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PROJECTIVE DUALITY AND THE RISE OF MODERN LOGIC

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Abstract. The symmetries between points and lines in planar projective geometry and between points and planes in solid projective geometry are striking features of these geometries that were extensively discussed during the nineteenth century under the labels "duality" or "reciprocity." The aims of this article are, first, to provide a systematic analysis of duality from a modern point of view, and, second, based on this, to give a historical overview of how discussions about duality evolved during the nineteenth century. Specifically, we want to see in which ways geometers' preoccupation with duality was shaped by developments that lead to modern logic towards the end of the nineteenth century, and how these developments in turn might have been influenced by reflections on duality.

§1. Introduction. One of the central developments in early nineteenth century geometry was the rise of projective geometry, which became the predominant foundational enterprise in geometry by the middle of the nineteenth century. A stunning feature of projective geometry, and one of the main sources of its attractiveness, is the principle of duality. Its plane version roughly says that for every theorem in plane projective geometry there is another theorem that arises from the first by interchanging the words "point" and "line." Similarly, the solid version says that for every theorem in solid projective geometry there is another theorem that arises from the first by interchanging the words "point" and "plane."

From a historical perspective, duality is interesting for several reasons. One of them is that it is one of the first 'metatheoretical' principles that have been discussed extensively in nineteenth century mathematics. It is not about points, lines, and relations among these objects, but about how certain theorems arise from others as a result of certain linguistic substitutions. The aims of this article are, first, to provide a self-contained, systematic discussion of duality from a modern perspective. Based on this, we then want to see how informal versions of duality gradually received sharper formulations through methodological advances in logic and mathematics, and how, conversely, preoccupation with duality might have influenced the development of modern logic and mathematics.[1]

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[1]The significance of duality for the development of modern logic has been discussed earlier, most notably, by Ernest Nagel in his classic [53], an article that is still one of the central works

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The plan for this article is as follows: In the next section, we will review some of the basics of projective geometry and the principle of duality informally. In Section 3, we will look at several precise versions of the informal principle of duality for plane projective geometry and study their mutual connections from a modern perspective. In Section 4, we will take a closer look at the history of duality during the nineteenth century and try to identify in which ways the preoccupation with duality is connected to the development of modern logic.

§2. A short introduction to projective duality. Projective geometry has its roots in the study of perspective drawing during the Renaissance when painters, architects, and mathematicians became interested in the mathematical study of central projections. Projective geometry is distinguished from Euclidean geometry in two respects. First, unlike Euclidean geometry, projective geometry is concerned with relations among points, lines, and planes only insofar as their relative positions are concerned. That is, metrical notions like distances, volumes, angles, etc., are ruled out. Secondly, projective geometry also rules out the notion of parallelism. Just like two points always determine a unique joining line, in projective geometry two lines always determine a unique intersection point. In the parlance of nineteenth century geometers, parallel lines are said to meet at a 'point at infinity', and parallel planes are said to meet in a 'line at infinity'.[2]

Girard Desargues is often considered to be one of the early founders of projective geometry, and it is to him that we owe the idea of elements at infinity. Desargues realized that several issues in geometry receive a smoother treatment if we think of parallel lines as meeting at a point at infinity. In Euclidean geometry, we often have to deal with exceptions that arise from various pairs of lines in a configuration being parallel. For example, suppose we have two triangles where the lines joining corresponding vertices meet in a point. If we want to prove something about configurations of this kind, we have to distinguish several cases according to whether certain pairs of corresponding sides of the two triangles are parallel, because intersection points that are present in one configuration might not be present in another (see Figure 1). But if we allow ourselves to think of parallel lines as meeting at some 'point at infinity', then the problem disappears. Thus, in the *extended* Euclidean plane we introduce for each set of parallel lines a new 'point' that lies 'at infinity' in the direction of all these lines, and all these new points are

on this topic. It was also a central source of inspiration for the current article. Still, partly due to its age and partly due to the fact that Nagel does not always clearly distinguish between different, though related, ideas, Nagel's article contains some shortcomings which, I hope, this article helps rectify.

[2]See [1] for more on the prehistory of projective geometry. An overview of projective geometry during the nineteenth century can be found in [14] and in Section 2.3 of [77]. Developments in geometry during the nineteenth century more generally are discussed in [31] .

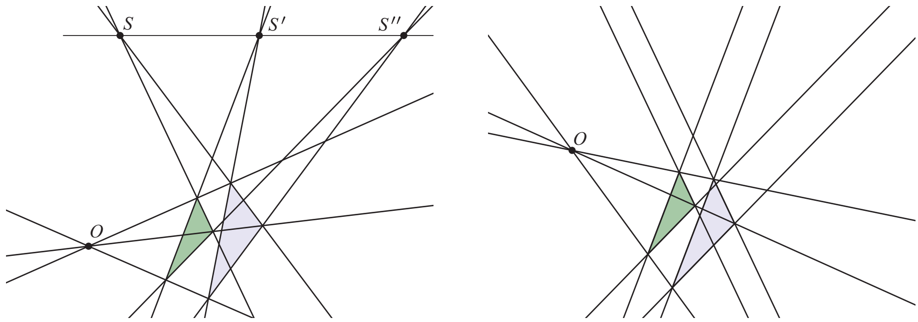

FIGURE 1. Two configurations where the lines joining corresponding vertices of two triangles meet in a point O. The intersection points S, S', S'' of corresponding sides of the triangles in the first configuration are not present in the second configuration.

assumed to lie on a 'line at infinity' which somehow surrounds the ordinary plane.

Partly because of the simultaneous rise of Descartes' analytic geometry, Desargues' work did not receive much attention in his time and it took another two centuries until his ideas were rediscovered and further developed by several French geometers, including Gaspard Monge, Joseph Diez Gergonne, and, especially, Jean-Victor Poncelet, who is often considered to be the proper father of modern projective geometry. Around that time, geometers started to notice some startling symmetries, where certain kinds of theorems come in pairs where one results from the other by systematically replacing certain words by others. Specifically, in the planar case, one obtains a theorem from another if one interchanges the words "point" and "line" and the relation of a point lying on a line by the relation of a line going through a point. This phenomenon came to be known as the (planar) *principle of duality*, sometimes also *principle of reciprocity*. To illustrate, here is an example concerning configurations of the type mentioned earlier:

Desargues' Theorem	**Dual of Desargues' Theorem**
If corresponding vertices of two triangles lie on three lines that meet in a point, then corresponding sides of the triangles meet in three points that lie on a line.	If corresponding sides of two triangles meet in three points that lie on a line, then corresponding vertices of the triangles lie on three lines that meet in a point.

Desargues' theorem is a basic truth about the real projective plane. It is a genuine *projective* truth, since it does not involve metrical notions and its formulation tacitly assumes that corresponding sides of the triangles always meet.[3] Now, what is striking is that after dualizing Desargues' theorem,

[3]See [18, 93 ff.] for more on Desargues' theorem. The mathematical significance and philosophical relevance of Desargues' theorem are also discussed in [3, 32].

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we end up with a proposition that is a theorem about the real projective plane just like Desargues' theorem itself. In this particular case, the dual proposition is simply the converse of Desargues' theorem.

Soon after people had realized that these symmetries are ubiquitous as long as we are dealing with projective relationships, duality was acknowledged as a general principle that is of central importance and many geometers put it right at the beginning of their treatises.[4] In the subsequent decades projective geometry developed into various directions. Two developments that also affected geometers' understanding of duality were the development of analytic projective geometry in the work of Plücker, Möbius, Cayley, and others, and the rise of axiomatic projective geometry that ultimately led to modern axioms systems for projective geometry at the turn of the twentieth century in the work of Italian geometers like Peano, Pieri, and Fano, as well as American 'postulate theorists' like Veblen and Young. By that time duality had been known for a century. And yet, as late as 1910, Veblen and Young would still write that "the treatment of this principle in many standard texts is far from convincing" [78, 29]. Part of the motivation for writing this article was the desire to understand why this is.

Just like projective geometry in general, duality can be studied in different settings and from different perspectives. The purpose of the next section is to introduce several precise versions of the intuitive principle of duality using the resources provided by modern logic. Two comments before we start: First, we will focus on planar projective geometry. Similar versions can be obtained though in an analogous way for the solid case. Secondly, we will be mainly concerned with the principle of duality as it applies to the *real projective plane* (rather than projective planes in general), because it is in the context of the study of the real projective plane in which duality was usually discussed by nineteenth century geometers.[5]

§3. A modern take on projective duality. Starting point for all the precise versions of duality we will consider here is a formal language, the language of plane projective geometry. Here, different choices are possible. For our purposes it will be convenient to think of projective geometry as being formulated in a two-sorted language, where distinct sorts of variables for points and lines are available.[6] We also assume that our full language contains second-order variables for *sets* of points and *sets* of lines respectively, though much of projective geometry can be formulated already

[4]See, for example, [72, IV], [62, IX], or [17, vii].

[5]In addition to the real projective plane and real projective space, nineteenth century geometers had been concerned with the *complex* projective plane and *complex* projective space. In fact, many important developments in nineteenth century geometry (e.g., Cayley's reduction of metric geometry to projective geometry) can only be understood in the context of complex projective geometry.

[6]As is well-known, many-sorted logic is little more than a notational variant of one-sorted logic. So all that's being said in what follows can be translated into the standard one-sorted case.

in the first-order fragment. The only primitive non-logical relation is the *incidence relation* I, which is the fundamental relation between points and lines. Informally, instead of saying that a point and a line are incident, we also say that the former *lies on* the latter or that the latter *passes through* the former. So, for example, the formula

$$\forall X \forall Y (X \neq Y \rightarrow \exists x (IxX \wedge IxY))$$

formally expresses that any two points lie on a line, which is a basic truth of virtually all geometries. Although this is by no means obvious, all other concepts of real projective geometry, including concepts like being a triangle, being a conic section, etc., are definable in the meager language set out so far.

A structure for this language is a triple $\langle \mathsf{P}, \mathsf{L}, \mathsf{I} \rangle$, where P is the domain of the point-variables, L is the domain of the line-variables, and I is a set of point–line-pairs which we assume to be symmetric. Structures of this kind are called *incidence structures*. Assuming that second-order quantifiers are interpreted standardly, second-order variables range over the entire powerset of P and L respectively. Two incidence structures are defined to be *isomorphic* if there is a bijective function i from the union of points and lines of the first structure to the union of points and lines of the second structure such that, first, the restriction of i to points of the first structure is a bijective function onto the points of the second structure, and the restriction of i to lines of the first structure is a bijective function onto the lines of the second structure, and, secondly, i preserves the incidence relation. That is, whenever a point of the first structure is incident with a line of the first structure, then the image of that point will be incident (in the sense of the second structure) with the image of that line in the second structure.

The structure we are particularly interested in is the *real projective plane*, which has a variety of different (though projectively isomorphic) representations. We have already mentioned an informal version of the extended Euclidean plane earlier, which can be made precise in various ways, for example, by identifying 'points at infinity' with equivalence classes of parallel lines.[7] In what follows we will introduce another model $\mathbf{H}$ that will facilitate our study of duality, and which is sometimes called the *homogeneous model* of the real projective plane. Since in modern projective geometry the homogeneous model is likely the one that is referred to as *the* real projective plane, we also refer to it as the *standard model* of the real projective plane. In this model the elements of $\mathsf{P}^{\mathbf{H}}$ (points) are triples $(u_1 : u_2 : u_3)_{\mathsf{P}}$ of real numbers (not all zero) of one sort, and the elements of $\mathsf{L}^{\mathbf{H}}$ (lines) are triples $(v_1 : v_2 : v_3)_{\mathsf{L}}$ of real numbers (not all zero) of another sort, where we assume that both kinds of triples are invariant under scaling. That is, both kinds of triples are really *ratios* of three quantities rather than ordinary triples. The interpretation of the incidence relation $\mathsf{I}^{\mathbf{H}}$ consists of the set of all pairs whose first component is a point $(u_1 : u_2 : u_3)_{\mathsf{P}}$, whose

[7] The construction is described in more detail, for example, in [63, 41].

second component is a line $(v_1 \colon v_2 \colon v_3)_{\mathsf{L}}$ (or vice versa), and which satisfy the equation

$$\text{(INC)} \quad u_1v_1 + u_2v_2 + u_3v_3 = 0.$$

Intuitively, one can think of **H** as a structure in three-dimensional Euclidean space. The points are Euclidean straight lines through the origin, the lines are Euclidean planes through the origin, and a point and a line are incident if the corresponding Euclidean line is contained in the corresponding Euclidean plane. A precise version of the intuitive picture of the extended Euclidean plane as a Euclidean plane that is 'surrounded' by a 'line at infinity' can be recovered from this model by suitable projections.[8]

All of the versions of duality that will be discussed in this section will be based on the *dual translation*. In general, a translation T from one formal language to another is defined as a function that maps each formula φ of the first language to a formula φ^T of the second language in such a way that logical form is preserved.[9] That is, the translation of a negated sentence is the negation of the translation of that sentence, the translation of a conjunction is the conjunction of the translations of the conjuncts, and so on. The dual translation is a particular translation that assigns to each formula φ of our formal language a formula φ^Δ of the same language. The central feature of this translation is that it interchanges point-(set-) and line-(set-)variables and that point-(set-)quantifications are interchanged with line-(set-)quantifications. Using this translation one can see, for example, that

$$[\forall X \forall Y (X \neq Y \rightarrow \exists x (IxX \wedge IxY))]^\Delta = \forall x \forall y (x \neq y \rightarrow \exists X (IXx \wedge IXy)),$$

where the translation formula expresses that any two lines pass through a common point, which is characteristic of projective geometry. In a precise sense, then, the dual of a sentence arises by simply interchanging lowercase and uppercase letters.

Note that for each structure **P** of our language, the dual translation determines another structure $\mathbf{P}^\Delta$, which we call the *dual structure*, where the roles of points and lines are reversed. So if P is the domain of the point variables in the original structure, P will be the domain of the line variables in the dual structure, and if L is the domain of the line variables in the original structure, L will be the domain of the point variables in the dual structure. Since we assume that incidence is symmetric, the set I remains the

[8]To see this, think of a sphere with its center in the origin. Each line through the origin intersects the sphere in two antipodal points and each plane through the origin traces out a great circle. Thus, projective points can be identified with pairs of antipodal points on the sphere and projective lines with great circles. By restricting oneself to, say, the upper half-sphere and projecting it onto the xy-plane, one gets a representation of the real projective plane that is a precise version of the intuitive picture of the projective plane as a Euclidean plane that is 'surrounded' by a line at infinity. See [2] for more on intuitive representations of the real projective plane.

[9]For more on syntactic translations and their role in mathematical logic (e.g., in the context of various concepts of theory-reduction) see [79].

same in the dual structure. It follows from these definitions that the dual φ^Δ of a formula is true in a structure $\mathbf{P}$ just in case the formula φ itself is true in the dual structure $\mathbf{P}^\Delta$.

In order to prove our first version of duality, we need an instance of a basic result from model theory, which is sometimes called the *isomorphism lemma*. Applied to our current setting, it can be stated as follows:

LEMMA 1. *If* $\mathbf{P}$ *and* $\mathbf{P}'$ *are isomorphic incidence structures, then for each formula* φ *in the language of projective geometry*

$$\mathbf{P} \models \varphi \quad \textit{if and only if} \quad \mathbf{P}' \models \varphi.$$

Thus, isomorphic structures satisfy exactly the same formulas.[10] The isomorphism lemma can be understood as a precise expression of the informal idea that structurally equivalent models are indistinguishable. Given this basic result, we can prove our first version of the planar principle of duality:

DUALITY 1. *For each sentence* φ *in the language of projective geometry*:

$$\mathbf{H} \models \varphi \quad \textit{if and only if} \quad \mathbf{H} \models \varphi^\Delta.$$

To see why DUALITY 1 holds, note that the function i which sends the point $(u_1 : u_2 : u_3)_\mathsf{P}$ to the line $(u_1 : u_2 : u_3)_\mathsf{L}$ and the line $(v_1 : v_2 : v_3)_\mathsf{L}$ to the point $(v_1 : v_2 : v_3)_\mathsf{P}$ is an isomorphism between $\mathbf{H}$ and $\mathbf{H}^\Delta$. Obviously, i is bijective, and, because of the symmetric contribution of points and lines in the incidence condition (INC), i also preserves incidence.[11] So by LEMMA 1 we have $\mathbf{H} \models \varphi$ if and only if $\mathbf{H}^\Delta \models \varphi$ and from the definitions of the dual of a formula and the dual of a structure it follows that $\mathbf{H}^\Delta \models \varphi$ if and only if $\mathbf{H} \models \varphi^\Delta$. Thus, $\mathbf{H} \models \varphi$ if and only if $\mathbf{H} \models \varphi^\Delta$.

A couple of comments. First, DUALITY 1 says that for each sentence in our formal language that is true in the standard model $\mathbf{H}$, its dual is also true in that model. So DUALITY 1 says something about *every* sentence that is true in the standard-model, regardless of how its truth is established or whether its truth can be established based on this or that system of axioms. Secondly, the justification of this version of duality crucially relies on the existence of an isomorphism, which ensures that the real projective plane and its dual are structurally identical. Since an isomorphism is a function of

[10] In the context of first order model theory, this is also expressed by saying that isomorphic structures are *elementarily equivalent*.

[11] There are many other isomorphisms. Identifying points and lines with three-dimensional vectors $\mathbf{u}, \mathbf{v}$, for each real, invertible 3×3-matrix A, the function i_A that maps each point $\mathbf{u}$ to $A\mathbf{u}$ and each line $\mathbf{v}$ to $(A^{-1})^T\mathbf{v}$ is bijective and incidence-preserving. (Here A^{-1} represents the inverse and A^T the transpose of a matrix A.) Thus, i_A is an isomorphism between $\mathbf{H}$ and $\mathbf{H}^\Delta$. Such an isomorphism is also called a *correlation*. If A is symmetric, then i_A is called a *polarity*. Since each symmetric matrix determines a (real or imaginary) conic, i_A can be geometrically interpreted as a function that maps each point to its *polar*, and each line to its *pole* with respect to that conic. In the special case where A is the unit matrix, i_A is the isomorphism mentioned in the main text, and the conic is an imaginary circle.

a certain kind, this version is sometimes called the *functional conception* of duality.[12]

The second conception of duality that can be found in the literature differs from the functional conception in both respects just mentioned. Because it relies on a system of axioms for projective geometry, it is also called the *axiomatic conception*.[13] A variety of axiom systems can be found in the literature.[14] The following three axioms are sometimes used to characterize the general concept of a projective plane:

(P1) *If X and Y are distinct points, then there is exactly one line x that is incident with both X and Y.*
(P2) *If x and y are distinct lines, then there is exactly one point X that is incident with both x and y.*
(P3) *There are four distinct points* X_1, X_2, X_3, X_4 *such that no line x is incident with more than two of them.*

Evidently, this system has many non-isomorphic models, so in order to characterize the standard model of the real projective plane, further axioms have to be added. Axiom systems for the real projective plane can be set up in close analogy to axiomatizations of the real Euclidean plane. Typically, a first group of axioms consists of basic incidence axioms like the ones just mentioned plus further axioms such as Desargues' or Pappus' theorem. A second group consists of axioms that describe the cyclic order of points on a projective line. These axioms can be stated in terms of a generalization of the Euclidean notion of *betweeness*, namely, the notion of *separation* of point pairs. As hinted at earlier, separation can be defined in terms of incidence, so the formulation of these axioms can be achieved in our official language.[15] Similar to the case of Euclidean geometry, the final group consists of one or more axioms that ensure the continuity of projective straight lines. A continuity axiom for projective lines can be formulated, for example, in terms of a suitably adapted version of the Euclidean axiom which says that every upwards bounded set of points on a line has a least upper bound. As

[12]Projective planes that are isomorphic to their duals are sometimes called *self-dual*. It can be shown that many natural projective planes can be coordinatized by a suitable field. Using the proof strategy from above, versions of DUALITY 1 can be established for all of them, including, for example, the complex projective plane or finite projective planes like the Fano plane.

[13]The terminology of "functional" and "axiomatic" conception of duality has been coined by Saunders MacLane in his [49] . Kromer and Corfield [47] also mention the labels "formal" or "Gergonne-type" conception for the axiomatic conception, and "concrete" or "Poncelet-type" conception for the functional conception. The reason for the labels "Poncelet-type conception" and "Gergonne-Type conception" will become clearer in the historical Section 4.

[14]See, for example, [15] for an informal and [37] for a formal approach to axiomatic real projective geometry. The *locus classicus* for axiomatic projective geometry is [78] .

[15]Since separation can be defined in terms of *harmonic* separation, which in turn can be defined in terms of the complete quadrilateral construction, which in turn is a concept of pure incidence geometry, separation can be defined in terms of incidence. This has been shown by Mario Pieri in his [57]. See [50, 129 ff.] for more on Pieri's definition and its historical context.

in the Euclidean case, it is at this point where higher-order quantification is required. In order to keep things general, we will not be concrete about axiom systems in what follows. All we assume is that the axioms are true in our standard model of the real projective plane, and so, in particular, consistent.

Given some system of axioms for the real projective plane, we can consider its logical consequences. There are different ways to define the concept of logical consequence. One is to proceed in terms of semantic notions. More precisely, a sentence φ is defined to be a *semantic consequence* of a set of sentences $\mathcal{P}$ ($\mathcal{P} \vDash \varphi$) just in case φ is true in every model in which all sentences in $\mathcal{P}$ are true. Call a set of axioms $\mathcal{P}$ *semantically closed under duals* if the dual of every sentence in $\mathcal{P}$ is a semantic consequence of $\mathcal{P}$. For example, the system of axioms that consists of (P1), (P2), and (P3) is semantically closed under duals: (P1) and (P2) are obviously duals of each other, and the dual of (P3), i.e.,

(P4) *There are four distinct lines x_1, x_2, x_3, x_4 such that no point X is incident with more than two of them.*

can be shown to be a semantic consequence of the other axioms.

With these preliminaries, our second precise version of the planar principle of duality can be stated as follows:

Duality 2. *If $\mathcal{P}$ is a system of axioms for the real projective plane that is semantically closed under duals, then for each sentence φ in the language of projective geometry*:

$$\mathcal{P} \vDash \varphi \quad \textit{if and only if} \quad \mathcal{P} \vDash \varphi^{\Delta}.$$

To see why Duality 2 holds, suppose that φ is a semantic consequence of $\mathcal{P}$ and that $\mathbf{P}$ is a model of $\mathcal{P}$. Since $\mathcal{P}$ is closed under duals it follows that the dual ψ^{Δ} of each sentence ψ in $\mathcal{P}$ is true in $\mathbf{P}$, and, thus, that ψ is true in $\mathbf{P}^{\Delta}$ for every ψ in $\mathcal{P}$. So $\mathbf{P}^{\Delta}$ is a model of $\mathcal{P}$. Since $\mathcal{P} \vDash \varphi$, φ must be true in $\mathbf{P}^{\Delta}$, and so φ^{Δ} must be true in $\mathbf{P}$. But because $\mathbf{P}$ was an arbitrary model of $\mathcal{P}$, this means that $\mathcal{P} \vDash \varphi^{\Delta}$. By what we've just shown, $\mathcal{P} \vDash \varphi^{\Delta}$ entails $\mathcal{P} \vDash (\varphi^{\Delta})^{\Delta}$. So the right-to-left direction follows from the observation that $(\varphi^{\Delta})^{\Delta} = \varphi$.

The last precise version of duality we will consider is similar to the previous one, except that instead of *semantic* consequence the effective notion here is that of *proof-theoretic* or *syntactic* consequence, which is determined by some formal proof procedure. For the sake of definiteness, let's assume that we have adopted some Hilbert-style calculus, which is characterized by a set of logical axioms and purely formal inference rules. A formula φ is then defined to be a proof-theoretic consequence of $\mathcal{P}$ ($\mathcal{P} \vdash \varphi$) just in case there exists a correct formal proof of φ from the assumptions in $\mathcal{P}$. Defining the notion of a system of axioms being *syntactically closed under duals* in the obvious way, we can prove the following version of duality:

DUALITY 3. *If $\mathcal{P}$ is a system of axioms for the real projective plane that is syntactically closed under duals, then for each sentence φ in the language of projective geometry*:

$$\mathcal{P} \vdash \varphi \quad \textit{if and only if} \quad \mathcal{P} \vdash \varphi^{\Delta}.$$

To see why DUALITY 3 holds, suppose that $\mathcal{P} \vdash \varphi$, i.e., that there exists a finite sequence $\varphi_1, \dots \varphi_n = \varphi$ of formulas each of which is either a logical axiom, an axiom in $\mathcal{P}$, or derived from earlier formulas in the sequence by means of some inference rule. It is straightforward to see that $\varphi_1, \dots \varphi_n = \varphi$ can be transformed into a proof of φ^{Δ} from $\mathcal{P}$. The idea is to replace each formula φ_i in this sequence by its dual and include a sub-proof of φ_i^{Δ} from $\mathcal{P}$ whenever φ_i is a non-logical axiom in $\mathcal{P}$ (which exists by the assumption that $\mathcal{P}$ is syntactically closed under duals). Because of the formal nature of the logical axioms and inference rules, the resulting sequence will be a correct proof of φ^{Δ} from $\mathcal{P}$. So $\mathcal{P} \vdash \varphi$ implies $\mathcal{P} \vdash \varphi^{\Delta}$. Hence, $\mathcal{P} \vdash \varphi^{\Delta}$ entails $\mathcal{P} \vdash (\varphi^{\Delta})^{\Delta}$, and so the right-to-left direction again follows from the fact that $(\varphi^{\Delta})^{\Delta} = \varphi$.

Each of the rigorous versions of duality we have discussed in this section is different from the others in that its formulation requires distinctive concepts and that its justification requires distinctive methods. The first is about truth in a structure and relies on the existence of certain structure-preserving mappings (isomorphisms) as well as the idea that two structures that are related by such a mapping 'satisfy' the same propositions. By contrast, the second and the third are about the logical consequences of certain systems of axioms and rely on the formality of logical consequence, the formality of semantic consequence being immediate from its characterization in terms of *all* interpretations of a given language, and the formality of proof-theoretic consequence being obvious from the purely syntactic nature of the axioms and rules of a formal proof procedure. So a natural questions is: how are these versions related to each other?

In order to compare our three versions of duality, let us call a formula of the form

$$\varphi \longleftrightarrow \varphi^{\Delta}$$

a *duality equivalence*. We say that a duality equivalence is *established* by DUALITY 1 (DUALITY 2 and DUALITY 3) if its truth in **H** is entailed by the corresponding principle, and we call the set of all duality equivalences that are established by a duality principle its *scope*. The question then becomes what the scopes of our principles are and how they are related to each other. So, first of all, since DUALITY 1 says that *every* duality equivalence is true in the standard model **H**, it follows that its scope contains the scope of the other two principles for any given system of axioms. Also, since we assume that our given proof procedure is sound with respect to semantic consequence, it follows that the scope of DUALITY 2 contains the scope of DUALITY 3. These containment relations may be proper or not depending on the particular choice of non-logical axioms and background logic. For example, suppose $\mathcal{P}$ is the system of axioms that consists of the following two axioms:

(P1') *If X and Y are distinct points, then there is at most one line x that is incident with both X and Y.*
(P2') *If x and y are distinct lines, then there is at most one point X that is incident with both x and y.*

Both axioms are true in the standard model. Since these axioms are evidently dual to each other, the system $\mathcal{P}$ is semantically closed under duals, and so Duality 2 holds for this system. However, the system clearly does not entail *all* true duality equivalences. For example, suppose that φ is a sentence that says that there are at least four points. Then the corresponding duality equivalence is not entailed by the axioms.[16] So, in general, the scope of Duality 1 is wider than that of Duality 2. Furthermore, unless the underlying proof procedure is complete, a sentence that is a semantic consequence of certain axioms is generally not a syntactic consequence of these axioms. So the scope of Duality 2 is generally wider than that of Duality 3.

On the other hand, scope identities can be proved if further assumptions are made about the given axiom system or the background logic. For example, if we restrict ourselves to first-order axiom systems and their consequences, then the scopes of Duality 2 and Duality 3 agree in virtue of the completeness theorem for first-order logic. Also, it is not hard to see that the scopes of Duality 1 and Duality 3 agree if we assume that the axiom system $\mathcal{P}$ is *syntactically complete*, i.e., if for each formula either the formula itself or its negation is a syntactic consequence of $\mathcal{P}$. Similarly, the scopes of Duality 1 and Duality 2 agree if we assume that the axiom system $\mathcal{P}$ is *semantically complete*, i.e., if for each formula either the formula itself or its negation is a semantic consequence of $\mathcal{P}$. Note that a sufficient condition for an axiom system to be semantically complete is that it is *categorical*, i.e., that any two of its models are isomorphic. As indicated earlier, categorical axiom systems for the real projective plane can be set up in close analogy to the Euclidean case. So for each such system, Duality 1 and Duality 2 agree in scope.

Obviously, there is much more to be said about all of our reconstructions of duality and their mutual relations, but this should be enough background for our historical discussion in the next section.

§4. Duality and the rise of modern logic. In the previous section we have introduced several precise versions of duality and discussed some of their interrelations. Of course, the formulation of these versions involves several

[16]Consider an incidence structure with five points, three lines, and an empty incidence relation. Then both axioms are trivially satisfied in this structure. But whereas φ is satisfied, its dual φ^{Δ} is not. The argument is mentioned in [71, 456 ff.], which also contains further results and generalizations relating to duality that are informed by modern logic.

arbitrary features (e.g., two-sorted language vs one-sorted language). Also, they rely on precisely defined concepts that could not have been anticipated in their current form by nineteenth century geometers. So, obviously, we shouldn't expect these streamlined versions of duality to appear in their work. Still, our precise versions illustrate some of the central ideas that are presupposed in the rigorous formulation and justification of duality. In this section we want to see how three specific ideas that were central to the development of modern logic evolved in connection with duality: first, the rise of formal notions of logical consequence, second, the emergence of the semantic idea of reinterpretation, and, third, the gradual formation of informal metatheoretical reasoning.

4.1. Duality and logical consequence. One of the central presuppositions that figure prominently in the two versions of the axiomatic conception of duality (Duality 2 and Duality 3) is the idea that logical consequence is *formal*. Of course, this idea is not new to modern logic. But while common among philosophers and logicians, it was not widespread among working mathematicians until the nineteenth century. The formality of logical consequence can be expressed in different ways, but it is often formulated as the requirement that the meanings of the non-logical terms involved in an argument must not affect its validity or correctness. It is this idea that we want to trace in connection with duality.

We start with Gergonne, who discusses duality in several articles during the Twenties of the nineteenth century.[17] Gergonne is usually considered to be the first to give a clear statement of both the solid and planar principle of duality in his *Considérations philosophiques sur les élémens de la science de l'étendue*:

> An extremely striking feature of the geometry which does not depend in any way upon metrical relations between parts of figures is that with the exception of some theorems which are themselves symmetrical, for example Euler's theorem on polyhedrons, all the theorems are dual. That is to say, to each theorem in plane geometry there necessarily corresponds another, deduced from it by simply interchanging the two words *points* and *lines*, while in solid geometry the words *points* and *planes* must be interchanged in order to deduce the correlative form of a given theorem.
>
> Among the many examples that arise from this sort of *duality* [*dualité*] of theorems which constitute the *geometry of position* we content ourselves with highlighting the two elegant theorems by M. Coriolis as especially significant. [27, 210]

After the principle had been enunciated, a bitter controversy between Gergonne and Poncelet ensued over the question of priority. But the debate is also about the actual content and proper justification of the principle

[17] See especially [27, 28].

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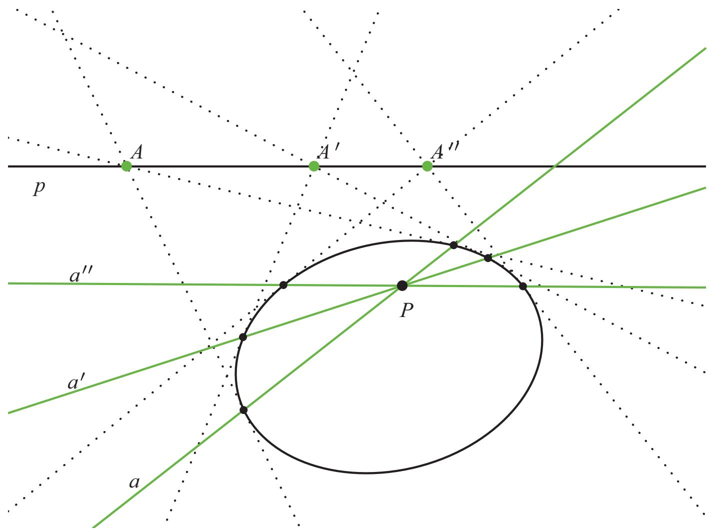

FIGURE 2. The line p is the polar of the point P, and each of the points on the line p has a corresponding polar that passes through P. As a point traverses the line p, its corresponding polar rotates around P.

of duality.[18] In his *Traité des propriétés projectives des figures* from 1822, Poncelet had laid the foundations of modern projective geometry, and he also discussed the phenomenon of "reciprocity" in the section titled *Théorie générale des polaires réciproques* of the second volume.[19] Poncelet's reasoning was this: Suppose we are given some arbitrary but fixed conic section, say, an ellipse. With respect to this ellipse, we can then associate with each point of the extended Euclidean plane a certain line, its *polar*, and with each line a certain point, its *pole* (see Figure 2). This is done in such a way that, first, the pole of the polar of a point is the point itself and the polar of the pole of a line is the line itself and, secondly, a point lies on a line just in case the polar of the former passes through the pole of the latter. Poncelet's argument, then, was that in virtue of this one-to-one correlation between points and lines, each theorem about a particular type of configuration can be effectively transformed into a theorem about the dual configuration by simply replacing each point with its polar and each line with its pole.

In contrast to Poncelet, Gergonne thought that duality is not tied to the theory of poles and polars and has nothing specifically to do with conic sections. Gergonne is not entirely clear and does not provide an explicit justification of duality. His understanding of duality becomes clearer though in the subsequent parts of the *Considérations*. He starts out in the first section, titled "Preliminary notions" ("Notions préliminaires") by presenting 15 basic propositions as well as their duals in the parallel column

[18]See [31, 53 ff.] for further discussion of the mathematical background of the duality controversy and the classic [53] for its larger bearings. The polemical nature of the dispute is discussed in [48] .

[19]See [60, 57, Vol. 2], and Jeremy Gray's [30] and Chapters 1–6 in [31] for more on Poncelet's *Traité* and its historical context.

format, where dual propositions are placed next to each other in two columns in order to highlight their symmetry, a way of presenting projective geometry that was copied by many geometers after Gergonne. In the second section, Gergonne then goes on to prove a variety of theorems of projective geometry. In each case he states the theorem as well as its dual in parallel columns. After the statement of the theorems, pairs of proofs are provided in parallel columns as well. As Gergonne says in the introduction, in this way "it is possible to highlight that there is the same correspondence between the proofs of two theorems as for their statements" [27, 211].[20]

Gergonne's discussion leaves a lot of things unclear, but the basic idea that underlies the axiomatic conception of duality in its proof-theoretic guise (DUALITY 3) seems to be in place. In particular, we have basic unproven propositions and we have an appreciation, albeit implicit, of the formality of logical deduction or proof, which is manifest in Gergonne's use of the dual column format, where valid proofs can be seen to be generated in a uniform and systematic way from others by the purely formal operation of substitution of words. However, Gergonne's discussion also contains some significant shortcomings. First, it is not clear whether his "notions préliminaires" are meant to constitute a system of axioms in anything like the modern sense or whether they are just illustrations of duality. Also, no reasons are given for why certain propositions are included among the "notions préliminaires" while others are not. Secondly, Gergonne does not explicitly delineate any primitive concepts. Partly because of this, it is unclear how complex concepts and propositions are to be dualized properly.[21] Finally, while Gergonne mentions that the dual column format highlights the similarity of dual proofs, his presentation contains no discussion of the relevant notion of proof or "logical deduction." Indeed, some of his remarks to the effect that duality was a consequence of the "nature of extension itself" [27, 231] seem to indicate that he was not yet fully aware of the logical significance of his own procedure.

Subsequently several advances had been made with regard to the basics of projective geometry, both in terms of clarifying its conceptual basis as well as its basic truths by people like Jakob Steiner and Karl Georg Christian von Staudt. Steiner and Von Staudt were tenacious advocates of a systematic approach to synthetic projective geometry. Freudenthal even credits Von Staudt with establishing projective geometry as a "purely deductive discipline" [26, 614]. However, despite their emphasis on the systematic nature of projective geometry, neither of them devised an explicit list of primitive concepts and axioms in the way that would become standard towards the end of the nineteenth century.[22] Real progress in this respect has

[20] In the introduction he notes that he avoids the use of diagrams because they are often "more embarrassing than useful," and he explicitly notes that it is all about the "logical deductions" ("déductions logiques"), which can be easily followed if the notation is well-chosen [27, 212].

[21] See [71] for further discussion of this point.

[22] Steiner's central work in projective geometry is his *Systematische Entwicklung der Abhängigkeit geometrischer Gestalten von einander* [74]. Von Staudt's major contributions

been made only with Pasch's *Vorlesungen über neuere Geometrie* from 1882, which also contains the most detailed study of duality of the time.

Pasch's *Vorlesungen* have been praised by later geometers and historians of mathematics as an important transitional piece between nineteenth century mathematics and modern axiomatics.[23] One of Pasch's central contributions is his insistence on explicitly stated axioms and gapless, deductive proofs in geometry, requirements that are meant to ensure that no assumption can creep into a proof that has not been acknowledged beforehand.[24] Closely related to this is Pasch's view that rigorous proofs in geometry can only be achieved if the use of diagrams is limited to mere illustration and deductive proof is understood in a way that abstracts from the meanings commonly associated with the geometrical terms. As Pasch says in one of the most-cited passages from the *Vorlesungen*, "if geometry is to be genuinely deductive, the process of inferring must everywhere be independent of the *meaning* of the geometrical concepts, just like as it has to be independent of the figures" [54, 98]. Pasch's general approach in the *Vorlesungen* therefore not only prepared the ground for modern axiomatizations of projective geometry but modern axiomatics in general. What is not always mentioned, however, is that this passage forms part of an extensive discussion of the principle of duality in Section 12 of the *Vorlesungen*. Here, Pasch puts together many of the conceptual components that are necessary for a proper understanding of the axiomatic conception of duality.

Pasch starts out in Section 12 by highlighting a certain "transfer" between various groups of propositions from previous sections and then provides an informal statement of the solid principle of duality, the main principle he considers in Section 12 [54, 98]. He then gradually gets more and more into the details. Earlier in the book, Pasch had already identified *stem concepts* ("Stammbegriffe") of solid projective geometry. These are the concepts

to projective geometry are his *Geometrie der Lage* [72] and the *Beiträge zur Geometrie der Lage* [73]. An overview of Von Staudt's work can be found in [31, 337 ff.] and [14]. Both Von Staudt and Steiner spent some effort to discuss duality in their central works. With respect to the dispute between Poncelet and Gergonne, Steiner claims that Gergonne was "closer to the source," noting that "the duality emerges together with the basic forms [Grundgebilde]" [74, VII]. Von Staudt too thinks that duality manifests itself in the "basic forms" of projective geometry. However, these "basic forms" are not obviously "primitive concepts" in the sense of modern axiomatics. Also, despite their systematic approaches, their works do not yet contain explicitly stated axioms.

[23]The importance of Pasch as the "father of rigor" is emphasized in [26], see also Section 3.1. of [77]. His philosophy of mathematics, especially his empiricist conception of geometry, is discussed in [68, 70].

[24]Pasch's position is actually more complicated when it comes to projective geometry. In his *Vorlesungen* Pasch distinguishes between, on the one hand, *basic concepts* and *basic propositions* of geometry (in later editions *core concepts* and *core propositions*), which are conceived in empiricist terms. On the other hand, Pasch also speaks of *stem concepts* and *stem propositions* of projective geometry, which are "derived" from the basic concepts and basic propositions and which account for the ideal elements of projective geometry. A more detailed discussion of these distinctions can be found in [55]. See [68, 100 ff.] for further discussion.

point, *line*, *plane*, the two-place relation of two elements *lying together*, and the four-place relation of *separation* of pairs of elements. Properties that are defined in terms of these notions are called *graphical properties* and propositions that only involve graphical properties are called *graphical propositions* [54, 74–75].[25] Pasch explicitly notes that in dualizing a graphical proposition we may restrict ourselves to the stem concepts, the dual of "point" being "plane" and vice versa, and "line," "lying together," and "separation" being self-dual. The *dual proposition* of a graphical proposition then results if we replace each concept with its dual, unpacking defined concepts if necessary.

Pasch then discusses how the solid principle of duality is proved. Before doing so, he concedes that, at this point, duality cannot be established for *all* graphical propositions. So first he goes on to delineate more precisely for which propositions duality is to be established, namely, all graphical propositions that are consequences of the propositions he set up in Sections 7–9 [54, 95]. Pasch notes that duality is immediate for the propositions in this group, because for each proposition in this group its dual also belongs to the group. In the final step of his argument he claims that from this it follows that for each proposition that is a *consequence* of this group, its dual would be a consequence as well. But what exactly justifies this last step? Pasch comes back to this question after he shows how to derive two further duality principles from the solid one, and then states his famous deductivist credo cited earlier, according to which the validity of a geometrical proof must be entirely independent of the meanings commonly associated with the geometrical terms, apparently, as a way to justify the last step in his justification of solid duality [54, 98]. Pasch quickly brings the discussion to a more general level and notes that "if one has deduced a theorem in full rigour from a group of propositions—call them the *stem propositions*—then the derivation has a value beyond its original purpose" [54]. This added value is that once we have deduced a proposition *P* from a group of propositions, the proposition that results from *P* by replacing certain words by others can be deduced from the transformed propositions in the group. Pasch notes that this "entitlement" was in fact used throughout the book without comment.

Let's take stock. First of all, in his *Vorlesungen* Pasch provides a clear description of an informal version of the axiomatic conception of duality along the lines of Duality 3 from Section 3. In particular, Pasch explicitly appeals to the formality of deductive proof in his justification. Pasch did not develop an explicit system of formal logic. But he seems to have welcomed the idea of formalizing the notion of mathematical proof and in some of his articles from the 1920s, he tries to set out some ideas on

[25] According to Pasch, *lying together* is a general concept that applies to point-line pairs, line-plane pairs, and point-plane pairs. Although Pasch does not use the term "incidence," which was used by others before, he mentions it in a footnote [54, 73]. Also, on Pasch's view, *separation of pairs of elements* is a single primitive notion, applying to points on a line as well as planes through a line.

this topic himself.[26] Pasch's interest in formalization is also witnessed in his correspondence with Frege, where Pasch makes clear that he considers questions about the nature of mathematical proof to be a natural outgrowth of investigations like the ones pursued in his own *Vorlesungen* from 1882. In a letter from around 1905 he notes that "I was interested to see that you discuss in depth the nature of mathematical proof, a subject on which there are indeed few clear ideas around" [24, 103]. One gets the sense that Pasch is not just having a few kind words for a colleague who felt neglected by the mathematical community, but that he really means it when he says that the kind of investigations pursued by Frege are "urgently desirable" [24, 104].[27]

What is important for our purposes is that in Pasch's *Vorlesungen* the formality of mathematical proof becomes an issue in the specific context of a discussion of duality. Moreover, it is not merely mentioned as a feature that may be philosophically interesting, but otherwise insignificant for actual mathematics. Rather, it plays a central justificatory role in his informal proof of the solid principle of duality. Now, we don't have to assume that the issue of duality was the *only* reason for Pasch to get sucked into questions of this kind. But the fact that Pasch discusses the formality of proof in this specific context certainly indicates that providing a proper understanding of duality was one of them.

From Pasch there is a clear trace to early proponents of modern axiomatics (Hilbert and Peano) and modern symbolic logic (Peano and Pieri), and at the beginning of the twentieth century, the axiomatic conception of duality was by many considered to be sufficiently clear.[28] In their *Projective Geometry* from 1910, after complaining about insufficient discussions of duality by other geometers, Veblen and Young even note that "the method of formal

---

[26]Pasch develops his ideas on this subject, e.g., in [55, 56], where he emphasizes that genuine insight into the "nature of mathematical proof" can only be gained by completely breaking down a proof into simple "proof steps" ("Beweisschritte") [55, 354]. Although Pasch's explanations are couched in natural language, he does adopt a 'formal' way of expressing himself by using words like "A-things," "B-things," etc. See again [68] for further discussion. Schlimm also notes, however, that "Pasch explicitly distanced himself from these approaches [that use symbolic language, G.E.] and promoted formalization only to the extent that it remained compatible with ordinary mathematical practice" [68, 104], referring to a passage from a letter by Pasch to Klein, where Pasch notes that "with regard to the external representation [of proof, G.E.], I do not want to go so far as, for example, Peano" [69, 193]. The distinction between 'formalization' and 'symbolization' is further discussed in [8].

[27]As an interesting aside, it is worth mentioning that in an article on the foundations of geometry from 1906, Frege discusses a method to prove independence of axioms that is presented in a way that closely resembles standard presentations of duality in terms of parallel columns. (This has been first highlighted in [75].) In this discussion, he also mentions a "new law" which he refers to as an "emanation of the formal nature of logical laws" [25, 337 ff.], which essentially states that logical consequence is preserved under substitutions of non-logical vocabulary. Given that Frege was corresponding with Pasch at the time, his "new law" may have well been inspired by Pasch's discussion of duality.

[28]Freudenthal notes that "Peano, when working on logistics, was greatly indebted to Pasch, who first showed how to formulate axioms" [26, 617]. Pasch's influence on Peano is discussed in more detail in [41], his influence on Hilbert in [76]. For connections to Pieri, see [50].

inference from explicitly stated assumptions makes the theorems [plane and solid duality, G.E.] appear almost self-evident," and they add that this "may well be regarded as one of the important advantages of this method" [78, 29].

To sum up: we have seen that the gradual clarification of the axiomatic conception of duality, especially in its proof-theoretic guise, is closely intertwined with the development of formal conceptions of logical consequence. Moreover, our discussion suggests that we have a connection in both directions: advances with respect to foundational issues led to a more sophisticated understanding of duality, and, conversely, considerations relating to duality sometimes triggered investigations into the formal nature of mathematical proof.

4.2. Duality and the idea of reinterpretation. We have seen how formal conceptions of consequence naturally emerged from geometer's reflections on duality. Such conceptions often went hand in hand with several related, but distinct ideas. One of them is the idea of *reinterpretation*, that is, the idea that the meanings or interpretations commonly associated with the non-logical terms of a mathematical discipline are not glued to them, that we can *assign* different interpretations to them. It has been highlighted by others that the idea of reinterpretation is closely connected to the development of modern semantics and logic. Hintikka, for example, notes that "the development of model theory worthy of its name presupposes the possibility of varying in a large scale the interpretation of the language in question, be it natural or formal" [40, 2–3]. It is important to emphasize though that the semantic idea of reinterpretation is different from the formality of consequence as discussed in the previous section and as formulated, say, by Pasch. It is one thing to require that the meanings commonly associated with geometrical terms should not affect the validity of a geometrical proof, and another thing to assume that these terms can be 'assigned' different meanings. Both these ideas, in turn, are different from the idea that geometrical terms *acquire* their meaning through axioms that are conceived as 'implicit definitions'. Finally, all of these conceptions must be distinguished from the crude formalist idea that geometrical language is 'meaningless' altogether.[29]

The origins of the idea that mathematical language is amenable to reinterpretation are manifold and can be found in various branches of nineteenth century mathematics. In geometry, this idea evolved in close connection with the rise of modern axiomatics and questions relating to the independence and consistency of axioms. [30] The idea of reinterpretation also

[29] For more on the pre-history of model theory, see [20, 40]. A recent discussion of implicit definitions is provided by [29].

[30] A well-known example of this concerns the independence of the axiom of parallels from the remaining axioms of Euclidean geometry, which was established by Felix Klein, Eugenio Beltrami, and others, by constructing geometries in which the axiom of parallel fails. See [7, 42, 44]. For a general discussion of the history of non-Euclidean geometries, see [31, 77]. The origins of model theory in connection with developments in nineteenth century geometry,

arises in an inchoate form in nineteenth century projective geometry in the context of discussions about 'extension elements', i.e., elements at infinity and imaginary elements.[31] In what follows we will try to identify more precisely how the idea of reinterpretation evolved in projective geometer's discussions of duality during the nineteenth century.

As indicated earlier, projective geometry can be approached in two different ways. On the one hand, there is the synthetic approach favoured by Poncelet and his followers. Here, projective geometry is concerned with geometrical figures, properties of figures that are preserved by projection, and general principles that guide such investigations, such as Poncelet's notorious *principle of continuity*. Poncelet's synthetic projective geometry was novel in several respects, but it was traditional in that its focus remained on geometrical figures that are given to us intuitively. On the other hand, ever since Descartes there is also *analytic* or *coordinate geometry*. If we restrict ourselves to the planar case, the basic idea here is to describe points of the Euclidean plane by means of two real numbers x and y which determine the position of a point relative to two fixed lines, the coordinate axes. Straight lines, circles, and other curves can then be described by means of equations that involve those quantities. Now, something similar has been developed towards the end of the 1820s for the *projective* plane. One of the key figures in the development of this analytic approach was Julius Plücker, who was among the first to introduce *homogeneous coordinates*.[32] As we saw in Section 3, homogeneous coordinates belong to the standard repertoire of modern projective geometry. As a reminder, the idea is that each point of the projective plane is assigned a ratio $(x_1 : x_2 : x_3)$ of three numbers. A straight line l can be conceived as the set of all point-triples that satisfy the linear equation

$$\text{(INC)} \quad u_1x_1 + u_2x_2 + u_3x_3 = 0,$$

where the ratio of the quantities u_1, u_2, u_3 uniquely determines the straight line l.

Plücker realized that the use of homogeneous coordinates has many advantages. First of all, they enable us to evade questions about the nature of 'elements at infinity' and deal with them in an unambiguous manner.

are discussed, e.g., in [9, 80]. The closely related "problem of the multiplicity of geometries" is discussed in [13]. I thank an anonymous referee for pointing out this reference.

[31] See, for example, Pasch's discussion in Section 6 of his *Vorlesungen*, where he remarks that "the expression 'proper point' will from now on mean exactly the same as was hitherto meant by point *simpliciter*; thereby the undetermined word 'point' becomes available for a more general application" [54, 40]. Although Schlimm correctly points out in [68, 111] that this is not yet reinterpretation in the sense discussed earlier, the idea that geometrical terminology can be 'extended' already points to the modern conception of reinterpretation. For a discussion of imaginary elements in (projective) geometry see [31, 43 ff.]. Wilson [81] contains an accessible presentation as well as a philosophical assessment of the approach developed in Von Staudt's [73].

[32] The basics of his approach are presented in [59, 4 ff.]. Around the same time, Möbius in his [52] had introduced the so-called *barycentric coordinates*, which are a particular kind of homogeneous coordinates. See [31, 143 ff.] for further discussion.

'Ordinary points' and 'points at infinity', for example, are both described by means of ratios of numbers. Because of this, several intuitive ideas that involve elements at infinity also receive a sharper formulation, for instance, the idea that the conic sections (circle, ellipse, parabola, and hyperbola) are projectively equivalent, distinguished from each other only by the number of points they have in common with the 'line at infinity'.[33] Secondly, the analytic approach naturally extends to the complex domain and enables us to deal with 'imaginary elements' in a precise way. Finally, Plücker's analytic approach also contains a new way to think about duality that was later exploited by Plücker and his followers on a much broader scale.

Usually, we think of the letters u_1, u_2, u_3 in an equation like (INC) as *constants* and the letters x_1, x_2, x_3 as *variables*. From this point of view, the equation is understood as the equation of a line in point-coordinates. Plücker realized that this is really just a matter of perspective. We might as well think of the x's as constants and the u's as variables. By doing so, (INC) becomes the equation of a point in 'line-coordinates'. Just like on the first perspective the equation (INC) represents a line, considered as a set of points that lie on it, from the second perspective it represents a point, considered as a set of lines that pass through it.[34]

Let's pause for a moment. Starting point of Plücker's version of projective geometry is a particular analytic representation of the real projective plane, essentially, what we've been calling the 'homogeneous model' in Section 3. Of course, Plücker does not think of this representation as a 'model' of some axiomatic theory that is formulated in some formal language with certain primitive terms. So, strictly speaking, 'reinterpretation' in the sense specified earlier is out of the picture. But the idea of alternative interpretations enters the picture once we start contemplating about the geometric meaning that is associated with *numerical variables* ("general symbols," to use Plücker's phrase). The fact that the contribution of numerical variables in the basic equation (INC) is symmetrical shows that one and the same number triple can be interpreted as either representing a *point* or a *line*. Hence, anything that can be established about points and lines must therefore have a dual correlate, and it is this fact that explains duality.

Plücker's point of view was subsequently taken up by other geometers in the analytic camp. Hesse, for example, notes in his *Vier Vorlesungen aus der Analytischen Geometrie* from 1866 that dual theorems emerge through a "dual geometrical interpretation of the same analytical formula by exchanging point coordinates and line coordinates" [36, 31]. Again, for Hesse it is a matter of *our interpretation* if certain symbols are to be conceived

[33]Desargues was the first to realize that the "most significant properties" of a conic section are projectively invariant, i.e., preserved under projection, and that, therefore, all conic sections are projectively equivalent. See [23, 53].

[34]In our presentation from Section 3, we effectively used line coordinates as interpretations of line-variables in the standard model. The idea of taking the coefficients in the characteristic equations of geometrical objects as their coordinates was later used by many other geometers. The idea was often couched in 'ontic' terms, i.e., in terms of the arbitrariness of 'space elements'. See, e.g., [45, 123 ff.], [46, 15 ff.], and [53, 189 ff.] for discussion.

as coordinates for points or lines. Another example for this point of view is provided by Arthur Cayley. After introducing the basic setup of analytic projective geometry, he writes in his *Introductory Memoir upon Quantics* from 1854:

> The theory includes as a very particular case, the ordinary theory of reciprocity in plane geometry; we have only to say that the word 'point' shall mean 'line', and the word 'line' shall mean 'point', and that expressions properly or primarily applicable to a point and a line respectively shall be construed to apply to a line and a point respectively, and any theorem (assumed of course to be a purely descriptive [i.e., projective, G.E.] one) relating to points and lines will become a corresponding theorem relating to lines and points. [10, 246]

What is remarkable about this passage is that Cayley uses quotation marks and explicit semantic terminology ("x means y") to describe the principle of duality. Cayley is not entirely clear here, but what he seems to be saying is that in a projective context we can take the word "line" to mean whatever is ordinarily meant by the word "point" provided that we take the word "point" to mean what is ordinarily meant by the word "line." If this reading is correct, then this suggests that Cayley thinks that the terms "point" and "line" can indeed be associated with 'interpretations' that differ from their ordinary ones. In his famous *Sixth Memoir* from 1859, referring back to the passage just quoted, he notes in a similar spirit that "the terms employed are not (unless this is done expressly or by the context) restricted to their ordinary significations" [11, 61]. Again, this is not yet the kind of 'reinterpretability' where different interpretations are assigned to primitive non-logical terms of a formal language on a large scale. But it was certainly part of what provided the ground for this point of view.[35]

We have seen that inchoate versions of the notion of reinterpretation naturally came up in connection with duality in the analytic tradition of projective geometry. But it also appears in a more direct way at the turn of the twentieth century in an axiomatic framework, where projective geometry is not based on a particular analytic representation of the real projective plane, but on basic truths or axioms. This can be seen, for example, in remarks by David Hilbert. Hilbert is considered to be one of the foremost proponents of a modern understanding of axioms, where primitive terms are understood as

[35]It is worth mentioning as an aside that in the analytic branch of projective geometry, the idea of reinterpretation is also linked to the formality of proof in a particular way. For example, Plücker notes that "to every proof that can be carried out through the connection of general symbols, there correspond two propositions that are connected to each other by the principle of reciprocity" [59, VIII]. See also [59, IX] and [36, 31 ff., 57]. While the formality of mathematical proof as we came to understand it in the wake of the rise of modern logic cannot be compared in a straightforward way to the kind of formality that is involved in the 'reinterpretation' of algebraic equations, still, this idea was certainly catalytic for the subsequent emergence of formal conceptions of mathematical proof and consequence. For further discussion of this point see [53, 192].

schematic terms that are amenable to reinterpretation, a point of view that was crucial for his investigations of the mutual independence and consistency of various groups of geometrical axioms in his seminal *Grundlagen der Geometrie* [39]. So it is interesting to see how Hilbert understands duality.[36]

Despite the central importance that many of his contemporaries were attaching to the principle of duality, Hilbert himself has remarkably little to say on this topic during the 1890s. One exception is provided by his notes for the lecture "Projektive Geometrie" from 1891. Here, Hilbert presents several "basic laws" of projective geometry in the introductory section in the familiar parallel column format and he gives an informal example of two theorems and their dual proofs. He notes that "through the way in which I ordered the eight propositions in pairs the principle of duality emerges clearly, a principle that is of great importance and fruitfulness as it portions the entire material in two groups of propositions" [33, 29]. As noted by Haubrich in his introduction to the lecture [33, 19], while Hilbert states several "basic laws" of projective geometry, these are not presented as a system of axioms in a way that compares to his axiomatic presentation of Euclidean geometry in his *Grundlagen* (Haubrich refers to them as "proto-axioms"). Still, given the basic setup that closely resembles Gergonne's, one would expect Hilbert to lean towards an axiomatic conception of duality. And yet, the lecture notes contain no indication of such a view. Hilbert mentions duality once again in Section 5, titled "Pol und Polare." After introducing the notions of Pole and Polar with respect to a conic, he then states the "Hauptsatz der Polarentheorie," and notes its importance because it would contain a proof of the principle of duality.[37]

Hilbert's lecture notes are incomplete and were addressed at students. But his sketchy remarks on duality suggest that, at least in the early 1890s, he thought that duality is to be justified in terms of polar reciprocity, essentially along the lines of Poncelet. In particular, dual axioms and proofs play no role in his justification of duality, and no reference is made to reinterpretation. This seems to have changed in the subsequent years. During the 1890s, perhaps under the influence of Pasch, Hilbert got more and more interested in foundational questions concerning geometry, culminating in his seminal *Grundlagen der Geometrie* from 1899.[38] In the ensuing controversy with

[36]The following discussion heavily relies on [65], where the authors focus on Hilbert's views on duality as a way to assess his methodology related to independence and consistency proofs.

[37]The *Hauptsatz* says that the polars of all points on a line p pass through the pole P of this line, and that, conversely, the poles of all lines that pass through a given point lie on the polars of these points. See [38, 78]. As noted by Haubrich in [33, 18 ff.], Hilbert's presentation of projective geometry was heavily influenced by Reye's *Geometrie der Lage*, in particular when it comes to duality. For example, Hilbert mentions exactly the same "proto-axioms" as Reye [62, 22]. In Section 8 of Reye's *Geometrie der Lage*, Reye also refers to the "Hauptsatz der Polarentheorie" [62, 80] and he infers from this the general principle of reciprocity.

[38]In a letter to Klein from 1893, Hilbert explicitly acknowledges Pasch's influence on him, noting that "I think that Pasch's ingenious book is the best way to gain insight about the controversy among geometers over the axioms." (Cited after [76, 44–45].) For more on Hilbert's foundational views during the 1890s, see [22, 76].

Frege, Hilbert elaborates on his conception of axioms and mathematical language in the following often-quoted passage:

> But it is surely obvious that every theory is only a scaffolding or schema of concepts together with their necessary relations to one another, and that the basic elements can be thought of in any way one likes. If in speaking of my points I think of some system of things, e.g., the system: love, law, chimneysweep ... and then assume all my axioms as relations between these things, then my propositions, e.g., Pythagoras' theorem, are also valid for these things. In other words: any theory can always be applied to infinitely many systems of basic elements. One only needs to apply a reversible one–one transformation and lay it down that the axioms shall be correspondingly the same for the transformed things. This circumstance is in fact frequently made use of, e.g., in the principle of duality, etc., and I have made use of it in my independence proofs. [24, 40]

This passage raises a number of questions, but it certainly contains a clear statement of the modern conception of axioms as schemas that are amenable to reinterpretation. For our purposes, the main question is how this idea of reinterpretation relates to Hilbert's understanding of the principle of duality, which is explicitly mentioned at the end of this passage. Hilbert doesn't elaborate on the connection, but two interpretations suggest themselves.

On both interpretations, Hilbert's talk of propositions being "valid" for different "systems of things" and of theories "applying" to "systems of basic elements" is understood as an informal expression of the concept of *truth in an interpretation*. More specifically, on a first reading, Hilbert in this passage appeals to an informal conception of semantic consequence, which is characterized in terms of truth-preservation with respect to all reinterpretations. Hilbert's reference to duality may then be explained by noting that by the symmetrical nature of projective geometry, any suitable system of axioms $\mathcal{P}$ for projective geometry that "applies" to some interpretation also "applies" to the interpretation in which the interpretations of "point" and "line" are interchanged. But then, since a proposition P is a consequence of $\mathcal{P}$ just in case it is "valid" with respect to every interpretation where the axioms in $\mathcal{P}$ are, it follows that the dual of P must also be "valid" in every interpretation of $\mathcal{P}$. In short, Hilbert's conception of duality, on this reading, would correspond to Duality 2.

On the second reading, by saying that "this circumstance is in fact frequently made use of", Hilbert is referring specifically to the use of "reversible one–one transformations." The point he is referring to, on this reading, is that structure-preserving transformations between different interpretations preserve "validity." That is, different interpretations that are related by an appropriate transformation "validate" the same propositions. Accordingly, Hilbert's remark about one–one transformations contains an informal statement of something like the isomorphism lemma mentioned in Section 3. If this is correct, then his remark that "this circumstance is in fact frequently made use of" seems to point to an informal version of duality along the lines of DUALITY 1, where duality is justified by exhibiting an isomorphism between the real projective plane and its dual.[39]

Hilbert was not the only one to think that the idea of reinterpretation is involved in an adequate explanation of duality. Another prominent example is Mario Pieri, one of the members in the group of mathematicians surrounding Giuseppe Peano in Turin, who have been acknowledged to have anticipated several ideas that are commonly attributed to Hilbert. In his *Della geometria elementare come sistema ipotetico deduttivo* from 1900, Pieri explains the "hypothetico-deductive" character of geometrical axioms as follows:

> For the goals of the purely deductive method, it is useful to preserve the greatest indetermination possible for the content of the primitive ideas, which must never appear nor be used other than by dint of the logical relations expressed in the postulates or primitive propositions. For that reason, let us not be obliged to associate with these two terms "point" and "motion" any concrete or even specific image—the system as a whole, which is generically affirmed about them in the primitive propositions just mentioned, suffices for the understanding of everything. It remains nevertheless in the faculty of the reader to attach to these words an arbitrary interpretation, provided that it should not be contradictory to our premises. [58, 178]

So Pieri is no less explicit than Hilbert when it comes to the idea that geometrical axioms are not tied to a particular interpretation of the primitive terms.[40] What is striking is that Pieri, just like Hilbert, specifically mentions duality in this context. In a footnote to the quoted passage, he notes that "[i]n this multiplicity and variety of possible interpretations (and I might almost say mutability of significance) of the primitive ideas, one notices a *law of plurality*, a clear example of which is provided by the *duality* that is encountered in projective geometry" [58, 178].[41] Similar to Hilbert, Pieri does not spell out what the precise connection between duality and the idea of reinterpretation is. But it is nonetheless striking that Pieri too mentions duality as a "clear example" where this idea is at work.

The discussion in this section should have made clear that, at the very least, duality was an important inspiration for the semantic idea of reinterpretation. We could observe this both in the analytic tradition of projective geometry and in the context of early axiomatics. Moreover, in

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[39]See [65, 80] for such an interpretation as well as further discussion.

[40]Marchisotto and Smith note that "[since] Pieri did not view the intuitive notion of space as the subject of geometry, but instead held a concept of space satisfying certain conditions, his idea of space was open to all interpretations that fulfill those conditions" [50, 127]. Pieri's views and those of other members of the Peano group (such as Fano and Padoa) are also appreciated in [26, 618]. For more on the role of the Italian geometers in establishing modern axiomatics see [26], [77, 218 ff.], [4], and [31, 261 ff.].

[41]I am indebted and extremely grateful to Prof. Elena Marchisotto for drawing my attention to this passage.

some instances, a connection is made between duality and the idea of reinterpretation as it figures in independence and consistency proofs, which is routinely taken to be one of the main roots of modern model theory.

4.3. Duality, transfer, and the harbingers of early metatheory. Modern formal logic as it emerged during the twentieth century is concerned with the mathematical study of formal systems of logic and formalized mathematical theories. As such, metatheoretical problems, like the consistency or various forms of completeness of mathematical theories or systems of logic, are built right into the modern conception of logic. Investigations of this kind require precisely defined formal systems, which is why the term 'early metatheory' is sometimes used to indicate that one is concerned with investigations in the early twentieth century, when people where just starting to delineate precise systems and concepts.[42] In what follows I want to make plausible that some of the issues dealt with in the context of discussions of duality had a catalytic effect on the formation of early metatheoretic concepts and problems. On a general level, we have seen this already in connection with the emergence of formal notions of logical consequence and the idea of reinterpretation. In this section we will be concerned with two further, more specific issues. First, we will look at nineteenth century geometer's discussions of *transfer principles*. Secondly, we want to see how reflections on the relative scopes of different versions of duality came to be conceptualized, specifically in the work of Pasch.

The idea that structure-preserving mappings give rise to systematic correlations between truths about structures that are related by such a mapping was well-entrenched in nineteenth century geometry. In the jargon of the time this was often discussed under labels like 'transfer', 'transformation', or 'translation'. Indeed, projective duality was considered to be a prime example of this phenomenon. In what follows we want to look more closely at how this came about and how this relates to the emergence of early forms of metatheoretic reasoning.[43]

As we saw earlier, according to Poncelet, duality is a consequence of the theory of poles and polars, where points and lines are correlated by means of a certain construction relative to some conic section. Michel Chasles had already observed in his *Aperçu historique sur l'origine et le développement des méthodes en géométrie* from 1837 that this specific correlation has "accidental" properties that are not required to justify duality and that (in the planar case) all that is required is that points on a line are mapped to

[42]See, e.g., [5, 6, 66, 67] for more on the development of various metatheoretical concepts during that period.

[43]The significance of transfer principles for the formation of metatheoretical conceptions has been emphasized in the context of a discussion of Frege's views on metatheory in [75], an article that was a major inspiration for the current paper. More recently, transfer principles and their significance for structuralist conceptions of mathematics have been discussed in [64].

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lines through a point, in modern terminology, that 'incidence is preserved'.[44] But while Chasles clearly saw that Poncelet was wrong in believing that the justification of duality requires polar theory, Chasles still agrees with Poncelet (and against Gergonne) that duality should be understood in terms of certain mappings rather than dual axioms and proofs. Neither Poncelet nor Chasles was entirely explicit about how such mappings support the transition from certain truths to others. Lacking the modern notions of a formal language, including a clear distinction between syntax and semantics, their discussions where usually cast in terms of figures and transformations that act on them. Poncelet, for example, maintains that relations among objects in one figure are "immediately translated into similar relations" in the other and that geometrical objects are "substituted" for others when passing from a figure to its reciprocal [60, 59, Vol. II].

The idea of transfer emerges in a sharper form in the analytic branch of projective geometry. Hesse's discussion of transfer ("Übertragung") in his *Vier Vorlesungen aus der analytischen Geometrie* from 1866 is an important example of this. According to Hesse, one of the main reasons for the importance of duality derives from the fact that it provides for more economy: we only have to know half the theorems plus the principle of duality in order to master the entire field, an advantage that was routinely emphasized in nineteenth century discussions of duality.[45] Hesse emphasizes that, because of this, transfer principles like duality are much more important than single theorems [36, 32]. Indeed, he realizes that duality is just one instance of a more general phenomenon and that transfer principles can arise between two domains that may at first appear to be entirely unrelated. One example is his own "Übertragungsprincip," which he had formulated in his [35].[46] The principle relates the complex projective plane with the complex projective line. Each point in the plane is associated with a unique pair of points on the line (also called the "fundamental line") and vice versa. According to Hesse, this gives us "a transfer principle, which reduces [zurückführt] the geometry in the plane to the geometry on the straight line and vice versa" [36, 50]. After determining the analytic "mode of transfer" by which points of the plane are associated with pairs of points on the fundamental line (a quadratic equation), Hesse states two "fundamental propositions," which state that three points of the plane that are collinear are mapped to three pairs of points on the fundamental line that are in involution and vice versa [36, 52]. After further developing the theory, Hesse explains in the concluding remarks: "As we have interpreted the stated equations doubly, one can interpret doubly all those symbolic equations through

[44]See [12, Chapter 5, Sections 29–37]. Chasles' views on the broader significance of duality are discussed in [53, 185 ff.].

[45]This point has been explored in a systematic way in [21] . I thank an anonymous referee for reminding me of this paper.

[46]The principle is also discussed in the third and fourth lecture of [36]. A modern discussion of Hesse's *Übertragungsprinzip* is provided in [63, 179 ff.]. An excellent historical discussion can be found in [34]. I thank an anonymous referee for pointing out this reference.

which propositions in the plane are proved, and this gives the opportunity to discover a great number of propositions of the geometry in the line" [36, 57]. He then gives two examples of this by placing two theorems next to each other in the parallel column format, which is clearly intended to be reminiscent of ordinary duality.

Hesse does not explain what exactly he means when he says that one geometry is "reduced" to another. Also, his discussion is shaped by his analytic approach to geometry. But Hesse clearly articulates in an analytic setting two important ideas. The first is the idea that one-to-one mappings between domains of objects that preserve certain properties give rise to systematic correlations between truths about these domains. The second is the notion that, in this way, two geometries can be *compared*, and sometimes *reduced* to one another. In the case at hand, Hesse takes his Übertragungsprincip to establish that the geometry of point pairs on a line is 'essentially the same' as the geometry of points in a plane. Hesse's discussion of transfer principles is an important instance where the preoccupation with duality inspired broadly structuralist conceptions of geometry.[47] But it is also interesting for its 'metatheoretical' style of reasoning. Geometrical theorems are not proved on the basis of this or that geometrical proposition or axiom. Rather, via transfer principles, entire classes of theorems are set into correspondence with others in virtue of general properties that relate to their logical form or the structures they describe.

It is worth mentioning in this context that the conception of transfer that is at issue here was not unique to geometry. In one of the most-discussed passages in Dedekind's *Was sind und was sollen die Zahlen*? from 1888, Dedekind too talks about "transfer" ("Übertragung") in a similar context. After showing that his conditions for simply infinite systems are categorical, Dedekind notes that a proposition about the natural numbers as defined by him "possesses general validity for every other simply infinite system Ω," and that the "transfer from N to Ω (e.g., also the translation of an arithmetic proposition from one language into another) is affected by the mapping ψ considered in (132) and (133), which transforms every element n of N into an element ν of Ω" [19, 42–43]. Here, the mapping ψ is the isomorphism mentioned in Dedekind's categoricity proof.[48] So Dedekind takes it that

[47] Klein's *Erlangen Programme* may be viewed as belonging into this ballpark. In the section titled "Übertragung durch Abbildung" in his [43], Klein explains how two geometries can be compared, and in the subsequent section he specifically mentions Hesse's Übertragungsprincip as an example that would establish the equivalence of the geometry of point pairs on the line (theory of binary forms) and the ordinary projective geometry of the plane. See again [34]. Klein's views are also discussed in [77, 190 ff.] and [53]; his structuralism is discussed in [64] and, from a category-theoretic perspective, in [51, 12 ff.].

[48] In the book, Dedekind had shown that any "simply infinite system" is "similar" (i.e., isomorphic) to the natural numbers N as defined by him (and, thus, that any two simply infinite systems are isomorphic to each other) by inductively defining an isomorphism ψ between N and any such system Ω. Dedekind's *Was sind und was sollen die Zahlen*?, especially his structuralism, is discussed in detail in [61].

the existence of an isomorphism enables us to "transfer" truths about one system to any other system that is isomorphic.

I think it is reasonably clear that Dedekind's argumentation is informed by the same kind of reasoning that also informed Hesse's discussion in the context of geometry. My contention is that reasoning in terms of transfer principles is something of a surrogate for genuine model-theoretic reasoning at a time when reinterpretable formal languages had not yet become the gold standard. Specifically, transfer principles enabled geometers to formulate something that is close in spirit to the isomorphism lemma without assuming a reinterpretable language in the modern sense. Similar to the isomorphism lemma, transfer principles relate truths about two domains of objects that are related by a structure-preserving mapping. But whereas from a modern point of view such a mapping establishes that certain reinterpretable sentences are 'true in' either structure, on the transfer-based view, the same mapping establishes a correlation between pairs of *distinct* sentences that are assumed to be interpreted over their respective domain of objects.[49]

For our second example where preoccupation with duality led to informal metatheoretic reasoning in a quite specific way, we will go back to Pasch's *Vorlesungen* to see how Pasch reflects on the *scopes* of different conceptions of duality. The first passage where the issue of scope is raised is right after the statement of the axiomatic conception of solid duality in Section 12, where he notes that, up to this point, duality has only been established "with a certain limitation" [54, 94]. Pasch then elaborates on what he means by this "limitation." He notes that every theorem that can be "accessed" at this point is a consequence of the theorems from Chapters 7–9, and that, therefore, graphical (i.e., projective) geometry must confine itself to drawing conclusions from the graphical propositions in Chapters 7–9 "until an increase of its material is possible by adding new basic principles" [54, 95]. Pasch then presents his axiomatic justification of solid duality (see Section 4.1) and notes that the duality between point and plane is "valid, at least within the indicated bounds, but it has to be examined anew later on" [54, 96]. After the discussion of two other "transfer laws," Pasch comes back to this point:

> In the justification of the duality between points and lines on a plane and the duality between the lines and planes on a point we have used the duality between points and planes. Since this could not be proved without a certain limitation, those are afflicted with a corresponding limitation. We will return to the general law of duality once further basic laws have been established (Sections 16 and 18) in order to

[49] A similar point is made in [51, 26–27, fn. 13], where Marquis notes that "[i]n the axiomatic framework, the principle of transference becomes the principle of isomorphism," which he thinks is expressed in the quote from Hilbert's letter to Frege cited earlier. Marquis adds that the principle works differently in the axiomatic framework, because "whereas the principle of isomorphism starts with one axiomatic theory and allows us to see how the models are basically the same, the principle of transference starts in a way with different theories and allows us to see how the theories are fundamentally the same."

> free it from the mentioned limitation. Once this has happened, the reservation concerning the two special duality laws will become empty. [54, 98]

Note that Pasch distinguishes between a "general law of duality" and restricted forms that are based on certain sets of propositions which would not, however, provide a basis for *all* of graphical geometry. As announced, Pasch returns to duality in Section 16. At the beginning of Section 16, he refers back to Section 12 and emphasizes once again the restriction to graphical propositions that can be deduced from those in Sections 7–9. He then adds two further "stem propositions" and notes that "the expanded group retains the property that 'point' and 'plane' are interchangeable, and since, in what follows, we don't want to pursue graphical geometry beyond this extended group of stem propositions, for our purposes the reciprocity between points and planes is already evident enough" [54, 127].

Pasch comes back to duality for one last time in Section 18, titled "Reciprocal Figures." The chapter is concerned with dual figures in space that are related by means of a one-to-one correlation between points and planes with respect to a "null system."[50] Pasch notes that such a correlation preserves graphical properties [54, 142] and concludes:

> From this it follows without exception that to every figure A that is arbitrarily composed of points, lines and planes there exists another figure A' that is composed of the reciprocal elements, and which possesses from all the graphical properties of A the reciprocal ones and no others. If we then denote by α and β the graphical properties of a figure, by α' and β' the reciprocal properties, which naturally presuppose the reciprocal elements, and if we assume that the property α always implies [nach sich zieht] the property β, then α' always entails [hat zur Folge] β'; because if one passes from a figure A with property α' to the figure A' by means of a null system, then A' possesses the property α and therefore also β, and therefore the figure A has the property β'.
>
> With this the law of duality between point and plane is established without any qualification, and therefore also the two other duality laws of projective geometry. Although it was certain that the three kinds of reciprocity are valid for all consequences of the stem propositions mentioned thus far, we now see that these laws must find general application in projective geometry, regardless of whether one can provide further stem propositions or not. [54, 142]

Now, it is not obvious how exactly Pasch's reasoning in this passage is to be translated into modern mathematical language. My reading is that, lacking the modern notions of a formal language and a structure for such a language,

[50]The concept of a 'null system' is a three-dimensional generalization of the planar concept of a conic section and has been introduced by Möbius. See [31, 347–348] for further discussion.

his talk about "figures" and "properties" is a pre-modern surrogate for talk about structures and reinterpretable sentences. Specifically, talk about structures and their duals is replaced by talk about "figures" and their "reciprocals," and talk about reinterpretable propositions that are true or false in a model is replaced by talk about fully interpreted "properties" of figures. In any case, even though Pasch's argumentation may not be entirely transparent from a modern point of view, two things seem to be clear. First, Pasch's aim here is to establish the "general law of duality," which says that *every* correct graphical proposition can be dualized, not just the consequences of some given set of "stem propositions." Secondly, this aim is achieved by considering a particular structure-preserving mapping. Hence, Pasch's "general law of duality" should be understood as an informal version of DUALITY 1, that is, our precise version of the functional conception.

What is interesting about Pasch's discussion is that, first, unlike earlier geometers, Pasch does not think of the axiomatic and the functional conception of duality as 'rivals', where one is 'better' or 'closer to the true state of affairs'. Rather, both are valid conceptions in their own right. Secondly, though, Pasch makes it clear that the axiomatic conception of duality is of *limited scope*, because it is tied to particular systems of axioms (or "stem propositions") that may or may not provide a complete description of the intended structure described by the axioms (whatever it may be). By contrast, the functional conception gives us the "general law of duality" that holds for *all* of projective geometry.[51] Finally, it is in the context of his discussion of the scopes of different versions of duality that Pasch alludes to the idea of 'completeness' of a system of axioms several times. Pasch does not state this explicitly, but his discussion suggests that he believed that *if* his system of axioms *were* extended to form a complete system, then the axiomatic justification in terms of dual proofs *would* give us the "general law of duality." As we saw in Section 3, this can be made precise for rigorous versions of duality that correspond to Pasch's informal versions of the axiomatic and functional conception. Of course, Pasch is still lacking the conceptual tools that are required to discuss such issues with full rigour. Still, it is striking that he came across such questions specifically in connection with duality.

§5. Conclusion. We have seen that discussions about duality, especially in the first-third of the nineteenth century, were often dominated by the contrast between informal versions of the functional and the axiomatic conception of duality. This changed as soon as modern axiomatics had entered the stage in the wake of Hilbert's *Grundlagen*. From this point on,

[51]Of course, lacking a fully modern conception of axiomatics, Pasch still was not entirely clear about the conceptual difference between *truth* with respect some intended model of a certain set of axioms and *theoremhood* with respect to that set. But he arguably came closer to making such a distinction than most geometers before him. This is certainly due to the fact that he came closer to a modern axiomatic treatment of geometry than others before him.

people tended to think of the axiomatic conception of duality as somehow prior to the functional conception. This can be observed throughout the twentieth century, and it is also reflected in the assessments of early historians of nineteenth century geometry, such as Nagel.[52] In this article we have seen that *both* versions were important for the further development of modern conceptions of logic and mathematics in quite specific ways. First of all, preoccupation with duality was instrumental to the development of modern conceptions of logical consequence as a relation that holds between sentences in virtue of their logical form. Secondly, debates about duality also furthered the formation of the semantic idea that mathematical language is amenable to reinterpretation and thereby contributed to the rise of modern formal languages and the modern study of axiom systems in terms of their models. Finally, while nineteenth century geometers did not yet have the sophisticated conceptual apparatus that is available nowadays, attempts to precisely formulate and justify different versions of duality, and relate them to each other, also involved informal metatheoretical reasoning that would later become part and parcel of modern logic. Obviously, duality is not the only issue that was important in these respects. But it does appear to be a significant one.

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[52] See, e.g., [53, 184]. The priority of the axiomatic conception seems to have been established early on. In their classic [78], Veblen and Young discuss the axiomatic conception on page 26, while functional duality is only mentioned in passing on page 268. Modern standard textbooks on projective geometry like [16, 16] and [15] have essentially followed their lead. For a discussion of this point see [47, 101].

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