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Equivalence of codes for countable sets of reals

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Abstract. A set $U \subseteq \mathbb{R} \times \mathbb{R}$ is universal for countable subsets of $\mathbb{R}$ if and only if for all $x \in \mathbb{R}$, the section $U_x = \{y \in \mathbb{R} : U(x,y)\}$ is countable and for all countable sets $A \subseteq \mathbb{R}$, there is an $x \in \mathbb{R}$ so that $U_x = A$. Define the equivalence relation E_U on $\mathbb{R}$ by $x_0 \; E_U \; x_1$ if and only if $U_{x_0} = U_{x_1}$, which is the equivalence of codes for countable sets of reals according to U. The Friedman–Stanley jump, $=^+$, of the equality relation takes the form E_{U^*} where U^* is the most natural Borel set that is universal for countable sets. The main result is that $=^+$ and E_U for any U that is Borel and universal for countable sets are equivalent up to Borel bireducibility. For all U that are Borel and universal for countable sets, E_U is Borel bireducible to $=^+$. If one assumes a particular instance of Σ^1_3-generic absoluteness, then for all $U \subseteq \mathbb{R} \times \mathbb{R}$ that are Σ^1_1 (continuous images of Borel sets) and universal for countable sets, there is a Borel reduction of $=^+$ into E_U.

1 Equivalence of Codes for Countable Sets of Reals

Let ${}^\omega 2$ be the collection of functions $f : \omega \to 2$. The elements of ${}^\omega 2$ are called *reals*. (Sometimes ${}^\omega 2$ will be denoted by $\mathbb{R}$ especially when typographically convenient.) Let $\mathsf{pair} : \omega \times \omega \to \omega$ be a recursive bijection. If $x \in {}^\omega 2$ and $n \in \omega$, let $\hat{x}_n \in {}^\omega 2$ be defined by $\hat{x}_n(k) = x(\mathsf{pair}(n,k))$. So a single real x naturally gives a countable set of reals $\{\hat{x}_n : n \in \omega\}$

Suppose $U \subseteq {}^\omega 2 \times {}^\omega 2$. For $x \in {}^\omega 2$, let $U_x = \{y \in {}^\omega 2 : U(x,y)\}$. Define an equivalence relation E_U on ${}^\omega 2$ by $x \; E_U \; y$ if and only if $U_x = U_y$. $U \subseteq {}^\omega 2 \times {}^\omega 2$ is universal for countable sets if and only if for all $x \in {}^\omega 2$, U_x is countable and for all countable $A \subseteq {}^\omega 2$, there exists an $x \in {}^\omega 2$ so that $U_x = A$. If $U_x = A$, then x is said to be a code for A according to U. E_U is essentially the equivalence relation stating two reals code the same countable set according to U.

Suppose $A \subseteq {}^\omega 2$ is a countable set. Let $\langle x_n : n \in \omega \rangle$ be an enumeration of A. Then $A = \bigcup_{n \in \omega} \{x_n\}$. Since singletons are Π^0_1 subsets of ${}^\omega 2$, this shows that every countable subset of ${}^\omega 2$ is Σ^0_2. Let $U^* \subseteq {}^\omega 2 \times {}^\omega 2$ be defined by $(x,y) \in U^*$ if and only if $(\exists n)(\hat{x}_n = y)$ if and only if $(\exists n)(\forall m)(x(\mathsf{pair}(n,m)) = y(m))$. Note that U^* is Σ^0_2 and universal for countable sets of reals. U^* is the most natural coding of countable sets. (The enumeration of a countable set is a code for that countable set.) Let $=^+$ be

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the equivalence relation on ${}^{\omega}2$ defined to be E_{U^*}. Note that for any $x, y \in {}^{\omega}2$, $x =^+ y$ if and only if $x \; E_{U^*} \; y$ if and only if $\{\hat{x}_n : n \in \omega\} = \{\hat{y}_n : n \in \omega\}$. The latter is the familiar definition of $=^+$ as the Friedman–Stanley jump of $=$. (See [5] and [6, Section 8.3] for more information concerning the Friedman–Stanley jump of a Borel equivalence relation.)

Equivalence relations on ${}^{\omega}2$ are compared by Borel reductions. That is, if E and F are two equivalence relations on ${}^{\omega}2$, one writes $E \leq_{\Delta^1_1} F$ if and only if there is a Borel function $\Phi : {}^{\omega}2 \to {}^{\omega}2$ so that for all $x, y \in {}^{\omega}2$, $x \; E \; y$ if and only if $\Phi(x) \; F \; \Phi(y)$. One writes $E \equiv_{\Delta^1_1} F$ if and only if $E \leq_{\Delta^1_1} F$ and $F \leq_{\Delta^1_1} E$. (For example, [6, Theorem 8.3.6] shows that $= \leq_{\Delta^1_1} =^+$ but $\neg(=^+ \leq_{\Delta^1_1} =)$. In fact, this relation holds more generally between a Borel equivalence relation E with more than one class and its Friedman–Stanley jump E^+.) Since $=^+$ is E_{U^*} where U^* is the most natural Σ^0_2 set universal for countable subsets of ${}^{\omega}2$, a natural question ([4, Question 2.5]) asked by Ding and Yu is whether for any Borel set $U \subseteq {}^{\omega}2 \times {}^{\omega}2$ that is universal for countable sets, is $E_U \equiv_{\Delta^1_1} =^+$? They showed that if U is Borel and universal for countable sets, then $E_U \leq_{\Delta^1_1} =^+$. Thus, the question becomes whether $=^+ \leq_{\Delta^1_1} E_U$ when U is Borel and universal for countable sets. They also asked if $=^+ \leq_{\Delta^1_1} E_U$ when U is Σ^1_1 (a continuous image of a Borel set) and universal for countable sets.

This article will answer these questions. It will be shown that if $U \subseteq {}^{\omega}2 \times {}^{\omega}2$ is Borel and universal for countable subsets of ${}^{\omega}2$, then $=^+$ is Borel bireducible to E_U. Intuitively, this means that every coding of countable sets via a Borel U that is universal for countable sets is indistinguishable from the natural coding of countable sets given by U^* via Borel procedures. The argument uses forcing ideas and absoluteness. Granting sufficient absoluteness of certain statements between the ground model and certain forcing extensions, the method in the Borel case can be extended to produce a Borel reduction from $=^+$ into E_U when $U \subseteq {}^{\omega}2 \times {}^{\omega}2$ is a Σ^1_1 set that is universal for countable sets. The end of the article has a broad overview of why forcing produces certain countable sets of reals for which one can easily search for the code for these countable sets according to the Borel set U that is universal for countable sets. In general, the search for a code for a countable set of reals seems quite complex.

Let $\forall^{\omega}$ and $\exists^{\omega}$ refer to universal and existential quantification over ω. Let $\forall^{\mathbb{R}}$ and $\exists^{\mathbb{R}}$ refer to universal and existential quantification over $\mathbb{R}$ (or ${}^{\omega}2$). Often, it is clear in context what type of objects are being quantified, and one will simply write $\forall$ or $\exists$.

A tree T on $2 \times \omega$ (or 2 or ω) is a subset of ${}^{<\omega}(2 \times \omega)$ (or ${}^{<\omega}2$ or ${}^{<\omega}\omega$, respectively), which is $\subseteq$-downward closed (here $\subseteq$ refers to string extension). Note that such trees are coded by reals. If T is a tree on $2 \times \omega$, then $[T] = \{f \in {}^{\omega}(2 \times \omega) : (\forall^{\omega} n)(f \upharpoonright n \in T)\}$, where $f \upharpoonright n$ refers to the length n initial segment of f. Let $\pi_1 : {}^{\omega}2 \times {}^{\omega}\omega \to {}^{\omega}2$ be the projection onto the first coordinate. A set B is $\Sigma^1_1(z)$ if and only if there is an z-recursive tree T in $2 \times \omega$ so that $B = \pi_1[\![T]\!]$. It is important to note that whenever one writes B in any universe of set theory containing T, it will always refer to the interpretation of $\pi_1[\![T]\!]$. A set B is $\Delta^1_1(z)$ if and only if there are z-recursive trees T and S so that $\pi_1[\![T]\!] = B$ and $\pi_1[\![S]\!] = {}^{\omega}2 \setminus B$. Note that the statement "$(\exists x)(T_x$ and S_x are ill-founded)" is $\Sigma^1_1(z)$. By Mostowski absoluteness, in any transitive set or class M satisfying an adequate amount of ZF with $\{z\} \cup \omega \subseteq M$, $M \models \pi_1[\![T]\!] \cap \pi_1[\![S]\!] = \varnothing$. Also, $(\forall x)(T_x$ or T_y is ill-founded) is $\Pi^1_2(z)$. By Shoenfield

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absoluteness, in any transitive set or class M satisfying adequate amount of ZF with $\{z\} \cup \omega_1 \subseteq M$, $M \models \pi_1[\![T]\!] = {}^\omega 2 \setminus \pi_1[\![S]\!]$. Thus, if trees T and S define a $\Delta^1_1(z)$ subset of ${}^\omega 2$, then in any transitive set or class model M such that $\{z\} \cup \omega_1 \subseteq M$, T and S will continue to represent a $\Delta^1_1(z)$ set. In this way, when one speaks of this $\Delta^1_1(z)$ set, one implicitly means $\pi_1[\![T]\!]$, and its complement is $\pi_1[\![S]\!]$.

Fact 1.1 *Suppose $U \subseteq {}^\omega 2 \times {}^\omega 2$ is $\Sigma^1_1(z)$. Then the statement "$(\forall x)(U_x$ is countable$)$" is $\Pi^1_1(z)$.*

Proof For all $x \in {}^\omega 2$, if U_x is countable, then U_x is a countable $\Sigma^1_1(x,z)$ set. The effective perfect set theorem of Mansfield ([11, 4F.1]) implies that U_x consists only of $\Delta^1_1(x,z)$ reals. By [11, 4D.2], there is a Π^1_1 relation $H \subseteq \mathbb{R} \times \mathbb{R}$ such that $H(x,y)$ if and only if $y \in \Delta^1_1(x)$. Thus, the statement "$(\forall x)(U_x$ is countable$)$" is equivalent to

$$(\forall x)(\forall y)\big(U(x,y) \Longrightarrow H(x \oplus z, y)\big),$$

where $(x \oplus z) \in {}^\omega 2$ is the recursive join defined by $(x \oplus z)(2n) = x(n)$ and $(x \oplus z)(2n+1) = z(n)$ for all $n \in \omega$. The latter statement is $\Pi^1_1(z)$. ∎

Fact 1.2 *(Ding-Yu, [4, Theorem 2.4]) If U is $\Delta^1_1(z)$ and universal for countable sets, then there is a $\Delta^1_1(z)$ reduction $\Phi : {}^\omega 2 \to {}^\omega 2$ witnessing $E_U \leq_{\Delta^1_1} =^+$. In particular, this implies that E_U is a $\Delta^1_1(z)$ equivalence relation.*

Proof First, one will show that $\mathrm{dom}(U) = \{x \in {}^\omega 2 : (\exists y)U(x,y)\}$ is $\Delta^1_1(z)$. (See [9, Lemma 18.12] for another argument.) It is clearly $\Sigma^1_1(z)$. By [11, 4D.2], there is a Π^1_1-recursive partial function $\mathbf{d} : \omega \times \mathbb{R} \to \mathbb{R}$ so that $y \in \Delta^1_1(x)$ if and only if $(\exists^\omega n)((n,x) \in \mathrm{dom}(\mathbf{d}) \wedge \mathbf{d}(n,x) = y)$. Since U is countable, the effective perfect set theorem implies $U_x \subseteq \Delta^1_1(x \oplus z)$. Thus, $x \in \mathrm{dom}(U)$ if and only if $(\exists^\omega n)((n,x) \in \mathrm{dom}(\mathbf{d}) \wedge U(x, \mathbf{d}(n, x \oplus z)))$. By [11, 4D.1(ii)], the latter expression is $\Pi^1_1(z)$.

If $\sigma \in {}^{<\omega}2$, then let $N_\sigma = \{f \in {}^\omega 2 : \sigma \subseteq f\}$ be the basic neighborhood determined by σ. Let $\Psi : {}^\omega 2 \to N_{\langle 0 \rangle}$ be a recursive bijection. By the Lusin–Novikov theorem ([11, 4F.17]), there is a $\Delta^1_1(z)$ relation $P \subseteq \omega \times {}^\omega 2 \times {}^\omega 2$ so that $U(x,y)$ if and only if $(\exists^\omega n)P(n,x,y)$, and for each $n \in \omega$, $P_n = \{(x,y) : P(n,x,y)\}$ uniformizes U.

Define $\Phi : {}^\omega 2 \to {}^\omega 2$ by $\Phi(x) = w$ if and only if the disjunction of the following holds:

- $x \in \mathrm{dom}(U) \wedge (\forall^\omega n)(\exists^{\mathbb{R}} y)[P(n,x,y) \wedge (\forall^\omega k)(w(\mathrm{pair}(n,k)) = \Psi(y)(k))]$,
- $x \notin \mathrm{dom}(U) \wedge (\forall^\omega n)(\forall^\omega k)(w(\mathrm{pair}(n,k)) = 1)$

if and only if the disjunction of the following holds:

- $x \in \mathrm{dom}(U) \wedge (\forall^\omega n)(\forall^{\mathbb{R}} y)[P(n,x,y) \Rightarrow (\forall^\omega k)(w(\mathrm{pair}(n,k)) = \Psi(y)(k))]$,
- $x \notin \mathrm{dom}(U) \wedge (\forall^\omega n)(\forall^\omega k)(w(\mathrm{pair}(n,k)) = 1)$.

By the properties of P stated above, these two definitions are equivalent. Since the first definition is $\Sigma^1_1(z)$ and the second definition is $\Pi^1_1(z)$, Φ is $\Delta^1_1(z)$. Intuitively, if $U_x \neq \varnothing$, $\Phi(x)$ has the property that $\{\widehat{\Phi(x)}_n : n \in \omega\} = \{\Psi(y) : U(x,y)\}$. If $U_x = \varnothing$, then $\Phi(x)$ has the property that $\{\widehat{\Phi(x)}_n : n \in \omega\} = \{\bar{1}\}$, where $\bar{1}$ is the constant function taking value 1. Since $\Psi : {}^\omega 2 \to N_{\langle 0 \rangle}$, one has that Φ is a $\Delta^1_1(z)$ reduction of E_U into $=^+$. ∎

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Lemma 1.3 *Suppose $U \subseteq {}^\omega 2 \times {}^\omega 2$ is $\Sigma^1_1(z)$; then the statement "U is universal for countable sets" is $\Pi^1_3(z)$. If $U \subseteq {}^\omega 2 \times {}^\omega 2$ is $\Delta^1_1(z)$, then the statement "U is universal for countable sets" is $\Pi^1_2(z)$.*

Proof Suppose U is $\Sigma^1_1(z)$. Then U is universal for countable set if and only if the conjunction of the following holds:

- $(\forall^{\mathbb{R}} x)(U_x \text{ is countable})$,
- $(\forall^{\mathbb{R}} z)(\exists^{\mathbb{R}} x)[(\forall^\omega n)U(x, \hat{z}_n) \wedge (\forall^{\mathbb{R}} y)(U(x, y) \Rightarrow (\exists^\omega n)(\hat{z}_n = y))]$.

The first condition is $\Pi^1_1(z)$ by Fact 1.1. The second condition is $\Pi^1_3(z)$. The entire expression is $\Pi^1_3(z)$.

Now suppose that U is $\Delta^1_1(z)$. If for all $x \in {}^\omega 2$, U_x is countable, then the Lusin–Novikov theorem ([11, 4F.17]) states that there is a $\Delta^1_1(z)$ relation $P \subseteq \omega \times {}^\omega 2 \times {}^\omega 2$ so that $U(x, y)$ if and only $(\exists^\omega n)P(n, x, y)$, and for all $n \in \omega$, $P_n = \{(x, y) : P(n, x, y)\}$ is a uniformization for U. U is universal for countable sets if and only if the conjunction of the following holds:

- $(\forall^{\mathbb{R}} x)(U_x \text{ is countable})$,
- $(\forall^{\mathbb{R}} z)(\exists^{\mathbb{R}} x)[(\forall^\omega n)U(x, \hat{z}_n) \wedge (\forall^\omega m)(\exists^{\mathbb{R}} y)(\exists^\omega n)(P(m, x, y) \wedge \hat{z}_n = y)]$.

Note that the second condition is $\Pi^1_2(z)$. Thus, the entire expression is $\Pi^1_2(z)$. ■

Next, one will produce a Borel reduction of $=^+$ into E_U using a technique involving countable models of set theory and forcing that is similar to those used in [10, Section 2.8] to prove the unpinnedness dichotomy in the Solovay model. Rather than using the Lévy collapse of a measurable, these reductions will be created using the finite support product of Cohen forcing. See the end for further discussions of these methods.

Definition 1.4 Let $\mathbb{C}$ be the set of finite partial functions $p : \omega \to 2$. Let $\leq_{\mathbb{C}}$ be reverse inclusion. The largest condition is $1_{\mathbb{P}} = \varnothing$. The forcing $\mathbb{C} = (\mathbb{C}, \leq_{\mathbb{C}}, 1_{\mathbb{C}})$ is called *Cohen forcing.*

For any $\varepsilon \in \mathrm{ON}$, let $\mathbb{C}_\varepsilon = \prod_{\alpha<\varepsilon} \mathbb{C}$ be the finite support product of $\mathbb{C}$. The conditions are $p : \varepsilon \to \mathbb{C}$ so that $\mathrm{supp}(p) = \{\alpha < \varepsilon : p(\alpha) \neq 1_{\mathbb{C}}\}$ is finite. If $p, q \in \mathbb{C}_\varepsilon$, $p \leq_{\mathbb{C}_\varepsilon} q$ if and only if for all $\alpha < \varepsilon$, $p(\alpha) \leq_{\mathbb{C}} q(\alpha)$. $1_{\mathbb{C}_\varepsilon}$ is the constant function on ε taking value $1_{\mathbb{C}}$.

Let $\mathrm{Coll}(\omega, \mathbb{R})$ be the forcing consisting of finite partial functions $p : \omega \to \mathbb{R}$. Let $\leq_{\mathrm{Coll}(\omega,\mathbb{R})}$ be reverse inclusion and $1_{\mathrm{Coll}(\omega,\mathbb{R})} = \varnothing$. Note that if $G \subseteq \mathrm{Coll}(\omega, \mathbb{R})$ is $\mathrm{Coll}(\omega, \mathbb{R})$-generic over the ground model, then the extension by G adds a surjection g from ω onto the reals of the ground model. Therefore, the set of ground model reals are countable in this forcing extension.

Throughout the article, one will need several effectiveness or uniformity observations concerning the forcing construction on countable models coded as reals. Some details will be provided without including too many burdensome coding notations. The authors of [10, Section 2.8a] also develop a framework for some of these coding results and observed the effectiveness of various forcing constructions.

Definition 1.5 If $x \in {}^\omega 2$, let $\mathcal{R}_x(m, n)$ if and only if $x(\mathrm{pair}(m, n)) = 1$. Then $\mathcal{R}_x$ is the binary relation on ω coded by x.

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Let WO be the collection of $x \in {}^{\omega}2$ so that $\mathcal{R}_x$ is a well-ordering. If $x \in \mathrm{WO}$, then let $\mathrm{ot}(x)$ be the ordertype $(\omega, \mathcal{R}_x)$.

If $\mathcal{R}_x$ is a set-like, extensional, and well-founded relation, then let the transitive set $(\mathcal{M}_x, \in)$ denote the Mostowski collapse of $(\omega, \mathcal{R}_x)$. Let $\mathrm{most}_x : (\omega, \mathcal{R}_x) \to (\mathcal{M}_x, \in)$ be the Mostowski collapse function.

Recall that the satisfaction relation Sat is defined by $(x, \varphi, \langle i_1, \dots, i_k \rangle) \in \mathrm{Sat}$ if and only if $(\omega, \mathcal{R}_x) \models \varphi(i_1, \dots, i_k)$ and is Δ^1_1. (Formulas are coded by integers in some recursive manner.) The fact that Sat is Δ^1_1 will often be implicitly used.

Let $\mathrm{AC}^{\mathbb{R}}_{\omega}$ denote countable choice for the reals, which is the statement that if $R \subseteq \omega \times \mathbb{R}$, then there is a function $\Phi : \mathrm{dom}(R) \to \mathbb{R}$ so that for all $n \in \mathrm{dom}(R)$, $R(n, \Phi(n))$. An important consequence of $\mathrm{AC}^{\mathbb{R}}_{\omega}$ is that ω_1 is a regular cardinal which will be used later in the argument.

Lemma 1.6 *Suppose $\mathfrak{m} \in {}^{\omega}2$ is such that $(\omega, \mathcal{R}_{\mathfrak{m}})$ is a set-like, well-founded, and extensional structure satisfying some adequate amount of $\mathsf{ZF} + \mathrm{AC}^{\mathbb{R}}_{\omega}$. Let $\mathrm{most}_{\mathfrak{m}} : (\omega, \mathcal{R}_{\mathfrak{m}}) \to (\mathcal{M}_{\mathfrak{m}}, \in)$ be the Mostowski collapse map. For notational simplicity, let $\mathcal{N} = \mathcal{M}_{\mathfrak{m}}$. Then there are Δ^1_1 functions $\mathrm{Gen}_0, \mathrm{GenMod}_0 : {}^{\omega}2 \to {}^{\omega}2$ so that the following hold:*

(i) *For all $x \in \mathbb{R}$, let $\mathcal{G}_x = \mathrm{most}_{\mathfrak{m}}[\{n \in \omega : \mathrm{Gen}_0(x)(n) = 1\}]$. $\mathcal{G}_x$ is $\mathbb{C}$-generic over $\mathcal{N}$.*

(ii) *For any $k \in \omega$ and injective sequence $\ell : k \to {}^{\omega}2$, $\prod_{i<k} \mathcal{G}_{\ell(k)}$ is $\prod_{i<k} \mathbb{C}$-generic over $\mathcal{N}$.*

(iii) *$(\omega, \mathcal{R}_{\mathrm{GenMod}_0(x)})$ is a set-like, well-founded, and extensional structure whose Mostowski collapse, $\mathcal{M}_{\mathrm{GenMod}_0(x)}$, is $\mathcal{N}[\mathcal{G}_x]$.*

Proof Using the fact that the satisfaction relation is Δ^1_1, one can obtain from $\mathfrak{m}$ in a Δ^1_1 manner a function $\mathfrak{d} : \omega \times \omega \to \omega$ with the following properties: For $1 \leq k < \omega$ and $i \in \omega$, let $D^k_i = \mathrm{most}_{\mathfrak{m}}(\mathfrak{d}(i,k))$. For each $1 \leq k < \omega$, $\{D^k_i : i \in \omega\}$ enumerates all of the dense open subsets of $\prod_{j<k} \mathbb{C}$ in the countable transitive set $\mathcal{N}$.

Next, one will sketch the standard construction of a perfect set of mutually $\mathbb{C}$-generics filters over $\mathcal{N}$. One will build a perfect tree $\langle p_\sigma : \sigma \in {}^{<\omega}2 \rangle$ of $\mathbb{C}$-conditions so that each path generates a $\mathbb{C}$-generic filter over $\mathcal{N}$.

Let $p_\varnothing = 1_{\mathbb{C}}$. Suppose for some $n \in \omega$, p_σ has been defined for all $\sigma \in {}^{n}2$. For each $\sigma \in {}^{n}2$, let n be least so that $n \notin \mathrm{dom}(p_\sigma)$. Let $q_{\sigma\hat{}i} = p_\sigma \cup \{(n,i)\}$ for $i \in \{0,1\}$. By repeatedly extending q_τ for all $\tau \in {}^{n+1}2$ as necessary to meet all the requisite dense open sets, one can find a collection $\{p_\tau : \tau \in {}^{n+1}2\}$ such that:

- For all $\tau \in {}^{n+1}2$, $p_\tau \leq_{\mathbb{C}} q_\tau$.
- For all $k < 2^{n+1}$, for all injections $B : k \to {}^{n+1}2$, and any dense open set D^k_i for $i \leq n$, $(p_{B(0)}, \dots, p_{B(k-1)}) \in D^k_i$.

This completes the construction. For each $x \in {}^{\omega}2$, let $\mathcal{G}_x$ be the $\leq_{\mathbb{C}}$-upward closure of $\{p_{x \restriction n} : n \in \omega\}$. One can check that each $\mathcal{G}_x$ is $\mathbb{C}$-generic over $\mathcal{N}$, and any finite collection has the mutual genericity property.

The reader can check that by coding using $\mathfrak{m}$ and $\mathfrak{d}$ (which is obtained from $\mathfrak{m}$), one can find a $\Delta^1_1(\mathfrak{m})$ function Gen_0 so that $\mathrm{Gen}_0(x)$ is a real that codes $\mathrm{most}^{-1}_{\mathfrak{m}}[\mathcal{G}_x]$.

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By the uniformity of the forcing construction, one can also find a $\Delta^1_1(\mathfrak{m})$ function GenMod_0 so that for all $x \in {}^\omega 2$, $\mathsf{GenMod}_0(x)$ codes a structure whose Mostowski collapse is $\mathcal{N}[\mathcal{G}_x]$. ■

Fact 1.7 *Suppose $\varepsilon < \omega_1$. Then the Cohen forcing $\mathbb{C}$ and the ε-length finite support product of Cohen forcing $\mathbb{C}_\varepsilon$ are isomorphic.*

Proof The following is a sketch of this well known basic fact: Let $\varepsilon < \omega_1$ and $B : \varepsilon \times \omega \to \omega$ be a bijection. Let $\Phi : \mathbb{C} \to \mathbb{C}_\varepsilon$ be defined by $\Phi(p)(\alpha)(n) = p(B(\alpha, n))$ whenever $B(\alpha, n)$ is in the domain of p. So for each $\alpha < \varepsilon$, $\Phi(p)(\alpha) \in \mathbb{C}$, and for only finitely many α, $\Phi(p)(\alpha) \neq 1_{\mathbb{C}} = \varnothing$. Recall that elements of $\mathbb{C}_\varepsilon$ are functions from ε into $\mathbb{C}$ with finite support. Thus, Φ is well defined and is an isomorphism. ■

Lemma 1.8 *Assume the notation of Lemma 1.6. There is a uniform procedure that takes an injective sequence $\langle G_n : n \in \omega\rangle$ of $\mathbb{C}$-generic filters over $\mathcal{N}$ with the property that any finite collection is mutually generic to a $\mathbb{C}_{\omega_1^{\mathcal{N}}}$-generic filter G^* over $\mathcal{N}$ so that $\mathbb{R}^{\mathcal{N}[G^*]} = \bigcup\{\mathbb{R}^{\mathcal{N}[\prod_{i<k} G_i]} : k \in \omega\}$.*

Proof (A similar property is obtained in [10, Lemma 2.8.12(1)] and [8, Claim 6.29].) Recall that $\mathcal{N} = \mathcal{M}_{\mathfrak{m}}$. Using the fact that the satisfaction relation is Δ^1_1, one can define, in a Δ^1_1 manner using $\mathfrak{m}$, a sequence $\Xi : \omega \to \omega$ by induction as follows: $\Xi(0)$ is the least element k of ω so that $\mathcal{N} \models \mathsf{most}_{\mathfrak{m}}(k) < \omega_1$. Suppose $\Xi(n)$ has been defined; let $\Xi(n+1)$ be the least integer $k > \Xi(n)$ so that $\mathcal{N} \models \mathsf{most}_{\mathfrak{m}}(k) < \omega_1$. Note that $\Xi[\omega] = \{n \in \omega : \mathcal{N} \models \mathsf{most}_{\mathfrak{m}}(n) \in \omega_1\}$; that is, Ξ enumerates all the integers n so that $(\omega, \mathcal{R}_{\mathfrak{m}}) \models$ "n is a countable ordinal". Let $\rho(n) = \sup\{\mathsf{most}_{\mathfrak{m}}(\Xi(k)) : k \leq n\}$. Note that $\rho : \omega \to \omega_1^{\mathcal{N}}$ is a cofinal increasing sequence. Let $I_0 = \{\alpha \in \omega_1^{\mathcal{N}} : 0 \leq \alpha < \rho(0)\}$ and for $n > 0$, $I_n = \{\alpha \in \omega_1^{\mathcal{N}} : \rho(n-1) \leq \alpha < \rho(n)\}$. Note that for all $n \in \omega$, $I_n \in \mathcal{N}$ and $\mathcal{N} \models |I_n| \leq \aleph_0$. As before in a Δ^1_1 manner from $\mathfrak{m}$, one can define $\Upsilon : \omega \to \omega$ by $\Upsilon(n)$ is the least integer k so that $\mathcal{N} \models$ " $\mathsf{most}_{\mathfrak{m}}(k)$ is a bijection from $I_n \times \omega \to \omega$". Let $B_n = \mathsf{most}_{\mathfrak{m}}(\Upsilon(n))$. Thus, for each $n \in \omega$, $B_n : I_n \times \omega \to \omega$ is a bijection and $B_n \in \mathcal{N}$ (however, the entire sequence $\langle B_n : n \in \omega\rangle$ does not belong to $\mathcal{N}$).

Next, the idea is to create a $\mathbb{C}_{\omega_1^{\mathcal{N}}}$-generic filter by "transferring" each G_n onto the interval I_n via isomorphisms $\Phi_n : \mathbb{C} \to \prod_{I_n} \mathbb{C}$ created from B_n as in the proof of Fact 1.7. More precisely, let $\Phi_n : \mathbb{C} \to \prod_{I_n} \mathbb{C}$ be defined by $\Phi_n(p)(\alpha)(k) = p(B_n(\alpha, k))$ whenever $\alpha \in I_n$ and $p(B_n(\alpha, k))$ is defined. Since $B_n \in \mathcal{N}$, $\Phi_n \in \mathcal{N}$ as well. For each $n \in \omega$, let $\Psi^*_n : \prod_{i \leq n} \mathbb{C} \to \mathbb{C}_{\rho(n)}$ by $\Psi^*_n(q)(\alpha)(k) = \Phi_j(q(j))(\alpha)(k)$ where $q \in \prod_{i \leq n} \mathbb{C}$ and $j \leq n$ is the unique j so that $\alpha \in I_j$. Note that $\Psi^*_n \in \mathcal{N}$ for each $n \in \omega$ and $\Psi^*_n : \prod_{i \leq n} \mathbb{C} \to \mathbb{C}_{\rho(n)}$ is an isomorphism. For each $n \in \omega$, let $\mathfrak{I}_n : \mathbb{C}_{\rho(n)} \to \mathbb{C}_{\omega_1^{\mathcal{M}}}$ be the canonical order preserving injection defined by

$$\mathfrak{I}_n(p)(\alpha) = \begin{cases} p(\alpha) & \alpha < \rho(n), \\ 1_{\mathbb{C}} & \alpha \geq \rho(n). \end{cases}$$

Observe that for each $n \in \omega$, $\mathfrak{I}_n \in \mathcal{N}$. Let $\Psi_n : \prod_{i \leq n} \mathbb{C} \to \mathbb{C}_{\omega_1^{\mathcal{N}}}$ be defined by $\mathfrak{I}_n \circ \Psi^*_n$. Note also that for all $n \in \omega$, $\Psi_n \in \mathcal{N}$. Define $G^* = \bigcup\{\Psi_n[\prod_{i \leq n} G_i] : n \in \omega\}$. Then $G^* \subseteq \mathbb{C}_{\omega_1^{\mathcal{N}}}$ is a filter.

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It remains to show that G^* is $\mathbb{C}_{\omega_1^{\mathcal{N}}}$-generic over $\mathcal{N}$. Recall that $\mathcal{N} \models \mathbb{C}_{\omega_1}$ satisfies the ω_1-chain condition. Let $A \in \mathcal{N}$ be such that $\mathcal{N}$ thinks A is a maximal antichain of $\mathbb{C}_{\omega_1^{\mathcal{N}}}$. Since the ω_1-chain condition holds, $\omega_1^{\mathcal{N}}$ is regular in $\mathcal{N} \models \mathsf{AC}_\omega^{\mathbb{R}}$, and each Ψ_n is an isomorphism, one has that there is an $n \in \omega$ so that $\Psi_n^{-1}[A]$ is a maximal antichain of $\prod_{i\leq n} \mathbb{C}$. Since $\prod_{i\leq n} G_i$ is $\prod_{i\leq n} \mathbb{C}$-generic over $\mathcal{N}$, $\Psi_n^{-1}[A] \cap \prod_{i\leq n} G_i \neq \varnothing$. Hence, $A \cap G^* \neq \varnothing$. This shows that G^* is $\mathbb{C}_{\omega_1^{\mathcal{N}}}$-generic over $\mathcal{N}$.

Since all $\Psi_n \in \mathcal{N}$ and are isomorphisms, one has that $\mathcal{N}[\prod_{i\leq n} G_i] = \mathcal{N}[\Psi_n[\prod_{i\leq n} G_i]]$. Since $\mathcal{N}[\prod_{i\leq n} G_i] \subseteq \mathcal{N}[G^*]$, one has that $\bigcup\{\mathbb{R}^{\mathcal{N}[\prod_{i\leq n} G_i]} : n \in \omega\} \subseteq \mathbb{R}^{\mathcal{N}[G^*]}$. Now suppose that $x \in \mathbb{R}^{\mathcal{N}[G^*]}$. There is a nice name $\tau \in \mathcal{N}$ of the form $\tau = \bigcup_{n\in\omega} \{\check{n}\} \times A_n$ (where A_n is an antichain of $\mathbb{C}_{\omega_1^{\mathcal{N}}}$) such that $\tau[G^*] = x$. Since $\mathcal{N}$ believes that $\mathbb{C}_{\omega_1^{\mathcal{N}}}$ has the $\omega_1^{\mathcal{N}}$-chain condition and $\omega_1^{\mathcal{N}}$ is regular, there is an $n \in \omega$ so that all conditions mentioned in the name τ occurs in $\mathbb{C}_{\rho(n)}$. Thus, $x = \tau[G^*] = \tau[\Psi_n[\prod_{i\leq n} G_i]]$, where one considers τ as a $\mathbb{C}_{\rho(n)}$-name in the natural way. Since $\mathcal{N}[\prod_{i\leq n} G_i] = \mathcal{N}[\Psi_n[\prod_{i\leq n} G_i]]$, $x \in \mathbb{R}^{\mathcal{N}[\prod_{i\leq n} G_i]}$. It has been shown that $\mathbb{R}^{\mathcal{N}[G^*]} \subseteq \bigcup\{\mathbb{R}^{\mathcal{N}[\prod_{i\leq n} G_i]} : n \in \omega\}$. Hence, these two sets are equal.

It is important to note that there is an explicit and uniform method to obtain G^* from $\langle G_n : n \in \omega\rangle$. One can check this procedure is $\Delta_1^1(\mathfrak{m})$ as a function in the codes in the sense of Lemma 1.9. ∎

Lemma 1.9 *Assume the setting from Lemma 1.6. Then there are $\Delta_1^1(\mathfrak{m})$ function* $\mathrm{Gen}_1, \mathrm{GenMod}_1 : {}^\omega 2 \to {}^\omega 2$ *with the following properties. Let* $\mathcal{H}_x = \mathrm{most}_{\mathfrak{m}}[\{n : \mathrm{Gen}_1(x)(n) = 1\}]$.

(i) *Suppose $\{\hat{x}_n : n \in \omega\}$ is finite. Let $E(x) : N \to {}^\omega 2$ be the enumeration of $\{\hat{x}_n : n \in \omega\}$ that removes the duplicates from $\langle \hat{x}_n : n \in \omega\rangle$ where $N \in \omega$. Then $\mathcal{H}_x$ is $\prod_{i<N} \mathbb{C}$-generic over $\mathcal{N}$ and $\mathcal{N}[\mathcal{H}_x] = \mathcal{N}[\prod_{i<N} \mathcal{G}_{E(x)(i)}]$.*

(ii) *Now suppose $x \in {}^\omega 2$ is such that $\{\hat{x}_n : n \in \omega\}$ is infinite. Let $E(x) : \omega \to {}^\omega 2$ be the enumeration of $\{\hat{x}_n : n \in \omega\}$ that removes the duplicate from $\langle \hat{x}_n : n \in \omega\rangle$. Then $\mathcal{H}_x$ is $\mathbb{C}_{\omega_1^{\mathcal{N}}}$-generic over $\mathcal{N}$ and $\mathbb{R}^{\mathcal{N}[\mathcal{H}_x]} = \bigcup\{\mathbb{R}^{\mathcal{N}[\prod_{i<n} \mathcal{G}_{E(x)(i)}]} : n \in \omega\}$.*

(iii) *$\mathcal{R}_{\mathrm{GenMod}_1(x)}$ is a set-like, well-founded, and extensional relation on ω whose Mostowski collapse $\mathcal{M}_{\mathrm{GenMod}_1(x)}$ is equal to $\mathcal{N}[\mathcal{H}_x]$.*

Proof In case (ii), the existence of the $\Delta_1^1(\mathfrak{m})$ function Gen_1 follows from the uniformity of the argument in the proof of Lemma 1.8. Case (i) is similar and somewhat easier. GenMod_1 again comes from the uniformity of the forcing construction. ∎

Lemma 1.10 *Assume the setting of Lemma 1.9. For all $x, y \in {}^\omega 2$, $x =^+ y$ if and only if $\mathbb{R}^{\mathcal{N}[\mathcal{H}_x]} = \mathbb{R}^{\mathcal{N}[\mathcal{H}_y]}$.*

Proof Without loss of generality (since the arguments are similar), assume that $\{\hat{x}_n : n \in \omega\}$ and $\{\hat{y}_n : n \in \omega\}$ are infinite. Let $E(x)$ and $E(y)$ enumerate without repetition $\langle \hat{x}_n : n \in \omega\rangle$ and $\langle \hat{y}_n : n \in \omega\rangle$, respectively. Since $\mathbb{R}^{\mathcal{N}[\mathcal{H}_x]} = \bigcup\{\mathbb{R}^{\mathcal{N}[\prod_{i<n} \mathcal{G}_{E(x)(i)}]} : n \in \omega\}$ and $\mathbb{R}^{\mathcal{N}[\mathcal{H}_y]} = \bigcup\{\mathbb{R}^{\mathcal{N}[\prod_{i<n} \mathcal{G}_{E(y)(i)}]} : n \in \omega\}$, it is clear that if $x =^+ y$, then $\mathbb{R}^{\mathcal{N}[\mathcal{H}_x]} = \mathbb{R}^{\mathcal{N}[\mathcal{H}_y]}$.

Now suppose that $\neg(x =^+ y)$. Without loss of generality, there is an n^* so that $E(x)(n^*) \notin \{\hat{y}_n : n \in \omega\}$. Let $g \in {}^\omega 2$ denote the $\mathbb{C}$-generic real associated with the $\mathbb{C}$-

generic filter $\mathcal{G}_{E(x)(n^*)}$. Suppose for the sake of contradiction that $\mathbb{R}^{\mathcal{N}[\mathcal{H}_x]} = \mathbb{R}^{\mathcal{N}[\mathcal{H}_y]}$. Since $\mathcal{N} \models$ " $\mathbb{C}_{\omega_1^{\mathcal{N}}}$ has the $\omega_1^{\mathcal{N}}$-chain condition", there is some $m \in \omega$ so that $g \in \mathbb{R}^{\mathcal{N}[\prod_{i<m} \mathcal{G}_{E(y)(i)}]}$, as argued in the proof Lemma 1.8. This is impossible, since by Lemma 1.6, $\{\mathcal{G}_{E(x)(n^*)}\} \cup \{\mathcal{G}_{E(y)(i)} : i < m\}$ is a collection of mutually $\mathbb{C}$-generic filters. ■

Lemma 1.11 *There are* $\Delta_1^1(\mathfrak{m})$ *functions* $\mathrm{Gen}_2, \mathrm{GenMod}_2 : {}^\omega 2 \to {}^\omega 2$ *with the following properties:*

(i) *Let* $\mathcal{K}_x = \mathrm{most}_{\mathrm{GenMod}_1(x)}[\{n : \mathrm{Gen}_2(x)(n) = 1\}]$. $\mathcal{K}_x$ *is a* $\mathrm{Coll}(\omega, \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]})$*-generic filter over* $\mathcal{N}[\mathcal{H}_x]$.

(ii) $\mathcal{R}_{\mathrm{GenMod}_2(x)}$ *is a set-like, well-founded, and extensional relation on* ω *whose Mostowski collapse* $\mathcal{M}_{\mathrm{GenMod}_2(x)}$ *is* $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$.

Proof The main ideas are the following: Fix $x \in {}^\omega 2$. Using that the satisfaction relation is Δ_1^1, one can obtain in a Δ_1^1 manner from the real $\mathrm{GenMod}_1(x)$ an enumeration of all the $\mathrm{Coll}(\omega, \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]})$-dense open subsets that belong to $\mathcal{N}[\mathcal{H}_x]$. From this enumeration of dense open sets, one can construct the code $\mathrm{Gen}_2(x)$ for $\mathcal{K}_x$, a $\mathrm{Coll}(\omega, \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]})$-generic filter over $\mathcal{N}[\mathcal{H}_x]$ and the code $\mathrm{GenMod}_2(x)$ for a structure on ω whose Mostowski collapse is $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$. (The construction is a simplified version of the argument in Lemma 1.6.) ■

Lemma 1.12 *Assume* $\mathsf{ZF} + \mathsf{AC}_\omega^{\mathbb{R}}$. *Let* $U \subseteq {}^\omega 2 \times {}^\omega 2$ *be a* $\Delta_1^1(z)$ *set that is universal for countable subsets of* ${}^\omega 2$. *Let* V *denote the real world. Let* $\mathfrak{m} \in {}^\omega 2$ *be such that the Mostowski collapse* $\mathcal{N} = \mathcal{M}_\mathfrak{m}$ *of* $(\omega, \mathcal{R}_\mathfrak{m})$ *is an elementary substructure of* V_κ *(for some cardinal* κ*) satisfying adequate amount of* $\mathsf{ZF} + \mathsf{AC}_\omega^{\mathbb{R}}$ *and* $z \in \mathcal{N}$. *Then there is a* $\Delta_1^1(\mathfrak{m})$ *function* $\Phi : {}^\omega 2 \to {}^\omega 2$ *so that* $U_{\Phi(x)} = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]}$.

Assume $\mathsf{ZF} + \mathsf{AC}_\omega^{\mathbb{R}}$ *and* $\Sigma_3^1(z)$*-generic absoluteness holds (specifically for the two step iteration* $\mathbb{C}_{\omega_1} * \mathrm{Coll}(\omega, \dot{\mathbb{R}})$*). Let* $U \subseteq {}^\omega 2 \times {}^\omega 2$ *be a* $\Sigma_1^1(z)$ *set that is universal for countable subsets of* ${}^\omega 2$. *Let* $\mathfrak{m} \in {}^\omega 2$ *be such that Mostowski collapse* $\mathcal{N} = \mathcal{M}_\mathfrak{m}$ *of* $(\omega, \mathcal{R}_\mathfrak{m})$ *is an elementary substructure of* V_κ *(for some cardinal* κ*) satisfying an adequate amount of* $\mathsf{ZF} + \mathsf{AC}_\omega^{\mathbb{R}} + \Sigma_3^1(z)$*-generic absoluteness and* $z \in \mathcal{N}$. *Then there is a* $\Delta_1^1(\mathfrak{m})$ *function* $\Phi : {}^\omega 2 \to {}^\omega 2$ *so that* $U_{\Phi(x)} = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]}$.

Proof Fix $x \in {}^\omega 2$. Since $\mathcal{K}_x$ is $\mathrm{Coll}(\omega, \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]})$-generic over $\mathcal{N}[\mathcal{H}_x]$, $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x] \models \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]}$ is countable.

If U is $\Delta_1^1(z)$, then Lemma 1.3 and the fact that $\mathcal{N}$ is an elementary substructure of V_κ imply that "U is universal for countable sets" holds in $\mathcal{N}$, and is a $\Pi_2^1(z)$ statement. By Schoenfield absoluteness, $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$ continues to believe that U is universal for countable sets. Thus, there is an $e \in \mathbb{R} \cap \mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$ so that $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x] \models U_e = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]}$.

If U is $\Sigma_1^1(z)$, then Lemma 1.3 and the fact that $\mathcal{N}$ is an elementary substructure of V_κ imply that $\mathcal{N}$ believes that U is universal for countable sets and that this statement is $\Sigma_3^1(z)$. Since $\mathcal{N}$ satisfies $\Sigma_3^1(z)$-generic absoluteness (for the forcing $\mathbb{C}_{\omega_1^{\mathcal{N}}} * \mathrm{Coll}(\omega, \dot{\mathbb{R}})$), $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$ continues to believe that U is universal for countable sets. Thus, there is an $e \in \mathbb{R} \cap \mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$ so that $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x] \models U_e = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]}$.

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Thus, in either the $\Delta^1_1(z)$ or $\Sigma^1_1(z)$ case, let $n^* \in \omega$ be least so that $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x] \models U_{\mathrm{most}_{\mathrm{GenMod}_2(x)}(n^*)} = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]}$. Let $\Phi(x) = \mathrm{most}_{\mathrm{GenMod}_2(x)}(n^*)$. Since the satisfaction relation is Δ^1_1, Φ is a $\Delta^1_1(\mathfrak{m})$ function.

It has only been shown that $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x] \models U_{\Phi(x)} = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]}$. One needs to show that this holds in the real world V. Note that since U is $\Delta^1_1(z)$ or $\Sigma^1_1(z)$, the effective perfect set theorem of Mansfield implies that $U_{\Phi(x)}$ consists only of $\Delta^1_1(z, \Phi(x))$ reals. Since the reals in $\Delta^1_1(z, \Phi(x))$ are exactly the reals that belong to every $z \oplus \Phi(x)$-admissible set (transitive model of Kripke–Platek set theory, KP), and $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$ is admissible (since admissibility is preserved by forcing and $\mathcal{N}$ is an elementary substructure of the admissible set V_κ), one has that $\Delta^1_1(z, \Phi(x)) \subseteq \mathbb{R} \cap \mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$. Thus, by Mostowski absoluteness between $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$ and V, one has that $\mathbb{R}^{\mathcal{N}[\mathcal{H}_x]} = (U_{\Phi(x)})^{\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]} = (U_{\Phi(x)})^V$. (Recall the Mostowski absoluteness states that Σ^1_1 statements are absolute between two transitive models with the same ω, and $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$ and V both have the same ω although they share very few other ordinals.) This completes the proof. ∎

Theorem 1.13 *Assume* ZF + $\mathrm{AC}^{\mathbb{R}}_\omega$. *Let $U \subseteq {}^\omega 2 \times {}^\omega 2$ be Δ^1_1 universal for countable sets. Then $=^+ \equiv_{\Delta^1_1} E_U$.*

Assume ZF + $\mathrm{AC}^{\mathbb{R}}_\omega$ + Σ^1_3*-generic absoluteness for the two step iteration* $\mathbb{C}_{\omega_1} * \mathrm{Coll}(\omega, \dot{\mathbb{R}})$. *Let $U \subseteq {}^\omega 2 \times {}^\omega 2$ be Σ^1_1 universal for countable sets. Then $=^+ \leq_{\Delta^1_1} E_U$.*

Proof Assume the setting of Lemma 1.12. Let $\Phi : {}^\omega 2 \to {}^\omega 2$ be the function given by Lemma 1.12. By Lemma 1.10, for any $x, y \in {}^\omega 2$, $x =^+ y$ if and only if $U_{\Phi(x)} = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]} = \mathbb{R}^{\mathcal{N}[\mathcal{H}_y]} = U_{\Phi(y)}$ if and only if $\Phi(x)\ E_U\ \Phi(y)$. Thus Φ is a reduction witnessing $=^+ \leq_{\Delta^1_1} E_U$. In the case where U is Δ^1_1, Fact 1.2 gives that $E_U \leq_{\Delta^1_1} =^+$, and therefore $E_U \equiv_{\Delta^1_1} =^+$. ∎

Finally, some comments on the arguments used in this article. Suppose $\Sigma : {}^\omega 2 \to \mathscr{P}_{\omega_1}({}^\omega 2)$ is a map from ${}^\omega 2$ to $\mathscr{P}_{\omega_1}({}^\omega 2)$, the collection of countable subsets of ${}^\omega 2$, with the property that $x =^+ y$ if and only $\Sigma(x) = \Sigma(y)$. Since U is assumed to be universal for countable sets, for each $x \in {}^\omega 2$, there is an e such that $U_e = \Sigma(x)$. Without any concrete knowledge of the definition of U, it seems that a function $\Phi : {}^\omega 2 \to {}^\omega 2$ so that $U_{\Phi(x)} = \Sigma(x)$ could be quite complex. Forcing and absoluteness allow for the simultaneous construction (for each for $x \in {}^\omega 2$) of a countable set of reals $\Sigma(x)$ and another countable set of reals $\mathcal{C}_x$ so that one can successfully search within $\mathcal{C}_x$ to find an e such that $U_e = \Sigma(x)$. Specifically in the above argument, $\Sigma(x) = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]}$ and $\mathcal{C}_x = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]}$. Since the search has been restricted to a countable set, one can produce a Δ^1_1 function Φ that essentially selects the least $e \in \mathcal{C}_x$ so that $U_e = \Sigma(x)$.

The use of forcing to study $=^+$ is quite natural in results involving producing $=^+$ reductions into other equivalence relations such as the following two examples: [8, Theorem 6.24] implies that if $B \subseteq {}^\omega 2$ is nonmeager, then $=^+$ Borel reduces into $=^+ \upharpoonright B$; [10, Theorem 2.8.11] showed a dichotomy result that states that in the Solovay model of a measurable cardinal, a Σ^1_1 equivalence relation E is unpinned (in the sense of [7] or [10]) if and only if $=^+ \leq_{\Delta^1_1} E$ or there is an almost Borel reduction of E_{ω_1} into E. (E_{ω_1} is isomorphisms of wellordering on ω with non-wellorderings put into a single class.

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An almost Borel reduction is a Borel function that fails to be a reduction on at most one class.)

Countable models of set theory are commonly used to produce Borel objects. The set of generics over a countable model for a forcing in that countable model is a Borel set. For instance, the set of Cohen generic reals over a countable model is a Borel comeager set. If the forcing is proper coming from a σ-ideal, then the set of generics has many interesting canonization properties for equivalence relations; see, for instance, [8, Theorem 6.24]. (See [12, 8, 1, 3] for other examples.) Variations of the idea of producing coherent families of mutual generics over a countable model and the Borelness of evaluating names by such generics over countable models are used to produce perfect sets and prove various dichotomy results for equivalence relations.

A common approach to creating a Borel reduction from $=^+$ into another equivalence relation involves defining a map Σ that assigns reals to countable sets of reals. In this situation for the equivalence relation E_U where U is Δ^1_1 universal for countable sets, the requirements are that there is a function $\Sigma : {}^\omega 2 \to \mathscr{P}_{\omega_1}({}^\omega 2)$ and a function assigning $x \mapsto \mathcal{M}_x$, where $\mathcal{M}_x$ is a countable transitive model of some adequate fragment of ZFC, which is Δ^1_1 in a suitable coding and satisfy the following two key properties.

(1) For all $x, y \in {}^\omega 2$, $x =^+ y$ if and only if $\Sigma(x) = \Sigma(y)$.

(2) $\Sigma(x) \in \mathcal{M}_x$, and $\mathcal{M}_x \models \Sigma(x)$ is countable.

For the purpose of this article, this can be done within ZFC using simple forcings such as $\mathbb{C}$, $\mathbb{C}_{\omega_1}$, and $\mathrm{Coll}(\omega, \mathbb{R})$. The following is a summary of this method used above to produce $=^+$ Borel reductions.

First, one creates a Δ^1_1 assignment of reals to generics for a countable model that satisfy a mutual genericity condition, which is accomplished here in Lemma 1.6 by Cohen forcing and the function Gen_0. (For the unpinnedness dichotomy ([10, Theorem 2.8.11]) in the Solovay model from a measurable cardinal κ, the suitable forcing is naturally $\mathrm{Coll}(\omega, < \kappa)$.) Then Σ is defined to be the reals that can appear in any finite product of mutual generics from certain countable collections of the assigned mutual generics. Here, $\Sigma(x) = \bigcup\{\mathbb{R}^{\mathcal{N}[\prod_{i<n} \mathcal{G}_{E(x)(i)}]} : n \in \omega\}$ in the notation of Lemma 1.9. The mutual genericity property of the assignment plays an essentially role in establishing the first key property as argued in Lemma 1.10. One needs $\Sigma(x)$ to be a countable set in an appropriate countable model. Here, one arranges that $\Sigma(x)$ is, uniformly in a Δ^1_1-manner, the set of reals of a generic extension of the original countable model, which is done in Lemma 1.9, which arranges $\Sigma(x) = \mathbb{R}^{\mathrm{GenMod}_1(x)} = \mathbb{R}^{\mathcal{N}[\mathcal{H}_x]}$. Then Lemma 1.11 uses a further forcing by $\mathrm{Coll}(\omega, \mathbb{R})$ to make $\Sigma(x)$ countable in $\mathrm{GenMod}_2(x)$, which Mostowski collapses to $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$. This creates the necessary objects with the two key properties.

By the absoluteness observations, U remains universal for countable sets of reals in $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$. Thus, one can search in this model for a real e so that $U_e = \Sigma(x)$ holds in $\mathcal{N}[\mathcal{H}_x][\mathcal{K}_x]$. Using some further absoluteness arguments, it is shown that $U_e = \Sigma(x)$ holds in the real world V. The map Φ taking x to this e is the desired reduction.

As observed by the referee, since the collection of Cohen generic reals over a countable model $\mathcal{M}$ is comeager, one can apply [8, Theorem 6.24] and use the absoluteness argument of Lemma 1.12 to search for a real e in $\mathcal{M}[x]$ such that $U_e = \{\hat{x}_n : n \in \omega\}$. This will avoid directly constructing the reduction of $=^+$ into E_U, although [8, Theorem 6.24] is proved using argument similar to what is outlined above.

Here, the forcing $\mathrm{Coll}(\omega, \mathbb{R})$ is important for obtaining a set of reals that is countable in the desired countable model. The use of this forcing is quite common in the study of $=^+$. The forcing $\mathrm{Coll}(\omega, \mathbb{R})$ and the canonical $\mathrm{Coll}(\omega, \mathbb{R})$-name for the generic surjection witness that $=^+$ is an unpinned equivalence relation in the sense of [7] or [10]. The unpinnedness of $=^+$ is often used in the study of this equivalence relation. For instance, the witness to the unpinnedness of $=^+$ is used in [2, Example 2.17] to show in $L(\mathbb{R}) \models \mathsf{AD}$ that $=^+$ has an OD equivalence class with no OD member. Under $\mathsf{ZF} + \mathsf{AD}^+ + \mathsf{V} = \mathsf{L}(\mathscr{P}(\mathbb{R}))$, unpinnedness of Σ^1_1 equivalence relations is in some sense the main obstacle to making definable selections from equivalence classes. (See [2, Corollary 2.14, Theorem 3.1, Example 3.5, and Example 3.6.].)

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