Bull. Aust. Math. Soc. **108** (2023), 69–80
doi:10.1017/S0004972722000946

ON p-ADIC INTERPOLATION IN TWO OF MAHLER'S PROBLEMS

BRUNO DE PAULA MIRANDA and JEAN LELIS ✉

(Received 11 July 2022; accepted 29 July 2022; first published online 22 September 2022)

Abstract

Motivated by the p-adic approach in two of Mahler's problems, we obtain some results on p-adic analytic interpolation of sequences of integers $(u_n)_{n\geq 0}$. We show that if $(u_n)_{n\geq 0}$ is a sequence of integers with $u_n = O(n)$ which can be p-adically interpolated by an analytic function $f : \mathbb{Z}_p \to \mathbb{Q}_p$, then $f(x)$ is a polynomial function of degree at most one. The case $u_n = O(n^d)$ with $d > 1$ is also considered with additional conditions. Moreover, if X and Y are subsets of $\mathbb{Z}$ dense in $\mathbb{Z}_p$, we prove that there are uncountably many p-adic analytic injective functions $f : \mathbb{Z}_p \to \mathbb{Q}_p$, with rational coefficients, such that $f(X) = Y$.

2020 *Mathematics subject classification*: primary 11J61; secondary 11S80.

Keywords and phrases: p-adic interpolation, p-adic analytic function, Mahler expansion, Strassmann's theorem, p-adic Liouville number.

1. Introduction

In what follows, p is a prime number, $\mathbb{Q}_p$ is the field of p-adic numbers and $\mathbb{Z}_p$ is the ring of p-adic integers. Let $(u_n)_{n\geq 0}$ be a sequence of integers. If there exists a continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ such that $f(n) = u_n$ for all nonnegative integers n, we say that f is a p-adic interpolation of $(u_n)_{n\geq 0}$. In addition, if f is analytic, we say that it is a p-adic analytic interpolation of this sequence. Since the set of nonnegative integers is a dense subset of $\mathbb{Z}_p$, any given sequence of integers admits at most one such interpolation, which will only exist under certain strong conditions on the sequence (for more details, see [17]).

Many authors have studied the problem of p-adic interpolation. Bihani *et al.* [2] considered the problem of p-adic interpolation of the Fibonacci sequence, they proved that the sequence $(2^n F_n)_{n\geq 0}$ can be interpolated by a p-adic hypergeometric function on $\mathbb{Z}_5$. Rowland and Yassawi in [16] studied p-adic properties of sequences of integers (or p-adic integers) that satisfy a linear recurrence with constant coefficients. For such a sequence, they obtained an explicit approximate twisted interpolation to $\mathbb{Z}_p$. In particular, they proved that for any prime $p \neq 2$, there is a twisted interpolation of the

Fibonacci sequence by a finite family of p-adic analytic functions with coefficients in some finite extension of $\mathbb{Q}_p$. Inspired by the Skolem–Mahler–Lech theorem on linear recurrent sequences, Bell [1] proved that for a suitable choice of a p-adic analytic function f and a starting point $\bar{x}$, the iterate-computing map $n \mapsto f^n(\bar{x})$ extends to a p-adic analytic function g defined for all $x \in \mathbb{Z}_p$. That is, the sequence $f^n(\bar{x})$ can be interpolated by the p-adic analytic function g.

Mahler [7] states that the polynomial functions

$$\binom{x}{n} := \frac{x(x-1)\cdots(x-n+1)}{n!},$$

with $n \geq 0$ integer, form an orthonormal basis, called the *Mahler basis*, for the space of p-adic continuous functions $C(\mathbb{Z}_p \to \mathbb{Q}_p)$. More precisely, he showed that every continuous function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ has a unique uniformly convergent expansion

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}, \tag{1.1}$$

where $a_n \to 0$ and $\|f\|_{\sup} = \max_{n\geq 0} \|a_n\|_p$. Conversely, every such expansion defines a continuous function. Furthermore, if $f \in C(\mathbb{Z}_p \to \mathbb{Q}_p)$ has a *Mahler expansion* given by (1.1), then the *Mahler coefficients* a_n can be reconstructed from f by the *inversion formula*

$$a_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(j) \quad (n = 0, 1, 2, \ldots). \tag{1.2}$$

Using the Mahler expansion (1.1) and the inversion formula (1.2), we conclude that the sequence $(u_n)_{n\geq 0}$ of integers can be p-adically interpolated if and only if

$$\Big\| \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} u_j \Big\|_p \to 0 \quad \text{as } n \to \infty.$$

We became interested in studying the p-adic analytic interpolation of sequences of integers with polynomial growth while studying a problem about *p-adic Liouville numbers*. Based on the classic definition of complex Liouville numbers, Clark [3] called a p-adic integer λ a *p-adic Liouville number* if

$$\liminf_{n\to\infty} \sqrt[n]{\|n-\lambda\|_p} = 0.$$

It is easily seen that all p-adic Liouville numbers are transcendental p-adic numbers. Moreover, if λ is a p-adic Liouville number and a, b are integers, with $a > 0$, then $a\lambda + b$ is also a p-adic Liouville number.

In his book, Maillet [10, Ch. III] discusses some arithmetic properties of complex Liouville numbers. One of them states that given a nonconstant rational function f with rational coefficients, if ξ is a Liouville number, then so is $f(\xi)$. Motivated by this fact, Mahler [9] posed the following question.

https://doi.org/10.1017/S0004972722000946 Published online by Cambridge University Press

QUESTION 1.1 (Mahler [9]). Are there transcendental entire functions $f : \mathbb{C} \to \mathbb{C}$ such that if ξ is any Liouville number, then $f(\xi)$ is also a Liouville number?

He pointed out: 'The difficulty of this problem lies of course in the fact that the set of all Liouville numbers is nonenumerable.' We are interested in studying the analogous question for p-adic Liouville numbers.

QUESTION 1.2. Are there p-adic transcendental analytic functions $f : \mathbb{Z}_p \to \mathbb{Q}_p$ such that if λ is a p-adic Liouville number, then so is $f(\lambda)$?

It is important to note that the analogue of Maillet's result is not true for p-adic Liouville numbers. In fact, Lelis and Marques [5] proved that the analogue of Maillet's result is true for a class of p-adic numbers called *weak p-adic Liouville numbers*, but not for all p-adic Liouville numbers.

Inspired by an argument presented by Marques and Moreira in [11] and discussed by Lelis and Marques in [6], we approached Question 1.2 as follows. If there were a positive integer sequence $(u_n)_{n\geq 0}$ satisfying $u_n \to \infty$ and $u_n = O(n)$ that could be interpolated by a p-adic transcendental analytic function $f : \mathbb{Z}_p \to \mathbb{Q}_p$, then f would answer Question 1.2 affirmatively. Indeed, assuming all that is true, if we get any p-adic Liouville number $\lambda \in \mathbb{Z}_p$, by definition there would be a sequence of integers $(n_k)_{k\geq 0}$ such that

$$\lim_{k\to\infty} \sqrt[n_k]{\|n_k - \lambda\|_p} = 0.$$

The function f being analytic would satisfy a Lipschitz condition (see [15, Ch. 5, Section 3]). Thus, there would be a constant $c > 0$ such that

$$\|u_{n_k} - f(\lambda)\|_p = \|f(n_k) - f(\lambda)\|_p \leq c\|n_k - \lambda\|_p,$$

and so

$$(\|u_{n_k} - f(\lambda)\|_p)^{1/u_{n_k}} \leq (c\|n_k - \lambda\|_p)^{1/u_{n_k}},$$

where $u_{n_k} \to \infty$ and $u_{n_k} = O(n_k)$. So $f(\lambda)$ would also be a p-adic Liouville number.

In light of this, it is natural to try to characterise the p-adic analytic functions which interpolate sequences of integers $(u_n)_{n\geq 0}$ of linear growth. There are other reasons for seeking such characterisations. Indeed, one may ask whether there exists a p-adic interpolation of some arithmetic function (many of which have linear growth) or, more generally, if polynomials with integer coefficients are the only p-adic analytic functions that take positive integers into positive integers with polynomial order.

THEOREM 1.3. *Let $(u_n)_{n\geq 0}$ be a sequence of positive integers such that $u_n = O(n^d)$ for some fixed $d \geq 0$ ($d \in \mathbb{R}$). Assume there exists a p-adic analytic function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ which interpolates the sequence $(u_n)_{n\geq 0}$.*

(i) *If $d \leq 1$, then f is a polynomial function of degree at most one.*
(ii) *If $d > 1$ and the Mahler expansion of f converges for all $x \in \mathbb{Q}_p$, then f is a polynomial function of degree at most $\lfloor d \rfloor$.*

We remark that the condition 'f is a p-adic analytic function on $\mathbb{Z}_p$' is fundamental in the result above. Indeed, if we write $n = \sum_{i=0}^{k} a_i p^i$ in base p, then the function $f : \{0\} \cup \mathbb{N} \to \mathbb{Q}_p$ given by

$$f(n) = \begin{cases} \displaystyle\sum_{i=0}^{k-1} a_i p^i & \text{if } n \geq p, \\ n & \text{if } 0 \leq n \leq p-1, \end{cases}$$

clearly can be extended in a unique way to a continuous function $\overline{f} : \mathbb{Z}_p \to \mathbb{Q}_p$ such that $\overline{f}(n) = O(n)$. However, $\overline{f}$ is nonanalytic and it is clearly not a polynomial function.

Moreover, consider the p-adic function $f_d : \mathbb{Z}_p \to \mathbb{Q}_p$ defined by

$$f_d(z) = \sum_{k=0}^{\infty} a_k p^{dk},$$

where $z = \sum_{k=0}^{\infty} a_k p^k$ is the p-adic expansion of $z \in \mathbb{Z}_p$. Then it is well known that f_d is a continuous function for all integers $d \geq 2$. In fact, if $d \geq 2$ is an integer, then

$$\|f_d(x) - f_d(y)\|_p \leq \|x - y\|_p^d.$$

In particular, we have $f_d'(x) = 0$ for all $x \in \mathbb{Q}_p$ and $f_d \in C^1(\mathbb{Z}_p \to \mathbb{Q}_p) \subset C(\mathbb{Z}_p \to \mathbb{Q}_p)$. Note that $f_d(n) = O(n^d)$, but f_d is not a polynomial function. However, since f_d is not a p-adic analytic function, its Mahler expansion does not converge for all $x \in \mathbb{Q}_p$.

Very strict conditions must be satisfied for a sequence $(u_n)_{n\geq 0}$ to be interpolated by a p-adic analytic function. However, if the set $A = \{u_0, u_1, \ldots\} \subseteq \mathbb{Z}$ is a dense subset of $\mathbb{Z}_p$, one may ask whether there is some re-enumeration $\sigma : \{0\} \cup \mathbb{N} \to \{0\} \cup \mathbb{N}$ such that $(u_{\sigma(n)})_{n\geq 0}$ can be interpolated by a p-adic analytic function.

In the complex case, Georg [4] established that for each countable subset $X \subset \mathbb{C}$ and each dense subset $Y \subseteq \mathbb{C}$, there exists a transcendental entire function f such that $f(X) \subset Y$. In 1902, Stäckel [18] used another construction to show that there is a function $f(z)$, analytic in a neighbourhood of the origin and with the property that both $f(z)$ and its inverse function assume, in this neighbourhood, algebraic values at all algebraic points. Based on these results, Mahler [8] suggested the following question about the set of algebraic numbers $\overline{\mathbb{Q}}$.

QUESTION 1.4 (Mahler, [8]). Are there transcendental entire functions $f(z) = \sum c_n z^n$ with rational coefficients c_n and such that $f(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}}$ and $f^{-1}(\overline{\mathbb{Q}}) \subset \overline{\mathbb{Q}}$?

This question was answered positively by Marques and Moreira [12]. Moreover, in a more recent paper [13], they proved that if X and Y are countable subsets of $\mathbb{C}$ satisfying some conditions necessary for analyticity, then there are uncountably many transcendental entire functions $f(z) = \sum a_n z^n$ with rational coefficients such that $f(X) \subset Y$ and $f^{-1}(Y) \subset X$. Keeping these results in mind, we prove the following theorem.

THEOREM 1.5. *Let X and Y be subsets of $\mathbb{Z}$ dense in $\mathbb{Z}_p$. Then there are uncountably many p-adic analytic injective functions $f : \mathbb{Z}_p \to \mathbb{Q}_p$ with*

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \in \mathbb{Q}[[x]]$$

such that $f(X) = Y$.

Note that by Theorem 1.5, if $Y = \{y_0, y_1, y_2, \ldots\} \subset \mathbb{Z}$ is a dense subset of $\mathbb{Z}_p$, that is, if Y contains a complete system of residues modulo any power of p, then there is a p-adic analytic function

$$f(x) = \sum_{n=0}^{\infty} c_n x^n, \quad c_n \in \mathbb{Q} \text{ for all } n \geq 0,$$

and a bijection $\sigma : \{0\} \cup \mathbb{N} \to \{0\} \cup \mathbb{N}$ such that $f(n) = u_{\sigma(n)}$, where we take $X = \{0\} \cup \mathbb{N}$. Moreover, the series above converges for all $x \in \mathbb{Z}_p$. Thus, if we consider the Mahler expansion, then we immediately obtain the following result.

COROLLARY 1.6. *Let $Y = \{y_0, y_1, y_2, \ldots\}$ be a subset of $\mathbb{Z}$ dense in $\mathbb{Z}_p$. Then there are $a_0, a_1, a_2, \ldots \in \mathbb{Z}$ and a bijection $\sigma : \{0\} \cup \mathbb{N} \to \{0\} \cup \mathbb{N}$ such that*

$$\sum_{i=0}^{n} a_i \binom{i}{n} = y_{\sigma(n)},$$

for all integers $n \geq 0$, where $v_p(a_n/n!) \to \infty$ as $n \to \infty$.

We end this section by presenting some questions which we are still unable to answer. One may ask whether Theorem 1.5 is still true if X and Y are free to contain elements outside $\mathbb{Z}$. What could one do to guarantee rational coefficients in f in a situation like that? Moreover, if we consider the algebraic closure of $\mathbb{Q}_p$, denoted by $\overline{\mathbb{Q}}_p$, and its completion $\mathbb{C}_p$, we may ask a probably more difficult question.

QUESTION 1.7. Are there p-adic transcendental entire functions $f : \mathbb{C}_p \to \mathbb{C}_p$ given by

$$f(z) = \sum_{n=0}^{\infty} c_n z^n, \quad c_n \in \mathbb{Q} \text{ for all } n \geq 0,$$

such that $f(\overline{\mathbb{Q}}_p) \subset \overline{\mathbb{Q}}_p$ and $f^{-1}(\overline{\mathbb{Q}}_p) \subset \overline{\mathbb{Q}}_p$?

Naturally, the main difficulty of this problem lies again in the fact that the set $\overline{\mathbb{Q}}_p$ is uncountable.

2. Proof of Theorem 1.3

We start by introducing the classic Strassmann's theorem about zeros of p-adic power series. This result says that a p-adic analytic function with coefficients in $\mathbb{Q}_p$ has finitely many zeros in $\mathbb{Z}_p$ and provides a bound for the number of zeros.

THEOREM 2.1 (Strassmann, [14]). *Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ be a nonzero power series with coefficients in $\mathbb{Q}_p$ and suppose that $\lim_{n\to\infty} c_n = 0$ so that $f(x)$ converges for all x in $\mathbb{Z}_p$. Let N be the integer defined by conditions*

$$\|c_N\|_p = \max \|c_n\|_p \quad \textit{and} \quad \|c_n\|_p < \|c_N\|_p \quad \textit{for all } n > N.$$

Then the function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ defined by $x \mapsto f(x)$ has at most N zeros.

PROOF OF THEOREM 1.3. Let $(u_n)_{n\geq 0}$ be a sequence of integers of linear or sublinear growth, that is, $u_n = O(n)$. Suppose that $(u_n)_{n\geq 0}$ can be interpolated by some p-adic analytic function

$$f(x) = \sum_{n=0}^{\infty} c_n x^n \in \mathbb{Q}_p[[x]].$$

Since $f(x)$ is a p-adic analytic function, $\lim_{n\to\infty} \|c_n\|_p = 0$. Thus, there exists an integer N defined by the conditions

$$\|c_N\|_p = \max \|c_n\|_p \quad \text{and} \quad \|c_n\|_p < \|c_N\|_p \quad \text{for all } n > N,$$

and Strassman's theorem guarantees that the function $f : \mathbb{Z}_p \to \mathbb{Q}_p$ has at most N zeros.

By hypothesis, $u_n = O(n)$, so there is a $C > 0$ such that $0 < u_n \leq Cn$ for all $n \geq 0$. Taking the subsequence $(u_{p^k})_{k\geq 0}$,

$$0 < u_{p^k} \leq Cp^k. \tag{2.1}$$

Since f is an analytic function, it is easily seen that it satisfies the Lipschitz condition

$$\|f(x) - f(y)\|_p \leq \|x - y\|_p$$

for all $x, y \in \mathbb{Z}_p$. In particular,

$$\|u_{p^k} - u_0\|_p = \|f(p^k) - f(0)\|_p \leq \|p^k\|_p,$$

and it follows that

$$u_{p^k} = u_0 + t_k p^k \tag{2.2}$$

with $t_k \in \mathbb{Z}_+$, because u_{p^k} is a positive integer. By (2.1) and (2.2), we conclude that $0 \leq t_k \leq C$. Hence, by the pigeonhole principle, there exists an integer t with $0 \leq t \leq C$ such that

$$u_{p^j} = u_0 + tp^j$$

for infinitely many $j \geq 0$. Thus, the function

$$f(x) - u_0 - tx = (c_1 - t)x + \sum_{n=2}^{\infty} c_n x^n$$

has infinitely many roots and by Strassman's theorem, we conclude that $f(x) = u_0 + tx$.

https://doi.org/10.1017/S0004972722000946 Published online by Cambridge University Press

Now suppose that $u_n = O(n^d)$ for some fixed positive real number $d > 1$. Let

$$f(x) = \sum_{n=0}^{\infty} a_n \binom{x}{n}$$

be the Mahler expansion of f. By hypothesis, the Mahler expansion of f converges for all $x \in \mathbb{Q}_p$, so the function $x \mapsto \sum_{n=0}^{\infty} a_n \binom{x}{n}$ is analytic on $\mathbb{C}_p$ and

$$\lim_{n\to\infty} r^n \|a_n\|_p = 0$$

for all real numbers $r > 0$ (see [17, Ch. 3]). Taking $r = p^2$, we find $v_p(a_n) \geq 2n$ for all n sufficiently large. Moreover, a_n is an integer for all $n \geq 0$. In fact, by the Mahler expansion,

$$a_n = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(j) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} u_j \quad (n = 0, 1, 2, \ldots),$$

where $u_j \in \mathbb{Z}_+$ for all $j \geq 0$. Hence, either $a_n = 0$ or

$$\|a_n\|_\infty \geq p^{2n}. \tag{2.3}$$

However,

$$\|a_n\|_\infty = \Big\| \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} u_j \Big\|_\infty \quad (n = 0, 1, 2, \ldots).$$

Since $\|u_j\|_\infty \leq j^d \leq n^d$ for all $j \leq n$, it follows that

$$\|a_n\|_\infty \leq D n^d 2^n, \tag{2.4}$$

where $D > 0$ is a fixed constant. It is easily seen that (2.3) and (2.4) cannot both be true for n sufficiently large. Hence, there exists an $N > 0$ such that $a_n = 0$ for all $n > N$. Consequently, f is a polynomial function. Furthermore, $f(n) = O(n^d)$, so its degree must be at most $\lfloor d \rfloor$. □

3. Proof of Theorem 1.5

Suppose that $X = \{x_0, x_1, x_2, \ldots\}$ and $Y = \{y_0, y_1, y_2, \ldots\}$ are subsets of $\mathbb{Z}$ dense in $\mathbb{Z}_p$. Our proof consists in determining a sequence of polynomial functions $f_0, f_1, \ldots$ such that $f_n \to f$ as $n \to \infty$, where f is a p-adic analytic injective function on $\mathbb{Z}_p$ with rational coefficients satisfying $f(X) = Y$. In addition, we will show that there are uncountably many such functions.

To be more precise, we will construct a sequence of polynomial functions $f_0, f_1, f_2, \ldots \in \mathbb{Q}[x]$ of degrees $t_0, t_1, t_2, \ldots \in \mathbb{Z}$, respectively, such that for all $m \geq 0$,

$$f_m(x) = \sum_{i=0}^{t_m} c_i x^i, \tag{3.1}$$

https://doi.org/10.1017/S0004972722000946 Published online by Cambridge University Press

where $c_0 = y_0 - x_0$, $c_1 = 1$ and $\|c_i\|_p \leq p^{-1}$ for all $2 \leq i \leq t_m$. Furthermore, our sequence will obey the recurrence relation

$$f_{m+1}(x) = f_m(x) + x^{t_m+1} P_m(x)(\delta_m + \epsilon_m(x - x_{m+1})), \tag{3.2}$$

where the polynomial functions $P_m \in \mathbb{Z}[x]$ are given by

$$P_m(x) = \prod_{k \in X_m \cup Y_m^{-1}} (x - k), \tag{3.3}$$

with $X_m = \{x_0, \ldots, x_m\}$ and $Y_m^{-1} = f_m^{-1}(\{y_0, \ldots, y_m\})$, and δ_m and ϵ_m are rational numbers such that

$$\max\{\|\delta_m\|_p, \|\epsilon_m\|_p\} \leq p^{-m}.$$

Finally, our sequence will also satisfy $f_m(x_k) \in Y$ and $f_m^{-1}(\{y_k\}) \cap X \neq \emptyset$ for all $0 \leq k \leq m$.

We make some remarks regarding such a sequence of polynomials. First, since f_m is a polynomial, Y_m^{-1} must be a finite subset of $\mathbb{Z}_p$ for each m, so the polynomials P_m are well defined. Second, by (3.1), $\|c_1\|_p > \|c_i\|_p$ for all $i \geq 2$, so each f_m is necessarily injective on $\mathbb{Z}_p$ by Strassmann's theorem. Lastly, since f_m is injective, there is only one $x_s \in X \cap f_m^{-1}(\{y_k\})$. The existence of such a sequence is guaranteed by the following lemma.

LEMMA 3.1. *Suppose that* $f_m(x) = c_0 + c_1 x + \cdots + c_{t_m} x^{t_m} \in \mathbb{Q}[x]$ *is a polynomial with*

$$\|c_i\|_p < \|c_1\|_p \quad \text{for } 2 \leq i \leq t_m \in \mathbb{Z},$$

such that $f_m(X_m) \subset Y$ *and* $Y_m^{-1} \subset X$. *Then there exist rational numbers* δ_m *and* ϵ_m *with*

$$\max\{\|\delta_m\|_p, \|\epsilon_m\|_p\} \leq p^{-m}$$

such that the function

$$f_{m+1}(x) = f_m(x) + x^{t_m+1} P_m(x)(\delta_m + \epsilon_m(x - x_{m+1}))$$

is a polynomial given by

$$f_{m+1}(x) = c_0 + c_1 x + \cdots + c_{t_{m+1}} x^{t_{m+1}} \in \mathbb{Q}[x]$$

satisfying $f_{m+1}(X_{m+1}) \subset X$ *and* $Y_{m+1}^{-1} \subset X$ *and, moreover,* $\|c_i\|_p < \|c_1\|_p$ *for all integers* i *with* $2 \leq i \leq t_{m+1}$.

PROOF. Suppose that for some $m \geq 0$, there is a function f_m satisfying the hypotheses of the lemma. We will show that we can choose rational numbers δ_m and ϵ_m such that

$$\max\{\|\delta_m\|_p, \|\epsilon_m\|_p\} \leq p^{-m}$$

in such a way that the polynomial f_{m+1} in (3.2) has the desired properties.

https://doi.org/10.1017/S0004972722000946 Published online by Cambridge University Press

First, we will determine $\delta_m \in \mathbb{Q}$ such that $f_{m+1}(x_{m+1}) \in Y$. Suppose that $f_m(x_{m+1}) \in \{y_0, y_1, \ldots, y_m\}$. Since $P_m(x_{m+1}) = 0$, we have $f_{m+1}(x_{m+1}) = f_m(x_{m+1}) \in Y$. Note that here we did not make direct use of δ_m to get $f_{m+1}(x_{m+1}) \in Y$. So we are free to choose any $\delta_m \in \mathbb{Q}$ and we do so by setting $\delta_m = p^m$. Now, suppose that $f_m(x_{m+1}) \notin \{y_0, y_1, \ldots, y_m\}$, which implies that $P_m(x_{m+1}) \neq 0$. Since Y is a dense subset of $\mathbb{Z}_p$, there exists $\hat{y} \in Y$ such that

$$0 < \left\| \frac{\hat{y} - f_m(x_{m+1})}{(x_{m+1})^{t_m+1} P_m(x_{m+1})} \right\|_p \leq p^{-m}.$$

Then, taking

$$\delta_m = \frac{\hat{y} - f_m(x_{m+1})}{(x_{m+1})^{t_m+1} P_m(x_{m+1})},$$

we obtain $f_{m+1}(x_{m+1}) = \hat{y} \in Y$ independently of ϵ_m. Observe that in both cases just analysed, $\|\delta_m\|_p \leq p^{-m}$.

Now we will choose $\epsilon_m \in \mathbb{Q}$ to get $f_{m+1}(\hat{x}) = y_{m+1}$ for some $\hat{x} \in X$. Since f_m is injective on $\mathbb{Z}_p$, there is at most one $\hat{x} \in X$ such that $f_m(\hat{x}) = y_{m+1}$. If there exists $\hat{x} \in X_m$ such that $f_m(\hat{x}) = y_{m+1}$, then $P_m(\hat{x}) = 0$ and we obtain $f_{m+1}(\hat{x}) = y_{m+1}$. In this case, ϵ_m does not play a role and we are free to set $\epsilon_m = p^m$. It remains to consider the case where there is no $\hat{x} \in X_m$ with $f_m(\hat{x}) = y_{m+1}$. Note that if we choose

$$\delta_m = \frac{y_{m+1} - f_m(x_{m+1})}{(x_{m+1})^{t_m+1} P_m(x_{m+1})},$$

then $f_{m+1}(x_{m+1}) = y_{m+1}$ and we have $\hat{x} = x_{m+1}$. Since we again did not use ϵ_m to ensure that $f_{m+1}(x_{m+1}) = y_{m+1}$, we are free to take $\epsilon_m = p^m$. However, if

$$\delta_m \neq \frac{y_{m+1} - f_m(x_{m+1})}{(x_{m+1})^{t_m+1} P_m(x_{m+1})},$$

we consider the polynomial equation

$$f_m(x) + \delta_m x^{t_m+1} P_m(x) = y_{m+1}.$$

Since $\|\delta_m\|_p \leq p^{-m}$ and $\|c_i\|_p < p^{-1}$ for $i \geq 2$,

$$f_m(x) + \delta_m x^{t_m+1} P_m(x) - y_{m+1} \equiv y_0 + x - y_{m+1} \pmod{p\mathbb{Z}_p}$$

for all $m \geq 2$. Thus, the congruence

$$f_m(x) + \delta_m x^{t_m+1} P_m(x) - y_{m+1} \equiv 0 \pmod{p\mathbb{Z}_p}$$

has a solution $\bar{x} \equiv y_{m+1} - y_0 \pmod{p\mathbb{Z}_p}$. Moreover, taking the formal derivative,

$$[f_m(x) + \delta_m x^{t_m+1} P_m(x) - y_{m+1}]' \equiv [y_0 + x - y_{m+1}]' \equiv 1 \pmod{p\mathbb{Z}_p}.$$

https://doi.org/10.1017/S0004972722000946 Published online by Cambridge University Press

Hence, by Hensel's lemma [14], there exists $b \in \mathbb{Z}_p$ such that

$$f_m(b) + \delta_m b^{t_m+1} P_m(b) = y_{m+1}.$$

Let $v_p(x)$ be the p-adic valuation of $x \in \mathbb{Z}_p$ and take

$$s = v_p(b^{t_m+1} P_m(b)(b - x_{m+1})).$$

Note that $s < +\infty$, since $P_m(b)(b - x_{m+1}) \neq 0$. Thus, we have a Lipschitz condition on $\mathbb{Z}_p$, namely

$$\|f_m(x) + \delta_m x^{t_m+1} P_m(x) - f_m(y) + \delta_m y^{t_m+1} P_m(y)\|_p \leq \|x - y\|_p$$

for all $x, y \in \mathbb{Z}_p$. Since X is a dense subset of $\mathbb{Z}_p$, there is an integer $\hat{x} \in X$ such that

$$\|\hat{x} - b\|_p \leq \frac{1}{p^{s+m}}$$

and $v_p(\hat{x}^{t_m+1} P_m(\hat{x})(\hat{x} - x_{m+1})) = s$. So,

$$\|f_m(\hat{x}) + \delta_m \hat{x}^{t_m+1} P_m(\hat{x}) - y_{m+1}\|_p \leq \frac{1}{p^{s+m}}.$$

Taking

$$\epsilon_m = \frac{y_{m+1} - f_m(\hat{x}) - \delta_m \hat{x}^{t_m+1} P_m(\hat{x})}{\hat{x}^{t_m+1} P_m(\hat{x})(\hat{x} - x_{m+1})},$$

we get $\epsilon_m \in \mathbb{Q}$, $\|\epsilon_m\|_p < p^{-m}$ and $f_{m+1}(\hat{x}) = y_{m+1}$. This completes the proof of the lemma. □

Proof of Theorem 1.5. If in Lemma 3.1 we start with $f_0(x) = (x - x_0) + y_0$, we get a sequence of polynomials as described in the beginning of this section. Furthermore, in each step, we have at least two options for the choice of δ_m and ϵ_m so we get uncountably many sequences. We will fix one of these sequences and prove that $f(x) = \lim_{m\to\infty} f_m(x)$ solves Theorem 1.5. Indeed,

$$f_m(x) = y_0 + (x - x_0) + \sum_{j=1}^{m-1} x^{t_j+1} P_j(x)[\delta_j + \epsilon_j(x - x_{j+1})] = \sum_{j=0}^{t_m} c_j x^j,$$

where $\|c_i\|_p \leq p^{-j}$ for $t_{j-1} < i \leq t_j$ and $1 \leq j \leq m$ (since $\max\{\|\delta_j\|_p, \|\epsilon_j\|_p\} \leq p^{-j}$). Therefore, $\lim_{i\to\infty} \|c_i\|_p = 0$ and

$$f(x) = \lim_{m\to\infty} f_m(x)$$

is a p-adic analytic function on $\mathbb{Z}_p$.

Moreover, $f(X) = Y$. Indeed, we are assuming that $f_k(x_k) \in Y$. By (3.3), $P_m(x_k) = 0$ for all $m \geq k \geq 0$ and, consequently, $f_m(x_k) = f_{m-1}(x_k) = \cdots = f_k(x_k)$. Thus, we conclude that

$$f(x_k) = \lim_{m\to\infty} f_m(x_k) = f_k(x_k) \in Y.$$

https://doi.org/10.1017/S0004972722000946 Published online by Cambridge University Press

However, by hypothesis, given an integer $j \geq 0$, there exists an integer $s \geq 0$ such that $f_j(x_s) = y_j$. Similarly,

$$f(x_s) = \lim_{m\to\infty} f_m(x_s) = f_j(x_s) = y_j \in Y$$

and we conclude $f(X) = Y$.

It remains to prove that f is injective. For this, suppose that there are a_1 and a_2 in $\mathbb{Z}_p$ such that $f(a_1) = f(a_2) = b \in \mathbb{Z}_p$ and note that by (3.1), $c_1 = 1$ satisfies

$$\|c_1\|_p = \max \|c_j\|_p \quad \text{and} \quad \|c_j\|_p < \|c_1\|_p \quad \text{for all } j > 1. \tag{3.4}$$

Now, consider the function

$$f(x) - b = (y_0 - x_0 - b) + x + \sum_{n=2}^{\infty} c_n x^n.$$

Note that in the equation above, $c_1 = 1$ still satisfies the conditions in (3.4). Hence, $f(x) - b$ has at most one zero (by Strassman's theorem), so we have $a_1 = a_2$. □

References

[1] J. P. Bell, 'A generalised Skolem–Mahler–Lech theorem for affine varieties', *J. Lond. Math. Soc. (2)* **73**(2) (2006), 367–379.

[2] P. Bihani, W. P. Sheppard and P. T. Young, 'p-Adic interpolation of the Fibonacci sequence via hypergeometric functions', *Fibonacci Quart.* **43**(3) (2005), 213–226.

[3] D. Clark, 'A note on the p-adic convergence of the solutions of linear differential equations', *Proc. Amer. Math. Soc.* **17** (1966), 262–269.

[4] F. Georg, 'Ueber arithmetische Eingenschaften analytischer Functionen', *Math. Ann.* **58**(4) (1904), 545–557.

[5] J. Lelis and D. Marques, 'Some results on arithmetic properties of p-adic Liouville numbers'. *p-Adic Numbers Ultrametric Anal. Appl.* **11**(3) (2019), 216–222.

[6] J. Lelis and D. Marques, 'On transcendental entire functions mapping $\mathbb{Q}$ into itself', *J. Number Theory* **206** (2020), 310–319.

[7] K. Mahler, 'An interpolation series for continuous functions of a p-adic variable', *J. reine angew. Math.* **199** (1958), 23–34.

[8] K. Mahler, *Lectures on Transcendental Numbers*, Lecture Notes in Mathematics, 546 (Springer-Verlag, Berlin, 1976).

[9] K. Mahler, 'Some suggestions for further research,' *Bull. Aust. Math. Soc.* **29**(1) (1984), 101–108.

[10] E. Maillet, *Introduction à la Théorie des Nombres Transcendants et des Propriétés Arithmétiques des Fonctions* (Gauthier–Villars, Paris, 1906).

[11] D. Marques and C. G. Moreira, 'On a variant of a question proposed by K. Mahler concerning Liouville numbers', *Bull. Aust. Math. Soc.* **91** (2015), 29–33.

[12] D. Marques and C. G. Moreira, 'A positive answer for a question proposed by K. Mahler', *Math. Ann.* **368** (2017), 1059–1062.

[13] D. Marques and C. G. Moreira, 'On a stronger version of a question proposed by K. Mahler', *J. Number Theory* **194** (2019), 372–380.

[14] M. R. Murty, *Introduction to p-Adic Analytic Number Theory*, Studies in Advanced Mathematics, 27 (American Mathematical Society, Providence, RI, 2009).

[15] A. M. Robert, *A Course in p-Adic Analysis*, Graduate Texts in Mathematics, 198 (Springer, New York, 2013).

[16] E. Rowland and E. Yassawi, 'p-Adic asymptotic properties of constant-recursive sequences', *Indag. Math. (N.S.)* **28**(1) (2017), 205–220.
[17] W. H. Schikhof, *Ultrametric Calculus* (Cambridge University Press, Cambridge, 2006).
[18] P. Stäckel, 'Arithmetische eingenschaften analytischer Functionen', *Acta Math.* **25** (1902), 371–383.

BRUNO DE PAULA MIRANDA, Instituto Federal de Goiás,
Avenida Saia Velha, Km 6, BR-040, s/n, Parque Esplanada V,
Valparaíso de Goiás, GO 72876-601, Brazil
e-mail: bruno.miranda@ifg.edu.br

JEAN LELIS, Faculdade de Matemática, Instituto de Ciências Exatas e Naturais,
Universidade Federal do Pará, Belém PA, Brazil
e-mail: jeanlelis@ufpa.br

https://doi.org/10.1017/S0004972722000946 Published online by Cambridge University Press