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On the quenching set for a fast diffusion equation: regional quenching

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We study positive solutions of a very fast diffusion equation, $u_t = (u^{m-1}u_x)_x$, $m < 0$, in a bounded interval, $0 < x < L$, with a quenching-type boundary condition at one end, $u(0,t) = (T-t)^{1/(1-m)}$ and a zero-flux boundary condition at the other, $(u^{m-1}u_x)(L,t) = 0$. We prove that for $m \geqslant -1$ regional quenching is not possible: the quenching set is either a single point or the whole interval. Conversely, if $m < -1$ single-point quenching is impossible, and quenching is either regional or global. For some lengths the above facts depend on the initial data. The results are obtained by studying the corresponding blow-up problem for the variable $v = u^{m-1}$.

1. Introduction and main results

We study the asymptotic behaviour, as $t \to T$, of positive solutions of

$$\left.\begin{aligned} u_t &= (u^{m-1}u_x)_x, && 0<x<L,\ 0<t<T,\\ u(0,t) &= (T-t)^{1/(1-m)}, && 0<t<T,\\ (u^{m-1}u_x)(L,t) &= 0, && 0<t<T,\\ u(x,0) &= u_0(x), && 0<x<L, \end{aligned}\right\} \tag{1.1}$$

where $m < 0$, $L > 0$ and $T > 0$. The initial data are continuous, bounded and strictly positive functions. Under appropriate limits, the equation in problem (1.1) models, for example, for $m = -1$, the diffusion of Cr in GaAs [21], and heat conduction in solid hydrogen [19]. Other values of $m < 0$ can be shown to arise from physically relevant limits of models for the diffusion of Zn and Be in GaAs [22].

Let us briefly motivate the choice of the parameters in problem (1.1). For $m < 1$, due to the singular character of the diffusivity u^{m-1} at level zero, solutions have a strong tendency towards positivity. Conversely, the condition on the left boundary forces the solution to be zero, at least at $x = 0$, at $t = T$. The competition between both facts produces two distinct behaviours depending on m. Indeed, if we take as the Dirichlet condition $u(0,t) = \varepsilon$ and consider the limit $\varepsilon \to 0$, we obtain a limit which is positive in $(0, L]$ for $0 < m < 1$ (fast diffusion) and the trivial limit for $m \leqslant 0$ (very fast diffusion) (see [13, 18]). We concentrate on the very fast diffusion case and consider a boundary condition which goes to zero as $t \to T$. We will show that, in some cases, $\liminf_{t\to T} u(x,t) > 0$ for some $x \in (0, L]$.

On the other hand, the boundary condition $u(0,t) = (T-t)^{\mu}$ provokes a singularity of *quenching* type, whenever $0 < \mu < 1$, in the sense of [12], i.e. $u_t(0,t) \to -\infty$ when $t \to T$. Quenching has deservedly received a great deal of attention in recent years (see, for example, [5, 15, 16]).

In most of the applications, quenching is produced by the presence of a singular nonlinear absorption term, for instance, fixing the flux at the boundary (see [9, 17]). Once the quenching rate at this boundary is known, we can apply the results obtained in this paper to deduce the quenching behaviour of the solution (usually via comparison arguments). A similar procedure is used in the study of blow-up problems with peaking (see [11]).

We are especially interested in studying the possibility of *regional quenching*, i.e. the situation in which the solution vanishes in some interval $[0, L']$ for $t = T$, while remaining positive in the complement $(L', L]$. To the authors' knowledge, there are no previous examples of this phenomenon.

As is often the case in parabolic problems, the asymptotic behaviour is expected to be described by means of special solutions in the self-similar form

$$U(x,t) = (T-t)^{\alpha} F(x(T-t)^{\beta}), \quad \alpha = \mu, \quad \beta = \tfrac{1}{2}(\mu(1-m) - 1).$$

The analogy with the peaking problem, $\mu < 0$ and $m > 1$ (see [11]) suggests that only $\beta = 0$ would make regional quenching possible. This corresponds to our choice of $\mu = 1/(1-m)$. Observe that the restriction $\mu < 1$, needed for quenching, implies $m < 0$. In the case $m = 0$ ($\mu = 1$) we have that u_t is bounded, and thus we have a non-singular extinction behaviour.

In order to perform our analysis, we transform (1.1) into a *blow-up* problem for the variable $v = u^{m-1}$, which is common in the literature on nonlinear diffusion. In terms of v, our problem reads

$$\left.\begin{aligned} v_t &= v v_{xx} - \gamma (v_x)^2, && 0 < x < L,\ 0 < t < T, \\ v(0,t) &= (T-t)^{-1}, && 0 < t < T, \\ v^{-\gamma} v_x(L,t) &= 0, && 0 < t < T, \\ v(x,0) &= v_0(x) \equiv u_0^{-1/\gamma}(x), && 0 < x < L, \end{aligned}\right\} \tag{1.2}$$

where $\gamma = 1/(1-m)$. According to the usual mathematical language (coming from the applications in the case $m > 1$) we shall call (1.2) the *pressure problem*, and v the *pressure*, while u is the *density*. The condition $m < 0$ implies that $0 < \gamma < 1$. The value $\gamma = \frac{1}{2}$, corresponding to $m = -1$, will be shown to be critical. For general references on blow-up see [10, 20]. Observe also that this problem admits more general data than those coming from problem (1.1). In particular, we will also consider compactly supported initial values v_0. This means that u_0 takes a value of infinity on a set of non-zero measure.

As mentioned above, the asymptotic behaviour is expected to be described by self-similar solutions, which in our case means separated variables, i.e.

$$V(x,t) = (T-t)^{-1}G(x). \tag{1.3}$$

Hence, a starting point will be the construction of self-similar profiles G, using the ideas of [6, 8]. In our analysis we have to distinguish between positive profiles and those that vanish somewhere. For a complete description of all possible profiles see § 3.

We will show in §§ 4 and 5 that the asymptotic behaviour of the solution v to problem (1.2) as t approaches T is described by the profiles G, when they exist. Following the standard technique, we introduce the rescaled function

$$g(x,\tau) = (T-t)v(x,t), \quad \tau = -\log(1-t/T).$$

Then g satisfies the parabolic problem

$$\left.\begin{aligned} g_\tau &= gg_{xx} - \gamma(g_x)^2 - g, && 0 < x < L,\ \tau > 0,\\ g(0,\tau) &= 1, && \tau > 0,\\ g^{-\gamma}g_x(L,\tau) &= 0, && \tau > 0,\\ g(x,0) &= g_0(x) \equiv Tv_0(x), && 0 < x < L. \end{aligned}\right\} \tag{1.4}$$

Therefore, the study of the asymptotic behaviour of $v(x,t)$ near the finite blow-up time $T > 0$ is reduced to the study of the stabilization of $g(x,\tau)$ as $\tau \to \infty$ to a stationary profile G. Such stabilization indeed occurs for $\gamma < \frac{1}{2}$. However, for $\gamma \geqslant \frac{1}{2}$ and large L, there are no self-similar profiles, and we prove that g tends to zero for $x > 0$.

Next we use the convergence of the rescaled variable g to study the set of points where v becomes unbounded at time T, the blow-up set, which is defined as follows:

$$B(v) = \{0 \leqslant x \leqslant L : \exists (x_n,t_n) \to (x,T) \text{ such that } v(x_n,t_n) \to \infty\}.$$

It is clear that there exist only three possibilities: global blow-up, $B(v) = [0,L]$; regional blow-up, $B(v) = [0,L']$ for some $0 < L' < L$; or single-point blow-up, $B(v) = \{0\}$. For $\gamma < \frac{1}{2}$ we obtain the following result.

Theorem 1.1. *Assume $\gamma < \frac{1}{2}$. There exists a critical length, $L_* = L_*(\gamma)$, such that:*

(i) *if $0 < L \leqslant L_*$, blow-up can be regional or global depending on the initial data;*

(ii) *if $L > L_*$, blow-up is always regional.*

We will prove in theorem 2.1 that u_t goes to $-\infty$ if and only if u goes to 0. Therefore, since $u^{m-1} = v$, blow-up results for v translate to quenching results for u. In particular, the quenching set of u,

$$\begin{aligned} Q(u) &= \{0 \leqslant x \leqslant L : \exists (x_n,t_n) \to (x,T) \text{ such that } u_t(x_n,t_n) \to -\infty\}\\ &= \{0 \leqslant x \leqslant L : \exists (x_n,t_n) \to (x,T) \text{ such that } u(x_n,t_n) \to 0\} \end{aligned}$$

coincides with the blow-up set of v.

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The unexpected phenomenon regarding the asymptotic behaviour for this value of the parameters ($m < -1$) is that, for a fixed length $L > 0$ in a certain range (see § 4), though there are solutions with regional quenching, other solutions exhibit global quenching. Moreover, we conclude from the stability properties of the profiles described in § 4 that there exists an open set of initial data with regional quenching and another open set of initial data with global quenching. Hence, regional and global quenching are not exceptional phenomena.

Theorem 1.2. *Assume* $\frac{1}{2} \leqslant \gamma < 1$. *There exists a critical length* $L_* = L_*(\gamma)$ *such that:*

(i) *if* $0 < L \leqslant L_*$, *single-point blow-up or global blow-up may occur depending on the initial condition;*

(ii) *if* $L > L_*$, *we have always single-point blow-up.*

The most striking feature, when the above result is translated to the density variable u, is that, for $-1 \leqslant m < 0$, regional quenching is not possible. As before, we can show that for $0 < L \leqslant L_*$ there exists an open set of initial data with single-point quenching and another open set of initial data with global quenching. This has to be contrasted with the case $m < -1$.

Remark 1.3. The blow-up results of the previous theorem also hold for $\gamma = 1$, i.e. $m = 0$. In this case we have described the extinction set of $u = v^{-1}$. That is, for $t = T$ we have the alternative: $u(x, T) = 0$ for every $0 \leqslant x \leqslant L$ or $u(x, T) > 0$ for every $0 < x \leqslant L$.

1.1. Organization of the paper

In § 2 we state some preliminaries. In § 3 we construct the self-similar profiles; § 4 is devoted to the proof of the convergence of the rescaled function g to one of the profiles G, the study of the stability of the profiles and the blow-up set for $\gamma < \frac{1}{2}$. In § 5 we deal with the case $\frac{1}{2} \leqslant \gamma < 1$. Finally, § 6 is devoted to some concluding remarks.

2. Preliminaries

We consider problem (1.1) with the initial data $u_0 \geqslant c_0 > 0$ continuous. In this case, existence and uniqueness of positive classical solutions can easily be established (see, for example, [14]).

Our first result concerning the solution to problem (1.1) shows that the quenching set coincides with the set of points where the solution vanishes.

Theorem 2.1. *Let* u *be the solution to (1.1). Given* $x \in [0, L]$ *and* $(x_n, t_n) \to (x, T)$, *we have that* $u(x_n, t_n) \to 0$ *if and only if* $u_t(x_n, t_n) \to -\infty$.

Proof. First we observe that, if $u(x_n, t_n) \geqslant c > 0$, by standard regularity theory, u_t is uniformly bounded in a sequence of neighbourhoods $B_n \ni (x_n, t_n)$. In order to prove the converse, we use the fact that, for some $t_0 > 0$, the minimum of $u(\cdot, t)$ is achieved at $x = 0$ for every $t_0 < t < T$. This is easily seen to hold for t_0 such that $(T-t_0)^{1/(1-m)} = \min(u_0)$, thanks to the maximum principle. Assume now by contradiction that $u_t(x_n, t_n) \geqslant -C > -\infty$, while $u(x_n, t_n) \to 0$. Then, integrating in $[t, t_n]$ we get

$$u(x_n, t) - u(x_n, t_n) = -\int_t^{t_n} u_t(x_n, s)\,\mathrm{d}s \leqslant C(t_n - t).$$

Taking limits, for $t > t_0$ we have

$$(T-t)^{1/(1-m)} = u(0,t) \leqslant u(x,t) \leqslant C(T-t),$$

which is impossible, since $m < 0$. □

As mentioned in § 1, the solutions to problem (1.1) are studied in terms of the pressure variable $v = u^{m-1}$. It is known that, for the pressure problem (1.2), the most delicate case occurs when v has compact support (see [3,4]). For that type of data there is no uniqueness in general in the class of weak solutions (defined in the standard way). Thus, we will use the concept of the viscosity solution given in [3], which will ensure uniqueness. The construction of this solutions is as follows.

Let w_ε be, for any $\varepsilon > 0$, the unique classical solution to the problem

$$\left.\begin{aligned} w_t &= (w+\varepsilon)w_{xx} - \gamma(w_x)^2, && 0 < x < L,\ 0 < t < T,\\ w(0,t) &= (T-t)^{-1} + \varepsilon, && 0 < t < T,\\ w^{-\gamma} w_x(L,t) &= 0, && 0 < t < T,\\ w(x,0) &= v_0(x) + \varepsilon, && 0 < x < L. \end{aligned}\right\} \tag{2.1}$$

Then the limit function

$$v(x,t) = \lim_{\varepsilon \to 0} w_\varepsilon(x,t) \tag{2.2}$$

is a solution to problem (1.2) in weak sense, and it is called a *viscosity solution*. Moreover, it is the maximal weak solution to that problem. The comparison principle between viscosity solutions is immediately deduced from the construction.

Another interesting property of viscosity solutions proved in [2,4] is the stationary character of the support. In fact, the support is non-expanding for every weak solution, and stationary for the maximal one.

3. The self-similar profiles

In this section we construct the profiles giving the asymptotic behaviour. For a complete description of the existence of the different types of self-similar profiles in the fast diffusion range see [8]. The characterization in terms of the length of the interval is of special importance for us (see below).

We consider the following problem:

$$\left.\begin{aligned} GG'' - \gamma(G')^2 - G = 0, &\quad \text{for } 0 < x < L,\\ G(0) = 1,&\\ G'(L) = 0.& \end{aligned}\right\} \tag{3.1}$$

We observe that if G is strictly positive, the boundary condition at $x = L$ is equivalent to the boundary condition of problem (1.4). Therefore, G is in this case

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a stationary solution of problem (1.4), i.e. a self-similar solution to problem (1.2). However, if $G(L) = 0$, these boundary conditions are not equivalent. Nevertheless, G will turn out to be a possible limit for the solutions of problem (1.4).

We begin with the case $\gamma < \frac{1}{2}$, $m < -1$.

THEOREM 3.1. *Assume* $\gamma < \frac{1}{2}$.

(i) *There exist two critical lengths,* $0 < L_0 < L_*$, *depending on* γ, *such that:*

(a) *if* $0 < L \leqslant L_0$ *or* $L = L_*$, *there exists a unique positive profile* G *solution to problem (3.1);*

(b) *if* $L_0 < L < L_*$, *there exist two positive profiles;*

(c) *if* $L > L_*$, *there exist no positive profiles.*

(ii) *For every* $0 < \bar{L} \leqslant L_0$ *and* $L > \bar{L}$, *there exists a unique profile with support* $[0, \bar{L}]$.

The constant

$$L_0 = \sqrt{2(1 - 2\gamma)}, \tag{3.2}$$

and the solutions with support exactly $[0, L_0]$,

$$G_0(x) = \left(1 - \frac{x}{L_0}\right)_+^2, \tag{3.3}$$

are explicit.

Proof. We consider the following variables:

$$X(\eta) = G(x), \qquad Y(\eta) = G'(x), \qquad \mathrm{d}\eta = \frac{\mathrm{d}x}{X}, \tag{3.4}$$

and study the trajectories in the fourth quadrant $\Theta = \{X \geqslant 0, Y \leqslant 0\}$, solving the autonomous system

$$\left.\begin{aligned} \frac{\mathrm{d}X}{\mathrm{d}\eta} &= XY, \\ \frac{\mathrm{d}Y}{\mathrm{d}\eta} &= X + \gamma Y^2. \end{aligned}\right\} \tag{3.5}$$

The equation in (3.1) shows that G is convex. Therefore, since $G'(L) = 0$, G is non-increasing, which implies that $Y \leqslant 0$. The condition at $x = 0$ is translated into shooting from the line $X = 1$. The condition at $x = L$ means that the trajectories end at the horizontal axis, $Y = 0$. System (3.5) has only one critical point in Θ, namely the origin. It is not hard to check that it is a saddle-node point, the separatrix between both behaviours being the explicit trajectory

$$\Gamma_* = \{X - \mu Y^2 = 0\}, \quad \mu = \tfrac{1}{2} - \gamma. \tag{3.6}$$

Note also that this trajectory exists if and only if $0 < \gamma < \frac{1}{2}$. The trajectory Γ_* intersects the line $X = 1$ at the point $(1, -1/\sqrt{\mu})$. Therefore, the corresponding profile G_0 is

$$G_0(x) = \left(1 - \frac{x}{L_0}\right)^2,$$

with $L_0 = 2\sqrt{\mu} = \sqrt{2(1-2\gamma)}$. Observe that this function satisfies

$$G_0'(L_0) = (G_0^{1-\gamma})'(L_0) = 0, \tag{3.7}$$

since $2(1-\gamma) > 1$. Therefore, if $L = L_0$, it satisfies the boundary condition at $x = L$ of problem (1.4), and thus it gives not only a solution of problem (3.1), but also a stationary solution of problem (1.4). Moreover, extending this function by zero for $x > L_0$, we obtain a profile for every $L > L_0$,

$$G_L(x) = \begin{cases} (1 - x/L_0)^2, & 0 \leqslant x \leqslant L_0, \\ 0, & L_0 \leqslant x \leqslant L. \end{cases} \tag{3.8}$$

Shooting now from points $(1, -r)$ with $r > r_0 \equiv 1/\sqrt{\mu}$, we see that all the trajectories are below Γ_* and enter the origin like the power $Y \approx -X^\gamma$. The length $\bar{L}$ corresponding to each one of these trajectories is

$$\bar{L} = \int_0^1 \frac{\mathrm{d}s}{|Y(s)|},$$

which is seen to be finite from the behaviour of Y at 0. Observe that this behaviour itself also implies that the corresponding profiles satisfy

$$|Y(X)| \approx X^\gamma \Rightarrow G_{\bar{L}}(x) \approx (\bar{L} - x)^{1/(1-\gamma)}. \tag{3.9}$$

This means that $G_{\bar{L}}$ satisfies $G_{\bar{L}}'(\bar{L}) = 0$. We obtain a solution for every $L \geqslant \bar{L}$ extending $G_{\bar{L}}$ by zero. We remark that, by (3.9), we obtain a solution of problem (1.4) if and only if $L > \bar{L}$.

Finally, since Y lies below the explicit trajectory Γ_*, it follows that the lengths corresponding to those trajectories are smaller than that of Γ_*, i.e. $0 < \bar{L} < L_0$.

We now shoot from points $(1, -r)$ with $0 < r < r_0$. All the trajectories obtained in this way intersect the horizontal axis at some positive point $X = X_r > 0$, and are admissible trajectories. They give rise to profiles bounded from below by X_r (see figure 1).

The above is equivalent to shooting in the plane $x - G$ from the point $(0, 1)$ with different negative slopes $G'(0) = -r$.

As to the characterization of the profiles in terms of the length, we observe that there exists a first integral equation of (3.1), giving a constant energy

$$E(x) = \tfrac{1}{2}(G^{-\gamma}G')^2(x) - \frac{1}{1-2\gamma}G^{1-2\gamma}(x) = E, \tag{3.10}$$

for every $x \in [0, \bar{L}] \equiv \operatorname{supp}(G)$. Positive profiles imply negative energies, since $\bar{L} = L$, and

$$E = E(L) = -\frac{1}{1-2\gamma}G^{1-2\gamma}(L) < 0.$$

The explicit profile gives zero energy and, finally, the compactly supported profiles give rise to positive energies with value $E = E(\bar{L}) = \frac{1}{2}(G^{-\gamma}G')^2(\bar{L}) > 0$. In fact, in terms of the slope $G'(0) = -r$, we have

$$E = E(0) = \frac{r^2}{2} - \frac{1}{1-2\gamma}.$$

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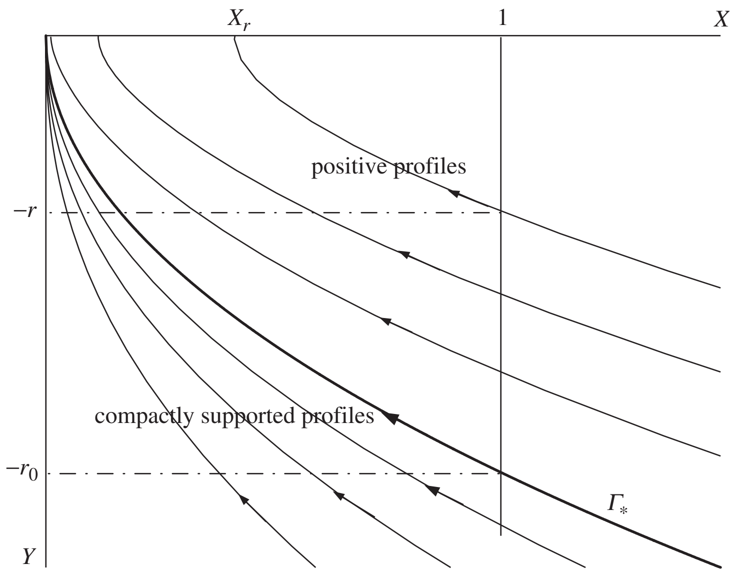

Figure 1. The trajectories in the (X, Y)-plane.

Therefore, if $r^2 > 2/(1-2\gamma)$, we obtain $E > 0$, which gives a compactly supported profile, while if $r^2 < 1/(1-2\gamma)$, the profile is positive since $E < 0$. The slope $r^2 = 1/(1-2\gamma)$ obviously corresponds to the explicit profile G_0.

Now let G be a positive profile. We want to describe the length of the interval in terms of the value $B = G(L) \in (0, 1)$. Since G must be non-increasing (see above), from (3.10) we obtain

$$G^{-\gamma}G'(x) = -\frac{2}{L_0}\sqrt{G^{1-2\gamma}(x) - B^{1-2\gamma}},$$

and thus the profile G is given by the implicit formula

$$\int_{G(x)}^{1} \frac{s^{-\gamma}}{\sqrt{s^{1-2\gamma} - B^{1-2\gamma}}}\,\mathrm{d}s = \frac{2}{L_0}x. \tag{3.11}$$

Putting $x = L$, we get the following expression for the length L:

$$L(B) = \tfrac{1}{2}L_0 \int_{B}^{1} \frac{s^{-\gamma}}{\sqrt{s^{1-2\gamma} - B^{1-2\gamma}}}\,\mathrm{d}s. \tag{3.12}$$

In order to finish the proof of theorem 3.1, we need only to give a precise description of the function $L(B)$ (see figure 2).

Lemma 3.2. *The function $L(B)$ defined in (3.12) increases from $L(0) = L_0$ until some value $L_* > 0$ and then decreases until $L(1) = 0$.*

Proof. The fact that the limit of $L(B)$ is 0 as B tends to 1 is immediate. On the other hand, $L(B)$ can be written in the form

$$L(B) = \tfrac{1}{2}L_0 B^{1/2} \int_{B}^{1} \frac{s^{-3/2}}{\sqrt{1 - s^{1-2\gamma}}}\,\mathrm{d}s. \tag{3.13}$$

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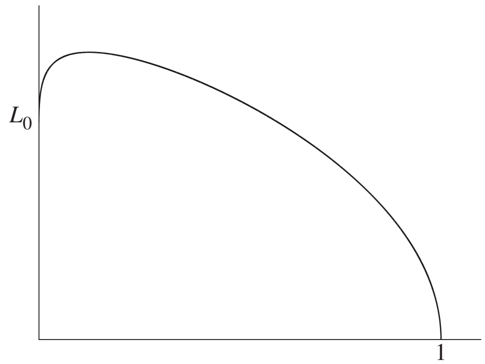

Figure 2. The function $L(B)$ for $\gamma < \frac{1}{2}$.

It is not hard to show the following behaviour of $L(B)$ for $B \approx 0$:

$$\frac{L(B)}{L_0} = \begin{cases} 1 + D_1(\gamma)B^{1/2} + O(B^{1-2\gamma}) & \text{if } \gamma < \frac{1}{4}, \\ 1 + \frac{1}{4}B^{1/2}\log(1/B) + O(B^{1/2}) & \text{if } \gamma = \frac{1}{4}, \\ q1 + D_2(\gamma)B^{1-2\gamma} + O(B^{(3-8\gamma)/2}) & \text{if } \gamma > \frac{1}{4}, \end{cases} \tag{3.14}$$

where

$$D_1(\gamma) = \sum_{k=0}^{\infty} \frac{(2k)!}{(2k(1-2\gamma)-1)2^{2k-1}(k!)^2}, \qquad D_2(\gamma) = \frac{1}{2(4\gamma-1)}. \tag{3.15}$$

Since $D_1(\gamma) > 0$ and $D_2(\gamma) > 0$, we have that $L(B) > L_0$ for all small B. We now prove that $L(B)$ has only one maximum. For this purpose we differentiate the expression for $L(B)$ to get

$$L'(B) = \frac{1}{2B}\left(L(B) - \frac{L_0}{\sqrt{1-B^{1-2\gamma}}}\right).$$

Thus, if $L'(B_0) = 0$ for some $B_0 \in (0,1)$, we must have

$$L(B_0) = \frac{L_0}{\sqrt{1-B_0^{1-2\gamma}}}.$$

We conclude with the observation that this last function is increasing in $(0,1)$, which implies that B_0 is unique. □

Now we deal with the case $\gamma \geqslant \frac{1}{2}$, $-1 \leqslant m < 0$.

THEOREM 3.3. *Assume* $\frac{1}{2} \leqslant \gamma < 1$.

(i) *There exists a critical length,* $L_* > 0$, *depending on* γ, *such that*

  (a) *if* $0 < L < L_*$, *then there exist two positive profiles;*

  (b) *if* $L = L_*$, *then there exists a unique positive profile;*

  (c) *if* $L > L_*$, *then there exist no positive profiles.*

(ii) *No compactly supported profiles exist in this case.*

Proof. In the case $\gamma \geqslant \frac{1}{2}$, in the phase plane X–Y given by (3.5) the separatix has moved to the vertical axis, and thus the parabolic sector of the saddle-node point $(0,0)$ contains the whole of the fourth quadrant. No trajectories in this quadrant enter the origin. In particular, this means that all the profiles G are strictly positive. We concentrate on the case $\gamma > \frac{1}{2}$. As before, the conservation of energy

$$E(x) = \tfrac{1}{2}(G^{-\gamma}G')^2(x) + \frac{1}{2\gamma - 1}G^{1-2\gamma}(x) = E, \tag{3.16}$$

holds for every $0 \leqslant x \leqslant L$. Note that the energy is always positive, since all the terms are positive. Observe also that, since $1 - 2\gamma < 0$, this implies again that G is positive. The function which gives the length of the interval in terms of the value $B = G(L)$ is here

$$L(B) = RB^{1/2}\int_B^1 \frac{s^{\gamma-2}}{\sqrt{1 - s^{2\gamma-1}}}\,\mathrm{d}s, \tag{3.17}$$

where $R = \sqrt{\frac{1}{2}(2\gamma - 1)}$.

Lemma 3.4. *The function $L(B)$ defined in (3.17) satisfies $L(0) = L(1) = 0$, and it has a unique maximum.*

Proof. The limits as B tends to 0 or 1 are immediate. The proof that $L(B)$ has a unique maximum follows the same argument as before since, at a point $B_0 \in (0,1)$ of maximum of $L(B)$, we must have

$$L(B_0) = \frac{2R}{\sqrt{B_0^{1-2\gamma} - 1}}.$$

□

Finally, we study the case $\gamma = \frac{1}{2}$. The energy equation is now given by

$$E(x) = \tfrac{1}{2}(G^{-1/2}G')^2(x) - \log G(x) = E \quad \text{for every } 0 \leqslant x \leqslant L. \tag{3.18}$$

Arguing as before, we need only easy modifications to obtain the desired result. □

The profiles constructed in the two theorems above are ordered in $(0, L]$. We call G_1 and G_2 the two different positive profiles that exist for $L_0 < L < L_*$ (putting $L_0 = 0$ when $\frac{1}{2} \leqslant \gamma < 1$), and we assume that $G_1 < G_2$. If $\gamma < \frac{1}{2}$ and $0 < L < L_0$, the profile G_1 disappears, while for $L = L_*$ we have $G_1 = G_2$. On the other hand, it is clear that the profiles with compact support (in the case $\gamma < \frac{1}{2}$) are ordered by their support $\bar{L}$, i.e. $G_{\bar{L}_1} < G_{\bar{L}_2} < G_0 < G_1$ whenever $0 < \bar{L}_1 < \bar{L}_2 < L_0$.

4. Asymptotic behaviour for $\gamma < \frac{1}{2}$

In this section we prove the stabilization result for the rescaled problem (1.4). The proof is based on the construction of a Lyapunov function. We also describe the quenching sets.

Theorem 4.1. *Let $\gamma < \frac{1}{2}$ and v be a solution to problem (1.2). The rescaled orbits g then tend to a stationary profile G. Thus, if $V(x,t) = (T-t)^{-1}G(x)$, we have*

$$\lim_{t \to T}(T - t)|v(x,t) - V(x,t)| = 0, \tag{4.1}$$

uniformly in $x \in [0, L]$.

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Proof. Thanks to the rescaling, g is bounded. Also, the behaviour of v near $t = T$ is translated into the behaviour of g as $\tau \to \infty$. Thus there exists a sequence $\tau_j \to \infty$ such that

$$\lim_{j\to\infty} g(x, \tau + \tau_j) = g_*(x, \tau) \tag{4.2}$$

uniformly in $[0, L]$. On the other hand, the compactness results of [3] imply that the limit function is also a viscosity solution to the equation in problem (1.4). We want to prove that the function g_* does not depend on τ, and therefore that it coincides with one of the stationary solutions constructed in the previous section.

To this end consider the function

$$L_g(\tau) = \frac{1}{2}\int_0^L |g^{-\gamma} g_x(x, \tau)|^2 \,\mathrm{d}x + \frac{1}{1-2\gamma}\int_0^L g^{1-2\gamma}(x, \tau)\,\mathrm{d}x. \tag{4.3}$$

Recall that in this case we have $0 < \gamma < \frac{1}{2}$. By differentiating and integrating by parts, we get

$$\frac{\mathrm{d}}{\mathrm{d}\tau} L_g(\tau) = -\frac{4}{(1-2\gamma)^2}\int_0^L |(g^{(1-2\gamma)/2})_\tau(x, \tau)|^2 \,\mathrm{d}x \leqslant 0.$$

Therefore, L_g is positive and non-increasing. This implies the convergence in a standard way (see, for example, [1]):

$$\begin{aligned}\|g^{(1-2\gamma)/2}(\cdot, \tau_j + \tau) - g^{(1-2\gamma)/2}(\cdot, \tau_j)\|^2_{L^2([0,L])} \\ = \int_0^L |g^{(1-2\gamma)/2}(x, \tau_j + \tau) - g^{(1-2\gamma)/2}(x, \tau_j)|^2 \,\mathrm{d}x \\ \leqslant \tau \int_0^L \int_{\tau_j}^{\tau_j+\tau} |(g^{(1-2\gamma)/2})_\tau(x, s)|^2 \,\mathrm{d}s\,\mathrm{d}x \to 0\end{aligned}$$

as $j \to \infty$, uniformly for bounded τ. Therefore, the sequence $g^{(1-2\gamma)/2}(x, \tau_j + \tau)$ converges in the space $L^\infty([0, \tau] : L^2([0, L]))$ for every $\tau > 0$. The limit does not depend on τ.

On the other hand, it is easy to check that the function $\underline{G}(x) = \lambda G_0(x/\sqrt{\lambda})$ is a subsolution of problem (1.4) if $\lambda > 0$ is small enough. Therefore, $g(x, \tau) \geqslant c > 0$ for $0 \leqslant x < \delta$, $\tau \geqslant 0$. Thus, by standard regularity theory, the limit g_* is continuous down to $x = 0$.

Let us now look at what happens at the right end, $x = L$. If the function g_* is strictly positive, again by regularity theory we have that $g_*^{-\gamma} g_*'(L) = g_*'(L) = 0$. If not, we cannot ensure that the flux at $x = L$ is zero in the limit, only that $g_*'(L) = 0$. Anyway, the ω-limit set is contained in the family of stationary solutions constructed in the previous section.

Next we prove that the ω-limit consists of a unique profile. To do that we argue by contradiction. Assume that there exist two different sequences τ_j and $\bar{\tau}_j$ such that

$$g(x, \tau_j) \to G(x), \quad g(x, \bar{\tau}_j) \to \bar{G}(x) \quad \text{as } j \to \infty,$$

with $\operatorname{supp}(G) = [0, \ell] \subsetneq [0, \bar{\ell}] = \operatorname{supp}(\bar{G})$. As before, we consider the subsolution $\bar{h}(x) = \lambda\bar{G}(x/\sqrt{\lambda})$ with $(\ell/\bar{\ell})^2 < \lambda < 1$. Now take j large enough that $g(x, \bar{\tau}_j) > \bar{h}(x)$. Then, by comparison, we have $g(x, t) > \bar{h}(x)$ for every $t > \bar{\tau}_j$, which is a contradiction with the existence of the sequence τ_j. □

Remark 4.2. Since compactly supported initial data give solutions with stationary support, if we begin with initial data with support $[0, A]$ with $A < L_0$, we get a solution that, when properly rescaled, has behaviour given by one of profiles G with support contained in $[0, A]$.

Next we analyse the stability of the profiles, G_0 (the maximal compactly supported profile), G_1 and G_2 (the two positive profiles), when they exist.

Theorem 4.3.

(i) *For $L_0 < L < L_*$ the compactly supported profile G_0 and the greater of the two positive profiles, G_2, are stable and the other positive profile, G_1, that lies between G_0 and G_2 is unstable.*

(ii) *For $L = L_0$ the unique positive profile is stable and the compactly supported profile G_0 is unstable from above.*

(iii) *For $L = L_*$, G_0 is stable and the unique positive profile $G_1 = G_2$ is unstable from below.*

Proof. First of all we remark that from the comparison principle G_2 is stable from above.

(i) In the case $L_0 < L < L_*$ we have that $G_0(x) < G_1(x) < G_2(x)$ for all $x \in (0, L]$. In order to prove that G_1 is unstable, we consider $\tilde{G}_1$, the profile with $\tilde{L} = L - \delta$. From lemma 3.2, we have that $\tilde{G}_1(\tilde{L}) < G_1(L)$. Therefore, $\tilde{G}_1(x) < G_1(x)$ for every $x \in (0, \tilde{L}]$. Moreover, $\tilde{G}_1'(x) > 0$ in $(\tilde{L}, L]$. Hence, $\tilde{G}_1$ is a supersolution of problem (1.4). In the same way we can prove that if we take $\hat{L} = L + \delta$, the corresponding profile $\hat{G}_1$ is a subsolution of (1.4). Therefore, if we take δ small enough and initial data $g_0(x)$, such that $g_0(x) < \tilde{G}_1$ we obtain, by the comparison principle, the result that the solution of (1.4) converges to $G_0(x)$. In the same way, if $g_0(x) > \hat{G}_1$, we have that $g(x, \tau)$ tends to $G_2(x)$.

(ii) For $L = L_0$, we only have two profiles G_0 and G_2. In fact, as L goes to L_0 the profile G_1 tends to G_0. We can argue as before, to obtain the result that G_0 is unstable from above and then G_2 is stable from below.

(iii) When L tends to L_* the profile G_1 tends to G_2. Therefore, for $L = L_*$, we have the result that G_0 is stable from above. □

We now proceed with the study of the blow-up set, i.e. we prove theorem 1.1. It is based on the following result, adapted from [7].

Lemma 4.4. *Let $\varepsilon > 0$ be small, $0 < \delta < B$ be two arbitrary constants, and let p be a solution to the problem*

$$\left.\begin{aligned} p_\tau &= \tfrac{1}{2}(g^2)_{xx} - p, && 0 < x < B,\ \tau > 0,\\ p_x(0,\tau) &= 0,\ p(B,\tau) = \varepsilon, && \tau > 0,\\ p(x,0) &= \varepsilon, && 0 < x < B. \end{aligned}\right\} \tag{4.4}$$

Then there exists a constant $C > 0$ such that for every $x \in [0, B - \delta]$, $\tau > 0$, it holds

$$p(x,\tau) \leqslant C\mathrm{e}^{-\tau}. \tag{4.5}$$

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With this result we obtain the following corollary.

Corollary 4.5. *If the solution g to problem (1.4) converges to 0 uniformly in $[a,b]$, then there exist constants $C, \tau_0, \delta' > 0$ such that*

$$g(x,\tau) \leqslant C\mathrm{e}^{-\tau} \tag{4.6}$$

for every $\tau > \tau_0$, $a + \delta' < x < b - \delta'$.

Proof. Take large $\tau_0 > 0$ such that $g(x,\tau) \leqslant \varepsilon$ for every $a \leqslant x \leqslant b$, $\tau \geqslant \tau_0$. Since g satisfies

$$\begin{aligned} g_\tau &= g g_{xx} - \gamma (g_x)^2 - g \\ &= \tfrac{1}{2}(g^2)_{xx} - (\gamma+1)(g_x)^2 - g \\ &\leqslant \tfrac{1}{2}(g^2)_{xx} - g, \end{aligned}$$

we have that, for every $\tau > 0$, it holds that

$$g(x,\tau+\tau_0) \leqslant \begin{cases} p(z,\tau), & \frac{1}{2}(a+b) < x < b, \\ p(-z,\tau), & a < x < \frac{1}{2}(a+b), \end{cases}$$

where

$$z = \frac{2B}{b-a}\biggl(x - \frac{a+b}{2}\biggr),$$

and p is the solution to problem (4.5). The conclusion follows. □

Proof of theorem 1.1. The fact that the blow-up set of v contains some interval comes from comparison with the explicit subsolution $\underline{G}(x) = (T-t)^{-1}(\lambda - x/L_0)_+^2$ for some $\lambda > 0$ small enough.

If the initial data v_0 have compact support $[0,\ell]$, then, since this support cannot expand in time, blow-up cannot be global. On the other hand, if v_0 is bigger than the positive self-similar solution $v_2(x,0) = T^{-1}G_2(x)$, then, again by comparison, we obtain global blow-up.

Moreover, from the stability properties of the self-similar profiles and the asymptotic result, we obtain, for $L_0 < L \leqslant L_*$ and $g_0(x) \leqslant G_1(x)$, that the rescaled orbit g converges to a compactly supported profile. Thus g converges to zero uniformly in any interval $[a,b]$ contained in $[L_0, L]$. Corollary 4.5 implies that v is bounded in $(L_0, L]$ for any $t \in (0,T)$. Hence, the blow-up set of v is contained in the interval $[0, L_0]$. In the case $L > L_*$, since there exist no positive profiles, blow-up is always regional. □

5. Asymptotic behaviour for $\frac{1}{2} \leqslant \gamma < 1$

In this case, for $L > L_*$ there is no self-similar profile (see theorem 3.3). Thus, we cannot have a general asymptotic result like the one given in the previous section. In fact, if g goes to zero in some interval, (4.3) becomes unbounded. Nevertheless, if the solution verifies that $g(x,\tau) \geqslant k > 0$ for $x \in [0,L]$, $\tau > 0$, then (4.3) is well defined and gives a Lyapunov functional for our problem. Therefore, if g is uniformly bounded away from zero, g converges to a positive profile as $\tau \to \infty$, and we get global blow-up for v. Observe that this is possible only if $0 < L \leqslant L_*$. Thus we have proved the following asymptotic result.

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THEOREM 5.1. *Let $\frac{1}{2} \leqslant \gamma < 1$ and v be the solution to problem (1.2). The rescaled orbits $g(x,\tau)$ then tend to a stationary profile G as long as they are strictly bounded away from zero. Therefore, we have global blow-up in this case. The latter holds only for $0 < L \leqslant L_*$.*

As in the case $\gamma < \frac{1}{2}$, we have the following stability properties for the positive self-similar profiles $0 < G_1 \leqslant G_2$:

(i) for $0 < L < L_*$, G_1 is unstable and G_2 is stable;

(ii) for $L = L_*$, $G_1 \equiv G_2$ is unstable from below and stable from above.

We now concentrate on studying the blow-up set for our solutions. To do that we observe that, if we replace γ by $\tilde{\gamma} < \gamma$ in problem (1.2), we obtain a supersolution to the original problem. More precisely, the following lemma holds.

LEMMA 5.2. *Let g_1 and g_2 be the solutions of problem (1.4) with $\gamma = \gamma_1$ and $\gamma = \gamma_2$, respectively. Then, if $\gamma_1 < \gamma_2$, we have $g_1(x,\tau) \geqslant g_2(x,\tau)$. Therefore, the blow-up sets of the corresponding pressures satisfy $B(v_1) \supset B(v_2)$.*

With this lemma, let us prove that there exist initial data such that the blow-up set is a single point, $B(v) = \{0\}$.

LEMMA 5.3. *If the initial data g_0 verify that the solution $\tilde{g}$ of (1.4) for some $\tilde{\gamma} < \frac{1}{2}$ converges to a compactly supported profile, then the solution g with initial data g_0 and $\frac{1}{2} \leqslant \gamma < 1$ tends to zero for all $x \in (0, L]$. Therefore, v has single-point blow-up for those initial data.*

Proof. By lemma 5.2 we obtain the result that, for every $\tilde{\gamma} < \hat{\gamma} < \frac{1}{2} < \gamma$ we have, for the corresponding solutions,

$$g(x,\tau) \leqslant \hat{g}(x,\tau) \leqslant \tilde{g}(x,\tau).$$

Thus the solution $\hat{g}$ of problem (1.4) with initial data g_0 and with $\hat{\gamma}$ instead of γ converges to a compactly supported profile, with support $[0,\ell]$. This implies that

$$g(x,\tau) \to 0, \quad x \in (\ell, L] \quad \text{as } \tau \to \infty.$$

The proof is completed merely by recalling that $\ell \leqslant L_0(\hat{\gamma}) \to 0$ as $\hat{\gamma} \nearrow \frac{1}{2}$. □

Examples of global blow-up and single-point blow-up are derived from theorem 5.1 and lemma 5.3, respectively. Moreover, the stability properties of the profiles that exist for $0 < L \leqslant L_*$ give the result that global blow-up indeed occurs for an open set of initial data. Also from lemma 5.3 we obtain the result that single-point blow-up holds for an open set of initial data.

LEMMA 5.4. *Regional blow-up is impossible for $\frac{1}{2} \leqslant \gamma < 1$.*

Proof. Assume that there exists a solution v for which blow-up is not global, i.e. $B(v) = [0, x_1]$ for some $0 \leqslant x_1 < L$. Then, since $v(x,t)$ is bounded for any $x_1 < x \leqslant L$, we obtain the result that $g(x,\tau)$ goes to zero for every $x_1 < x \leqslant L$. In order to prove that $x_1 = 0$, we argue as follows: given $\eta > 0$, let $\tau_0 > 0$ be such that $g(x_1+\delta,\tau)\leqslant\eta$ for $x_1+\delta<L$ and every $\tau>\tau_0$. Let $w=w_\eta$ be the solution of

$$\left.\begin{aligned} w_\tau &= ww_{xx}-\gamma(w_x)^2-w, && 0<x<x_1+\delta,\ \tau>\tau_0,\\ w(0,\tau) &= 1, && \tau>\tau_0,\\ w(x_1+\delta,\tau) &= \eta, && \tau>\tau_0,\\ w(x,\tau_0) &= w_0(x), && 0<x<x_1+\delta. \end{aligned}\right\} \tag{5.1}$$

First of all we observe that there exists a stationary positive solution G_η with a minimum located at a point $y_\eta\in(0,x_1+\delta)$. Indeed, if G_η were monotone, we would obtain stationary profiles for problem (3.1) defined in an interval $[0,\tilde{L}]$, with $\tilde{L}>x_1+\delta$ and $G(\tilde{L})\leqslant\eta$ small, which is a contradiction with the properties of the function $L(B)$ in lemma 3.4. Now assume that $w_0(x)\geqslant G_\eta(x)$. Since $w_\eta(x,\tau)\geqslant G_\eta(x)$, we can use the functional defined by (4.3) to obtain the result that w_η converges to G_η as τ tends to infinity. Finally, it is easy to check that G_η tends to zero for every $x>0$ as η tends to zero. Hence, if w_0 is chosen to also satisfy $w_0(x)\geqslant g(x,\tau_0)$, we obtain

$$\lim_{\tau\to\infty} g(x,\tau)=0 \quad \text{for every } x>0,$$

proving that $x_1=0$. Therefore, the blow-up set is a single point whenever it is not the whole interval $[0,L]$. □

Proof of theorem 1.2. For $L>L_*$, there exist no self-similar profiles. Thus, from lemma 5.4 we find that the blow-up set is a single point for every initial datum. If $L\leqslant L_*$, lemma 5.3 shows that there is an open set of initial data such that the corresponding solutions have single-point blow-up. On the other hand, we have the existence of positive profiles and, from its stability properties, we obtain that there exists an open set of initial data for which we have global blow-up, for instance, if $g_0(x)\geqslant G_1(x)$.

In summary, we find that for $L>L_*$ the blow-up set is always a single point, while for $L<L_*$ it can be either a single point or the whole interval $[0,L]$. □

6. Concluding remarks

As we mentioned in § 1, the study of parabolic problems that develop singularities in finite time is attracting a great deal of attention. Many questions are posed in these studies concerning, for example, the asymptotic behaviour of the solution near the singularity. In particular, the structure of the set where the solution becomes singular seems not to be completely understood.

Throughout this paper we have looked for a simple example (based in physical models) where *regional* quenching may appear. To this end we have taken a diffusion equation with a quenching boundary condition such that the natural scaling law of the problem does not change the spatial variable. This may lead to regional phenomena (cf. the case of blow-up problems [7, 20]).

We have shown that regional quenching is possible only in the range $m<-1$ (this corresponds to $\gamma<\frac{1}{2}$). Moreover, we have proved that, for any given length $L>L_0$, the set of initial data that exhibit regional quenching contains an open set

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in L^∞; therefore this phenomenon is not exceptional. In this case ($m < -1$), we also prove that there exists global quenching only if the interval under consideration is small enough ($0 < L \leqslant L_*$).

In the case $m \geqslant -1$ (which corresponds to $\gamma \geqslant \frac{1}{2}$), we find that regional quenching is not possible; we may have single-point quenching (again for a non-empty set of initial data for any $L > 0$) or global quenching (if $0 < L \leqslant L_*$). Note that no compactly supported (in pressure variable) self-similar profiles exist in this case.

The results contained here are restricted to one spatial dimension. There are natural extensions to higher dimensions; however, the analysis of the existence of the profiles is much more complicated. Another possible extension is to consider more general problems, including a different diffusion (not just a power) or different boundary conditions, for example, prescribing a flux at $x = 0$.

A possible improvement of our results is to determine the exact regional quenching set for the case $m < -1$. Recall that we have proved that the pressure v blows up in a set strictly included in $[0, L_0]$ for initial data v_0 whose support is contained in $[0, A]$ with $A < L_0$. These compactly supported initial data v_0 give initial data for the quenching problem u_0 that are infinite in the complement of their support. We conjecture that, if we restrict ourselves to positive and finite initial data u_0 with regional quenching, then the quenching set must be $[0, L_0]$ and the behaviour must be given by the explicit profile G_0. We have proved this fact for initial data u_0 such that the rescaled function g_0 lies above G_0. Nevertheless, we have not been able to prove this statement for general initial data. Numerical experiments support the conjecture. The main obstacle to prove it lies in the separation of the limit profile G_0 from the other infinitely many compactly supported profiles that exist in this range of exponents. This separation could come from fine regularity properties that G_0 has but the other profiles do not.

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References

1 D. Aronson, M. G. Crandall and L. A Peletier. Stabilization of solutions of a degenerate nonlinear diffusion problem. *Nonlin. Analysis* **6** (1982), 1001–1022.

2 M. Bertsch and M. Ughi. Positivity properties of viscosity solutions of a degenerate parabolic equation. *Nonlin. Analysis* **14** (1990), 571–592.

3 M. Bertsch, R. Dal Passo and M. Ughi. Discontinuous 'viscosity' solutions of a degenerate parabolic equation. *Trans. Am. Math. Soc.* **320** (1990), 779–798.

4 M. Bertsch, R. dal Passo and M. Ughi. Nonuniqueness of solutions of a degenerate parabolic equation. *Annli Mat. Pura Appl.* **161** (1992), 57–81.

5 C. Y. Chan. Recent advances in quenching phenomena. In *Proc. Dynamic Systems and Applications, Atlanta, GA, 1995*, vol. 2, pp. 107–113 (Atlanta, GA: Dynamic, 1996).

6 M. Chipot, M. Fila and P. Quittner. Stationary solutions, blow up and convergence to stationary solutions for semilinear parabolic equations with nonlinear boundary conditions. *Acta Math. Univ. Comenian.* **60** (1991), 35–103.

7 C. Cortázar, M. del Pino and M. Elgueta. On the blow-up set for $u_t = \Delta u^m + u^m$, $m > 1$. *Indiana Univ. Math. J.* **47** (1998), 541–561.

8 R. Ferreira and J. L. Vazquez. Study of self-similarity for the fast diffusion equation. *Adv. Diff. Eqns* **8** (2003), 1125–1152.

9 R. Ferreira, A. de Pablo, F. Quirós and J. D. Rossi. Superfast quenching. *J. Diff. Eqns* **199** (2004), 189–209.

10 V. A. Galaktionov and J. L. Vazquez. The problem of blow-up in nonlinear parabolic equations. *Discrete Contin. Dynam. Syst.* A **8** (2002), 399–433.

11 B. H. Gilding and M. A. Herrero. Localization and blow-up of thermal waves in nonlinear heat conduction with peaking. *Math. Ann.* **282** (1988), 223–242.

12 H. Kawarada. On solutions of initial-boundary problem for $u_t = u_{xx} + 1/(1-u)$. *Publ. Res. Inst. Math. Sci.* **10** (1974), 729–736.

13 J. R. King. Asymptotic results for nonlinear outdiffusion. *Eur. J. Appl. Math.* **5** (1994), 359–390.

14 O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Ural'ceva. *Linear and quasilinear equations of parabolic type*. Mathematical Monographs, vol. 23 (Providence, RI: American Mathematical Society, 1968).

15 H. A. Levine. The phenomenon of quenching: a survey. In *Trends in the theory and practice of nonlinear analysis*, North-Holland Mathematics Studies, vol. 110, pp. 275–286 (Amsterdam: North-Holland, 1985).

16 H. A. Levine. Quenching and beyond: a survey of recent results. In *Nonlinear mathematical problems in industry, II*, GAKUTO International Series on Mathematical Science and Applications, vol. 2, pp. 501–512 (Tokyo: Gakkōtosho, 1993).

17 F. J. Mancebo and J. M. Vega. A model of porous catalyst accounting for incipiently non-isothermal effecty. *J. Diff. Eqns* **121** (1999), 79–110.

18 A. Rodríguez and J. L. Vazquez. Obstructions to existence in fast-diffusion equations. *J. Diff. Eqns* **184** (2002), 348–385.

19 G. Rosen. Nonlinear heat conduction in solid H_2. *Phys. Rev.* B **19** (1979), 2398–2399.

20 A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov and A. P. Mikhailov. *Blow-up in problems for quasilinear parabolic equations* (Moscow: Nauka, 1987) (In Russian.) (English transl. Berlin: Walter de Gruyter, 1995).

21 S. Yu, T. Y. Tan and U. Gösele. Diffusion mechanism of cromium in GaAs. *J. Appl. Phys.* **70** (1991), 4827–4836.

22 S. Yu, T. Y. Tan and U. Gösele. Diffusion mechanism of zinc and beryllium in gallium arsenide. *J. Appl. Phys.* **69** (1991), 3547–3565.

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