Canad. J. Math. Vol. 73 (1), 2021 pp. 108–130
http://dx.doi.org/10.4153/S0008414X19000531

SL(n) Invariant Valuations on Super-Coercive Convex Functions

Fabian Mussnig

Abstract. All non-negative, continuous, SL(n), and translation invariant valuations on the space of super-coercive, convex functions on $\mathbb{R}^n$ are classified. Furthermore, using the invariance of the function space under the Legendre transform, a classification of non-negative, continuous, SL(n), and dually translation invariant valuations is obtained. In both cases, different functional analogs of the Euler characteristic, volume, and polar volume are characterized.

1 Introduction and Main Results

At the Paris ICM in 1900, David Hilbert asked the following question: Given two polytopes with equal volume, can one of them be cut into finitely many pieces that can be used to yield the other? It was already known that this is possible in the 2-dimensional case, but the higher dimensional cases were still open. In the same year, Max Dehn was able to construct two polytopes that have the same volume but cannot be cut and reassembled to yield each other. Thus, the answer to Hilbert's question is no for dimensions greater than or equal to 3. In his proof, Dehn used the so-called Dehn invariant and made substantial use of its valuation property. To be more precise, let $\mathcal{K}^n$ denote the space of convex bodies, *i.e.,* compact, convex sets, in $\mathbb{R}^n$. A map $\mu\colon \mathcal{Q}^n \subseteq \mathcal{K}^n \to \mathbb{R}$ is called a *valuation* whenever

$$\mu(K) + \mu(L) = \mu(K \cup L) + \mu(K \cap L)$$

for every $K, L \in \mathcal{Q}^n$ such that $K \cup L, K \cap L \in \mathcal{Q}^n$. Since Dehn's proof, valuations have been studied extensively in convex and discrete geometry, and a first classification result was established by Blaschke in the 1930s [11]. He proved that linear combinations of the Euler characteristic and the n-dimensional volume are the only continuous, SL(n), and translation invariant valuations on $\mathcal{K}^n$. Here, continuity is understood with respect to the Hausdorff metric.

Important generalizations of Blaschke's result have been obtained since then [1, 30, 35, 36]. Recently, Haberl and Parapatits generalized Blaschke's result to valuations defined on $\mathcal{K}^n_{(o)}$, the set of convex bodies in $\mathbb{R}^n$ that contain the origin in their interiors. Note that by restricting to a smaller space, it is possible that more valuations appear in a classification result. In this case, not only the n-dimensional volume, V_n, and the Euler characteristic, V_0, were characterized, but also the

Received by the editors June 25, 2019; revised October 9, 2019.
Published online on Cambridge Core October 25, 2019.
AMS subject classification: **26B25**, 46A40, 52A20, 52A41, 52B45.
Keywords: valuation, convex function, super-coercive, Legendre transform, SL(n) invariance, polar volume.

polar volume $V_n^*(K) := V_n(K^*)$. Here, $K^* = \{x \in \mathbb{R}^n : x \cdot y \le 1 \ \forall y \in K\}$ is the *polar body* of $K \in \mathcal{K}_{(o)}^n$, where $x \cdot y$ denotes the inner product of $x, y \in \mathbb{R}^n$.

Theorem 1.1 ([27]) *For $n \ge 2$, a map $\mu: \mathcal{K}_{(o)}^n \to \mathbb{R}$ is a continuous and* SL(n) *invariant valuation if and only if there exist constants $c_0, c_1, c_2 \in \mathbb{R}$ such that*

$$\mu(K) = c_0 V_0(K) + c_1 V_n(K) + c_2 V_n^*(K) \tag{1.1}$$

for every $K \in \mathcal{K}_{(o)}^n$.

Here, a valuation $\mu: \mathcal{K}_{(o)}^n \to \mathbb{R}$ is said to be SL(n) *invariant* if $\mu(\phi K) = \mu(K)$ for every $\phi \in \mathrm{SL}(n)$ and $K \in \mathcal{K}_{(o)}^n$.

In recent years, the notion of valuation was extended to function spaces. Let $\mathcal{S}$ be a space of (extended) real-valued functions on $\mathbb{R}^n$. We say that a map $\mathrm{Z}: \mathcal{S} \to \mathbb{R}$ is a *valuation* whenever

$$\mathrm{Z}(u) + \mathrm{Z}(v) = \mathrm{Z}(u \vee v) + \mathrm{Z}(u \wedge v)$$

for every $u, v \in \mathcal{S}$ such that also $u \vee v, u \wedge v \in \mathcal{S}$. Here, $u \vee v$ and $u \wedge v$ denote the pointwise maximum and minimum of the functions $u, v \in \mathcal{S}$, respectively. In particular, valuations on Sobolev spaces [31, 33, 37], L^p spaces [34, 42, 52, 53], on definable functions [8] and on quasi-concave functions [12, 14–16, 40] were studied and characterized. See also [2, 17, 18, 32, 50, 51, 54, 55].

For convex functions, an analog of Blaschke's characterization of continuous, SL(n), and translation invariant valuations was established in [19]. More recently, this result was improved, and a functional analog of Theorem 1.1 was found in [41]. Thereby, functional versions of the Euler characteristic and volume together with a new analog of the polar volume were characterized. In order to state this result, let $\mathrm{Conv_c}(\mathbb{R}^n, \mathbb{R})$ denote the space of all convex, coercive functions $u: \mathbb{R}^n \to \mathbb{R}$. Here, a function u is said to be *coercive* if

$$\lim_{|x| \to \infty} u(x) = +\infty.$$

We equip $\mathrm{Conv_c}(\mathbb{R}^n, \mathbb{R})$ with the topology associated with pointwise convergence (see also Section 2). If $\mathcal{S}$ is closed under translations, that is, $u \circ \tau^{-1} \in \mathcal{S}$ for every $u \in \mathcal{S}$ and translation τ on $\mathbb{R}^n$, then a map $\mathrm{Z}: \mathcal{S} \to \mathbb{R}$ is called *translation invariant* if $\mathrm{Z}(u \circ \tau^{-1}) = \mathrm{Z}(u)$ for every $u \in \mathcal{S}$ and translation τ on $\mathbb{R}^n$. Furthermore, if $\mathcal{S}$ is closed under SL(n) transforms, then Z is said to be SL(n) invariant if $\mathrm{Z}(u \circ \phi^{-1}) = \mathrm{Z}(u)$ for every $u \in \mathcal{S}$ and $\phi \in \mathrm{SL}(n)$.

Theorem 1.2 ([41]) *For $n \ge 2$, a map $\mathrm{Z}: \mathrm{Conv_c}(\mathbb{R}^n, \mathbb{R}) \to [0, \infty)$ is a continuous,* SL(n) *and translation invariant valuation if and only if there exist continuous functions $\zeta_0, \zeta_1, \zeta_2: \mathbb{R} \to [0, \infty)$ where ζ_1 has finite moment of order $n-1$ and $\zeta_2(t) = 0$ for all $t \ge T$ with some $T \in \mathbb{R}$ such that*

$$\mathrm{Z}(u) = \zeta_0\big(\min_{x \in \mathbb{R}^n} u(x)\big) + \int_{\mathbb{R}^n} \zeta_1\big(u(x)\big)\,\mathrm{d}x + \int_{\operatorname{dom} u^*} \zeta_2\big(\nabla u^*(x) \cdot x - u^*(x)\big)\,\mathrm{d}x \tag{1.2}$$

for every $u \in \mathrm{Conv_c}(\mathbb{R}^n, \mathbb{R})$.

https://doi.org/10.4153/S0008414X19000531 Published online by Cambridge University Press

Here, a function $\zeta\colon \mathbb{R} \to [0,\infty)$ has *finite moment of order* $n-1$ if

$$\int_0^\infty t^{n-1}\zeta(t)\,\mathrm{d}t < +\infty.$$

Note that for functions $u \in \mathrm{Conv_c}(\mathbb{R}^n,\mathbb{R})$, the minimum is attained and hence finite. For a convex function u on $\mathbb{R}^n$,

$$u^*(x) = \sup_{y\in\mathbb{R}^n} \big(x\cdot y - u(y)\big), \qquad x \in \mathbb{R}^n$$

denotes the *Legendre transform* or *convex conjugate* of u. Moreover, $\operatorname{dom} u^* = \{x \in \mathbb{R}^n : u^*(x) < +\infty\}$ denotes the *domain of* u^*, which is needed, since u^* might attain the value $+\infty$. Lastly, ∇u^* denotes the *gradient* of u^*. Note that it follows from Rademacher's theorem (see for example [21, Theorem 3.1.6]) that the convex function u^* is differentiable almost everywhere on the interior of its domain.

Remark Observe, that (1.1) can be retrieved from (1.2) if u is chosen to be $\|\cdot\|_K$, the norm with unit ball $K \in \mathcal{K}^n_{(o)}$.

We will show that the statement of Theorem 1.2 is still true on the space

$$\mathrm{Conv_{sc}}(\mathbb{R}^n,\mathbb{R}) := \{u\colon \mathbb{R}^n \to \mathbb{R} : u \text{ is convex and super-coercive}\},$$

where we say that a function u, defined on $\mathbb{R}^n$, is *super-coercive* if

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|} = +\infty.$$

It is a priori not clear that no new valuations appear on $\mathrm{Conv_{sc}}(\mathbb{R}^n,\mathbb{R})$. Note that the proof of Theorem 1.2 made extensive use of functions that are coercive but not super-coercive. Furthermore, to the best of the author's knowledge, there does not seem to be an easy way to generalize the proof to the setting of super-coercive functions.

Moreover, we want to point out that the space $\mathcal{K}^n_{(o)}$ is invariant under the polarity transform, that is, $\{K^* : K \in \mathcal{K}^n_{(o)}\} = \mathcal{K}^n_{(o)}$. Results of Artstein-Avidan and Milman [4] show that the Legendre transform is the only natural functional analog of the polarity transform on the space of proper, lower semicontinuous convex functions on $\mathbb{R}^n$. In contrast to the space $\mathrm{Conv_c}(\mathbb{R}^n,\mathbb{R})$, the space $\mathrm{Conv_{sc}}(\mathbb{R}^n,\mathbb{R})$ is invariant under the Legendre transform, that is,

$$\{u^* : u \in \mathrm{Conv_c}(\mathbb{R}^n,\mathbb{R})\} \neq \mathrm{Conv_c}(\mathbb{R}^n,\mathbb{R}),$$
$$\{u^* : u \in \mathrm{Conv_{sc}}(\mathbb{R}^n,\mathbb{R})\} = \mathrm{Conv_{sc}}(\mathbb{R}^n,\mathbb{R}).$$

In that sense, $\mathrm{Conv_{sc}}(\mathbb{R}^n,\mathbb{R})$ seems to be a better functional analog of the space $\mathcal{K}^n_{(o)}$. For further details, see Section 2.

Theorem 1.3 *For $n \geq 2$, a map $\mathrm{Z}\colon \mathrm{Conv_{sc}}(\mathbb{R}^n,\mathbb{R}) \to [0,\infty)$ is a continuous, $\mathrm{SL}(n)$ and translation invariant valuation if and only if there exist continuous functions $\zeta_0,\zeta_1,\zeta_2\colon \mathbb{R}\to[0,\infty)$ where ζ_1 has finite moment of order $n-1$ and $\zeta_2(t) = 0$ for all $t \geq T$ with some $T \in \mathbb{R}$ such that*

$$\mathrm{Z}(u) = \zeta_0\big(\min_{x\in\mathbb{R}^n} u(x)\big) + \int_{\mathbb{R}^n} \zeta_1\big(u(x)\big)\,\mathrm{d}x + \int_{\mathbb{R}^n} \zeta_2\big(\nabla u^*(x)\cdot x - u^*(x)\big)\,\mathrm{d}x \tag{1.3}$$

for every $u \in \mathrm{Conv_{sc}}(\mathbb{R}^n,\mathbb{R})$.

https://doi.org/10.4153/S0008414X19000531 Published online by Cambridge University Press

By using the invariance of $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n, \mathbb{R})$ under the Legendre transform, we also obtain the following equivalent result. Let $\mathcal{S}$ be a space of (extended) real-valued functions on $\mathbb{R}^n$ such that $u + l \in \mathcal{S}$ for every $u \in \mathcal{S}$ and linear functional l on $\mathbb{R}^n$. A map $\mathrm{Z}\colon \mathcal{S} \to \mathbb{R}$ is said to be *dually translation invariant* if $\mathrm{Z}(u + l) = \mathrm{Z}(u)$ for every $u \in \mathcal{S}$ and every linear functional l on $\mathbb{R}^n$. Equivalently, Z is dually translation invariant if and only if $u \mapsto \mathrm{Z}(u^*)$ is translation invariant, where u is such that $u^* \in \mathcal{S}$. Note that for $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n, \mathbb{R})$, also $u + l \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n, \mathbb{R})$ for every linear functional l on $\mathbb{R}^n$.

Theorem 1.3* *For $n \geq 2$, a map $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n, \mathbb{R}) \to [0, \infty)$ is a continuous,* SL(n) *and dually translation invariant valuation if and only if there exist continuous functions $\zeta_0, \zeta_1, \zeta_2\colon \mathbb{R} \to [0, \infty)$ where ζ_1 has finite moment of order $n-1$ and $\zeta_2(t) = 0$ for all $t \geq T$ with some $T \in \mathbb{R}$ such that*

$$\mathrm{Z}(u) = \zeta_0\big(u(0)\big) + \int_{\mathbb{R}^n} \zeta_1\big(u^*(x)\big)\,\mathrm{d}x + \int_{\mathbb{R}^n} \zeta_2\big(\nabla u(x)\cdot x - u(x)\big)\,\mathrm{d}x \tag{1.4}$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n, \mathbb{R})$.

The outline of the paper is as follows. After collecting background material on convex functions in Section 2, we will discuss the operators that appear in Theorem 1.3 and Theorem 1.3* in Section 3. In Section 4, we will introduce the embedding that will be needed in order to prove the main results, which happens in Section 5. A discussion concerning future research on functional inequalities can be found in Section 6. Lastly, the Appendix contains the proof of Lemma 4.2.

2 Convex Functions

We will work in n-dimensional Euclidean space, $\mathbb{R}^n$. Let $\mathrm{Conv}(\mathbb{R}^n)$ denote the space of all convex, proper, lower semicontinuous functions $u\colon \mathbb{R}^n \to (-\infty, \infty]$, where we call a function u on $\mathbb{R}^n$ *proper* if $u \not\equiv +\infty$. We will consider the following subsets of $\mathrm{Conv}(\mathbb{R}^n)$:

$$\begin{aligned}
\mathrm{Conv}_{\mathrm{c}}(\mathbb{R}^n) &= \{u \in \mathrm{Conv}(\mathbb{R}^n) : u \text{ is coercive}\},\\
\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n) &= \{u \in \mathrm{Conv}(\mathbb{R}^n) : u \text{ is super-coercive}\},\\
\mathrm{Conv}_{(o)}(\mathbb{R}^n) &= \{u \in \mathrm{Conv}(\mathbb{R}^n) : 0 \in \operatorname{int}\operatorname{dom} u\},
\end{aligned}$$

where $\operatorname{int} A$ denotes the *interior* of the set $A \subseteq \mathbb{R}^n$. Furthermore, $\mathrm{Conv}_{\mathrm{c}}(\mathbb{R}^n, \mathbb{R})$ and $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n, \mathbb{R})$ will denote the sets of functions in $\mathrm{Conv}_{\mathrm{c}}(\mathbb{R}^n)$ and $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$, respectively, that only take values in $\mathbb{R}$, *i.e.,* functions that do not attain the value $+\infty$.

For $u \in \mathrm{Conv}(\mathbb{R}^n)$ and $t \in \mathbb{R}$, we will write

$$\{u \leq t\} = \{x \in \mathbb{R}^n : u(x) \leq t\}$$

for the *sublevel sets* of u. Since u is convex and lower semicontinuous, the sets $\{u \leq t\}$ are convex and closed. Moreover, since u is proper, there exists $t \in \mathbb{R}$ such $\{u \leq t\} \neq \varnothing$. If, in addition, u is coercive, then all sublevel sets of u are bounded. In particular $\{u \leq t\} \in \mathcal{K}^n$ for all $t \geq \min_{x\in\mathbb{R}^n} u(x)$.

https://doi.org/10.4153/S0008414X19000531 Published online by Cambridge University Press

For $K \in \mathcal{K}^n$, we will denote by

$$\mathrm{I}_K(x) = \begin{cases} 0, & x \in K, \\ +\infty, & x \notin K, \end{cases}$$

the *(convex) indicator function of* K. Observe that $\{\mathrm{I}_K \le t\} = K$ for all $t \ge 0$ and $\mathrm{I}_K \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n)$ for all $K \in \mathcal{K}^n$.

The space $\mathrm{Conv}(\mathbb{R}^n)$ and its subspaces will be equipped with the topology associated with epi-convergence. Here, we say that a sequence $u_k \in \mathrm{Conv}(\mathbb{R}^n)$, $k \in \mathbb{N}$ is *epi-convergent* to $u \in \mathrm{Conv}(\mathbb{R}^n)$ if the following two conditions hold for all $x \in \mathbb{R}^n$:

(a) For every sequence x_k that converges to x,

$$u(x) \le \liminf_{k\to\infty} u_k(x_k).$$

(b) There exists a sequence x_k that converges to x such that

$$u(x) = \lim_{k\to\infty} u_k(x_k).$$

If a sequence u_k is epi-convergent to u, we will write

$$u = \text{epi-lim}_{k\to\infty} u_k \quad \text{and} \quad u_k \xrightarrow{epi} u.$$

As the next result shows, on the space $\mathrm{Conv}(\mathbb{R}^n)$ epi-convergence coincides with local uniform convergence a.e. The only exceptions occur at the boundary of the domain of the limit function.

Theorem 2.1 ([46], Theorem 7.17) *For any epi-convergent sequence of convex functions $u_k\colon \mathbb{R}^n \to (-\infty,\infty]$ the limit function $u = \text{epi-lim}_{k\to\infty} u_k$ is convex. Moreover, under the assumption that $u\colon \mathbb{R}^n \to (-\infty,\infty]$ is convex and lower semicontinuous such that* dom *u has nonempty interior, the following are equivalent:*

(i) $u = \text{epi-lim}_{k\to\infty} u_k$;
(ii) *$u_k(x) \to u(x)$ for all $x \in D$, where D is a dense subset of $\mathbb{R}^n$;*
(iii) *u_k converges uniformly to u on every compact set $C \subset \mathbb{R}^n$ that does not contain a boundary point of* dom *u.*

Remark 2.2 It is a consequence of Theorem 2.1 that epi-convergence coincides with pointwise convergence on $\mathrm{Conv}_{\mathrm{c}}(\mathbb{R}^n,\mathbb{R})$ and $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n,\mathbb{R})$. See also [20, Example 5.13].

For functions in $\mathrm{Conv}_{\mathrm{c}}(\mathbb{R}^n)$, epi-convergence also corresponds to Hausdorff convergence of sublevel sets. In the following, we say that $\{u_k \le t\} \to \varnothing$ as $k \to \infty$ if there exists $k_0 \in \mathbb{N}$ such that $\{u_k \le t\} = \varnothing$ for all $k \ge k_0$.

Lemma 2.3 ([19, Lemma 5] and [9, Theorem 3.1]) *Let u_k, $u \in \mathrm{Conv}_{\mathrm{c}}(\mathbb{R}^n)$. If $u_k \xrightarrow{epi} u$, then $\{u_k \le t\} \to \{u \le t\}$ as $k \to +\infty$ for every $t \in \mathbb{R}$ with $t \ne \min_{x\in\mathbb{R}^n} u(x)$. Furthermore, if for every $t \in \mathbb{R}$ there exists a sequence $t_k \to t$ such that $\{u_k \le t_k\} \to \{u \le t\}$, then $u_k \xrightarrow{epi} u$.*

Next, we want to recall some results about the convex conjugate or Legendre transform

$$u^*(x) = \sup_{y \in \mathbb{R}^n} \big(x \cdot y - u(y)\big)$$

for every $x \in \mathbb{R}^n$ and $u \in \operatorname{Conv}(\mathbb{R}^n)$.

Lemma 2.4 ([49, Theorem 1.6.13]) *If $u \in \operatorname{Conv}(\mathbb{R}^n)$, then also $u^* \in \operatorname{Conv}(\mathbb{R}^n)$ and $u^{**} = u$.*

The following is easy to see and follows directly from the definition of the convex conjugate. See also [41, Section 3]

Lemma 2.5 *Let $\mathrm{Z}\colon \mathcal{S} \to \mathbb{R}$, with $\mathcal{S} \subseteq \operatorname{Conv}(\mathbb{R}^n)$. If $\mathcal{S}$ is closed under translations, then Z is translation invariant if and only if $u \mapsto \mathrm{Z}(u^*)$ is dually translation invariant, where u is such that $u^* \in \mathcal{S}$. Furthermore, if $\mathcal{S}$ is closed under* SL(n) *transforms, then Z is* SL(n) *invariant if and only if $u \mapsto \mathrm{Z}(u^*)$ is* SL(n) *invariant, where u is such that $u^* \in \mathcal{S}$.*

The next lemma shows that the Legendre transform is compatible with the valuation property.

Lemma 2.6 ([18, Lemma 3.4, Proposition 3.5]) *Let $u, v \in \operatorname{Conv}(\mathbb{R}^n)$. If $u \wedge v$ is convex, then so is $u^* \wedge v^*$. Furthermore,*

$$(u \wedge v)^* = u^* \vee v^* \quad \text{and} \quad (u \vee v)^* = u^* \wedge v^*.$$

The following result establishes a connection between coercivity properties of a function and the domain of its conjugate.

Lemma 2.7 ([46, Theorem 11.8]) *For $u \in \operatorname{Conv}(\mathbb{R}^n)$, the following hold true:*

- *u is coercive if and only if $0 \in \operatorname{int} \operatorname{dom} u^*$;*
- *u is super-coercive if and only if $\operatorname{dom} u^* = \mathbb{R}^n$.*

We will also need the following theorem due to Wijsman, which shows that the Legendre transform is a continuous operation (see, for example, [46, Theorem 11.34]).

Theorem 2.8 *If $u_k, u \in \operatorname{Conv}(\mathbb{R}^n)$, then $u_k \xrightarrow{epi} u$ if and only if $u_k^* \xrightarrow{epi} u^*$.*

Let $u \in \operatorname{Conv}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. We call a vector $y \in \mathbb{R}^n$ a *subgradient of u at x* if

$$u(z) \geq u(x) + (z - x) \cdot y$$

for every $z \in \mathbb{R}^n$. The set of all subgradients of u at x is called the *subdifferential of u at x* and is denoted by $\partial u(x)$. Note that $\partial u(x)$ might be empty. Furthermore, if u is differentiable at x, then the only possible subgradient of u at x is the gradient itself and $\partial u(x) = \{\nabla u(x)\}$.

Lemma 2.9 ([45, Theorem 23.5]) *For $u \in \operatorname{Conv}(\mathbb{R}^n)$ and $x, y \in \mathbb{R}^n$, the following are equivalent:*

- $y \in \partial u(x)$,
- $x \in \partial u^*(y)$,
- $x \cdot y = u(x) + u^*(y)$,
- $x \in \operatorname{argmax}_{z\in\mathbb{R}^n}(y \cdot z - u(z))$
- $y \in \operatorname{argmax}_{z\in\mathbb{R}^n}(x \cdot z - u^*(z))$.

Here, $\operatorname{argmax}_{z\in V} f(z)$ denotes the points in the set V at which the function values of f are maximized on V.

For further results on convex functions as well as convex geometry in general we refer to the books of Gruber [26], Rockafellar and Wets [46], and Schneider [49].

3 Valuations on Convex Functions

In this section we discuss the operators that appear in Theorem 1.3 and Theorem 1.3*. Let $\mathcal{S}$ be a space of (extended) real-valued functions on $\mathbb{R}^n$ such that $x \mapsto u_\lambda(x) = u(x/\lambda) \in \mathcal{S}$ for every $u \in \mathcal{S}$ and $\lambda > 0$. In the following, we say that a valuation $\mathrm{Z}\colon \mathcal{S} \to \mathbb{R}$ is *homogeneous of degree* $p \in \mathbb{R}$ if $\mathrm{Z}(u_\lambda) = \lambda^p\,\mathrm{Z}(u)$ for every $u \in \mathcal{S}$ and $\lambda > 0$.

The following operator is a functional analog of the Euler characteristic.

Lemma 3.1 ([19, Lemma 12]) *For a continuous function $\zeta\colon \mathbb{R} \to \mathbb{R}$, the map*

$$u \longmapsto \zeta\big(\min_{x\in\mathbb{R}^n} u(x)\big)$$

defines a continuous, $\mathrm{SL}(n)$*, and translation invariant valuation on* $\mathrm{Conv_c}(\mathbb{R}^n)$ *that is homogeneous of degree* 0.

By combining the last operator with the Legendre transform, we obtain a dually translation invariant valuation.

Lemma 3.2 *For a continuous function $\zeta\colon \mathbb{R} \to \mathbb{R}$, the map*

$$u \longmapsto \zeta\big(\min_{x\in\mathbb{R}^n} u^*(x)\big) = \zeta\big(-u(0)\big) \tag{3.1}$$

defines a continuous, $\mathrm{SL}(n)$*, and dually translation invariant valuation on* $\mathrm{Conv}_{(o)}(\mathbb{R}^n)$ *that is homogeneous of degree 0.*

Proof By Lemma 2.7, $u \in \mathrm{Conv}_{(o)}(\mathbb{R}^n)$ if and only if $u^* \in \mathrm{Conv_c}(\mathbb{R}^n)$. Hence, by Lemma 3.1, the map (3.1) is well defined, and it is easy to see that it is homogeneous of degree 0. The further properties are direct consequences of Lemmas 2.5 and 2.6 and Theorem 2.8. ∎

Remark 3.3 Note that $u \mapsto \zeta\big(-u(0)\big)$ is also well defined on

$$\{u \in \mathrm{Conv}(\mathbb{R}^n) : 0 \in \operatorname{dom} u\} \supset \mathrm{Conv}_{(o)}(\mathbb{R}^n),$$

and in [41, Lemma 4.9] it is wrongfully claimed that even on this larger space (3.1) still defines a continuous, $\mathrm{SL}(n)$, and dually translation invariant valuation. To see that

https://doi.org/10.4153/S0008414X19000531 Published online by Cambridge University Press

this valuation is not continuous anymore, let $\ell_K \in \mathrm{Conv_c}(\mathbb{R}^n)$ be the gauge function associated with K, defined via

$$\{\ell_K \le t\} = tK$$

for every $K \in \mathcal{K}^n$ with $0 \in K$ and $t \ge 0$. Let $P_k := [-1/k, 1] \times [-1,1]^{n-1}$ and let $u_k = \ell_{P_k} \circ \tau_k^{-1}$, where $\tau_k(x) = x + e_1/k$ for $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$, where e_1 denotes the first vector of the standard basis of $\mathbb{R}^n$. Observe, that $u_k(0) = 1$ for every $k \in \mathbb{N}$. By Lemma 2.3 it is easy to see that $u_k \xrightarrow{epi} \ell_P$ as $k \to \infty$, where $P := [0,1] \times [-1,1]^{n-1}$ but $\ell_P(0) = 0$. In particular, $0 \in \operatorname{dom} \ell_P$ but $\ell_P \notin \mathrm{Conv}_{(o)}(\mathbb{R}^n)$.

Lemma 3.4 ([19, Lemma 16]) *For a continuous function $\zeta \colon \mathbb{R} \to [0,\infty)$ with finite moment of order $n-1$, the map*

$$u \longmapsto \int_{\operatorname{dom} u} \zeta\big(u(x)\big)\,\mathrm{d}x$$

defines a non-negative, continuous, SL(n)*, and translation invariant valuation on* $\mathrm{Conv_c}(\mathbb{R}^n)$ *that is homogeneous of degree n.*

The next lemma shows that the moment condition for the function ζ is necessary, even if one restricts to super-coercive, convex functions.

Lemma 3.5 *If $\zeta \in C(\mathbb{R})$ is non-negative such that $\int_0^\infty t^{n-1}\zeta(t)\,\mathrm{d}t = \infty$, then there exists $u_\zeta \in \mathrm{Conv_{sc}}(\mathbb{R}^n, \mathbb{R})$ such that*

$$\int_{\mathbb{R}^n} \zeta\big(u_\zeta(x)\big)\,\mathrm{d}x = \infty.$$

Proof Let $t_0 = 0$. By the assumption on ζ, there exists numbers $t_k > 0$, $k \in \mathbb{N}$ such that $\int_{t_{k-1}}^{t_k} t^{n-1}\zeta(t)\,\mathrm{d}t \ge k$ and $t_k - t_{k-1} \ge 1$. Let $r_0 = 0$ and let

$$r_k = \frac{t_k - t_{k-1}}{k^{1/n}} + r_{k-1}$$

for every $k \ge 1$. We now set $v_\zeta(r) = k^{1/n}(r - r_{k-1}) + t_{k-1}$ for every $r_{k-1} \le r < r_k$ and for every $k \ge 1$. Note that by the choice of t_k, we have $\lim_{k\to\infty} r_k = +\infty$, which shows that $v_\zeta(r)$ is well defined on $[0,\infty)$. Furthermore, it is easy to see that $v_\zeta(r_{k-1}) = t_{k-1}$ and

$$\lim_{r\to r_k} v_\zeta(r) = k^{1/n}\Big(\Big(\frac{t_k - t_{k-1}}{k^{1/n}} + r_{k-1}\Big) - r_{k-1}\Big) + t_{k-1} = t_k.$$

Hence, v_ζ is continuous, and furthermore, $v_\zeta'(r) = k^{1/n}$ for $r_{k-1} < r < r_k$ and every $k \ge 1$. In particular, v_ζ' is unbounded, and therefore $x \mapsto u_\zeta(x) := v_\zeta(|x|)$ defines a super-coercive, convex function on $\mathbb{R}^n$. This gives

$$\begin{aligned}
\int_{\mathbb{R}^n} \zeta\big(u_\zeta(x)\big)\,\mathrm{d}x &= n v_n \int_0^{+\infty} r^{n-1}\zeta\big(v_\zeta(r)\big)\,\mathrm{d}r \\
&= n v_n \sum_{k=1}^{+\infty} \int_{r_{k-1}}^{r_k} r^{n-1}\zeta\big(v_\zeta(r)\big)\,\mathrm{d}r = \left| \begin{matrix} t = v_\zeta(r) \\ \mathrm{d}t/\,\mathrm{d}r = k^{1/n} \end{matrix} \right| \\
&= n v_n \sum_{k=1}^{+\infty} \int_{t_{k-1}}^{t_k} \Big(t\frac{1}{k^{1/n}} + r_{k-1} - \frac{t_{k-1}}{k^{1/n}}\Big)^{n-1} \zeta(t)\frac{1}{k^{1/n}}\,\mathrm{d}t,
\end{aligned}$$

where v_n is the volume of the n-dimensional unit ball. If

$$r_{k-1} - \frac{t_{k-1}}{k^{1/n}} \geq 0, \tag{3.2}$$

then the expression above diverges, since

$$\begin{aligned}\int_{\mathbb{R}^n} \zeta\big(u_\zeta(x)\big)\,\mathrm{d}x &= n v_n \sum_{k=1}^{+\infty} \int_{t_{k-1}}^{t_k} \Big(t\frac{1}{k^{1/n}} + r_{k-1} - \frac{t_{k-1}}{k^{1/n}}\Big)^{n-1} \zeta(t) \frac{1}{k^{1/n}}\,\mathrm{d}t \\ &\overset{(3.2)}{\geq} n v_n \sum_{k=1}^{+\infty} \frac{1}{k} \int_{t_{k-1}}^{t_k} t^{n-1}\zeta(t)\,\mathrm{d}t \\ &\geq n v_n \sum_{k=1}^{+\infty} 1,\end{aligned}$$

where the last inequality follows from the definition of t_k. Hence, it remains to prove (3.2), which we will do by induction on k. The statement is obviously true for $k = 1$, since $r_0 = t_0 = 0$. Furthermore, it is easy to see that $r_1 = t_1$ and hence the statement also holds true for $k = 2$. Assume now that the statement holds for a $k \in \mathbb{N}$, that is, $r_{k-1} - \frac{t_{k-1}}{k^{1/n}} \geq 0$. By definition of r_k, we have

$$\begin{aligned} r_k - \frac{t_k}{(k+1)^{1/n}} &= \frac{t_k - t_{k-1}}{k^{1/n}} + r_{k-1} - \frac{t_k}{(k+1)^{1/n}} \\ &= t_k\Big(\frac{1}{k^{1/n}} - \frac{1}{(k+1)^{1/n}}\Big) + r_{k-1} - \frac{t_{k-1}}{k^{1/n}},\end{aligned}$$

which is positive by the induction hypothesis. ∎

To complete this section, we shall list the following lemmas that address the properties of the remaining operators from Theorems 1.3 and 1.3*.

Lemma 3.6 ([41, Lemma 4.3]) *For a continuous function $\zeta\colon \mathbb{R} \to [0, \infty)$ with finite moment of order $n-1$, the map*

$$u \longmapsto \int_{\operatorname{dom} u^*} \zeta\big(u^*(x)\big)\,\mathrm{d}x$$

defines a non-negative, continuous, $\mathrm{SL}(n)$, and dually translation invariant valuation on $\mathrm{Conv}_{(o)}(\mathbb{R}^n)$ that is homogeneous of degree $-n$.

Lemma 3.7 ([41, Lemma 4.6]) *For a continuous function $\zeta\colon \mathbb{R} \to \mathbb{R}$ such that $\zeta(t) = 0$ for all $t \geq T$ with some $T \in \mathbb{R}$, the map*

$$u \mapsto \int_{\operatorname{dom} u^*} \zeta\big(\nabla u^*(x)\cdot x - u^*(x)\big)\,\mathrm{d}x$$

defines a continuous, $\mathrm{SL}(n)$, and translation invariant valuation on $\mathrm{Conv}_{\mathrm{c}}(\mathbb{R}^n, \mathbb{R})$ that is homogeneous of degree $-n$.

Lemma 3.8 ([41, Lemma 4.12]) *For a continuous function $\zeta\colon \mathbb{R} \to \mathbb{R}$ such that $\zeta(t) = 0$ for all $t \geq T$ with some $T \in \mathbb{R}$, the map*

$$u \mapsto \int_{\operatorname{dom} u} \zeta\big(\nabla u(x)\cdot x - u(x)\big)\,\mathrm{d}x$$

defines a continuous, SL(n)*, and dually translation invariant valuation on* $\text{Conv}_{\text{sc}}(\mathbb{R}^n) \cap \text{Conv}_{(o)}(\mathbb{R}^n)$ *that is homogeneous of degree n.*

4 Super-Coercive Approximations

The main idea of the proof of Theorem 1.3 is to utilize a sequence of real-valued functions that can be used to embed $\text{Conv}_{\text{c}}(\mathbb{R}^n)$ into $\text{Conv}_{\text{sc}}(\mathbb{R}^n)$. We will define and study this sequence in the sequel.

For $k \in \mathbb{N}$, let $\text{sf}_k = \sum_{i=1}^{k} i!$ be the sum of the first k factorials and let $g_k : \mathbb{R} \to \mathbb{R}$ be defined as

$$g_k(r) = \begin{cases} r, & r \le \text{sf}_k, \\ \text{sf}_k + (k+1)(r - \text{sf}_k), & \text{sf}_k < r \le \text{sf}_k + k!, \\ \text{sf}_{k+j} + (k+j+1)(r - \text{sf}_{k+j-1} - k!), & \text{sf}_{k+j-1} + k! < r \le \text{sf}_{k+j} + k!,\ j \in \mathbb{N}, \end{cases}$$

or equivalently

$$g_k(r) = \begin{cases} r, & r \le \text{sf}_k \\ \text{sf}_{k+j} + (k+j+1)(r - \text{sf}_{k+j-1} - k!), & \text{sf}_{k+j-1} + k! < r \le \text{sf}_{k+j} + k!,\ j \in \mathbb{N}_0. \end{cases}$$

We will need the following properties of the sequence g_k.

Lemma 4.1 *For the sequence $g_k : \mathbb{R} \to \mathbb{R}$, the following properties hold true for every $k \in \mathbb{N}$:*

- (i) $g_k(\text{sf}_{k+j-1} + k!) = \text{sf}_{k+j}$ *for every* $j \in \mathbb{N}_0$.
- (ii) g_k *is continuous.*
- (iii) g_k *is strictly increasing and convex.*
- (iv) $g_k(r) \to r$ *as* $k \to +\infty$ *for every* $r \in \mathbb{R}$.
- (v) *If* $u \in \text{Conv}_{\text{c}}(\mathbb{R}^n)$*, then* $g_k(u) \in \text{Conv}_{\text{sc}}(\mathbb{R}^n)$*, and furthermore, if* $u \in \text{Conv}_{\text{c}}(\mathbb{R}^n, \mathbb{R})$*, then* $g_k(u) \in \text{Conv}_{\text{sc}}(\mathbb{R}^n, \mathbb{R})$.
- (vi) $g_k(u(x)) \ge u(x)$ *for every* $u \in \text{Conv}_{\text{c}}(\mathbb{R}^n)$ *and* $x \in \mathbb{R}^n$.
- (vii) *For* $u \in \text{Conv}_{\text{c}}(\mathbb{R}^n)$ *and* $s \le \text{sf}_k$ *we have* $\{g_k(u) \le s\} = \{u \le s\}$ *and for* $\text{sf}_{k+j-1} + k! < s \le \text{sf}_{k+j} + k!$, $j \in \mathbb{N}_0$ *we have*

$$\{g_k(u) \le s\} = \Big\{u \le \frac{s - \text{sf}_{k+j}}{k+j+1} + \text{sf}_{k+j-1} + k!\Big\}.$$

- (viii) $g_k(u) \xrightarrow{epi} u$ *as* $k \to +\infty$ *for every* $u \in \text{Conv}_{\text{c}}(\mathbb{R}^n)$.
- (ix) *For every translation* τ *on* $\mathbb{R}^n$, $\phi \in \text{SL}(n)$ *and* $u \in \text{Conv}_{\text{c}}(\mathbb{R}^n)$, $g_k \circ (u \circ \tau^{-1}) = (g_k \circ u) \circ \tau^{-1}$ *and* $g_k \circ (u \circ \phi^{-1}) = (g_k \circ u) \circ \phi^{-1}$.
- (x) $g_k(u \vee v) = g_k(u) \vee g_k(v)$ *and* $g_k(u \wedge v) = g_k(u) \wedge g_k(v)$ *for every* $u, v \in \text{Conv}_{\text{c}}(\mathbb{R}^n)$.
- (xi) *If* $u_j \xrightarrow{epi} u$ *in* $\text{Conv}_{\text{c}}(\mathbb{R}^n)$*, then* $g_k(u_j) \xrightarrow{epi} g_k(u)$ *as* $j \to +\infty$.

https://doi.org/10.4153/S0008414X19000531 Published online by Cambridge University Press

Proof (i) This follows directly from the definition of g_k, since

$$\begin{aligned} g_k(\mathrm{sf}_{k+j-1}+k!) &= \mathrm{sf}_{k+j-1} + (k+j)(\mathrm{sf}_{k+j-1}+k! - \mathrm{sf}_{k+j-2} - k!) \\ &= \mathrm{sf}_{k+j-1} + (k+j)\big((k+j-1)!\big) \\ &= \mathrm{sf}_{k+j-1} + (k+j)! \\ &= \mathrm{sf}_{k+j}. \end{aligned}$$

(ii) Since $\sum_{i=0}^{\infty} i! = \infty$, it follows that for every $k \in \mathbb{N}$ and $r \in \mathbb{R}$, either $r \le \mathrm{sf}_k + k!$ or there exists $j \in \mathbb{N}$ such that $\mathrm{sf}_{k+j-1} + k! < r \le \mathrm{sf}_{k+j} + k!$. Furthermore, it is easy to check that for $j \in \mathbb{N}$,

$$\lim_{r \to \mathrm{sf}_{k+j-1}+k!} g_k(r) = \mathrm{sf}_{k+j}.$$

Hence, g_k is continuous, since $g_k(\mathrm{sf}_{k+j-1}+k!) = \mathrm{sf}_{k+j}$.

(iii) This is easy to see, since g_k is a continuous, piecewise linear function with positive and non-decreasing slope.

(iv) This property is immediate, since for every $r \in \mathbb{R}$, there exists $k_0 \in \mathbb{N}$ such that $r \le \mathrm{sf}_k$ for every $k \ge k_0$, and therefore $g_k(r) = r$.

(v) Since g_k is increasing and convex,

$$\begin{aligned} (g_k \circ u)\big(\lambda x + (1-\lambda)y\big) &\le g_k\big(\lambda u(x) + (1-\lambda)u(y)\big) \\ &\le \lambda g_k\big(u(x)\big) + (1-\lambda) g_k\big(u(y)\big), \end{aligned}$$

for every $x, y \in \mathbb{R}^n$ and $0 \le \lambda \le 1$, which shows that $g_k(u)$ is a convex function. Furthermore, since $\lim_{r\to+\infty} (g_k(r))/r = +\infty$ for every $k \in \mathbb{N}$, the function $g_k(u)$ is super-coercive. The claim now follows, since $\operatorname{dom} g_k(u) = \operatorname{dom} u$.

(vi) This can be easily seen, since $g_k(r) = r$ for every $r \le \mathrm{sf}_k$ and g_k is a strictly increasing, convex function.

(vii) This follows directly from the definition of g_k.

(viii) This follows from the last property together with Lemma 2.3.

(ix) This is immediate.

(x) This is a direct consequence of the monotonicity of g_k.

(xi) This follows from (vii) together with Lemma 2.3. ∎

By property (iii) of the last lemma, the function g_k is strictly increasing. Hence, there exists an inverse function $g_k^{-1}\colon \mathbb{R} \to \mathbb{R}$. Furthermore, the particular choice of g_k allows us to construct a function v_l^t as in the next lemma. We shall make use of it in Lemma 5.3, where we need to find a sequence of functions $u_{k,l} \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n, \mathbb{R})$, $k, l \in \mathbb{N}$, such that (among other properties) $g_k \circ u_{k,l}$ is independent of k and $u_{k,l}$ is epi-convergent as $l \to \infty$.

Lemma 4.2 *For every $t \in \mathbb{R}$ there exists a sequence of functions $v_l^t\colon [0,\infty) \to \mathbb{R}$, $l \in \mathbb{N}$ such that the functions $x \mapsto g_k^{-1}(v_l^t(|x|))$ and $x \mapsto v_l^t(|x|)$ are convex, super-coercive and finite on $\mathbb{R}^n$ for every $k \in \mathbb{N}$. Furthermore,*

$$\operatorname{epi-lim}_{l\to\infty} g_k^{-1}(v_l^t(|\cdot|)) = \mathrm{I}_{B^n} + g_k^{-1}(t),$$

where B^n denotes the Euclidean unit ball in $\mathbb{R}^n$.

The construction of v_l^t together with the proof of Lemma 4.2 can be found in the Appendix.

5 Classification of Valuations on $\mathrm{Conv}_{sc}(\mathbb{R}^n, \mathbb{R})$

The basic idea of the proof of our main result is to embed $\mathrm{Conv}_c(\mathbb{R}^n)$ into $\mathrm{Conv}_{sc}(\mathbb{R}^n, \mathbb{R})$ by using the sequence g_k that was introduced in the last section and applying Theorem 1.2.

Lemma 5.1 *Let $(t_k)_{k\in\mathbb{N}}$ and $(b_k)_{k\in\mathbb{N}}$ be strictly monotone sequences of real numbers such that $b_k > 0$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} t_k = \lim_{k\to\infty} b_k = \infty$. Furthermore, let $v\colon [0,\infty) \to [0,\infty]$ be the strictly increasing piecewise linear function such that $v(r) = t_1$ if and only if $r = 0$ and $v'(r) = b_k$ if $r > 0$ is such that $t_k < v(r) < t_{k+1}$. If $u\colon \mathbb{R}^n \to [0,\infty]$ is defined by $u(y) := v(|y|)$ for $y \in \mathbb{R}^n$, then $u \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$, and furthermore,*

$$\nabla u^*(x)\cdot x - u^*(x) = \begin{cases} t_1 & \text{for a.e. } x \in \mathbb{R}^n \text{ s.t. } |x| < b_1, \\ t_k & \text{for a.e. } x \in \mathbb{R}^n \text{ s.t. } b_{k-1} < |x| < b_k,\ k \ge 2. \end{cases}$$

Proof Since $\lim_{k\to\infty} b_k = \infty$, the function v is convex, increasing, and super-coercive on $[0,\infty)$. Hence, it is easy to see that $u(\,\cdot\,) = v(|\cdot|) \in \mathrm{Conv}_{sc}(\mathbb{R}^n)$. Since u^* is a convex function, it is differentiable almost everywhere on the interior of its domain and since u is super-coercive, it follows from Lemma 2.7 that $\mathrm{dom}\, u^* = \mathbb{R}^n$. Therefore, without loss of generality, let $x \in \mathbb{R}^n$ be such that $\nabla u^*(x)$ exists. By Lemma 2.9,

$$\nabla u^*(x)\cdot x - u^*(x) = u\big(\nabla u^*(x)\big),$$

and furthermore,

$$x \in \partial u\big(\nabla u^*(x)\big). \tag{5.1}$$

Hence, it is enough to show that $u(\nabla u^*(x)) = t_1$ if $|x| < b_1$ and $u(\nabla u^*(x)) = t_k$ if $b_{k-1} < |x| < b_k$ for $k \ge 2$.

Since v is a piecewise linear function with slopes $(b_k)_{k\in\mathbb{N}}$, it is easy to see that

$$\{|x| : x \in \partial u(y)\} = \begin{cases} [0, b_1] & \text{if } y = 0, \\ \{b_k\} & \text{if } y \in \mathbb{R}^n \text{ is s.t. } t_k < u(y) < t_{k+1}, \\ [b_{k-1}, b_k] & \text{if } y \in \mathbb{R}^n \text{ is s.t. } u(y) = t_k,\ k \ge 2. \end{cases} \tag{5.2}$$

Hence, if $|x| < b_1$, it follows from (5.1) and (5.2) that $\nabla u^*(x) = 0$ and $u(\nabla u^*(x)) = t_1$. Similarly, if $b_{k-1} < |x| < b_k$ with $k \ge 2$, it follows that $u(\nabla u^*(x)) = t_k$, which concludes the proof. ∎

Lemma 5.2 *For $n \ge 2$, let $\mathrm{Z}\colon \mathrm{Conv}_{sc}(\mathbb{R}^n, \mathbb{R}) \to [0,\infty)$ be a continuous, $\mathrm{SL}(n)$ and translation invariant valuation. For $k \in \mathbb{N}$, there exist continuous functions ζ_0^k, ζ_1^k, $\zeta_2^k\colon \mathbb{R} \to [0,\infty)$ such that ζ_1^k has finite moment of order $n-1$ and $\zeta_2^k(t) = 0$ for every*

https://doi.org/10.4153/S0008414X19000531 Published online by Cambridge University Press

$t \geq T_k$ with some $T_k \in \mathbb{R}$ such that

$$\mathrm{Z}\big(g_k(u)\big) = \zeta_0^k\big(\min_{x\in\mathbb{R}^n} u(x)\big) + \int_{\mathbb{R}^n} \zeta_1^k\big(u(x)\big)\,\mathrm{d}x + \int_{\operatorname{dom} u^*} \zeta_2^k\big(\nabla u^*(x)\cdot x - u^*(x)\big)\,\mathrm{d}x$$

for every $u \in \mathrm{Conv_c}(\mathbb{R}^n,\mathbb{R})$. Furthermore, the limits

$$\lim_{k\to\infty}\int_{\mathbb{R}^n} \zeta_1^k\big(u(x)\big)\,\mathrm{d}x \quad \text{and} \quad \lim_{k\to\infty}\int_{\mathbb{R}^n} \zeta_2^k\big(\nabla u^*(x)\cdot x - u^*(x)\big)\,\mathrm{d}x$$

exist and are finite for every $u \in \mathrm{Conv_{sc}}(\mathbb{R}^n,\mathbb{R})$. Moreover,

$$\zeta_0^k\big(\min_{x\in\mathbb{R}^n} u(x)\big) = \zeta_0\big(\min_{x\in\mathbb{R}^n} g_k(u(x))\big)$$

$$\int_{\mathbb{R}^n} \zeta_1^k\big(u(x)\big)\,\mathrm{d}x = \lim_{m\to\infty}\int_{\mathbb{R}^n} \zeta_1^m\big(g_k(u(x))\big)\,\mathrm{d}x$$

$$\int_{\operatorname{dom} u^*} \zeta_2^k\big(\nabla u^*(x)\cdot x - u^*(x)\big)\,\mathrm{d}x = \lim_{m\to\infty}\int_{\mathbb{R}^n} \zeta_2^m\big(\nabla(g_k\circ u)^*(x)\cdot x - (g_k\circ u)^*(x)\big)\,\mathrm{d}x$$

for every $k\in\mathbb{N}$ and $u \in \mathrm{Conv_c}(\mathbb{R}^n,\mathbb{R})$, where $\zeta_0 : \mathbb{R} \to [0,\infty)$ is a continuous function such that $\zeta_0^k(t) = \zeta_0(t)$ for every $t \leq \sum_{i=1}^k i!$.

Proof By Lemma 4.1, the map

$$u \longmapsto \mathrm{Z}\big(g_k(u)\big)$$

defines a continuous, $\mathrm{SL}(n)$, and translation invariant valuation on $\mathrm{Conv_c}(\mathbb{R}^n,\mathbb{R})$ for every $k\in\mathbb{N}$. Hence, by Theorem 1.2, there exist continuous functions $\zeta_0^k,\zeta_1^k,\zeta_2^k : \mathbb{R} \to [0,\infty)$ such that ζ_1^k has finite moment of order $n-1$ and $\zeta_2^k(t) = 0$ for every $t \geq T_k$ with some $T_k\in\mathbb{R}$ such that

$$\mathrm{Z}\big(g_k(u)\big) = \zeta_0^k\big(\min_{x\in\mathbb{R}^n} u(x)\big) + \int_{\mathbb{R}^n} \zeta_1^k\big(u(x)\big)\,\mathrm{d}x + \int_{\operatorname{dom} u^*} \zeta_2^k\big(\nabla u^*(x)\cdot x - u^*(x)\big)\,\mathrm{d}x$$

for every $u \in \mathrm{Conv_c}(\mathbb{R}^n,\mathbb{R})$.

Next, fix an arbitrary $v \in \mathrm{Conv_{sc}}(\mathbb{R}^n,\mathbb{R})$ and let $t_0 = \min_{x\in\mathbb{R}^n} v(x)$. Furthermore, for $\lambda > 0$ let $v_\lambda(x) = v(x/\lambda)$ for $x\in\mathbb{R}^n$. Note that by Lemma 4.1 we have $g_k(v_\lambda) \xrightarrow{epi} v_\lambda$ as $k\to\infty$. Hence, by the continuity of Z, Lemmas 3.4 and 3.7,

$$\begin{aligned}
\mathrm{Z}(v_\lambda) &= \lim_{k\to\infty} \mathrm{Z}\big(g_k(v_\lambda)\big)\\
&= \lim_{k\to\infty}\Big(\zeta_0^k(t_0) + \int_{\mathbb{R}^n} \zeta_1^k\big(v_\lambda(x)\big)\,\mathrm{d}x + \int_{\mathbb{R}^n} \zeta_2^k\big(\nabla v_\lambda^*(x)\cdot x - v_\lambda^*(x)\big)\,\mathrm{d}x\Big)\\
&= \lim_{k\to\infty}\Big(\zeta_0^k(t_0) + \lambda^n\int_{\mathbb{R}^n} \zeta_1^k\big(v(x)\big)\,\mathrm{d}x + \lambda^{-n}\int_{\mathbb{R}^n} \zeta_2^k\big(\nabla v^*(x)\cdot x - v^*(x)\big)\,\mathrm{d}x\Big).
\end{aligned}$$

In particular, the limit on the right-hand side exists and is finite. Considering linear combinations of the last equation with different values of λ shows that

$$\lim_{k\to\infty}\int_{\mathbb{R}^n} \zeta_1^k\big(v(x)\big)\,\mathrm{d}x \quad \text{and} \quad \lim_{k\to\infty}\int_{\mathbb{R}^n} \zeta_2^k\big(\nabla v^*(x)\cdot x - v^*(x)\big)\,\mathrm{d}x$$

exist and are finite. Moreover, the limit $\lim_{k\to\infty}\zeta_0^k(t_0)$ exists and is finite. Since $v\in\operatorname{Conv_{sc}}(\mathbb{R}^n,\mathbb{R})$ and therefore also $t_0\in\mathbb{R}$ were arbitrary, there exists a function $\zeta_0\colon\mathbb{R}\to[0,\infty)$ such that $\zeta_0^k\to\zeta_0$ pointwise as $k\to\infty$. Since $t\mapsto \mathrm{Z}(u+t)$ is continuous for every $u\in\operatorname{Conv_{sc}}(\mathbb{R}^n,\mathbb{R})$, the function ζ_0 must be continuous as well.

Next, let $u\in\operatorname{Conv_c}(\mathbb{R}^n,\mathbb{R})$ and let $k\in\mathbb{N}$ be arbitrary. By Lemma 4.1, we have $g_k(u)\in\operatorname{Conv_{sc}}(\mathbb{R}^n,\mathbb{R})$, and therefore

$$\begin{aligned}
\zeta_0^k\Big(\min_{x\in\mathbb{R}^n}u(x)\Big)+\int_{\mathbb{R}^n}\zeta_1^k\big(u(x)\big)\,\mathrm{d}x+\int_{\operatorname{dom}u^*}\zeta_2^k\big(\nabla u^*(x)\cdot x-u^*(x)\big)\,\mathrm{d}x&\\
=\mathrm{Z}\big(g_k(u)\big)&\\
=\lim_{m\to\infty}\mathrm{Z}\big(g_m(g_k(u))\big)&\\
=\lim_{m\to\infty}\zeta_0^m\Big(\min_{x\in\mathbb{R}^n}g_k(u(x))\Big)+\lim_{m\to\infty}\int_{\mathbb{R}^n}\zeta_1^m\big(g_k(u(x))\big)\,\mathrm{d}x&\\
+\lim_{m\to\infty}\int_{\mathbb{R}^n}\zeta_2^m\big(\nabla(g_k\circ u)^*(x)\cdot x-(g_k\circ u)^*(x)\big)\,\mathrm{d}x.&
\end{aligned}$$

By homogeneity and the definition of ζ_0, we therefore obtain

$$\begin{aligned}
\zeta_0^k\Big(\min_{x\in\mathbb{R}^n}(u(x))\Big)&=\zeta_0\Big(\min_{x\in\mathbb{R}^n}g_k(u(x))\Big)\\
\int_{\mathbb{R}^n}\zeta_1^k\big(u(x)\big)\,\mathrm{d}x&=\lim_{m\to\infty}\int_{\mathbb{R}^n}\zeta_1^m\big(g_k(u(x))\big)\,\mathrm{d}x\\
\int_{\operatorname{dom}u^*}\zeta_2^k\big(\nabla u^*(x)\cdot x-u^*(x)\big)\,\mathrm{d}x&=\\
\lim_{m\to\infty}\int_{\mathbb{R}^n}\zeta_2^m\big(\nabla(g_k\circ u)^*(x)\cdot x&-(g_k\circ u)^*(x)\big)\,\mathrm{d}x. \qquad\blacksquare
\end{aligned}$$

In the sequel we will call the sequence ζ_1^k that appears in Lemma 5.2 the *volume growth function sequence* of the valuation Z. Furthermore, we will refer to the sequence ζ_2^k as the *dual volume growth function sequence*.

Lemma 5.3 *For $n\geq 2$, let $\mathrm{Z}\colon\operatorname{Conv_{sc}}(\mathbb{R}^n,\mathbb{R})\to[0,\infty)$ be a continuous,* SL(n) *and translation invariant valuation with volume growth function sequence ζ_1^k, $k\in\mathbb{N}$. There exists a continuous function $\zeta_1\colon\mathbb{R}\to[0,\infty)$ such that*

$$\zeta_1\big(g_k(t)\big)=\zeta_1^k(t)$$

for every $k\in\mathbb{N}$ and $t\in\mathbb{R}$.

Proof Let $t\in\mathbb{R}$ be arbitrary. Lemma 4.2 shows that $x\mapsto g_k^{-1}(v_l^t(|x|))\in\operatorname{Conv_{sc}}(\mathbb{R}^n,\mathbb{R})$ for every $k,l\in\mathbb{N}$, and by Lemma 5.2, we have

$$\begin{aligned}
\int_{\mathbb{R}^n}\zeta_1^k\big(g_k^{-1}(v_l^t(|x|))\big)\,\mathrm{d}x&=\lim_{m\to\infty}\int_{\mathbb{R}^n}\zeta_1^m\big(g_k(g_k^{-1}(v_l^t(|x|)))\big)\,\mathrm{d}x\\
&=\lim_{m\to\infty}\int_{\mathbb{R}^n}\zeta_1^m\big(v_l^t(|x|)\big)\,\mathrm{d}x.
\end{aligned}$$

https://doi.org/10.4153/S0008414X19000531 Published online by Cambridge University Press

Since $g_k^{-1}(v_l^t(|\cdot|)) \xrightarrow{epi} \mathrm{I}_{B^n} + g_k^{-1}(t)$ as $l \to \infty$, we therefore have by Lemma 3.4,

$$\begin{aligned} V_n(B^n)\zeta_1^k\big(g_k^{-1}(t)\big) &= \int_{\mathbb{R}^n} \zeta_1^k\big(\mathrm{I}_{B^n}(x) + g_k^{-1}(t)\big)\,\mathrm{d}x \\ &= \lim_{l\to\infty} \int_{\mathbb{R}^n} \zeta_1^k\big(g_k^{-1}(v_l^t(|x|))\big)\,\mathrm{d}x \\ &= \lim_{l\to\infty}\lim_{m\to\infty} \int_{\mathbb{R}^n} \zeta_1^m\big(v_l^t(|x|)\big)\,\mathrm{d}x. \end{aligned}$$

Since the right-hand side of this equation is independent of k and only depends on t, this defines a non-negative, continuous function $\zeta_1\colon \mathbb{R} \to [0,\infty)$ such that $\zeta_1(t) = \zeta_1^k(g_k^{-1}(t))$ or equivalently $\zeta_1(g_k(t)) = \zeta_1^k(t)$ for every $t \in \mathbb{R}$. ∎

For a continuous, $\mathrm{SL}(n)$, and translation invariant valuation $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n,\mathbb{R}) \to [0,\infty)$, we call the function ζ_1 from Lemma 5.3 the *volume growth function* of Z.

Lemma 5.4 *For $n \geq 2$, let $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n,\mathbb{R}) \to [0,\infty)$ be a continuous, $\mathrm{SL}(n)$ and translation invariant valuation. The volume growth function ζ_1 has finite moment of order $n-1$, and furthermore,*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \zeta_1^k\big(u(x)\big)\,\mathrm{d}x = \int_{\mathbb{R}^n} \zeta_1\big(u(x)\big)\,\mathrm{d}x$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n,\mathbb{R})$, where ζ_1^k denotes the volume growth function sequence.

Proof Assume that ζ_1 does not have finite moment of order $n-1$. By Lemma 3.5, there exists $u_\zeta \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n,\mathbb{R})$ such that $\int_{\mathbb{R}^n} \zeta_1(u_\zeta(x))\,\mathrm{d}x = +\infty$. For $k \in \mathbb{N}$ let $A_k := \{u_\zeta \leq \sum_{i=1}^k k!\}$. Note that by the properties of u_ζ we have $\bigcup_{k=1}^\infty A_k = \mathbb{R}^n$. Furthermore, by Lemmas 4.1 and 5.3, we have $\zeta_1^k(u_\zeta(x)) = \zeta_1(g_k(u(x))) = \zeta_1(u(x))$ for every $x \in A_k$, and therefore

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \zeta_1^k\big(u_\zeta(x)\big)\,\mathrm{d}x = \lim_{k\to\infty} \Big(\int_{A_k} \zeta_1\big(u_\zeta(x)\big)\,\mathrm{d}x + \int_{\mathbb{R}^n\setminus A_k} \zeta_1^k\big(u_\zeta(x)\big)\,\mathrm{d}x\Big),$$

which must be finite by Lemma 5.2. Since both integrals on the right-hand side are non-negative for every $k \in \mathbb{N}$ and $\int_{A_k} \zeta_1(u_\zeta(x))\,\mathrm{d}x$ is increasing in k, the limit

$$\lim_{k\to\infty} \int_{A_k} \zeta_1\big(u_\zeta(x)\big)\,\mathrm{d}x$$

exists and is finite, which contradicts the choice of $u_\zeta \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n,\mathbb{R})$. Hence, ζ_1 must have finite moment of order $n-1$.

By Lemma 3.4, the map

$$u \longmapsto \int_{\mathbb{R}^n} \zeta_1\big(u(x)\big)\,\mathrm{d}x$$

defines a continuous valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n,\mathbb{R})$, and therefore

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \zeta_1^k\big(u(x)\big)\,\mathrm{d}x = \lim_{k\to\infty} \int_{\mathbb{R}^n} \zeta_1\big(g_k(u(x)\big)\,\mathrm{d}x = \int_{\mathbb{R}^n} \zeta_1\big(u(x)\big)\,\mathrm{d}x,$$

since $g_k(u) \xrightarrow{epi} u$ for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n,\mathbb{R})$. ∎

Lemma 5.5 *For $n \geq 2$, let* $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n, \mathbb{R}) \to [0, \infty)$ *be a continuous,* SL(n), *and translation invariant valuation with dual volume growth function sequence ζ_2^k, $k \in \mathbb{N}$. There exists $T \in \mathbb{R}$ such that $\zeta_2^k(t) = 0$ for every $k \in \mathbb{N}$ and $t \geq T$.*

Proof We will prove the statement by contradiction and assume that there exists a subsequence $\zeta_2^{k_j}$ and monotone increasing numbers $t_{k_j} \in \mathbb{R}$, $j \in \mathbb{N}$ with $\lim_{j\to\infty} t_{k_j} = +\infty$ such that $0 < \zeta_2^{k_j}(t_{k_j}) < 1$, which is possible by the properties of $\zeta_2^{k_j}$. By possibly restricting to another subsequence, we can choose the numbers t_{k_j} as follows. Let $t_{k_1} \in \mathbb{R}$ be arbitrary and set $a_{k_1} = 0$. If t_{k_j} and a_{k_j} are given, let

$$a_{k_{j+1}} = \sqrt[n]{\frac{j}{v_n \zeta_2^{k_j}(t_{k_j})} + a_{k_j}^n},$$

where v_n is the volume of the n-dimensional unit ball and choose $t_{k_{j+1}}$ large enough such that

$$t_{k_{j+1}} - t_{k_j} \geq \max\{1, a_{k_{j+1}}\}.$$

This implies that t_{k_j} and a_{k_j} are strictly monotone increasing sequences such that $\lim_{j\to\infty} t_{k_j} = \lim_{j\to\infty} a_{k_j} = \infty$, and furthermore

$$\frac{t_{k_{j+1}} - t_{k_j}}{a_{k_{j+1}}} \geq \frac{a_{k_{j+1}}}{a_{k_{j+1}}} = 1. \tag{5.3}$$

Next, let $w\colon [0, \infty) \to \mathbb{R}$ be the strictly increasing piecewise affine function such that $w(0) = t_{k_1}$ and $w'(r) = a_{k_{j+1}}$ for every $r \in [0, \infty)$ with $t_{k_j} < w(r) < t_{k_{j+1}}$, $j \in \mathbb{N}$. Note that it follows from (5.3) that $\sum_{j=1}^{\infty} (t_{k_{j+1}} - t_{k_j})/a_{k_{j+1}} = \infty$, which ensures that w is well defined and finite. Furthermore, since a_{k_j} is strictly increasing with $\lim_{j\to\infty} a_{k_j} = \infty$, the function w is super-coercive. Therefore, the function $u\colon \mathbb{R}^n \to \mathbb{R}$, defined by $u(x) = w(|x|)$ for $x \in \mathbb{R}^n$ is also well defined and finite. Furthermore, it follows from Lemma 5.1 that $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n, \mathbb{R})$ and

$$\nabla u^*(x) \cdot x - u^*(x) = t_{k_j}$$

for almost everywhere $x \in \mathbb{R}^n$ such that $a_{k_j} < |x| < a_{k_{j+1}}$, $j \in \mathbb{N}$. Since the maps $\zeta_2^{k_j}$ are non-negative, this gives

$$\begin{aligned} \int_{\mathbb{R}^n} \zeta_2^{k_j}\big(\nabla u^*(x) \cdot x - u^*(x)\big)\,\mathrm{d}x &\geq \int_{a_{k_j} < |x| < a_{k_{j+1}}} \zeta_2^{k_j}(t_{k_j})\,\mathrm{d}x \\ &= v_n(a_{k_{j+1}}^n - a_{k_j}^n)\zeta_2^{k_j}(t_{k_j}) \\ &= v_n\Big(\frac{j}{v_n \zeta_2^{k_j}(t_{k_j})} + a_{k_j}^n - a_{k_j}^n\Big)\zeta_2^{k_j}(t_{k_j}) \\ &= j \end{aligned} \tag{5.4}$$

for every $j \in \mathbb{N}$.

On the other hand, the limit

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \zeta_2^k\big(\nabla u^*(x) \cdot x - u^*(x)\big)\,\mathrm{d}x$$

exists and is finite by Lemma 5.2, which contradicts (5.4). Hence, the initial assumption must be false. ∎

https://doi.org/10.4153/S0008414X19000531 Published online by Cambridge University Press

Lemma 5.6 *For $n \geq 2$, let* $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n, \mathbb{R}) \to [0, \infty)$ *be a continuous,* $\mathrm{SL}(n)$*, and translation invariant valuation with dual volume growth function sequence* ζ_2^k, $k \in \mathbb{N}$. *There exists a continuous function* $\zeta_2\colon \mathbb{R} \to [0,\infty)$ *such that* $\zeta_2(t) = 0$ *for every* $t \geq T$ *with some* $T \in \mathbb{R}$ *and*

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \zeta_2^k\big(\nabla u^*(x)\cdot x - u^*(x)\big)\,\mathrm{d}x = \int_{\mathbb{R}^n} \zeta_2\big(\nabla u^*(x)\cdot x - u^*(x)\big)\,\mathrm{d}x$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n, \mathbb{R})$.

Proof Let $T \in \mathbb{R}$ be as in Lemma 5.5 and let $k_0 \in \mathbb{N}$ be such that $T + 1 \leq \mathrm{sf}_{k_0}$. Furthermore, for $t \leq T$, let $u_t(x) = |x| + t$ for $x \in \mathbb{R}^n$. Note that $u_t^* = \mathrm{I}_B - t$. Moreover, by the definition of u_t and g_k, we can write $g_k(u_t(x)) = w_k^t(|x|)$ with a piecewise linear function $w_k^t\colon [0,\infty) \to \mathbb{R}$ such that $w_k^t(0) = t$, $(w_k^t)'(r) = 1$ for every $r > 0$ such that $t < w_k^t(r) < \mathrm{sf}_k$ and $(w_k^t)'(r) \geq k+1$ for almost everywhere every r such that $w_k^t(r) > \mathrm{sf}_k > T$ for every $k \geq k_0$. Hence, by Lemma 5.1 we have for every $k \geq k_0$,

$$\nabla(g_k \circ u_t)^*(x)\cdot x - (g_k\circ u_t)^*(x) = t$$

for almost everywhere $x \in \mathbb{R}^n$ with $|x| < 1$, and furthermore,

$$\nabla(g_k \circ u_t)^*(x)\cdot x - (g_k\circ u_t)^*(x) > T$$

for almost everywhere $x \in \mathbb{R}^n$ with $|x| > 1$. Therefore, by Lemma 5.2,

$$\begin{aligned} V_n(B^n)\zeta_2^k(t) &= \int_{B^n} \zeta_2^k\big(\mathrm{I}_{B^n}(x) + t\big)\,\mathrm{d}x \\ &= \int_{\mathrm{dom}\, u_t^*} \zeta_2^k\big(\nabla u_t^*(x)\cdot x - u_t^*(x)\big)\,\mathrm{d}x \\ &= \lim_{m\to\infty} \int_{\mathbb{R}^n} \zeta_2^m\big(\nabla(g_k\circ u_t)^*(x)\cdot x - (g_k\circ u_t)^*(x)\big)\,\mathrm{d}x \\ &= \lim_{m\to\infty} V_n(B^n)\zeta_2^m(t) \end{aligned}$$

for every $t \leq T$ and every $k \geq k_0$. Since $\zeta_2^k(t) = 0$ for every $t > T$ and $k \in \mathbb{N}$, this shows that the sequence ζ_2^k does not change for $k \geq k_0$. Hence, there exists a function $\zeta_2\colon \mathbb{R} \to [0,\infty)$ such that

$$\zeta_2(t) = \lim_{m\to\infty} \zeta_2^m(t) = \zeta_2^k(t)$$

for every $k \geq k_0$ and $t \in \mathbb{R}$. In particular, ζ_2 is continuous and $\zeta_2(t) = 0$ for every $t \geq T$. Furthermore,

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \zeta_2^k\big(\nabla u^*(x)\cdot x - u^*(x)\big)\,\mathrm{d}x = \int_{\mathbb{R}^n} \zeta_2\big(\nabla u^*(x)\cdot x - u^*(x)\big)\,\mathrm{d}x$$

for every $u \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n, \mathbb{R})$. ■

5.1 Proof of Theorem 1.3

By Theorem 1.2, equation (1.3) defines a continuous, $\mathrm{SL}(n)$, and translation invariant valuation on $\mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n, \mathbb{R})$.

Conversely, let $\mathrm{Z}\colon \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n, \mathbb{R}) \to [0,\infty)$ be a continuous, $\mathrm{SL}(n)$ and translation invariant valuation. By Lemma 5.2, there exist continuous functions ζ_0^k, ζ_1^k,

https://doi.org/10.4153/S0008414X19000531 Published online by Cambridge University Press

$\zeta_2^k\colon \mathbb{R} \to [0,\infty)$ such that ζ_1^k has finite moment of order $n-1$ and $\zeta_2^k(t) = 0$ for every $t \geq T_k$ with some $T_k \in \mathbb{R}$ such that

$$\begin{aligned} \mathrm{Z}\big(g_k(u)\big) = \zeta_0^k\big(\min_{x\in\mathbb{R}^n} u(x)\big) + \int_{\mathbb{R}^n} \zeta_1^k\big(u(x)\big)\,\mathrm{d}x \\ + \int_{\operatorname{dom} u^*} \zeta_2^k\big(\nabla u^*(x)\cdot x - u^*(x)\big)\,\mathrm{d}x \end{aligned}$$

for every $u \in \mathrm{Conv_c}(\mathbb{R}^n,\mathbb{R})$ and $k \in \mathbb{N}$. By continuity of Z and Lemma 4.1

$$\begin{aligned} \mathrm{Z}(u) &= \lim_{k\to\infty} \mathrm{Z}\big(g_k(u)\big) \\ &= \lim_{k\to\infty} \Big(\zeta_0^k\big(\min_{x\in\mathbb{R}^n} u(x)\big) + \int_{\mathbb{R}^n} \zeta_1^k\big(u(x)\big)\,\mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^n} \zeta_2^k\big(\nabla u^*(x)\cdot x - u^*(x)\big)\,\mathrm{d}x\Big) \end{aligned}$$

for every $u \in \mathrm{Conv_{sc}}(\mathbb{R}^n)$. By Lemma 5.2, there exists a continuous function $\zeta_0\colon \mathbb{R} \to [0,\infty)$ such that

$$\lim_{k\to\infty} \zeta_0^k\big(\min_{x\in\mathbb{R}^n} u(x)\big) = \zeta_0\big(\min_{x\in\mathbb{R}^n} u(x)\big)$$

for every $u \in \mathrm{Conv_{sc}}(\mathbb{R}^n,\mathbb{R})$. By Lemma 5.4, there exists a continuous function $\zeta_1\colon \mathbb{R} \to [0,\infty)$ that has finite moment of order $n-1$ such that

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \zeta_1^k\big(u(x)\big)\,\mathrm{d}x = \int_{\mathbb{R}^n} \zeta_1\big(u(x)\big)\,\mathrm{d}x$$

for every $u \in \mathrm{Conv_{sc}}(\mathbb{R}^n,\mathbb{R})$. Furthermore, by Lemma 5.6,

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} \zeta_2^k\big(\nabla u^*(x)\cdot x - u^*(x)\big)\,\mathrm{d}x = \int_{\mathbb{R}^n} \zeta_2\big(\nabla u^*(x)\cdot x - u^*(x)\big)\,\mathrm{d}x,$$

where $\zeta_2\colon \mathbb{R} \to [0,\infty)$ is a continuous function such that such that $\zeta_2(t) = 0$ for every $t \geq T$ with some $T \in \mathbb{R}$. Hence, Z must be as in (1.3). ■

5.2 Proof of Theorem 1.3*

Lemmas 3.2, 3.6, and 3.8 show that (1.4) defines a continuous, SL(n) and dually translation invariant valuation on $\mathrm{Conv_{sc}}(\mathbb{R}^n,\mathbb{R})$.

Conversely, let $\mathrm{Z}\colon \mathrm{Conv_{sc}}(\mathbb{R}^n,\mathbb{R}) \to [0,\infty)$ be a continuous, SL(n) and dually translation invariant valuation. By Lemmas 2.5, 2.6, and 2.7, and Theorem 2.8, the map $u \mapsto \mathrm{Z}^*(u) := \mathrm{Z}(u^*)$ defines a continuous, SL(n) and translation invariant valuation on $\mathrm{Conv_{sc}}(\mathbb{R}^n,\mathbb{R})$. Hence, by Theorem 1.3 and Lemma 2.4,

$$\begin{aligned} \mathrm{Z}(u) &= \mathrm{Z}\big((u^*)^*\big) \\ &= \mathrm{Z}^*(u^*) \\ &= \widetilde{\zeta}_0\big(\min_{x\in\mathbb{R}^n} u^*(x)\big) + \int_{\mathbb{R}^n} \zeta_1\big(u^*(x)\big)\,\mathrm{d}x \\ &\qquad + \int_{\mathbb{R}^n} \zeta_2\big(\nabla(u^*)^*(x)\cdot x - (u^*)^*(x)\big)\,\mathrm{d}x \\ &= \widetilde{\zeta}_0\big(-u(0)\big) + \int_{\mathbb{R}^n} \zeta_1\big(u^*(x)\big)\,\mathrm{d}x + \int_{\mathbb{R}^n} \zeta_2\big(\nabla u(x)\cdot x - u(x)\big)\,\mathrm{d}x \end{aligned}$$

https://doi.org/10.4153/S0008414X19000531 Published online by Cambridge University Press

for every $u \in \mathrm{Conv_{sc}}(\mathbb{R}^n, \mathbb{R})$, where $\widetilde{\zeta_0}, \zeta_1, \zeta_2 \colon \mathbb{R} \to [0,\infty)$ are continuous functions such that ζ_1 has finite moment of order $n-1$ and $\zeta_2(t) = 0$ for every $t \geq T$ with some $T \in \mathbb{R}$. The statement now follows by setting $\zeta_0(t) = \widetilde{\zeta_0}(-t)$ for $t \in \mathbb{R}$. ∎

6 Discussion

For convex bodies K that contain the origin in their interiors, the *volume product* $V_n(K)V_n(K^*)$ is of significant interest in convex geometric analysis. In particular,

(6.1) $$c^n V_n(B^n)^2 \leq V_n(K)V_n(K^*) \leq V_n(B^n)^2$$

for every origin symmetric $K \in \mathcal{K}^n_{(o)}$, *i.e.*, $K = -K$, where B^n denotes the Euclidean unit ball and $c > 0$ is an absolute constant. The right side of (6.1) is sharp with the maximizers being ellipsoids [43] and is also known as the Blaschke–Santaló inequality [10, 48]. The left side is due to Bourgain and Milman [13], but the optimal constant c is still not known. The famous Mahler conjecture states that the volume product is minimized for affine transforms of cubes (among others) and a proof for the two-dimensional case is due to Mahler [38]. More recently, the conjecture was confirmed for the three-dimensional case [28], but the general case remains open. Furthermore, the conjecture is known to be true for some special classes of convex bodies, *e.g.*, a proof for 1-unconditional bodies can be found in [47] and a characterization of the corresponding equality cases in [39]; see also [7, 44].

Functional versions of (6.1) for log-concave functions were obtained in [3, 5, 6, 29]. In particular, it was shown that

(6.2) $$\left(\frac{2\pi}{c}\right)^n \leq \int_{\mathbb{R}^n} \exp\big(-u(x)\big)\,\mathrm{d}x \int_{\mathbb{R}^n} \exp\big(-u^*(x)\big)\,\mathrm{d}x \leq (2\pi)^n$$

for suitable convex functions u on $\mathbb{R}^n$, where $c > 0$ is again an absolute constant. Furthermore, Fradelizi and Meyer [24, 25] could prove that the sharp lower bound of this functional volume product is 4^n in the case of 1-unconditional functions u, that is, $u(x_1,\dots,x_n) = u(|x_1|,\dots,|x_n|)$, and the corresponding equality cases were characterized in [22]. Moreover, in [23] Fradelizi and Meyer could also provide upper bounds for similar inequalities where $t \mapsto e^{-t}$ is replaced by more general log-concave functions.

Considering Theorems 1.3 and 1.3* as well as the results mentioned above, the question arises if an inequality similar to (6.2) can be achieved, where $t \mapsto e^{-t}$ is replaced by some continuous function $\zeta \colon \mathbb{R} \to [0,\infty)$ that has finite moment of order $n-1$. Furthermore, one can ask for inequalities using the quantities

$$u \longmapsto \int_{\mathrm{dom}\, u^*} \zeta\big(\nabla u^*(x)\cdot x - u^*(x)\big)\,\mathrm{d}x$$

and/or

$$u \longmapsto \int_{\mathrm{dom}\, u} \zeta\big(\nabla u(x)\cdot x - u(x)\big)\,\mathrm{d}x$$

for suitable functions $\zeta \colon \mathbb{R} \to [0,\infty)$ and convex functions u on $\mathbb{R}^n$.

A Appendix

We will give the construction of the function v_l^t from Lemma 4.2 and discuss its properties.

By definition of the function g_k, $k \in \mathbb{N}$, we can write its inverse function g_k^{-1} as

$$g_k^{-1}(s) = \begin{cases} s, & s \le \mathrm{sf}_k, \\ \mathrm{sf}_{k+j-1} + k! + \frac{s - \mathrm{sf}_{k+j}}{k+j+1}, & \mathrm{sf}_{k+j} < s \le \mathrm{sf}_{k+j+1},\ j \in \mathbb{N}_0. \end{cases} \tag{A.1}$$

Next, for $t \in \mathbb{R}$, let $m_t = \min\{m \in \mathbb{N} : t \le \mathrm{sf}_m\}$ and let

$$A_{l,m}^t = 1 + \frac{\mathrm{sf}_{m_t} - t + (m - m_t)}{l}$$

for $m \ge m_t$. Note that $A_{l,m}^t - A_{l,m-1}^t = \frac{1}{l}$, and therefore $\lim_{m\to\infty} A_{l,m}^t = +\infty$. For $l \in \mathbb{N}$, we define the piecewise linear function $v_l^t \colon [0,\infty) \to \mathbb{R}$ as

$$v_l^t(r) = \begin{cases} t, & 0 \le r \le 1 \\ t + l(r-1), & 1 < r \le A_{l,m_t}^t \\ \mathrm{sf}_m + (m+1)!\,l(r - A_{l,m}^t), & A_{l,m}^t < r \le A_{l,m+1}^t,\ m \ge m_t. \end{cases} \tag{A.2}$$

Note that by this definition,

$$\begin{aligned} v_l^t(A_{l,m}^t) &= \mathrm{sf}_{m-1} + m!\,l(A_{l,m}^t - A_{l,m-1}^t) \\ &= \mathrm{sf}_m = \lim_{r\to (A_{l,m}^t)^-} v_l^t(r) \end{aligned} \tag{A.3}$$

for every $m \ge m_t$, $l \in \mathbb{N}$, and $t \in \mathbb{R}$. Hence, v_l^t is continuous. Moreover, it follows immediately that v_l^t is convex, increasing, super-coercive, and finite on $[0,\infty)$, and, in particular, $x \mapsto v_l^t(|x|) \in \mathrm{Conv}_{\mathrm{sc}}(\mathbb{R}^n, \mathbb{R})$. Furthermore, by Lemma 2.3 it is easy to see that $\text{epi-lim}_{l\to\infty} v_l^t(|\cdot|) = \mathrm{I}_{B^n} + t$.

Next, we will consider the composition $g_k^{-1} \circ v_l^t$. Therefore, fix $t \in \mathbb{R}$ and $k \in \mathbb{N}$. If $t \le \mathrm{sf}_k$, we have by (A.2) and (A.3) that $v_l^t(r) \le \mathrm{sf}_k$ for every $r \le A_{l,k}^t$. Hence, by (A.1),

$$g_k^{-1}(v_l^t(r)) = \begin{cases} t, & 0 \le r \le 1, \\ t + l(r-1), & 1 < r \le A_{l,m_t}^t, \\ \mathrm{sf}_m + (m+1)!\,l(r - A_{l,m}^t), & A_{l,m}^t < r \le A_{l,m+1}^t,\ m_t \le m < k. \end{cases}$$

Furthermore, if $A_{l,m}^t < r \le A_{l,m+1}^t$, $k \le m$, we have by (A.3) that $\mathrm{sf}_m < v_l^t(r) \le \mathrm{sf}_{m+1}$. Therefore, using (A.1) with $k + j = m$ gives

$$\begin{aligned} g_k^{-1}(v_l^t(r)) &= g_k^{-1}\big(\mathrm{sf}_m + (m+1)!\,l\big(r - A_{l,m}^t\big)\big) \\ &= \mathrm{sf}_{m-1} + k! + \frac{\mathrm{sf}_m + (m+1)!\,l\big(r - A_{l,m}^t\big) - \mathrm{sf}_m}{m+1} \\ &= \mathrm{sf}_{m-1} + k! + m!\,l(r - A_{l,m}^t) \end{aligned}$$

https://doi.org/10.4153/S0008414X19000531 Published online by Cambridge University Press

for $A^t_{l,m} < r \le A^t_{l,m+1}$ and $k \le m$. In particular,

$$\begin{aligned} g_k^{-1}\big(v_l^t(A^t_{l,k})\big) &= \mathrm{sf}_{k-1} + k!l(A^t_{l,k} - A^t_{l,k-1}) \\ &= \mathrm{sf}_k \\ &= \lim_{r\to (A^t_{l,k})^-} g_k^{-1}\big(v_l^t(r)\big), \end{aligned}$$

which shows that $g_k^{-1} \circ v_l^t$ is continuous. Furthermore,

$$\frac{\mathrm{d}}{\mathrm{d}r} g_k^{-1}(v_l^t(r)) = k!l$$

for $A^t_{l,k-1} < r < A^t_{l,k+1}$. In particular, the slope of $g_k^{-1} \circ v_l^t$ is increasing, despite the fact that the slope of g_k^{-1} is decreasing. Hence, it is easy to see that $g_k^{-1} \circ v_l^t$ is a convex, increasing, super-coercive, and finite function on $[0, \infty)$. Moreover, if follows from Lemma 2.3 that $\text{epi-lim}_{l\to\infty} g_k^{-1}(v_l^t(|\cdot|)) = \mathrm{I}_{B^n} + t = \mathrm{I}_{B^n} + g_k^{-1}(t)$.

In the case $t > \mathrm{sf}_k$, there exists $j_t \in \mathbb{N}_0$ such that $\mathrm{sf}_{k+j_t} < t \le \mathrm{sf}_{k+j_t+1}$ and therefore, similarly to the case above,

$$g_k^{-1}(v_l^t(r)) = \begin{cases} \mathrm{sf}_{k+j_t-1} + k! + \frac{t-\mathrm{sf}_{k+j_t}}{k+j_t+1}, & 0 \le r \le 1 \\ \mathrm{sf}_{k+j_t-1} + k! + \frac{t-\mathrm{sf}_{k+j_t}+l(r-1)}{k+j_t+1}, & 1 < r \le A^t_{l,m_t} \\ \mathrm{sf}_{m-1} + k! + m!l(r - A^t_{l,m}), & A^t_{l,m} < r \le A^t_{l,m+1},\ m_t \le m. \end{cases}$$

Again, this is a convex, increasing, super-coercive, and finite function on $[0, \infty)$ with $\text{epi-lim}_{l\to\infty} g_k^{-1}(v_l^t(|\cdot|)) = \mathrm{I}_{B^n} + g_k^{-1}(t)$.

Acknowledgments The author would like to thank Andrea Colesanti for pointing out the idea of Lemma 3.5 as well as Florian Besau for his helpful remarks regarding Lemma 5.1. Furthermore, the author wishes to thank the referees for their valuable comments and suggestions.

References

[1] S. Alesker, *Continuous rotation invariant valuations on convex sets*. Ann. of Math. (2) 149(1999), 997–1005. https://doi.org/10.2307/121078

[2] S. Alesker, *Valuations on convex functions and convex sets and Monge–Ampère operators*. Adv. Geom. 19(2019), 313–322. https://doi.org/10.1515/advgeom-2018-0031

[3] S. Artstein-Avidan, B. Klartag, and V. D. Milman, *The Santaló point of a function, and a functional form of the Santaló inequality*. Mathematika 51(2004), 33–48. https://doi.org/10.1112/S0025579300015497

[4] S. Artstein-Avidan and V. D. Milman, *The concept of duality in convex analysis, and the characterization of the Legendre transform*. Ann. of Math. (2) 169(2009), 661–674. https://doi.org/10.4007/annals.2009.169.661

[5] S. Artstein-Avidan and B. A. Slomka, *A note on Santaló inequality for the polarity transform and its reverse*. Proc. Amer. Math. Soc. 143(2015), 1693–1704. https://doi.org/10.1090/S0002-9939-2014-12390-2

[6] K. M. Ball, *Isometric problems in ℓ_p and sections of convex sets*. PhD thesis, Cambridge, 1986.

[7] F. Barthe and M. Fradelizi, *The volume product of convex bodies with many hyperplane symmetries*. Amer. J. Math. 135(2013), 311–347. https://doi.org/10.1353/ajm.2013.0018

[8] Y. Baryshnikov, R. Ghrist, and M. Wright, *Hadwiger's Theorem for definable functions*. Adv. Math. 245(2013), 573–586. https://doi.org/10.1016/j.aim.2013.07.001

[9] G. Beer, R. T. Rockafellar, and R. J.-B. Wets, *A characterization of epi-convergence in terms of convergence of level sets*. Proc. Amer. Math. Soc. 116(1992), 753–761. https://doi.org/10.2307/2159443

[10] W. Blaschke, *Über affine Geometrie VII: Neue Extremeingenschaften von Ellipse und Ellipsoid*. Ber. Verh. Sächs. Akad. Wiss., Math. Phys. Kl. 69(1917), 412–420.

[11] W. Blaschke, *Vorlesungen über Integralgeometrie. H. 2*. Teubner, Berlin, 1937.

[12] S. G. Bobkov, A. Colesanti, and I. Fragalà, *Quermassintegrals of quasi-concave functions and generalized Prékopa–Leindler inequalities*. Manuscripta Math. 143(2014), 131–169. https://doi.org/10.1007/s00229-013-0619-9

[13] J. Bourgain and V. D. Milman, *New volume ratio properties for convex symmetric bodies in* $\mathbb{R}^n$. Invent. Math. 88(1987), 319–340. https://doi.org/10.1007/BF01388911

[14] A. Colesanti and I. Fragalà, *The first variation of the total mass of log-concave functions and related inequalities*. Adv. Math. 244(2013), 708–749. https://doi.org/10.1016/j.aim.2013.05.015

[15] A. Colesanti and N. Lombardi, *Valuations on the space of quasi-concave functions*. In: *Geometric aspects of functional analysis*. Lecture Notes in Math., 2169, Springer, Cham, 2017, pp. 71–105.

[16] A. Colesanti, N. Lombardi, and L. Parapatits, *Translation invariant valuations on quasi-concave functions*. Studia Math. 243(2018), 79–99. https://doi.org/10.4064/sm170323-7-7

[17] A. Colesanti, M. Ludwig, and F. Mussnig, *Minkowski valuations on convex functions*. Calc. Var. Partial Differential Equations 56(2017), Art. 162. https://doi.org/10.1007/s00526-017-1243-4

[18] A. Colesanti, M. Ludwig, and F. Mussnig, *Hessian valuations*. Indiana Univ. Math. J., to appear.

[19] A. Colesanti, M. Ludwig, and F. Mussnig, *Valuations on convex functions*. Int. Math. Res. Not. IMRN 2019 no. 8, 2384–2410. https://doi.org/10.1093/imrn/rnx189

[20] G. Dal Maso, *An Introduction to* Γ*-Convergence*. Progress in Nonlinear Differential Equations and their Applications, 8, Birkhäuser Boston, Inc., Boston, MA, 1993. https://doi.org/10.1007/978-1-4612-0327-8

[21] H. Federer, *Geometric measure theory*. Classics in Mathematics, Springer-Verlag, Berlin, Heidelberg, 1996.

[22] M. Fradelizi, Y. Gordon, M. Meyer, and S. Reisner, *The case of equality for an inverse Santaló inequality*. Adv. Geom. 10(2010), 621–630. https://doi.org/10.1515/ADVGEOM.2010.026

[23] M. Fradelizi and M. Meyer, *Some functional forms of Blaschke–Santaló inequality*. Math. Z. 256(2007), 379–395. https://doi.org/10.1007/s00209-006-0078-z

[24] M. Fradelizi and M. Meyer, *Some functional inverse Santaló inequalities*. Adv. Math. 218(2008), 1430–1452. https://doi.org/10.1016/j.aim.2008.03.013

[25] M. Fradelizi and M. Meyer, *Increasing functions and inverse Santaló inequality for unconditional functions*. Positivity 12(2008), 407–420. https://doi.org/10.1007/s11117-007-2145-z

[26] P. M. Gruber, *Convex and discrete geometry*. Grundlehren der mathematischen Wissenschaften, 336, Springer-Verlag, Berlin, 2007.

[27] C. Haberl and L. Parapatits, *The centro-affine Hadwiger theorem*. J. Amer. Math. Soc. 27(2014), 685–705. https://doi.org/10.1090/S0894-0347-2014-00781-5

[28] H. Iriyeh and M. Shibata, *Symmetric Mahler's conjecture for the volume product in the* 3*-dimensional case*. Duke Math. J., to appear. https://doi.org/doi:10.1215/00127094-2019-0072

[29] B. Klartag and V. Milman, *Geometry of log-concave functions and measures*. Geom. Dedicata 112(2005), 169–182. https://doi.org/10.1007/s10711-004-2462-3

[30] M. Ludwig, *Valuations on polytopes containing the origin in their interiors*. Adv. Math. 170(2002), 239–256. https://doi.org/10.1006/aima.2002.2077

[31] M. Ludwig, *Fisher information and matrix-valued valuations*. Adv. Math. 226(2011), 2700–2711. https://doi.org/10.1016/j.aim.2010.08.021

[32] M. Ludwig, *Valuations on function spaces*. Adv. Geom. 11(2011), 745–756. https://doi.org/10.1515/ADVGEOM.2011.039

[33] M. Ludwig, *Valuations on Sobolev spaces*. Amer. J. Math. 134(2012), 827–842. https://doi.org/10.1353/ajm.2012.0019

[34] M. Ludwig, *Covariance matrices and valuations*. Adv. in Appl. Math. 51(2013), 359–366. https://doi.org/10.1016/j.aam.2012.12.003

[35] M. Ludwig and M. Reitzner, *A classification of* SL(n) *invariant valuations*. Ann. of Math. (2) 172(2010), 1219–1267. https://doi.org/10.4007/annals.2010.172.1223

[36] M. Ludwig and M. Reitzner, SL(n) *invariant valuations on polytopes*. Discrete Comput. Geom. 57(2017), 571–581. https://doi.org/10.1007/s00454-016-9838-7

[37] D. Ma, *Real-valued valuations on Sobolev spaces*. Sci. China Math. 59(2016), 921–934. https://doi.org/10.1007/s11425-015-5101-6

[38] K. Mahler, *Ein Minimalproblem für konvexe Polygone*. Mathematica (Zutphen) B7(1939), 118–127.

[39] M. Meyer, *Une caractérisation volumique de certains espaces normés de dimension finie*. Israel J. Math. 55(1986), 317–326. https://doi.org/10.1007/BF02765029

[40] V. D. Milman and L. Rotem, *Mixed integrals and related inequalities*. J. Funct. Anal. 264(2013), 570–604. https://doi.org/10.1016/j.jfa.2012.10.019

[41] F. Mussnig, *Volume, polar volume and Euler characteristic for convex functions*. Adv. Math. 344(2019), 340–373. https://doi.org/10.1016/j.aim.2019.01.012

[42] M. Ober, *L_p-Minkowski valuations on L^q-spaces*. J. Math. Anal. Appl. 414(2014), 68–87. https://doi.org/10.1016/j.jmaa.2013.12.048

[43] C. M. Petty, *Affine isoperimetric problems*. In: *Discrete geometry and convexity (New York, 1982)*. Ann. New York Acad. Sci., 440, New York Acad. Sci., New York, 1985, pp. 113–127. https://doi.org/10.1111/j.1749-6632.1985.tb14545.x

[44] S. Reisner, *Minimal volume-product in Banach spaces with a 1-unconditional basis*. J. London Math. Soc. (2) 36(1987), 126–136. https://doi.org/10.1112/jlms/s2-36.1.126

[45] R. T. Rockafellar, *Convex analysis*. Princeton Mathematical Series, 28, Princeton University Press, Princeton, NJ, 1970.

[46] R. T. Rockafellar and R. J.-B. Wets, *Variational analysis*, Third ed., Grundlehren der mathematischen Wissenschaften, 317, Springer-Verlag, Berlin, 2009.

[47] J. Saint-Raymond, *Sur le volume des corps convexes symétriques*. In: Initiation Seminar on Analysis: G. Choquet-M. Rogalski-J. Saint-Raymond, 20th Year: 1980/1981, Exp. No. 11, 25 pp., Publ. Math. Univ. Pierre et Marie Curie, 46, Univ. Paris VI, Paris, 1981.

[48] L. A. Santaló, *Un invariante afin para los cuerpos convexos del espacio des n dimensiones*. Portugal. Math. 8(1949), 155–161.

[49] R. Schneider, *Convex Bodies: The Brunn–Minkowski theory*, Second expanded ed., Encyclopedia of Mathematics and its Applications, 151, Cambridge University Press, Cambridge, 2014.

[50] P. Tradacete and I. Villanueva, *Continuity and representation of valuations on star bodies*. Adv. Math. 329(2018), 361–391. https://doi.org/10.1016/j.aim.2018.02.021

[51] P. Tradacete and I. Villanueva, *Valuations on Banach lattices*. Int. Math. Res. Not. IMRN 2020 no. 1, 287–319. https://doi.org/10.1093/imrn/rny129

[52] A. Tsang, *Valuations on L_p-spaces*. Int. Math. Res. Not. IMRN 2010 no. 20, 3993–4023. https://doi.org/10.1090/S0002-9947-2012-05681-9

[53] A. Tsang, *Minkowski valuations on L_p-spaces*. Trans. Amer. Math. Soc. 364(2012), 6159–6186. https://doi.org/10.1090/s0002-9947-2012-05681-9

[54] I. Villanueva, *Radial continuous rotation invariant valuations on star bodies*. Adv. Math. 291(2016), 961–981. https://doi.org/10.1016/j.aim.2015.12.030

[55] T. Wang, *Semi-valuations on* $\mathrm{BV}(\mathbb{R}^n)$. Indiana Univ. Math. J. 63(2014), 1447–1465. https://doi.org/10.1512/iumj.2014.63.5365

School of Mathematical Sciences, Tel Aviv University, 69978 Tel Aviv, Israel
e-mail: mussnig@gmail.com