Adv. Appl. Prob. **22**, 749–750 (1990)
Printed in N. Ireland

LETTERS TO THE EDITOR

ON A SIMPLE PROOF OF UNIFORMIZATION FOR CONTINUOUS AND DISCRETE-STATE CONTINUOUS-TIME MARKOV CHAINS

NICO M. VAN DIJK*, *Free University, Amsterdam*

Uniformization has proven to be a convenient tool for modeling and computational purposes to analyze continuous-time Markov chain applications (e.g. [4], [5], [7], [8]). This method was introduced by Jensen [6] and further exploited such as in the above references in the case of discrete state space. For the case of continuous-state space, uniformization is also intuitively obvious and most likely employed in practice. A formal justification does not, however, seem to be available.

This note merely aims to show that uniformization for both the discrete- and continuous state cases is directly formalized by a simple uniqueness result from the literature.

Model. Consider the 3-tuple (S, q, H) where
S: is a separable and complete metric space with Borel field β
$q: S \to \mathbf{R}$ is a measurable function representing jump rate $q(x)$ in state $x \in S$
$H: S \times \beta \to [0, 1]$ is a transition probability measure representing the conditional transition probability $H(x; B)$ for a set $B \in \beta$ upon a transition out of state $x \in S$, where $H(x; S) = 1$ and $H(x; \{x\}) = 0$ for all $x \in S$.

Now assume that for some constant Q:

$$q(x) \leqq Q < \infty \qquad (x \in S). \tag{1}$$

The following lemma, adopted from Gihman and Skorohod [3], p. 25, proves that there exists a unique family of transition probabilities $\{P_t \mid t \geqq 0\}$ with $P_t: S \times \beta \to [0, 1]$, which satisfies the Markov property:

$$P_{t+s}(x; B) = \int P_s(y; B) P_t(x; dy) \tag{2}$$

for all t, s, x and $B \in \beta$, hereafter called a Markov semigroup, with infinitesimal jump characteristics $q(\cdot)$ and $H(\cdot; \cdot)$.

Lemma 1. There exists a unique Markov semigroup $\{P_t \mid t \geqq 0\}$ such that

$$[P_h(x; B) - 1_{\{B\}}(x)]h^{-1} \to q(x)[H(x; B) - 1_{\{B\}}(x)] \tag{3}$$

as $h \to 0$, uniformly in all $x \in S$ and $B \in \beta$, where $1_{\{B\}}(x) = 1$ if $x \in B$ and $1_{\{B\}}(x) = 0$ if $x \notin B$.

Proof. Write $a(x; B) = q(x)H(x; B)$ and $a(x) = q(x)$ for all $x \in S$ and $B \in \beta$. The conditions (a) and (b) on p. 25 of Gihman and Skorohod [3] are then satisfied. By Theorem 5 on p. 27 of this reference the proof is now concluded.

Remark. By Theorem 4 on p. 364 of Gihman and Skorohod [2], one can also construct a corresponding Markov jump process $\{Z_t \mid t \geqq 0\}$ with transition probabilities $\{P_t \mid t \geqq 0\}$. (The

Received 30 January 1990.
* Postal address: Department of Econometrics, Free University, De Boelelaán 1105, 1081 HV Amsterdam, The Netherlands.

so-called minimal construction.) By Theorem 14.5 of Billingsley [1] its corresponding probability measure at $D[0, 1]$ is unique.

Uniformization. Define the transition matrices $\bar{H}^n : S \times \beta \to [0, 1]$ by

$$(4)\qquad \begin{cases} \bar{H}^0(x; B) = 1_{\{B\}}(x), \text{ and for } n \geqq 0: \\ \bar{H}^{n+1}(x; B) = \int \bar{H}^n(y; B)\bar{H}(x; dy), \text{ where} \\ \bar{H}(x; B) = [1 - q(x)Q^{-1}]1_{\{B}(x) + q(x)Q^{-1}H(x; B) \end{cases}$$

for all $x \in S$ and $B \in \beta$, where $1_{\{B\}}$ is as in Lemma 1. The following result now formalizes the well-known uniformization technique for arbitrary S and β as defined and thus for both the discrete- and the continuous-state case.

Result 1. For all $x \in S$, $B \in \beta$ and $t \geqq 0$:

$$(5)\qquad P_t(x; B) = \sum_{n=0}^{\infty} \frac{(tQ)^n}{n!} e^{tQ} \bar{H}^n(x; B).$$

Proof. One can directly verify the convergence relation (3) as h tends to 0, uniformly in $x \in S$ and $B \in \beta$. Lemma 1 thus completes the proof.

Acknowledgement

This note has been written during a short visit to Bellcore, Morristown.

References

[1] Billingsley, P. (1968) *Convergence of Probability Measures.* Wiley, New York.

[2] Gihman, I. I. and Skorohod, A. V. (1969) *Introduction to the Theory of Random Processes.* W. B. Saunders, Philadelphia.

[3] Gihman, I. I. and Skorohod, A. V. (1975) *The Theory of Stochastic Processes* II. Springer-Verlag, Berlin.

[4] Grassman, W. (1990) Finding transient solutions in Markovian event systems through randomization. *Proc. First Internat. Conf. on the Numerical Solution of Markov Chains,* January 8–10, Raleigh, North Carolina, 375–385.

[5] Gross, D. and Miller, D. R. (1984) The randomization technique as a modeling tool and solution procedure for transient Markov processes. *Operat. Res.* **32,** 343–361

[6] Jensen, A. (1953) Markoff chains as an aid in the study of Markoff processes. *Skand. Aktuarietidskr.* **36,** 87–91.

[7] Heidelberger, P. and Goyal, A. (1988) Sensitivity analysis of continuous time Markov chains using uniformization. In *Computer Performance and Reliability,* ed. G. Iazeolla, P. J. Courtois, and O. J. Boxma, Elsevier North-Holland, Amsterdam, 93–104.

[8] Melamed, B. and Yadin, M. (1984) Randomization procedures in the computation of cumulative-time distributions over discrete state Markov processes. *Operat. Res.* **32,** 926–943.