^1$ are given by the H-invariant functions $f : T^n \to T^1$, that is, $f(z, hw) = f(z, w)$ for all $h \in H$.

If L is unipotent, then there exists a minimal affine diffeomorphism of T^n (see [**2**]). Thus, it suffices to consider L quasi-unipotent but not unipotent.

Suppose that $n = 2$. We may assume [**3**]

$$L = \begin{pmatrix} 1 & 0 \\ s & -1 \end{pmatrix}. \tag{11}$$

Let $\varphi(x, y) = (x + \alpha, sx - y + F(x))$. Thus,

$$\psi(x, y) = \varphi^2(x, y) = (x + 2\alpha, s\alpha + y - F(x) + F(x + \alpha)).$$

It is easy to see that (T^2, ψ) is simple free T^1-extension of the translation of T^1 given on the covering $\mathbb{R}^1$ by

$$\psi_0(x) = x + 2\alpha.$$

We choose a Liouville number α and a sequence $\{k_j\}_{j \in N}$ so that

$$|e^{2\pi i k_j \alpha} + 1| < \frac{4\pi}{(k_j)^j}$$

(see Appendix A) and $F : T^1 \to \mathbb{R}$ given by the Fourier transform

$$\hat{F}(k) = \begin{cases} 0 & \text{if } k \neq \pm k_j, \\ 1 + e^{\pm 2\pi i k_j \alpha} & \text{if } k = \pm k_j. \end{cases} \tag{12}$$

By (10) it suffices to show that the cohomological equation

$$\frac{f(\psi_0(x))}{f(x)} = \frac{f_\gamma(\psi(x, y))}{f_\gamma(x, y)}, \tag{13}$$

where f is given on the covering $\mathbb{R}^1$ by $f(x) = \ell x + G(x)$ and f_γ is given on the covering $\mathbb{R}^2$ by $f_\gamma(x, y) = \ell_\gamma y$, ℓ and ℓ_γ in $\mathbb{Z}$, has no continuous solution $f : T^1 \to \mathbb{R}^1$. This is equivalent to show that the equation

$$\ell\alpha + G(x + 2\alpha) - G(x) = \ell_\gamma s\alpha + \ell_\gamma[F(x + \alpha) - F(x)] \tag{14}$$

has no continuous solution G. If $\ell \neq \ell_\gamma$, then one sees that (14) has no continuous solution G. If $\ell = s\ell_\gamma$, (14) becomes

$$G(x + 2\alpha) - G(x) = \ell_\gamma[F(x + \alpha) - F(x)]$$

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and gives the Fourier coefficient equations

$$\hat{G}(k) = \begin{cases} 0 & \text{if } k \neq \pm k_j, \\ \ell_\gamma & \text{if } k = \pm k_j, \end{cases} \tag{15}$$

which by the choice of F does not give a L^1-solution G.

Suppose now that $n = 3$. There are two possibilities for the characteristic polynomial $p(x)$ of L, $p(x) = (x-1)q(x)$ or $p(x) = (x-1)^2(x+1)$ where $q(1) \neq 0$.

If $p(x) = (x-1)q(x)$, then we assume that [**3**]

$$L = \begin{pmatrix} 1 & 0 \\ C & B \end{pmatrix},$$

where $B \in GL(2, \mathbb{Z})$ is quasi-unipotent and 1 is not an eigenvalue of B. Thus, B is either periodic with period $m = 3, 4, 6$ or B is conjugate to the matrix

$$B = \begin{pmatrix} -1 & 0 \\ s & -1 \end{pmatrix}, \quad s \in \mathbb{Z}.$$

To see this note that the characteristic polynomial of B is $p(x) = x^2 - \operatorname{tr} Bx + \det B$, then if B is quasi-unipotent the only possibilities are $\operatorname{tr} B = -2, -1, 0, 1, 2$ for $\det B = 1$ and $\operatorname{tr} B = 0$ for $\det B = -1$. The $\det B = -1$ case and the $\det B = 1$, $\operatorname{tr} B = 2$ case can be ignored since they have 1 as eigenvalue. The remaining cases for $\det B = 1$ give periods 3, 4, 6, and eigenvalue -1.

By (5) the diffeomorphism φ is given by

$$\varphi(x, Y) = (x + \alpha, Cx + BY + F(x)), \tag{16}$$

where $Y = (y, z)$ and $F : \mathbb{R} \to \mathbb{R}^2$ is a $\mathbb{Z}$-periodic function.

If B is periodic, then by (6)

$$\psi(x, Y) = \varphi^m(x, Y) = (x + m\alpha, Y + \alpha(m) + F_m(x))$$

since $C_m = 0$.

Hence, (T^3, ψ) is a simple free T^2-extension of the translation of T^1 given on the covering $\mathbb{R}^1$ by

$$\psi_0(x) = x + \alpha.$$

Using Appendix A choose a Liouville number α and a sequence of integers $\{k_j\}_{j \in N}$ such that

$$|e^{2\pi i k_j \alpha} - e^{2\pi i/m}| < \frac{4\pi}{(k_j)^j}. \tag{17}$$

By [**5**] ψ is a minimal diffeomorphism if (10) has no continuous solution $f(x) = e^{2\pi i[\ell x + G(x)]}$, where $f_\gamma(x, y, z) = e^{2\pi i \langle \ell_\gamma, (y,z)\rangle}$ or, equivalently, the equation

$$m\ell\alpha + G(x + \alpha) - G(x) = \langle \ell_\gamma, \alpha(m) + F_m(x)\rangle \tag{18}$$

has no continuous solution G. If $m\ell\alpha \neq \langle \ell_\gamma, \alpha(m)\rangle$, then one sees that (18) has no continuous solutions G. If $m\ell\alpha = \langle \ell_\gamma, \alpha(m)\rangle$, equation (18) becomes

$$G(x + m\alpha) - G(x) = \langle \ell_\gamma, F_m(x)\rangle \tag{19}$$

or, in Fourier coefficients,

$$\hat{G}(k) = \langle \ell_\gamma, (e^{2\pi i k\alpha} I - B)^{-1} \hat{F}(k) \rangle. \tag{20}$$

Consider $F : T^1 \to \mathbb{R}^2$ the smooth function given by the Fourier transform

$$\hat{F}(k) = \begin{cases} 0, & k \neq \pm k_j, \\ (e^{2\pi i k_j \alpha} - e^{2\pi i/m}) V, & k = k_j, \\ (e^{-2\pi i k_j \alpha} - e^{-2\pi i/m}) \overline{V}, & k = -k_j, \end{cases} \tag{21}$$

where V is the eigenvector of B associate to the eigenvalue $e^{((2\pi i)/m)}$. Then

$$\hat{G}(k_j) = \langle \ell_\gamma, (e^{2\pi i k_j \alpha} I - B)^{-1} \hat{F}(k_j) \rangle = \langle \ell_\gamma, V \rangle,$$
$$\hat{G}(-k_j) = \langle \ell_\gamma, (e^{-2\pi i k_j \alpha} I - B)^{-1} \hat{F}(-k_j) \rangle = \langle \ell_\gamma, \overline{V} \rangle.$$

As the period of B is $m \geq 3$, then $\langle \ell_\gamma, V \rangle \neq 0$ for all $\ell_\gamma \in \mathbb{Z}^2 - \{0\}$. Thus, by choice of F, (19) has no L^1-solution.

If

$$B = \begin{pmatrix} -1 & 0 \\ s & -1 \end{pmatrix}, \quad s \in \mathbb{Z}$$

then by (16)

$$\varphi(x, Y) = (x + \alpha, Cx + BY + F(x)). \tag{22}$$

Thus,

$$\psi(x, y, z) = \varphi^2(x, y, z) = (x + 2\alpha, C_2 x + \alpha(2) + B^2 Y + F_2(x)).$$

If $s = 0$, (T^3, ψ) is a simple free T^2-extension of the translation (T_1, ψ_0), given in covering $\mathbb{R}^1$ by $\psi_0(x) = x + 2\alpha$.

By [**5**] ψ is a minimal diffeomorphism if (10) has no continuous solution $f(x) = e^{2\pi i[\ell x + G(x)]}$, where $f_\gamma(x, y, z) = e^{2\pi i \langle \ell_\gamma, (y,z) \rangle}$ or, equivalently, the equation

$$m\ell\alpha + G(x + 2\alpha) - G(x) = \langle \ell_\gamma, \alpha(2) + F_2(x) \rangle \tag{23}$$

has no continuous solution G.

If $2\ell\alpha \neq \langle \ell_\gamma, \alpha(2) \rangle$, then one sees that (23) has no continuous solutions G. If $2\ell\alpha = \langle \ell_\gamma, \alpha(2) \rangle$ the equation becomes

$$G(x + 2\alpha) - G(x) = \langle \ell_\gamma, F_2(x) \rangle$$

or, in Fourier coefficients,

$$\hat{G}(k) = \langle \ell_\gamma, (e^{2\pi i k\alpha} + 1)^{-1} \hat{F}(k) \rangle.$$

Consider $F : T^1 \to \mathbb{R}^2$ the smooth function given by the Fourier transform

$$\hat{F}(k) = \begin{cases} \mathbf{0}, & k \neq \pm k_j, \\ (e^{\pm 2\pi i k_j \alpha} + 1) V, & k = \pm k_j. \end{cases}$$

If the vector $V = (a, b) \in \mathbb{R}^2$ and a and b are linearly independent over the rational numbers, then by the choice of F, (23) has no L^1-solution.

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If $s \neq 0$, consider the diffeomorphism

$$\varphi(x, y, z) = (x + \alpha, px - y + F_1(x), qx + sy - z)$$

such that $\varphi_0(x, y) = (x + \alpha, px - y + F_1(x))$ is minimal. Hence, $(T^3, \psi = \varphi^2)$ is a simple free T^1-extension of the diffeomorphism (T^2, ψ_0), given on covering $\mathbb{R}^2$ by $\varphi_0^2(x, y) = \psi_0(x, y) = (x + 2\alpha, y + p\alpha - F_1(x) + F_1(x + \alpha))$. By [**5**] ψ is a minimal diffeomorphism if (10) has no continuous solution $f(x, y) = e^{2\pi i[\langle \ell,(x,y)\rangle + G(x,y)]}$, where $f_\gamma(x, y, z) = e^{2\pi i \ell_\gamma, z}$ or, equivalently, the equation

$$\begin{aligned}&\langle \ell, (2\alpha, p\alpha + F_1(x + \alpha) - F_1(x))\rangle + G(\psi_0(x, y)) - G(x, y)\\ &\quad = \ell_\gamma[spx - 2sy + q\alpha + sF_1(x)]\end{aligned}$$

has no continuous solution G. This is because the right-hand side of the above equation is not a periodic function for ℓ_γ since $s \neq 0$.

If $p(x) = (x - 1)^2(x + 1)$, then we assume that [**3**]

$$L = \begin{pmatrix} A & 0 \\ C & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ p & 1 & 0 \\ q & s & -1 \end{pmatrix}$$

by (5) the diffeomorphism φ is given by

$$\varphi(X, z) = (AX + \delta, CX - z + F(x)) = (a(X), CX - z + F(x)),$$

where $X = (x, y)$ and $\delta = (\alpha, \beta)$ and F a periodic smooth function, then

$$\psi(X, z) = \varphi^2(X, z) = (a^2(X), C_2X + z + \alpha(2) + F_2(x)).$$

Hence, (T^3, ψ) is a simple free T^1-extension of the minimal affine transformation (T^2, ψ_0), given on covering $\mathbb{R}^2$ by $\psi_0(X) = a^2(X)$. Again, by [**5**] ψ is a minimal diffeomorphism if (10) has no continuous solution $f(X) = e^{2\pi i[\langle \ell, X\rangle + G(X)]}$, where $f_\gamma(X, z) = e^{2\pi i \ell_\gamma z}$ or, equivalently, the equation

$$\langle \ell, a^2(X) - X\rangle + G(\psi_0(X)) - G(X) = \ell_\gamma[C_2X + \alpha(2) + F_2(x)] \tag{24}$$

has no continuous solution G. If $\langle \ell, a^2(X) - X\rangle \neq \ell_\gamma[C_2X + \alpha(2)]$, then one sees that (24) has no continuous solutions G.

If $\langle \ell, a^2(X) - X\rangle = \ell_\gamma[C_2X + \alpha(2)]$, equation (24) becomes

$$G(\psi_0(X)) - G(X) = \ell_\gamma F_2(x). \tag{25}$$

If $F : T^1 \to \mathbb{R}$ is given in Fourier coefficients as in (12), then (25) has no continuous solution G.

Finally suppose that $n = 4$. There are three possibilities for the characteristic polynomial $p(x)$ of L, $p(x) = (x - 1)q_1(x)$, $p(x) = (x - 1)^2q_2(x)$ or $p(x) = (x - 1)^3$ $(x + 1)$, where q_1 and q_2 are irreducible over $\mathbb{Q}(x)$.

If $p(x) = (x - 1)q_1(x)$, then we assume that [**3**]

$$L = \begin{pmatrix} 1 & 0 \\ C & B \end{pmatrix}$$

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where $B \in GL(3, \mathbb{Z})$ is quasi-unipotent and 1 is not an eigenvalue of B. Thus,

$$B = \begin{pmatrix} -1 & 0 \\ C_0 & B_0 \end{pmatrix}$$

where $B_0 \in GL(2, \mathbb{Z})$ is quasi-unipotent by (5) the diffeomorphism φ is given by

$$\varphi(x, Y) = (x + \alpha, Cx + BY + F(x))) \tag{26}$$

where $Y = (y, z, w)$ and $F : \mathbb{R} \to \mathbb{R}^3$, $F(x) = (F_1(x), F_2(x), F_3(x))$ is $\mathbb{Z}$-periodic smooth function.

If -1 is not an eigenvalue of B_0 then B_0 is periodic with period $m = 3$, 4 and 6, thus $B^{2m} = I$, then by (6) we have

$$\psi(x, Y) = \varphi^{2m}(x, y, Y) = (x + 2m\alpha, C_{2m}x + \alpha(2m) + Y + F_{2m}(x)). \tag{27}$$

Choose by (Appendix A) a Liouville number α and two sequences of integers $\{k_j\}_{j\in N}$ and $\{k'_j\}_{j\in N}$ such that

$$|e^{2\pi i k_j \alpha} - e^{2\pi i/m}| < \frac{4\pi}{(k_j)^j} \quad \text{and} \quad |e^{2\pi i k'_j \alpha} + 1| < \frac{4\pi}{(k'_j)^j}.$$

Again, by [**3**]. It is easy to see that (T^4, ψ) is simple free T^3-extension of the translation of T^1 given on the covering $\mathbb{R}^1$ by

$$\psi_0(x) = x + 2m\alpha.$$

Now ψ is a minimal diffeomorphism if (10) has no continuous solution $f(x) = e^{2\pi i[\ell x + G(x)]}$, where $f_\gamma(x, Y) = e^{2\pi i\langle \ell_\gamma, (y, Y)\rangle}$ or, equivalently, the following equation has no continuous solution G:

$$2m\ell\alpha + G(x + 2m\alpha) - G(x) = \langle \ell_\gamma, C_{2m}x + \alpha(2m) + F_{2m}(x)\rangle. \tag{28}$$

If $2m\ell\alpha \neq \langle \ell_\gamma, \alpha(2m) + C_{2m}x\rangle$, then one sees that (28) has no continuous solutions G. If $2m\ell\alpha = \langle \ell_\gamma, \alpha(2m) + C_{2m}x\rangle$ becomes

$$G(x + 2m\alpha) - G(x) = \langle \ell_\gamma, F_{2m}(x)\rangle \tag{29}$$

or, in Fourier coefficients,

$$\hat{G}(k) = \langle \ell_\gamma, (e^{2\pi i k\alpha} I - B)^{-1}\hat{F}(k)\rangle.$$

Consider a smooth function $F_1 : \mathbb{R}^1 \to \mathbb{R}^1$ given by the Fourier transform

$$\hat{F}_1(k) = \begin{cases} 0, & k \neq \pm k'_j, \\ (e^{\pm 2\pi i k'_j \alpha} + 1), & k = \pm k'_j, \end{cases}$$

as in (12) and the smooth function $F : \mathbb{R}^1 \to \mathbb{R}^2$, $F(x) = (F_2(x), F_3(x))$ given by the Fourier transform

$$\hat{F}(k) = \begin{cases} \mathbf{0}, & k \neq \pm k_j, \\ (e^{2\pi i k_j \alpha} - e^{2\pi i/m})V, & k = k_j, \\ (e^{-2\pi i k_j \alpha} - e^{-2\pi i/m})\overline{V}, & k = -k_j, \end{cases}$$

as in (21), then (29) has no continuous solution G.

Now if

$$B_0 = \begin{pmatrix} -1 & 0 \\ s & -1 \end{pmatrix},$$

then

$$L = \begin{pmatrix} A & \mathbf{0} \\ C_0 & B_0 \end{pmatrix},$$

where

$$A = \begin{pmatrix} 1 & 0 \\ p & -1 \end{pmatrix}$$

and the diffeomorphism given in (26) we can written as

$$\varphi(X, Y) = (AX + F, C_0X + B_0Y + H(x)),$$

where $X = (x, y)$, $Y = (z, w)$, $F(x) = (\alpha, F_1(x))$ and $H(x) = (F_2(x), F_3(x))$ are smooth $\mathbb{Z}$-periodic functions. These functions are determined by the Fourier transforms

$$\hat{F}_1(k) = \begin{cases} 0 & \text{if } k \neq \pm k_j, \\ 1 + e^{\pm 2\pi i k_j \alpha} & \text{if } k = \pm k_j, \end{cases} \tag{30}$$

and

$$\hat{H}(k) = \begin{cases} 0 & \text{if } k \neq \pm k_j, \\ (1 + e^{\pm 2\pi i k_j \alpha})V & \text{if } k = \pm k_j, \end{cases} \tag{31}$$

where $V = (a, b)$. Let us consider the diffeomorphism

$$\psi(X, Y) = \varphi^2(X, Y) = (X + AF(x) + F(x + \alpha),$$
$$C_0(2)X + B_0^2Y + C_0F(x) + B_0H(x) + H(x + \alpha))$$

and suppose that $\varphi_0(x, y) = (x + \alpha, px - y + F_1(x))$ is minimal. Hence, $(T^4, \psi = \varphi^2)$ is a simple free T^2-extension of the minimal diffeomorphism (T^2, ψ_0), given on covering $\mathbb{R}^2$ by $\varphi_0^2(x, y) = \psi_0(x, y) = (x + 2\alpha, y + p\alpha - F_1(x) + F_1(x + \alpha))$. By **[5]** ψ is a minimal diffeomorphism if (10) has no continuous solution $f(X) = e^{2\pi i[\langle \ell, X\rangle + G(X)]}$, where $f_\gamma(X, Y) = e^{2\pi i \langle \ell_\gamma, Y\rangle}$. This is equivalent to the equation

$$\langle \ell, AF(x + \alpha) + F(x)\rangle + G(\psi_0(X)) - G(X)$$
$$= \langle \ell_\gamma, C_0(2)X + [B_0^2 - I]Y + C_0F(x) + B_0H(x) + H(x + \alpha)\rangle. \tag{32}$$

If $\langle \ell_\gamma, C_0(2)X + [B_0^2 - I]Y\rangle \neq 0$, then one sees that (32) has no continuous solution G. If $\langle \ell_\gamma, C_0(2)X + [B_0^2 - I]Y\rangle = 0$, them (32) becomes

$$\langle \ell, AF(x) + F(x + \alpha)\rangle + G(\psi_0(X)) - G(X)$$
$$= \langle \ell_\gamma, C_0F(x) + B_0H(x) + H(x + \alpha)\rangle. \tag{33}$$

Integrating (33) along the fibres of the bundle $(x, y) \to y$ we obtain

$$\langle \ell, AF(x) + F(x + \alpha)\rangle + g(x + \alpha) - g(x)$$
$$= \langle \ell_\gamma, C_0F(x) + B_0H(x) + H(x + \alpha)\rangle, \tag{34}$$

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where $g(x) = \int_{T^1} G(x, y)\, dy$, note that $g(x+\alpha) = \int_{T^1} G(\psi(x, y))\, dy$. Hence, (34) becomes in Fourier coefficients $k \neq 0$

$$\begin{aligned}&\langle \ell, (0, \hat{F}_1(k))(e^{2\pi i k\alpha} - 1)\rangle + \hat{g}(k)(e^{2\pi i k 2\alpha} - 1)\\ &\quad = \langle \ell_\gamma, C_0(0, \hat{F}_1(k)) + [B_0 + e^{2\pi i k\alpha} I]\hat{H}(k)\rangle.\end{aligned} \tag{35}$$

A simple computation using (30) and (31) gives

$$\ell_1 + \hat{g}(k) = \langle \ell_\gamma, C_0(e^{2\pi i k\alpha} - 1)^{-1} e_2 + V + a(e^{2\pi i k\alpha} - 1)^{-1} s e_2\rangle. \tag{36}$$

Hence, (34) has no continuous solution g by the Riemann Lebesgue lemma. This implies necessarily that (33) has no continuous solution G. This finishes this case.

Now let $p(x) = (x-1)^2 q_2(x)$. We may assume that [**3**]

$$L = \begin{pmatrix} A & 0 \\ C & B \end{pmatrix},$$

where 1 is not an eigenvalue of B,

$$A = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \quad \text{and} \quad C \in M(2, \mathbb{Z}).$$

The diffeomorphism φ is given by

$$\varphi(X, Y) = (AX + \delta, CX + BY + F(x)), \tag{37}$$

where $X = (x, y)$, $Y = (z, w)$, $F(x) = (F_1(x), F_2(x))$ and $\delta = (\alpha, \beta)$. Denote by $a(X) = AX + \delta$ the minimal affine transformation.

Suppose that B is periodic with period m. Then

$$\begin{aligned}\varphi^m(X, Y) = (a^m(X), Y + C(m)X + \sum_{j=1}^{m} C(m-j-1)(\alpha, \beta)\\ + \sum_{j=1}^{m} B^{m-j-1} F(x + (j-1)\alpha)).\end{aligned}$$

Hence, $(T^4, \varphi^m = \psi)$ is a simple free T^2-extension of the diffeomorphism $(T^2, a^m = \psi_0)$, given on covering $\mathbb{R}^2$ by $\psi_0(X) = a^m(X)$. By [**5**] ψ is a minimal diffeomorphism if (10) has no continuous solution $f(X) = e^{2\pi i[\langle \ell, X\rangle + G(X)]}$, where $f_\gamma(X, Y) = e^{2\pi i\langle \ell_\gamma, Y\rangle}$ or, equivalently, the equation

$$\begin{aligned}&\langle \ell, a^m(X) - X\rangle + G(a^m(X)) - G(X)\\ &= \left\langle \ell_\gamma, C(m)X + \sum_{j=1}^{m} C(m-j-1)\delta + \sum_{j=1}^{m} B^{m-j-1} F(x + (j-1)\alpha)\right\rangle\end{aligned} \tag{38}$$

has no continuous solution G. If

$$\langle \ell, a^m(X) - X\rangle \neq \left\langle \ell_\gamma, C(m)X + \sum_{j=1}^{m} C(m-j-1)\delta\right\rangle,$$

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then one sees that (38) has no continuous solution G. If

$$\langle \ell, a^m(X) - X \rangle = \left\langle \ell_\gamma, C(m)X + \sum_{j=1}^{m} C(m-j-1)\delta \right\rangle,$$

then (38) becomes

$$G(a^m(X)) - G(X) = \left\langle \ell_\gamma, \sum_{j=1}^{m} B^{m-j-1} F(x + (j-1)\alpha) \right\rangle \tag{39}$$

or, in Fourier coefficients,

$$\hat{G}(k_1 + nk_2, k_2) e^{2\pi i \langle a^m(0,0), (k_1,k_2) \rangle} - \hat{G}(k_1, k_2) = \left\langle \ell_\gamma, \sum_{j=1}^{m} B^{m-j-1} \hat{F}(k_1) e^{2\pi i k \alpha} \right\rangle. \tag{40}$$

If $k_2 \neq 0$, then by the Riemann Lebesgue lemma $\hat{G}(k_1, k_2) = 0$ then (40) becomes

$$\hat{G}(k_1, 0) = \langle \ell_\gamma, (e^{2\pi i k \alpha} I - B)^{-1} \hat{F}(k_1) \rangle \tag{41}$$

take F as in (31) then (38) has no continuous solution G.

If B is not periodic, then

$$B = \begin{pmatrix} -1 & \mathbf{0} \\ s & -1 \end{pmatrix}$$

with $s \neq 0$. Consider the diffeomorphism as in (37)

$$\begin{aligned} \varphi(x, y) &= (AX + \delta, CX + BY + F(x)) \\ &= (AX + \delta, C_1 X - z + F_1(x), C_2 X + sz - w + F_2(x)) \end{aligned} \tag{42}$$

such that $\varphi_0(x, y, z) = (AX + \delta, C_1 X - z + F_1(x))$ is minimal. Hence, $(T^4, \psi = \varphi^2)$ is a simple free T^1-extension of the minimal diffeomorphism (T^3, ψ_0), given on covering $\mathbb{R}^3$ by $\psi = \varphi_0^2$. By [**5**] ψ is a minimal diffeomorphism if (10) has no continuous solution $f(X) = e^{2\pi i [\langle \ell, X \rangle] + G(X)}$, where $f_\gamma(X, Y) = e^{2\pi i \ell_\gamma w}$ this equation is equivalent to the equation

$$\begin{aligned} &\langle \ell, (a^2(X) - X, C_1(2)X + \delta(2) - F_1(x) + F_1(x + \alpha)\rangle + G(\psi_0(X)) - G(X) \\ &\quad = \ell_\gamma [C_2(2)X - 2sz + sF_1(x) - F_2(x) + F_2(x + \alpha)]. \end{aligned} \tag{43}$$

The above equation has no continuous solution G because $s \neq 0$.

If $p(x) = (x-1)^3(x+1)$, then we assume that [**3**]

$$L = \begin{pmatrix} A & 0 \\ C & -1 \end{pmatrix},$$

where $A \in GL(3, \mathbb{Z})$ is unipotent, that is, $A = I + N$. Consider the diffeomorphism φ given by

$$\varphi(X, w) = (AX + \alpha, CX + -w + F(x)), \tag{44}$$

where $X = (x, y, z)$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and $F : \mathbb{R} \to \mathbb{R}$ is $\mathbb{Z}$-periodic smooth function. Then

$$\begin{aligned} \psi(X, w) = \varphi^2(X, w) = (&A^2 X + A\alpha + \alpha, \\ &[CA - C]X + w + C\alpha - F(x) + F(x + \alpha_1)). \end{aligned} \tag{45}$$

Thus, (T^4, ψ) is a simple free T^1-extension of the minimal affine transformation (T^3, ψ_0), given on covering $\mathbb{R}^3$ by $\psi_0(X) = A^2X + A\alpha + \alpha$. Now by [**5**] ψ is minimal diffeomorphism if (10) has no continuous solution $f(X) = e^{2\pi i[\langle \ell, X\rangle + G(X)]}$, where $f_\gamma(X, w) = e^{2\pi i \ell_\gamma w}$ or, equivalently, the following equation has no continuous solution G:

$$\begin{aligned}\langle \ell, [A^2 - I]X + A\alpha + \alpha\rangle + G(\psi_0(X)) - G(X)\\ = \ell_\gamma[[CA - C]X + C\alpha - F(x) + F(x + \alpha_1)].\end{aligned} \tag{46}$$

If $\langle \ell, [A^2 - I]X + A\alpha \neq \ell_\gamma[[CA - C]X + C\alpha]$, then one sees that (46) has no continuous solution G. If $\langle \ell, [A^2 - I]X + A\alpha = \ell_\gamma[[CA - C]X + C\alpha]$, then (46) becomes

$$G(\psi_0(X)) - G(X) = \ell_\gamma[-F(x) + F(x + \alpha_1)]. \tag{47}$$

Now if α_1 is a Liouville number and F is a smooth function given by Fourier coefficients as in (12), then (47) has no continuous solution G. □

We now give a sufficient condition for a smooth skew-product transformation of a torus to be smoothly conjugate to an affine transformation. We present in the Appendix A the definition of Diophantine vectors. We recall that we are restricted to smooth skew-product diffeomorphism φ of the torus $T^n = T^p \times T^{n-p}$ given on the covering $\mathbb{R}^n$ by

$$\varphi(X, Y) = (X + \alpha, CX + BY + F(X)), \tag{48}$$

where $X \in \mathbb{R}^p$, $Y \in \mathbb{R}^{n-p}$ and $F : \mathbb{R}^p \longrightarrow \mathbb{R}^{n-p}$ is a smooth $\mathbb{Z}^p$-periodic function. We call $\alpha \in \mathbb{R}^p$ the translation vector of φ. To φ there naturally corresponds an affine transformation φ_0 given on the covering $\mathbb{R}^n$ by

$$\varphi_0(X, Y) = (X + \alpha, CX + BY + \beta_1),$$

where $\beta_1 = \mathrm{Proj}(\beta)$, $\mathrm{Proj} : \mathbb{R}^n \longrightarrow \ker(I - B)^t$ and $\beta = \int_{T^p} F\, d\mu$, μ being the Haar measure of T^p.

THEOREM 3. *Every smooth skew-product diffeomorphism φ of T^n of type (48) quasi-unipotent on homology whose translation vector is Diophantine is smoothly conjugate to its corresponding affine transformation. Moreover, if φ is minimal, then* 1 *is only eigenvalue of its linear part.*

Proof. Since the translation vector α is Diophantine and B is quasi-unipotent then the cohomological equation

$$F(x) = \beta_1 + G(x + \alpha) - BG(x)$$

has a smooth solution G (see [**4**, Theorem 2.6]). Thus, the diffeomorphism $h : T^n \longrightarrow T^n$ given on the covering $\mathbb{R}^n$ by

$$h(X, Y) = (x, Y + G(X))$$

conjugate φ with the affine transformation φ_0. If φ is minimal, then so is φ_0 and 1 is the only eigenvalue of B (see [**2**]). □

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A. *Appendix*

In this appendix we recall the definition of Diophantine and Liouville numbers and we show that for every nth root of unity there exists a 'fast approximation' by iterations of a Liouville rotation of the circle.

Definition A.1. Given $C > 0$ and $r \geq 0$, we say that $\alpha \in \mathbb{R} - \mathbb{Q}$ verifies a Diophantine condition of exponent r and constant C if and only if for all $q \in \mathbb{Z}$, one has $\|q\alpha\| \geq C|q|^{-1-r}$.

We have $\|x\| = \inf\{|x - p| \mid p \in \mathbb{Z}\}$. Note that the inequality

$$4s \leq |e^{2\pi i s} - 1| \leq 2\pi s \tag{49}$$

with $s \in [0, 1]$ implies that the orbit of 1 by the rotation R_α is a bad approximation of number 1 in the sense that

$$|e^{2\pi i q\alpha} - 1| \geq 4C|q|^{-1-r} \tag{50}$$

for all $q \in \mathbb{Z}$.

The irrational algebraic numbers are examples of Diophantine numbers.

More generally we say that a vector $\alpha \in \mathbb{R}^n$, $n > 1$ is a Diophantine vector if there are constants $r > 0$ and $C > 0$ such that

$$\|\langle k, \alpha\rangle\| \geq C|k|^{-r} \tag{51}$$

for every $k \in \mathbb{Z}^n$.

We say that an irrational number is Liouville if it is not Diophantine. In this case the orbit of 1 by the rotation R_α has a good approximation of the number 1 in the sense that there exists a sequence $\{q_j\}_{j\in\mathbb{N}} \subset \mathbb{Z}$ such that $|\alpha - (p_j/q_j)| < C|q_j|^{-(j+1)}$ by (49).

The number $\alpha = \sum_{k=1}^{\infty} q^{-k!}$ is a Liouville number.

The following proposition shows that for every root of unity ξ_n there exists a rotation R_α whose orbit by 1 has fast approximation to ξ_n.

Proposition A.2. *Given a family $\{e^{2\pi i(p_s/q_s)}\}_{s=1}^{m}$ of roots of unity there exits a Liouville number α and a family $\{\{k_j^s\}_{j\in\mathbb{N}}\}_{s=1}^{m}$ of sequences such that*

$$\left\| k_j^s \alpha - \frac{p_s}{q_s} \right\| < \frac{2}{(k_j^s)^j} \tag{52}$$

for all $j \in \mathbb{N}$ and $s \in \{1, 2, \ldots, m\}$.

Proof. We may assume that $q_s > 0$ and $0 < p_s < q_s$. We consider the following Liouville number

$$\alpha = \sum_{k=1}^{\infty} \frac{1}{q^{k!}},$$

where $q = q_1 q_2 \ldots q_m$. Now consider the sequence $k_j^s = p_s q_1^{j!} \ldots (q_s)^{j!-1} \ldots q_m^{j!}$, with $j \in \mathbb{N}$ and $s \in \{1, 2, \ldots, m\}$. We show that this sequence satisfies (52). In fact, it is easy to see that

$$\left\| k_j^s \alpha - \frac{p_s}{q_s} \right\| = \inf_{\ell \in \mathbb{Z}} \left| k_j^s \alpha - \frac{p_s}{q_s} + \ell \right| \leq \sum_{k=j+1}^{\infty} \frac{k_j^s}{q^{k!}}.$$

Thus,

$$\begin{aligned} \left\| k_j^s \alpha - \frac{p_s}{q_s} \right\| &\leq \frac{1}{(k_j^s)^j} \sum_{k=j+1}^{\infty} \frac{(k_j^s)^{j+1}}{q^{k!}} \\ &\leq \frac{1}{(k_j^s)^j} \sum_{k=j+1}^{\infty} \frac{q^{(j+1)!}}{q^{k!}} < \frac{1}{(k_j^s)^j} \sum_{k=0}^{\infty} \frac{1}{q^k} \leq \frac{2}{(k_j^s)^j}. \end{aligned} \tag{53}$$

□

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