# OPTIMAL REINSURANCE REVISITED – A GEOMETRIC APPROACH

BY

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ABSTRACT

In this paper, we reexamine the two optimal reinsurance problems studied in Cai et al. (2008), in which the objectives are to find the optimal reinsurance contracts that minimize the value-at-risk (VaR) and the conditional tail expectation (CTE) of the total risk exposure under the expectation premium principle. We provide a simpler and more transparent approach to solve these problems by using intuitive geometric arguments. The usefulness of this approach is further demonstrated by solving the VaR-minimization problem when the expectation premium principle is replaced by Wang's premium principle.

KEYWORDS

Value-at-risk, conditional tail expectation, reinsurance, expectation premium principle, comonotonicity, Wang's premium principle, increasing convex function.

## 1. INTRODUCTION

The problem of designing optimal (re)insurance contracts has a long history, starting from Borch (1960), Arrow (1963), Mossin (1968), Smith (1968), etc. Most of the early analysis was based on the assumption that decision makers are expected-utility maximizers. In more recent research, other optimization criteria have been proposed. Combined with different premium principles, various optimality results of optimal (re)insurance have been obtained. See for example Gerber (1979), Waters (1983), Goovaerts et al. (1989), Bowers et al. (1997), Young (1999), Schmitter (2001), Verlaak and Beirlant (2003), Kaluszka, (2001, 2004a, 2004b, 2005), Guerra and Centeno (2008), Balbás et al. (2008), amongst others.

In a recent paper by Cai et al. (2008), the authors studied the problems of minimizing the value-at-risk (VaR) and the conditional tail expectation (CTE) of the total retained loss under the expectation premium principle. The first objective of this paper is to give an alternative way to analyze and solve these problems. A detailed description of the model is as follows. Fix an integrable non-negative random variable $X$ which represents the loss initially assumed by

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the insurer. Its survival function is denoted as $S_X$. Following Cai et al. (2008), we assume that $S_X$ is strictly decreasing and continuous on $(0, \infty)$, with a possible jump at 0. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ denote the reinsurance policy in which the reinsurer pays $f(x)$ to the insurer if the insurer suffers a loss of size $x$. The function $f$ is called the ceded loss function. It is assumed that $f$ is increasing and convex and satisfies $0 \le f(x) \le x$ for $x \ge 0$. The restriction that $f(x) \le x$ is often referred to as an indemnity constraint. In Cai et al. (2008), the possibility that $f \equiv 0$ is excluded from consideration. However, we do admit this null function, which corresponds to the case that no reinsurance protection is purchased, as a legitimate ceded loss function in the present analysis. The collection of all possible ceded loss functions is denoted as $\mathcal{F}$.

Let $\delta_f(X)$ be the reinsurance premium when $f \in \mathcal{F}$ is chosen. The total cost or the total retained loss $T_f(X)$ of the insurer is the sum of the retained loss $I_f(X) = X - f(X)$ and the reinsurance premium $\delta_f(X)$, that is, $T_f(X) = I_f(X) + \delta_f(X)$. In Cai et al. (2008), the reinsurance premium is determined by the expectation premium principle, so that $\delta_f(X) = (1 + \rho)\mathbb{E}[f(X)]$. Here, $\rho$ is a positive constant known as the safety loading. A higher $\rho$ means that reinsurance is more expensive. The VaR and the CTE of a random variable $Y$ at a confidence level $1 - \alpha \in (0, 1)$ are defined as

$$\mathrm{VaR}_Y(\alpha) := S_Y^{-1}(\alpha) := \inf\{y; \mathbb{P}(Y > y) \le \alpha\},$$

and

$$\mathrm{CTE}_Y(\alpha) := \mathbb{E}(Y \mid Y \ge \mathrm{VaR}_Y(\alpha))$$

respectively. The optimal reinsurance problems studied by Cai et al. (2008) can now be formally stated as

$$\min_{f \in \mathcal{F}} \mathrm{VaR}_{T_f(X)}(\alpha), \tag{1}$$

and

$$\min_{f \in \mathcal{F}} \mathrm{CTE}_{T_f(X)}(\alpha). \tag{2}$$

Functions in $\mathcal{F}$ that minimize the above objective functions are called optimal ceded loss functions. As in Cai et al. (2008), we henceforth assume that $\alpha \in (0, S_X(0))$ to avoid trivial cases.

Cai et al. (2008) provided complete solutions to the above problems by complicated approximation and convergence arguments. They first proved that every function in $\mathcal{F}$ can be approximated by the subclass $\mathcal{F}^*$ of piecewise-linear increasing and convex functions of the form $f(x) = \sum_{j=1}^{n} c_j (x - d_j)_+$ where $c_j > 0$, $d_j \ge 0$ with $\sum c_j \le 1$. Then by utilizing some convergence properties of VaR and CTE, they proved that the optimal functions in the subclass $\mathcal{F}^*$ also optimally minimize the VaR and CTE of the total cost in $\mathcal{F}$. As a result, they can deduce optimal cede loss functions by confining attention to $\mathcal{F}^*$.

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In this paper, we present an alternative approach to solve the above optimal reinsurance problems. In the first step, we use a simple geometric argument to show that optimal ceded loss functions must take the form $f(x) = c(x - d)_+$. Since every such function is specified by only two parameters (the slope $c$ and the deductible $d$), the infinite-dimensional optimization problems (1) and (2) are reduced to two-dimensional problems, which can be solved explicitly by standard calculus method. Using this approach, we can not only avoid complicated convergence and approximation arguments but also gain geometric insight about the nature of the optimal ceded loss functions. In Sections 2 and 3 we study the VaR-minimization problem and the CTE-minimization problem respectively using this alternative approach. It is interesting to note that our approach remains applicable when the expectation premium principle is replaced by Wang's premium principle. The problem of minimizing the VaR of the total cost under Wang's premium principle is studied in Section 4. Concluding remarks are given in Section 5.

## 2. Optimal reinsurance under VaR risk measure

In this section, we solve (1), the VaR-minimization problem. Denote the objective function as $H(f) := \mathrm{VaR}_{T_f(X)}(\alpha)$. We rewrite $H(f)$ as follows:

$$\begin{aligned} H(f) &= \mathrm{VaR}_{I_f(X)}(\alpha) + (1+\rho)\,\mathbb{E}[f(X)] \\ &= I_f(\mathrm{VaR}_X(\alpha)) + (1+\rho)\,\mathbb{E}[f(X)] \\ &= S_X^{-1}(\alpha) - f(S_X^{-1}(\alpha)) + (1+\rho)\,\mathbb{E}[f(X)], \end{aligned} \tag{3}$$

in which the first equality follows from the translation invariance property of VaR and the second equality follows from the fact that $I_f$ is increasing and continuous (c.f. Lemma A.1 of Cai et al. (2008)). To simplify our notation, we write $S_X^{-1}(\alpha)$ as $a$ throughout this paper. Thus problem (1) can be rewritten as

$$\min_{f\in\mathcal{F}} H(f) = \min_{f\in\mathcal{F}} \{a - f(a) + (1+\rho)\,\mathbb{E}[f(X)]\}. \tag{4}$$

The following lemma excludes the possibility that $f$ is non-null but is identically zero on the interval $[0, a]$.

**Lemma 1.** *A ceded loss function $f \in \mathcal{F}$ that is not null but identically zero on $[0, a]$ is not optimal for problem (4).*

**Proof:** Let $f$ be such a function. Consider $h := \frac{1}{2} f \in \mathcal{F}$. Since $H(f) = a + (1 + \rho)\mathbb{E}[f(X)] > a + (1 + \rho)\mathbb{E}[h(X)] = H(h)$, $f$ is not optimal. □

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The situation depicted in Lemma 1 is described in Figure 1:

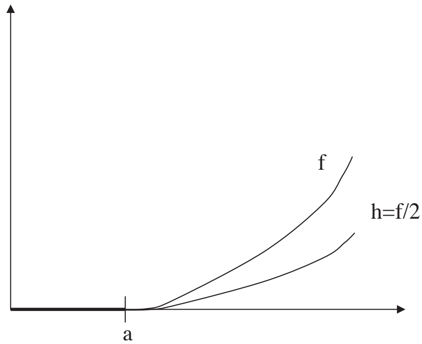


FIGURE 1: Geometric meaning of Lemma 1.

Because of this result, we assume in the remainder of this section that $\mathcal{F}$ does not contain non-null ceded loss functions that are identically zero on $[0, a]$. The next result shows that optimal ceded loss functions that minimize $H$ must take the form $f_{c,d}(x) = c(x - d)_+$ for some $(c, d) \in [0, 1] \times [0, a)$. The collection of all such functions is denoted as $\mathcal{G}$. Note that $\mathcal{G} \subset \mathcal{F}$, and $\mathcal{G}$ contains the linear function $f_{c,0}(x) = cx$, $x \geq 0$, $c \in (0,1]$, as well as the null function $f_{0,d}(x) \equiv 0$.

**Lemma 2.** *Let $f \in \mathcal{F}$ be a non-null ceded loss function. There always exists a function $h \in \mathcal{G}$ such that $H(h) \leq H(f)$.*

**Proof:** Let $f'_+(a)$ and $f'_-(a)$ be the right-hand derivative and left-hand derivative of $f$ at $a$ respectively. Let $c$ be an arbitrary number in $\partial f(a)$, the subdifferential of $f$ at $a$, which is defined as the interval $[f'_-(a), f'_+(a)]$. Then the straight line passing through $(a, f(a))$ with slope $c$ constitutes a supporting line of the convex function $f$, and hence always lies below the graph of $f$. For details, see Section 23 of Rockafellar (1970). Since $0 \leq f(x) \leq x$ for all $x \geq 0$ and $f$ is not identically zero on $[0, a]$, we have $c \in (0,1]$. Let $d$ be the unique intersection of this straight line and the $x$-axis. Then $d = a - f(a)/c \in [0, a)$. Define $h(x) = c(x - d)_+$, $x \geq 0$. Then $h \in \mathcal{G}$ and $f(a) = h(a)$. Hence

$$H(h) = a - h(a) + (1 + \rho)\mathbb{E}[h(X)] \leq a - f(a) + (1 + \rho)\mathbb{E}[f(X)] = H(f).$$

Therefore, $h$ is the desired function. □

The construction used in the proof of Lemma 2 is illustrated in Figure 2.

The geometric meaning of this lemma is clear. When minimizing $H(f) = a - f(a) + (1 + \rho)\mathbb{E}[f(X)]$, there are two opposing forces to consider. While the term $-f(a)$ requires that $f$ be as large as possible at $a$, the term $(1 + \rho)\mathbb{E}[f(X)]$ requires that the whole $f$ be as small as possible. If the value of $f(a)$ is fixed,

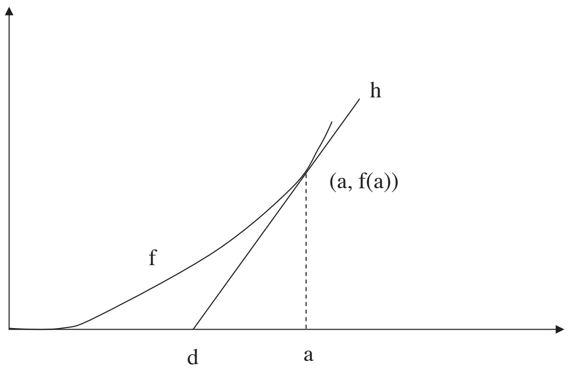


FIGURE 2: Geometric meaning of Lemma 2.

then $f$ is forced to be a straight line passing through the point $(a, f(a))$. Thus it suffices to consider the subclass $\mathcal{G} \subset \mathcal{F}$ when solving problem (4). As every ceded loss function $f_{c,d} \in \mathcal{G}$ is completely specified by parameters $c$ and $d$, the problem can be solved by straightforward applications of calculus.

To state the main result of this section, we follow Cai et al. (2008) by defining the following notation: $\rho^* = \frac{1}{1+\rho}$, $d^* = S_X^{-1}(\rho^*)$, and

$$g(x) = x + \frac{1}{\rho^*}\int_x^\infty S_X(t)\,dt, \quad u(x) = S_X^{-1}(x) + \frac{1}{\rho^*}\int_{S_X^{-1}(x)}^\infty S_X(t)\,dt.$$

Note that $g(d^*) = u(\rho^*)$, and $g(0) = (1+\rho)\mathbb{E}[X]$.

The following theorem gives the solution to the VaR-minimization problem (4). It is slightly different from the one presented in Cai et al. (2008) in that the null function is also included for consideration.

**Theorem 1.** *For a given $\alpha \in (0, S_X(0))$, the following statements hold true.*

*(a) If $\rho^* < S_X(0)$ and $a > u(\rho^*)$, then the minimum value of $H$ over $\mathcal{F}$ is $g(d^*)$, and the optimal ceded loss function is $f^*(x) = (x - d^*)_+$.*

*(b) If $\rho^* < S_X(0)$ and $a = u(\rho^*)$, then the minimum value of $H$ over $\mathcal{F}$ is $g(d^*)$, and the optimal ceded loss function is $f^*(x) = c(x - d^*)_+$ for any constant $c \in [0, 1]$.*

*(c) If $\rho^* \geq S_X(0)$ and $a > g(0)$, then the minimum value of $H$ over $\mathcal{F}$ is $g(0)$, and the optimal ceded loss function is $f^*(x) = x$.*

*(d) If $\rho^* \geq S_X(0)$ and $a = g(0)$, then the minimum value of $H$ over $\mathcal{F}$ is $g(0)$, and the optimal ceded loss function is $f^*(x) = cx$ for any constant $c \in [0, 1]$.*

*(e) For all other cases, the minimum value of $H$ over $\mathcal{F}$ is $a$, and the optimal ceded loss function is $f^*(x) \equiv 0$.*

**Proof:** We only prove (a) here. The proofs of the other parts are similar and are omitted. Let $f_{c,d}(x) = c(x-d)_+$ with $(c,d) \in [0,1] \times [0,a)$ be a ceded loss function in $\mathcal{G}$. Then

$$H(f_{c,d}) = a - c(a-d) + c(1+\rho)\int_d^\infty S_X(t)\,dt. \tag{5}$$

To minimize $H(f_{c,d})$ over all possible $(c,d)$ in the region $[0,1] \times [0,a)$, we first consider $c > 0$. Taking partial derivative of $H(f_{c,d})$ with respect to $d$ on $(0,a)$ yields

$$\frac{\partial H(f_{c,d})}{\partial d} = c\big[1-(1+\rho)\,S_X(d)\big],$$

which is increasing in $d$. So $H(f_{c,d})$ is convex in $d$. Moreover, $\frac{\partial H(f_{c,d})}{\partial d} = 0$ at $d = d^*$. Under the assumptions that $\rho^* < S_X(0)$ and $a > u(\rho^*)$, $d^* = S_X^{-1}(\rho^*) < a$. Thus $H(f_{c,d})$ attains its minimum value at $d = d^*$ no matter what $c$ is. So our two-dimensional minimization problem is equivalent to a repeated one-dimensional minimization problem. Next we consider the derivative of $H(f_{c,d})$ with respect to $c$:

$$\frac{\partial H(f_{c,d^*})}{\partial c} = -(a-d^*) + (1+\rho)\int_{d^*}^\infty S_X(t)\,dt = g(d^*) - a = u(\rho^*) - a < 0.$$

Thus the optimal value of $c$ is 1, and hence the optimal ceded loss function is given by $f_{1,d^*}$. From (5), the corresponding value of $H$ is

$$H(f_{1,d^*}) = a - (a-d^*) + (1+\rho)\int_{d^*}^\infty S_X(t)\,dt = g(d^*).$$

Now the result follows from the final observation that if $c = 0$, then $H(f_{0,d}) = a > u(\rho^*) = g(d^*) = H(f_{1,d^*})$ for every $d \in [0,a)$. □

We illustrate the above result using a simple numerical example. Suppose that $X$ is exponentially distributed with mean 1000, so that $S_X(t) = e^{-0.001t}$ for $t \geq 0$, $S_X^{-1}(\alpha) = -1000 \ln \alpha$ for $0 < \alpha < 1$, and $S_X(0) = 1$. We consider the following cases:

**Case 1.** $\rho = 0$: In this case, $\rho^* = 1$ and $g(0) = \mathbb{E}[X] = 1000$. By Theorem 1, the optimal reinsurance plan depends on whether the risk tolerance level $\alpha$ is higher or lower than a certain threshold level:

(1) If $a > g(0)$, that is, $\alpha < 0.3679$, then $f^*(x) = x$. [This is Case (c) of Theorem 1.]

(2) If $a = g(0)$, that is, $\alpha = 0.3679$, then $f^*(x) = cx$ for any $c \in [0,1]$. [This is Case (d) of Theorem 1.]

(3) If $a < g(0)$, that is, $\alpha > 0.3679$, then $f^*(x) \equiv 0$. [This is Case (e) of Theorem 1.]

The threshold level of $\alpha$ is 0.3679. If the risk tolerance level is high (so that $\alpha$ is higher than the threshold level), there is no need to purchase any reinsurance; otherwise, it is optimal to purchase full reinsurance for the whole loss $X$.

**Case 2.** $\rho = 0.2$: In this case, $\rho^* = 0.833$, $d^* = S_X^{-1}(\rho^*) = 182.32$, and $u(\rho^*) = g(d^*) = 1182.32$. By Theorem 1, there are several possibilities:

(1) If $a > u(\rho^*)$, that is, $\alpha < 0.3066$, then $f^*(x) = (x - 182.32)_+$. [This is Case (a) of Theorem 1.]

(2) If $a = u(\rho^*)$, that is, $\alpha = 0.3066$, then $f^*(x) = c(x - 182.32)_+$ for any $c \in [0,1]$. [This is Case (b) of Theorem 1.]

(3) If $a < u(\rho^*)$, that is, $\alpha > 0.3066$, then $f^*(x) \equiv 0$. [This is Case (e) of Theorem 1.]

When the reinsurance premium is higher ($\rho$ increases from 0 to 0.2), the threshold level of $\alpha$ decreases from 0.3679 to 0.3066. When the risk tolerance level is high so that $\alpha$ is higher than 0.3066, it is optimal not to purchase any reinsurance. However, if $\alpha$ is smaller than 0.3066, it is no longer optimal to purchase full reinsurance. Instead, one should buy a stop-loss reinsurance with deductible 182.32.

## 3. Optimal reinsurance under CTE risk measure

In this section, we solve (2), the CTE-minimization problem. Denote the objective function as $K(f) := \mathrm{CTE}_{T_f(X)}(\alpha)$. We rewrite $K(f)$ as follows:

$$
\begin{aligned}
K(f) &= \mathrm{VaR}_{T_f(X)}(\alpha) + \frac{\mathbb{E}(T_f(X) - \mathrm{VaR}_{T_f(X)}(\alpha))_+}{\mathbb{P}(T_f(X) \geq \mathrm{VaR}_{T_f(X)}(\alpha))} \\
&= \mathrm{VaR}_{T_f(X)}(\alpha) + \frac{\mathbb{E}(I_f(X) - \mathrm{VaR}_{I_f(X)}(\alpha))_+}{\mathbb{P}(I_f(X) \geq \mathrm{VaR}_{I_f(X)}(\alpha))} \\
&= a - f(a) + (1+\rho)\,\mathbb{E}[f(X)] + \frac{\mathbb{E}(I_f(X) - \mathrm{VaR}_{I_f(X)}(\alpha))_+}{\mathbb{P}(I_f(X) \geq \mathrm{VaR}_{I_f(X)}(\alpha))}
\end{aligned} \tag{6}
$$

in which the first equality follows from (3.2) of Cai and Tan (2007).

### 3.1. General Considerations

Before deriving optimal ceded loss functions $f \in \mathcal{F}$ that minimize $K(f)$ in (6), which will be done in the next three subsections, we first study some qualitative properties that optimal ceded loss functions should possess. This enables us to confine to a very small class of ceded loss functions when solving the minimization problem.

Notice that the denominator of the last term in (6) may not equal $\alpha$. In Cai et al. (2008), it was proved that the distribution function of $I_f(X)$ has at most one point of discontinuity on $[0, \infty)$. If such a discontinuity exists, then $I_f$ must take the form

$$I_f(x) = \begin{cases} v(x), & 0 \le x \le e_0, \\ v(e_0), & x > e_0, \end{cases}$$

for some strictly increasing and continuous function $v$ and constant $e_0 \in [0, \infty)$. In this case, $v(e_0)$ is the only point of discontinuity of the distribution function of $I_f(X)$, and the corresponding ceded loss function $f$ is a straight line with slope one from $e_0$ onward. If $\alpha \ge \mathbb{P}(X \ge e_0)$, or equivalently $a \le e_0$, then $\mathbb{P}(I_f(X) \ge \mathrm{VaR}_{I_f(X)}(\alpha)) = \alpha$ and hence (6) becomes

$$K(f) = a - f(a) + (1+\rho)\,\mathbb{E}[f(X)] + \frac{\mathbb{E}(I_f(X) - \mathrm{VaR}_{I_f(X)}(\alpha))_+}{\alpha}.$$

Of course, this equation also holds true if the distribution function of $I_f(X)$ is always continuous. On the other hand, if $\alpha < \mathbb{P}(X \ge e_0)$, or equivalently $a > e_0$, then

$$\mathbb{P}(I_f(X) \ge \mathrm{VaR}_{I_f(X)}(\alpha)) = \mathbb{P}(X \ge e_0) = S_X(e_0-) > \alpha,$$

and hence

$$K(f) = a - f(a) + (1+\rho)\,\mathbb{E}[f(X)] + \frac{\mathbb{E}(I_f(X) - \mathrm{VaR}_{I_f(X)}(\alpha))_+}{S_X(e_0-)}.$$

We remark that $S_X(e_0-)$ in the denominator of the last term equals 1 if $e_0 = 0$ (i.e. if $f(x) = x$) and $S_X(e_0)$ if $e_0 \in (0, a)$.

Furthermore, observe that the pair $(f(X), I_f(X))$ is comonotonic because both $f$ and $I_f$ are increasing. Recall that for any fixed $\alpha \in (0,1)$ the function $Y \mapsto \mathbb{E}(Y - S_Y^{-1}(\alpha))_+$, which is commonly called the expected shortfall of the random variable $Y$, is comonotonic additive (cf. Theorem 4.2.1 of Dhaene et al. (2006)). Thus we have

$$\mathbb{E}(I_f(X) - \mathrm{VaR}_{I_f(X)}(\alpha))_+ + \mathbb{E}(f(X) - \mathrm{VaR}_{f(X)}(\alpha))_+ = \mathbb{E}(X - \mathrm{VaR}_X(\alpha))_+.$$

Moreover, we know from Theorem 2.1 of Dhaene et al. (2006) that

$$\begin{aligned}\mathbb{E}(f(X)-\mathrm{VaR}_{f(X)}(\alpha))_+ &= \int_0^\alpha \mathrm{VaR}_{f(X)}(p)\,dp-\alpha\mathrm{VaR}_{f(X)}(\alpha)\\ &= \int_0^\alpha f(S_X^{-1}(p))\,dp-\alpha f(a).\end{aligned}$$

The following lemma summarizes all the above considerations. The collection of all ceded loss functions that are straight line with slope one from $e_0$ onward for some $e_0 \in [0,a)$ is denoted as $\hat{\mathcal{F}}$.

**Lemma 3.** *If $f \in \hat{\mathcal{F}}$ such that it is a straight line with slope one from $e_0$ onward for some $e_0 \in [0,a)$, then*

$$\begin{aligned}K(f) &= a-f(a)+(1+\rho)\mathbb{E}[f(X)]\\ &\quad+\frac{\mathbb{E}(X-\mathrm{VaR}_X(\alpha))_+-\int_0^\alpha f(S_X^{-1}(p))\,dp+\alpha f(a)}{S_X(e_0-)};\end{aligned} \tag{7}$$

*otherwise, we have for $f \in \mathcal{F}\setminus\hat{\mathcal{F}}$ that*

$$\begin{aligned}K(f) &= a-f(a)+(1+\rho)\,\mathbb{E}[f(X)]\\ &\quad+\frac{\mathbb{E}(X-\mathrm{VaR}_X(\alpha))_+-\int_0^\alpha f(S_X^{-1}(p))\,dp+\alpha f(a)}{\alpha}\\ &= a+(1+\rho)\,\mathbb{E}[f(X)]+\frac{\mathbb{E}(X-\mathrm{VaR}_X(\alpha))_+-\int_0^\alpha f(S_X^{-1}(p))\,dp}{\alpha}.\end{aligned} \tag{8}$$

Because of the discrepancy between (7) and (8), functions from $\hat{\mathcal{F}}$ require separate treatment. Let us remark that in (8), the term $-\int_0^\alpha f(S_X^{-1}(p))\,dp$ is decreasing in $f$ and depends only on the "tail" part of $f$, i.e. the value of $f(x)$ for $x \geq a$. On the other hand, the term $(1+\rho)\mathbb{E}[f(X)]$ is increasing in $f$. Thus the argument used in Lemma 2 implies that if the optimal ceded loss function belongs to $\mathcal{F}\setminus\hat{\mathcal{F}}$, it must take the form $f(x) = c(x-d)_+$ for some $(c,d) \in [0,1)\times[0,a]$ when $x$ is restricted to the interval $[0,a]$. Observe that non-null ceded loss functions that are identically zero on $[0,a]$, which correspond to the case where $c > 0$ and $d = a$, cannot be excluded here because Lemma 1 is no longer valid. Observe also that $c = 1$ but $d < a$ is not allowed here because in this case $f \in \hat{\mathcal{F}}$.

Now we investigate the optimal "tail" behavior of ceded loss functions in $\mathcal{F}\setminus\hat{\mathcal{F}}$.

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**Lemma 4.** *If $f$, $h$ are two ceded loss functions in $\mathcal{F} \setminus \hat{\mathcal{F}}$ such that $f = h$ on* $[0, a]$, *then*

$$K(f) - K(h) = \left(1 + \rho - \frac{1}{\alpha}\right)\int_a^\infty [f(t) - h(t)]dF_X(t).$$

**Proof:** By (8),

$$\begin{aligned} K(f) - K(h) &= (1+\rho)\,\mathbb{E}[f(X) - h(X)] - \frac{1}{\alpha}\int_0^\alpha \left[f(S_X^{-1}(p)) - h(S_X^{-1}(p))\right]dp \\ &= (1+\rho)\int_a^\infty [f(t) - h(t)]dF_X(t) - \frac{1}{\alpha}\int_a^\infty [f(t) - h(t)]dF_X(t) \\ &= \left(1 + \rho - \frac{1}{\alpha}\right)\int_a^\infty [f(t) - h(t)]dF_X(t), \end{aligned}$$

as desired. □

We infer from this lemma that if $\rho^* > \alpha$, the tail of the ceded loss function should be as large as possible in order to minimize $K$; if $\rho^* < \alpha$, the tail of the ceded loss function should be as small as possible; if $\rho^* = \alpha$, then $K(f) = K(h)$ and so the tail of the ceded loss function has no effect on $K$. Together with the remark immediately after Lemma 3 which specifics the shape of the optimal ceded loss function on $[0, a]$, we already have a very detailed qualitative description of the optimal ceded loss function if it is lying in $\mathcal{F} \setminus \hat{\mathcal{F}}$.

Now we turn our focus to $\hat{\mathcal{F}}$. The following result reveals that if the optimal ceded loss function belongs to $\hat{\mathcal{F}}$, it must be a stop-loss function.

**Lemma 5.** *For every $f \in \hat{\mathcal{F}}$, there exists a stop-loss function $f'$ in $\hat{\mathcal{F}}$ of the form $f'(x) = (x - d)_+$ for some $d \in [0, a)$ such that $K(f') \leq K(f)$.*

**Proof:** If $f(x) = x$, there is nothing to prove. So we assume that $f$ has slope one from $e$ onward for some $e \in (0, a)$. Construct another ceded loss function

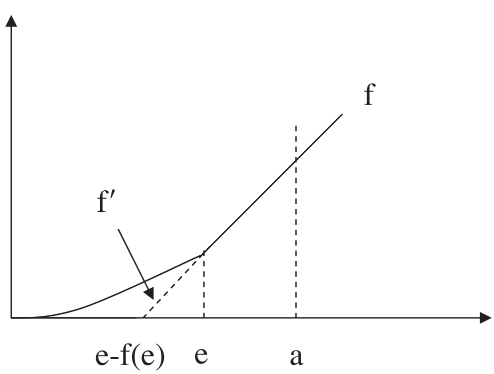


FIGURE 3: Construction of $f'$.

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$f'(x) = (x-(e-f(e)))_+ \in \hat{\mathcal{F}}$, which is the stop-loss function that coincides with $f$ from $e$ onward. See Figure 3.

By construction, we have $f(a) = f'(a)$, $\int_0^\alpha f(S_X^{-1}(p))\,dp = \int_0^\alpha f'(S_X^{-1}(p))\,dp$. Moreover, the convexity of $f$ implies that $f' \le f$ on $[0,e]$ and hence $\mathbb{E}[f'(X)] \le \mathbb{E}[f(X)]$. By Lemma 3,

$$\begin{aligned} K(f') &= a - f'(a) + (1+\rho)\mathbb{E}[f'(X)] + \frac{\mathbb{E}(X-\mathrm{VaR}_X(\alpha))_+ - \int_0^\alpha f'(S_X^{-1}(p))\,dp + \alpha f'(a)}{S_X(e-f(e))} \\ &= a - f(a) + (1+\rho)\mathbb{E}[f'(X)] + \frac{\mathbb{E}(X-\mathrm{VaR}_X(\alpha))_+ - \int_0^\alpha f(S_X^{-1}(p))\,dp + \alpha f(a)}{S_X(e-f(e))} \\ &\le a - f(a) + (1+\rho)\mathbb{E}[f(X)] + \frac{\mathbb{E}(X-\mathrm{VaR}_X(\alpha))_+ - \int_0^\alpha f(S_X^{-1}(p))\,dp + \alpha f(a)}{S_X(e)} \\ &= K(f), \end{aligned}$$

in which the inequality follows from $S_X(e) \le S_X(e-f(e))$ and the positivity of the numerator. Hence $f'$ is the desired function. □

Motivated by Lemma 5 and to prepare for future analysis, we first consider the problem of minimizing $K$ over the class $\tilde{\mathcal{F}}$ of all stop-loss functions $f_d(x) = (x-d)_+$, with deductible $d \in [0,a]$. Note that $f_d \in \hat{\mathcal{F}}$ if $d \in [0,a)$, but $f_a \notin \hat{\mathcal{F}}$.

**Lemma 6.** *For a given $\alpha \in (0, S_X(0))$, the following statements hold true.*

*(a) If $S_X(0) \le \rho^*$, then the minimum value of $K$ over $\tilde{\mathcal{F}}$ is $g(0)$, and the optimal ceded loss function is $f^*(x) = x$.*

*(b) If $\rho^* < S_X(0)$, then the minimum value of $K$ over $\tilde{\mathcal{F}}$ is $g(d^* \wedge a)$, and the optimal ceded loss function is $f^*(x) = (x - d^* \wedge a)_+$.*

**Proof:** For $f_d(x) = (x-d)_+$ for some $d \in [0,a]$, it is readily verified from (7) and (8) that

$$K(f_d) = d + (1+\rho)\,\mathbb{E}(X-d)_+ = g(d). \tag{9}$$

Taking partial derivative with respect to $d$ on $(0,a)$ yields

$$\frac{\partial K(f_d)}{\partial d} = \left[1-(1+\rho)\,S_X(d)\right], \tag{10}$$

which is increasing in $d$. Thus $K(f_d)$ is convex in $d$. Note that $\frac{\partial K(f_d)}{\partial d} = 0$ when $d = S_X^{-1}(\rho^*) = d^*$. If $S_X(0) \le \rho^*$, then $\frac{\partial K(f_d)}{\partial d}(0+) \ge 0$ and hence $K$ attains its minimum value $g(0)$ at $d = 0$. If $\rho^* < S_X(0)$, $K(f_d)$ attains its minimum value over $\tilde{\mathcal{F}}$ either at $d = a$ or at $d = d^*$, whichever smaller. So the optimal value of $d$ is $d^* \wedge a$, and the corresponding minimum value of $K$ is $g(d^* \wedge a)$. □

In the following subsections, we derive the optimal ceded loss functions and the corresponding minimum value of $K$ under different orderings of $\alpha$, $\rho^*$, and $S_X(0)$.

### 3.2. Case 1: $\alpha < \rho^*$

Fix $f \in \mathcal{F} \setminus \hat{\mathcal{F}}$. From the remark immediately after Lemma 3, we may assume that $f(x) = c(x-d)_+$ for some $(c,d) \in [0,1) \times [0,a]$ when restricted to $[0,a]$. Define another ceded loss function $f_1(x) = c(x-d)_+ + (1-c)(x-a)_+$, $x \geq 0$. See Figure 4 for the geometric meaning of this construction. Since $f, f_1 \in \mathcal{F} \setminus \hat{\mathcal{F}}$ and $\alpha < \rho^*$, it follows directly from Lemma 4 that $K(f_1) \leq K(f)$. Hence $f_1$ represents a better reinsurance contract than $f$. Define another ceded loss function $f_2(x) = (x-e)_+ := (x-(a-f_1(a)))_+$, $x \geq 0$, which is the stop-loss function that coincides with $f_1$ on $[a,\infty]$. Note that $f_2 \in \hat{\mathcal{F}}$ if $e < a$ (or equivalently $d < a$). See Figure 4.

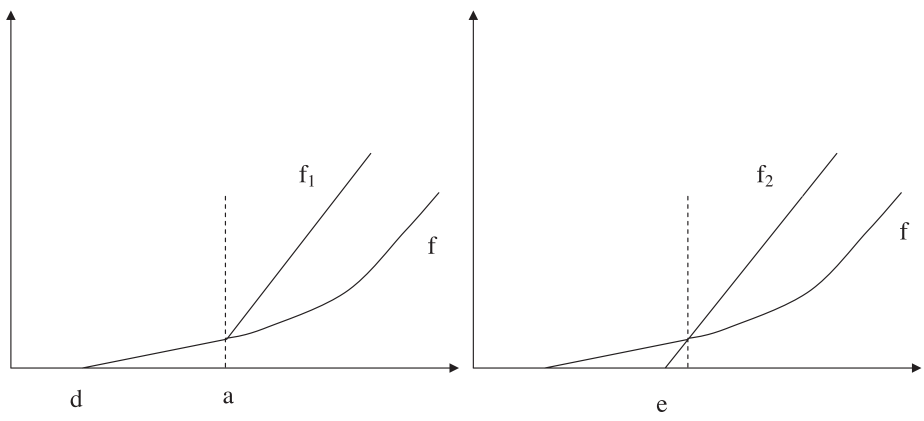


FIGURE 4: Construction of $f_1$ and $f_2$.

We claim that when $e < a$, $f_2$ is better than $f_1$ in the sense that $K(f_2) \leq K(f_1)$. To see this, we note that by Lemma 3,

$$K(f_2) = a - f_2(a) + (1+\rho)\mathbb{E}[f_2(X)] + \frac{\mathbb{E}(X - \mathrm{VaR}_X(\alpha))_+ - \int_0^\alpha f_2(S_X^{-1}(p))\,dp + \alpha f_2(a)}{S_X(e)},$$

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and

$$K(f_1) = a - f_1(a) + (1+\rho)\mathbb{E}[f_1(X)] + \frac{\mathbb{E}(X - \mathrm{VaR}_X(\alpha))_+ - \int_0^\alpha f_1(S_X^{-1}(p))\,dp + \alpha f_1(a)}{\alpha}.$$

Since $f_1 = f_2$ on $[a, \infty)$, $f_1(a) = f_2(a)$ and $\int_0^\alpha f_2(S_X^{-1}(p))\,dp = \int_0^\alpha f_1(S_X^{-1}(p))\,dp$. However, $\mathbb{E}[f_2(X)] \le \mathbb{E}[f_1(X)]$ and $S_X(e) \ge \alpha$, thus $K(f_2) \le K(f_1)$.

The above consideration, together with Lemma 5, implies that every $f \in \mathcal{F}$, belonging to $\hat{\mathcal{F}}$ or not, is always inferior to some stop-loss function $f(x) = (x-d)_+$ with $d \in [0, a]$. Combined with Lemma 6, we obtain the following conclusion:

1. If $\alpha < S_X(0) \le \rho^*$, then the optimal ceded loss function is given by $f^*(x) = x$, and the minimum value of $K$ over $\mathcal{F}$ is $g(0)$.
2. If $\alpha < \rho^* < S_X(0)$, then the optimal ceded loss function is given by $f^*(x) = (x - d^*)_+$, and the minimum value of $K$ over $\mathcal{F}$ is $g(d^*)$. Note that $\alpha < \rho^*$ implies that $0 < d^* < a$.

### 3.3. Case 2: $\alpha = \rho^*$

We first consider $f \in \mathcal{F} \setminus \hat{\mathcal{F}}$. As in Case 1, we assume that $f(x) = c(x-d)_+$ for some $(c, d) \in [0,1) \times [0, a]$ when restricted to $[0, a]$. When $\alpha = \rho^*$, it is clear from Lemma 4 that the value of $f(x)$ for $x > a$ has no influence on the value of $K(f)$. Hence we may further assume that $f$ takes the form $f_{c,d}(x) = c(x-d)_+$, $x \ge 0$ for some $(c, d) \in [0,1) \times [0, a]$. By Lemma 3,

$$K(f_{c,d}) = c(1+\rho)\,\mathbb{E}(X-d)_+ + \frac{1-c}{\alpha}\int_0^\alpha S_X^{-1}(p)\,dp + cd.$$

Differentiating it with respect to $d$ on $(0, a)$ yields

$$\begin{aligned}\frac{\partial K(f_{c,d})}{\partial d} &= c\big[1-(1+\rho)\,S_X(d)\big] \le c\big[1-(1+\rho)\,S_X(a)\big] \\ &= c\big[1-(1+\rho)\alpha\big] = 0.\end{aligned} \tag{11}$$

Therefore, the optimal value of $d$ is $a$, and hence optimal ceded loss functions in $\mathcal{F} \setminus \hat{\mathcal{F}}$ should be identically zero on $[0, a]$.

For ceded loss functions in $\hat{\mathcal{F}}$, we only need to consider stop-loss functions $f(x) = (x-d)_+$ with deductible $d \in [0, a)$ because of Lemma 5. However, Lemma 6 implies that all these stop-loss functions are not as good as $f(x) = (x-a)_+$, which is identically zero on $[0, a]$. Combined with the previous paragraph, we conclude that any $f \in \mathcal{F}$ that is identically zero on $[0, a]$ would be optimal in the present case where $\alpha = \rho^*$. Substituting $f \equiv 0$ into (8) shows that the corresponding minimum value of $K$ is $g(a) = g(d^*)$.

### 3.4. Case 3: $\alpha > \rho^*$

For $f \in \hat{\mathcal{F}}$, the situation is the same as in Case 2: ceded loss functions in $\hat{\mathcal{F}}$ are not as good as $f(x) = (x - a)_+$. For $f \in \mathcal{F} \setminus \hat{\mathcal{F}}$, we again assume that $f(x) = c(x - d)_+$ for some $(c, d) \in [0,1) \times [0, a]$ when restricted to $[0, a]$. From Lemma 4, the tail of $f$ should be as small as possible. Thus we may further assume that $f(x) = c(x - d)_+$, $x \geq 0$ for some $(c, d) \in [0,1) \times [0, a]$, just as in Case 2. By (11), we have

$$\frac{\partial K(f)}{\partial d} = c\big[1 - (1+\rho)S_X(d)\big] \leq c\big[1 - (1+\rho)S_X(a)\big] = c\big[1 - (1+\rho)\alpha\big] < 0.$$

Thus the optimal value of $d$ equals $a$.

As the tail of the optimal ceded loss function should be as small as possible, we conclude that the optimal ceded loss function is the null function $f \equiv 0$, and the corresponding minimum value of $K$ is $g(a)$.

### 3.5. Summary and numerical example

Combining all these three cases, we obtain the following theorem, which can also be found in Cai et al. (2008).

**Theorem 2.** *For a given $\alpha \in (0, S_X(0))$, the following statements hold true.*

*(a) If $\alpha < S_X(0) \leq \rho^*$, then the minimum value of $K$ over $\mathcal{G}_1$ is $g(0)$*[1]*, and the optimal ceded loss function is $f^*(x) = x$.*

*(b) If $\alpha < \rho^* < S_X(0)$, then the minimum value of $K$ over $\mathcal{G}_1$ is $g(d^*)$, and the optimal ceded loss function is $f^*(x) = (x - d^*)_+$.*

*(c) If $\alpha = \rho^* < S_X(0)$, then the minimum value of $K$ over $\mathcal{G}_2$ is $g(a)$, and the optimal ceded loss function can be any function that is identically zero on $[0, a]$.*

*(d) If $\alpha > \rho^*$, the minimum value of $K$ over $\mathcal{G}_2$ is $g(a)$, and the optimal ceded loss function is the null function $f(x) \equiv 0$.*

As an illustration, we again consider the example in which $X$ is exponentially distributed with mean 1000, so that $S_X(0) = 1$. If $\rho = 0$, then $\rho^* = 1$ and $f^*(x) = x$. No matter what the risk tolerance level is, it is always optimal to

[1] The minimum value stated in Theorem 4.1 of Cai et al. (2008) was mistyped as $u(\rho^*)$. The correct value should be $g(0)$.

purchase full reinsurance in this case. If reinsurance is getting more expensive, (say $\rho = 0.2$ and hence $\rho^* = 0.833$), then the optimal reinsurance plan depends on the value of $\alpha$. When the risk tolerance level is high so that $\alpha$ is higher than $\rho^*$, it is optimal not to purchase any reinsurance. However, if $\alpha$ is smaller than $\rho^*$, it is optimal to purchase a stop-loss reinsurance with deductible $d^* = 182.32$.

## 4. Optimal reinsurance under VaR risk measure with Wang's premium principle

In Wang et al. (1997), several natural axioms for pricing insurance contracts that characterize the premium principle of Wang (1996) were proposed. The premium principle is closely related to the dual theory of choice proposed by Yaari (1987). Under these axioms, it was proved that the price to insure a risk is given by the expectation of the risk with respect to a distorted probability. More precisely, Wang's premium principle is given by

$$H_w(X) = \int_0^\infty w(S_X(t))\,dt, \tag{12}$$

where the function $w$ is a non-decreasing, concave function such that $w(0) = 0$ and $w(1) = 1$. The function $w$ is called a distortion. In this formulation, the risk $X$ has to be non-negative, otherwise the formula has to be modified by including an extra term. The integral in (12) is a special case of the Choquet integral for non-additive measures. We refer to Denneberg (1994) for more information on the theory of of non-additive measure, and to Denuit et al. (2005) for an overview of the various aspects of the dual theory of choice and Wang's premium principle.

Wang's premium principle $H_w(\cdot)$ satisfies many convenient properties. In particular, the following are relevant to our subsequent analysis:

**Lemma 7.** *Wang's premium principle $H_w$ is positively homogeneous and monotone, in the sense that*

$$H_w[cY] = cH_w[Y], \quad c \geq 0$$

and

$$Y \leq Z \implies H_w[Y] \leq H_w[Z].$$

In this section, we study the problem of choosing the optimal ceded loss function that minimizes the VaR of the total cost $T_f(X)$ under Wang's premium principle. In this case $T_f(X) = I_f(X) + H_w(f(X))$. Denote the objective function as $L(f) := \mathrm{VaR}_{T_f(X)}(\alpha)$, we have

$$L(f) = S_X^{-1}(\alpha) - f(S_X^{-1}(\alpha)) + H_w(f(X)).$$

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Therefore, the minimization problem we want to study is

$$\min_{f\in\mathcal{F}} L(f) := \min_{f\in\mathcal{F}}\{a-f(a)+H_w(f(X))\}, \tag{13}$$

where $a := S_X^{-1}(\alpha)$ as before.

To solve minimization problem (13), we first observe that the arguments used in Lemmas 1 and 2 are still valid in the present situation because $H_w$ is monotone. It follows that we can confine our attention to the subclass $\mathcal{G}\subset\mathcal{F}$ of ceded loss functions of the form $f_{c,d}(x) = c(x-d)_+$ for some $(c,d)\in[0,1]\times[0,a)$.

**Theorem 3.** *For a given $\alpha\in(0,S_X(0))$, the following statements hold true.*

*(a) If $H_w(X) < a$, then the minimum value of $L$ over $\mathcal{F}$ is $H_w(X)$, and the optimal ceded loss function is $f^*(x) = x$.*

*(b) If $H_w(X) = a$, then the minimum value of $L$ over $\mathcal{F}$* is *$H_w(X)$, and the optimal ceded loss function is $f^*(x) = cx$ for any constant $c\in(0,1]$.*

*(c) If $H_w(X) > a$, then the minimum value of $L$ over $\mathcal{F}$ is $a$, and the optimal ceded loss function* is *$f^*(x)\equiv 0$.*

**Proof:** Let $f_{c,d}(x) = c(x-d)_+$ for some $(c,d)\in[0,1]\times[0,a)$. Then

$$L(f_{c,d}) = a-c(a-d)+cH_w[(X-d)_+] = a-c(a-d)+c\int_0^\infty w(S_{(X-d)_+}(t))\,dt.$$

Since $S_{(X-d)_+}(t) = \mathbb{P}((X-d)_+ > t) = \mathbb{P}(X > t+d) = S_X(t+d)$, we rewrite the above as

$$L(f_{c,d}) = a-c(a-d)+c\int_d^\infty w(S_X(t))\,dt.$$

If $c = 0$, then $f\equiv 0$ and the corresponding value of $L$ is $a$. Now we assume that $c > 0$. Taking partial derivative of $L(f_{c,d})$ with respect to $d$ on $(0,a)$ yields

$$\frac{\partial L(f_{c,d})}{\partial d} = c[1-w(S_X(d))].$$

This partial derivative is always positive because $w\le 1$. Thus the optimal value of $d$ is 0. Thus if suffices to consider ceded loss functions of the form $f_{c,0}(x) = cx$ for some $c\in(0,1]$. In this case,

$$L(f_{c,0}) = a-ca+c\int_0^\infty w(S_X(t))\,dt = a+c(H_w[X]-a).$$

Now the theorem follows directly from this equation. □

Unfortunately, this approach does not work for the minimization of CTE under Wang's premium principle. While we are still able to deduce that if the optimal ceded function does not belong to $\hat{\mathcal{F}}$, it must take the form $f(x) = c(x-d)_+$ for some $(c,d) \in [0,1) \times [0,a]$ on the interval $[0,a]$, we are unable to deduce its precise properties on $[a,\infty)$ using the method adopted in Section 3. Further research is necessary in this direction.

## 5. Conclusion

In this paper, we first examined the two optimal reinsurance problems studied in Cai et al. (2008): to minimize the value-at-risk and the conditional tail expectation of the total cost under the expectation premium principle. By using simple geometric arguments, we were able to restrict the class of all possible ceded loss functions to a very small class containing stop-loss functions and linear functions only. Since every such function involves at most two parameters, the optimization problems can be solved easily. This technique not only can greatly simplify the arguments used in Cai et al. (2008), but is also applicable when the expectation premium principle is replaced by Wang's premium principle in the VaR-minimization problem, as demonstrated in Section 4.

Finally, we note the following general principles that can be observed from Sections 2, 3, and 4. These principles are essentially Lemma 2 in a more general setting. Recall that a premium principle $\mathbb{H}$ is called monotone if $\mathbb{H}[Y_1] \leq \mathbb{H}[Y_2]$ when $Y_1 \leq Y_2$. Examples of monotone premium principle include the expectation premium principle and Wang's premium principle.

**General principle of VaR-minimization:** When minimizing the VaR of the total cost $T_f(X)$ over the class $\mathcal{F}$ under a monotone premium principle $\mathbb{H}$, optimal ceded loss functions must take the form $f(x) = c(x-d)_+$, $x \geq 0$ for some $c \in [0,1] \times [0,a)$. Therefore, the optimal form of reinsurance is either a full reinsurance (when $d = 0$ and $c = 1$), a stop-loss insurance (when $d \in (0,a]$ and $c = 1$), a quota-share reinsurance (when $d = 0$, $c \in (0,1)$), a change-loss reinsurance (when $d \in (0,a)$ and $c \in (0,1)$), or finally no reinsurance should be purchased (when $c = 0$).

**General principle of CTE-minimization:** When minimizing the CTE of the total cost $T_f(X)$ over the class $\mathcal{F}$ under a monotone premium principle $\mathbb{H}$, optimal ceded loss functions on the interval $[0,a]$ must take the form $f(x) = c(x-d)_+$, $x \geq 0$ for some $(c,d) \in [0,1) \times [0,a]$.

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