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Topological complexity

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Abstract. In a topological dynamical system (X, T) the complexity function of a cover $\mathcal{C}$ is the minimal cardinality of a sub-cover of $\bigvee_{i=0}^{n} T^{-i}\mathcal{C}$. It is shown that equicontinuous transformations are exactly those such that any open cover has bounded complexity. Call scattering a system such that any finite cover by non-dense open sets has unbounded complexity, and call 2-scattering a system such that any such 2-set cover has unbounded complexity: then all weakly mixing systems are scattering and all 2-scattering systems are totally transitive. Conversely, any system that is not 2-scattering has covers with complexity at most $n + 1$. Scattering systems are characterized topologically as those such that their cartesian product with any minimal system is transitive; they are consequently disjoint from all minimal distal systems. Finally, defining (x, y), $x \neq y$, to be a complexity pair if any cover by two non-trivial closed sets separating x from y has unbounded complexity, we prove that 2-scattering systems are disjoint from minimal isometries; that in the invertible case the complexity relation is contained in the regionally proximal relation and, when further assuming minimality, coincides with it up to the diagonal.

0. *Introduction*

The complexity function of symbolic systems, especially those with entropy zero, has been studied by many researchers in the past few years; a survey is given in [**F2**]. The complexity function is the non-decreasing map p from $\mathbb{N}$ to itself such that $p(n)$ is the number of allowed words of length n of a symbolic system. It must not be mistaken for the notion of algorithmic and structural complexity, which can be given a completely distinct meaning in Dynamics.

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It was remarked long ago that given a symbolic system (S, σ), the complexity function $p_S(n)$ is either ultimately constant (and then S is a finite union of periodic orbits), or greater than n for all n; the symbolic systems having a bounded complexity function are exactly the equicontinuous ones. In [**F1**] Ferenczi defined a natural complexity function for ergodic measured systems (independently introduced in [**KT**] for other purposes), and proved that it is bounded in a certain sense if and only if the system is isomorphic to an isometry, i.e. to a minimal rotation of a compact group equipped with its Haar measure; in the topological setting all minimal equicontinuous systems are known to be conjugate to isometries. There remained to see whether a suitable definition of topological complexity allows one to single out equicontinuous systems among all other topological systems.

Consider a topological dynamical system (X, T), where X is a compact metric space and T a continuous surjective self-map of X. Given a cover $\mathcal{C}$, usually open or closed, its complexity function $c(\mathcal{C}, n)$ is the minimal cardinality of a sub-cover of the refinement $\mathcal{C}_0^n = \bigvee_{i=0}^n T^{-i}\mathcal{C}$—the exponential growth rate of this function was called topological entropy as early as 1965 in [**AKM**], but here we are mainly concerned with the fact that it goes to infinity or not. When $\mathcal{C}$ is the natural generating partition-cover of a symbolic system $c(\mathcal{C}, n)$ is just the usual symbolic complexity. We do not try to define any global complexity for topological systems, as Ferenczi did in the measured setting [**F1**]; instead we consider the complexities of all finite covers of a system belonging to some convenient subclass.

The first result in this article (Proposition 2.2) establishes that topological complexity discriminates equicontinuous systems, among them isometries, from all other dynamical systems. We show that (X, T) is equicontinuous if and only if any open cover $\mathcal{C}$ has bounded complexity function: there is $k > 0$ such that $c(\mathcal{C}, n) \leq k$ for all n.

In the opposite direction we consider systems where in some sense complexity occurs everywhere, and we study the links between these 'scattering' properties and classical notions of Topological Dynamics, among them so-called chaotic properties; let us mention that several of the ideas in this article arose while reading [**GW1**].

Precisely we introduce the notions of scattering and 2-scattering: (X, T) is called scattering if any non-trivial closed cover (or any cover by non-dense open sets) has unbounded complexity, and 2-scattering if the same is true for 2-set covers only. Scattering evidently implies 2-scattering, but we do not know whether the converse is true.

Some results concern weak mixing and transitivity. Any topologically weakly mixing system is scattering (Proposition 3.4), and in the other direction if any non-trivial closed cover of (X, T) has complexity at least $n + 2$ then (X, T) is weakly mixing (Proposition 3.6). We did not prove that weak mixing is strictly stronger than scattering; recently Akin and Glasner found an example of a scattering non-weakly mixing system [**AG**].

When (X, T) has an equicontinuity point, there are particular open covers with complexity at most $n + 1$ (Proposition 3.11). However, there are not always covers with bounded complexity: Akin and Glasner's scattering non-weakly mixing system contains equicontinuity points. A corollary is that scattering does not imply sensitivity to initial conditions.

We then give a series of topological statements equivalent to the existence of a closed cover with bounded complexity for (X, T) (Proposition 4.1). The most striking is the following: (X, T) is scattering if and only if its cartesian product with any minimal system is transitive. An interesting consequence is that scattering systems are disjoint from all minimal distal systems (Proposition 4.2); the proof is almost identical to that given by Furstenberg for a similar theorem, when assuming weak mixing instead of scattering [**Fu**].

In [**Bl2**] one of the authors introduced entropy pairs. Complexity pairs are defined similarly: (x, x'), $x \neq x'$, forms a complexity pair if any closed 2-set cover separating x and x' has unbounded complexity; a system is 2-scattering if and only if any pair of distinct points is a complexity pair. With their help we prove that minimal equicontinuous systems are disjoint from 2-scattering systems (Proposition 5.4) and that the maximal equicontinuous factor of (X, T) is the one associated with the smallest closed invariant equivalence relation containing the set $\mathrm{Com}(X, T)$ of complexity pairs (Proposition 5.7). Finally when (X, T) is invertible $\mathrm{Com}(X, T)$ is included in the regionally proximal relation $\mathrm{RP}(X, T)$ (Proposition 5.7), and equal to it when (X, T) is also minimal (Example 5.10).

We do not address the measure-theoretic side of complexity, except by remarking that, almost obviously, the existence of an invariant probability measure with full support and trivial Kronecker factor implies scattering (Remark 5.12).

The results we obtain prove that the significance of complexity in Topological Dynamics is not restricted to its exponential growth rate. For minimal systems its meaning is clear. It mainly provides new equivalent definitions of classical properties: when (X, T) is minimal the weak mixing, scattering and 2-scattering properties are equivalent (Proposition 3.8); when T is also invertible the complexity relation is identical to the regionally proximal relation up to the diagonal. It is only for non-minimal zero-entropy systems that complexity properties introduce new elements of classification.

This research was partly motivated by, and may also shed new light on, the question of chaos. Since this is not the main concern of the article we have gathered all discussions of this topic in the last section.

After the introduction and definitions, §2 establishes the characterization of equicontinuity. §3 describes the relations of complexity with weak mixing, transitivity, and the existence of equicontinuous points. §4 is devoted to a topological characterization of the scattering property and its consequence in terms of disjointness. In §5 we introduce complexity pairs, establish a disjointness theorem for 2-scattering systems, prove that $\mathrm{Com}(X, T) \subset \mathrm{RP}(X, T)$ and that the two relations are equal in the minimal case. Finally, in §6 we make a few remarks about chaoticity and briefly discuss the connection between sensitive dependence on initial conditions and complexity properties.

1. *Definitions and notation*

1.1. *Topological dynamical systems.* A (topological) dynamical system (X, T) is a compact metric set X endowed with a continuous, onto map T. When T is invertible it is called a homeomorphism. An *invariant set* E is one such that $TE = E$: when E is closed the restriction (E, T_E) is also a dynamical system.

A *factor map* $\phi : (X, T) \to (Y, S)$ is a continuous onto map such that $S \circ \phi = \phi \circ T$; when such a map exists (Y, S) is called a factor of (X, T) and (X, T) an extension of (Y, S). A conjugacy map is a bijective factor map.

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A *joining* J between (X, T) and (Y, S) is a closed $T \times S$-invariant subset of the cartesian product $X \times Y$, projecting onto both factors; it is said to be *non-trivial* when it is a proper subset. Call $J^*(y)$ the fiber $\{x \in X : (x, y) \in J\}$ of J over y, and $J_*(x)$ the fiber of J over x; these two sets are closed.

A system (X, T) is called *transitive* if for any two non-empty open sets U, V there is $n > 0$ such that $U \cap T^{-n}V \neq \emptyset$; equivalent properties are for (X, T) to have transitive points, i.e. points having a dense positive orbit; or the fact that all non-empty invariant open sets are dense. *Total transitivity* means that (X, T^n) is transitive for any $n > 0$. A system (X, T) is said to be *weakly mixing* if its cartesian square $(X^2, T \times T)$ is transitive, or, equivalently, if for any four non-empty open sets A, B, C and D there is $n > 0$ such that $(A \cap T^{-n}C) \times (B \cap T^{-n}D) \neq \emptyset$. If (X, T) is weakly mixing then it is weak mixing of all orders, i.e. any of its cartesian powers is transitive; in other words, given two families of non-empty open sets $(A_1, \ldots, A_k)$ and $(B_1, \ldots, B_k)$, there is $n > 0$ such that all the intersections $A_i \cap T^{-n}B_i$ are simultaneously non-empty [**Fu**, Proposition II.3].

A *minimal* system is one that contains no proper closed T-invariant subset except $\emptyset$.

An *isometry* is a system (X, T) such that T preserves distances. An extension $\pi : (Y, S) \to (X, T)$ is said to be *isometric* if $d(Sy, Sy') = d(y, y')$ whenever $y, y' \in Y$ and $\pi(y) = \pi(y')$.

A slightly weaker property is equicontinuity. A system (X, T) is said to be *equicontinuous* if for any $\epsilon > 0$ there is $\eta > 0$ such that if $x, y \in X$ with $d(x, y) < \eta$, then for any $n \in \mathbb{N}$ one has $d(T^n x, T^n y) < \epsilon$. By compactness, if (X, T) is not equicontinuous there are $\epsilon > 0$ and a point $x \in X$ such that for any $\eta > 0$ one can find $y \in X$ with $d(x, y) < \eta$ and $n \in \mathbb{N}$ such that $d(T^n x, T^n y) > \epsilon$. A point $x \in X$ is called an *equicontinuity point* if for any $\epsilon > 0$ there is $\eta > 0$ such that if $y \in X$ with $d(x, y) < \eta$ then for any $n \in \mathbb{N}$ one has $d(T^n x, T^n y) < \epsilon$; obviously a system is equicontinuous if all its points are equicontinuity points. Minimal equicontinuous systems are conjugate to minimal isometries, i.e. minimal rotations of compact monothetic groups, and play a central rôle in the theory of Topological Dynamics as developed in Auslander's book [**Au**].

The property of distality is still weaker: a *distal system* (X, T) is one such that if $x \neq y \in X$, there is $\epsilon > 0$ with $d(T^n x, T^n y) \geq \epsilon$, $n \in \mathbb{N}$. Minimal distal systems form the smallest class containing the trivial system and stable under factor maps, minimal isometric extensions and inverse limits [**Fu**].

A system (X, T) is said to have *sensitive dependence on initial conditions*, or to be sensitive for short, if there exists $\epsilon > 0$ such that for every $x \in X$, every $\eta > 0$ there are $y \in B(x, \eta)$ and $n \geq 0$ with $d(T^n x, T^n y) > \epsilon$. Obviously a sensitive system has no equicontinuity points; on the other hand a transitive, non-sensitive system contains equicontinuity points [**AAB**].

One particular class is that of symbolic systems. Let A be a finite set of symbols or alphabet; finite sequences of elements of A are called *words*. Consider the set $A^{\mathbb{N}}$ (or $A^{\mathbb{Z}}$) endowed with the usual product topology; for $x \in A^{\mathbb{N}}$ let $x(i)$ (respectively $x(i, j)$ for $i < j$) denote the ith coordinate of x (respectively the block of its coordinates from i to j). The shift transformation, defined on $A^{\mathbb{N}}$ by $\sigma(x(0)x(1)x(2)\ldots) = x(1)x(2)x(3)\ldots$, is continuous and onto; its two-sided version is also invertible on $A^{\mathbb{Z}}$. A symbolic system or *subshift* (S, σ) is a closed σ-invariant subset of $A^{\mathbb{N}}$, endowed with the restriction of the

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shift to S. A subshift S is completely determined by the set $L(S)$ of all words of the form $w = \omega(i, j), \omega \in S, i, j \in \mathbb{N}, i < j$; one sometimes considers the sets $L_n(S)$ of allowable words of length n of S. The cylinder set associated to $w \in L_i(S)$ is

$$[w] = \{\omega \in S : \omega(0, i-1) = w\}.$$

1.2. *Covers and names.* The covers we are dealing with in this article are always finite, but not always open. A cover is said to be *trivial* if one of its elements is equal to the whole set X.

Given two covers $\mathcal{C} = (C_1, \ldots, C_n)$ and $\mathcal{D} = (D_1, \ldots, D_m)$ define their refinement as $\mathcal{C} \vee \mathcal{D} = (C_i \cap D_j : i = 1, \ldots, n, j = 1, \ldots, m)$. A cover $\mathcal{C}$ is said to be *coarser than* another cover $\mathcal{C}'$ (or $\mathcal{C}'$ to be *finer than* $\mathcal{C}$) if any element of $\mathcal{C}'$ is included in an element of $\mathcal{C}$; this property is denoted by $\mathcal{C} \prec \mathcal{C}'$.

For any finite cover $\mathcal{C}$ of a compact space X, let $r(\mathcal{C})$ be the minimal cardinality of a subcover of $\mathcal{C}$: if $\mathcal{C} \prec \mathcal{C}'$ then $r(\mathcal{C}) \le r(\mathcal{C}')$. One has the inequality $r(\mathcal{C} \vee \mathcal{D}) \le r(\mathcal{C}) \times r(\mathcal{D})$.

Standard covers were introduced for the study of the topological entropy of covers; it is not surprising they also play a part in the study of complexity. A *standard cover* is a cover by two non-dense open sets. A standard cover (A, B) is said to *separate* two different points x and y if $x \in \mathrm{Int}(A^c)$ and $y \in \mathrm{Int}(B^c)$.

With the help of a finite cover $\mathcal{C} = (C_1, \ldots, C_m)$ of (X, T) one defines $\mathcal{C}$-names for the points of X. Put $A = \{1, \ldots, m\}$ and $\Omega = A^{\mathbb{N}}$. A sequence $\omega \in \Omega$ is said to be a *$\mathcal{C}$-name of* $x \in X$ if

$$(x, \omega) \in J = \left\{ (x, \omega) \in X \times \Omega : x \in \bigcap_{i=0}^{\infty} T^{-i}(C_{\omega(i)}) \right\}.$$

Note that if ω is a $\mathcal{C}$-name of x then $\sigma(\omega)$ is a $\mathcal{C}$-name of Tx; consequently J is a $T \times \sigma$-invariant set, and when $\mathcal{C}$ is a closed cover J is a joining between (X, T) and a closed invariant subset of (Ω, σ). As in the case of joinings denote by $J^*(\omega) = \bigcap_{i=0}^{\infty} T^{-i}(C_{\omega(i)})$ the fiber of J over ω, here equal to the set of all points x of X such that ω is a $\mathcal{C}$-name of x, and by $J_*(x) = \{\omega \in \Omega : (x, \omega) \in J\}$ the set of all possible $\mathcal{C}$-names of x; when $\mathcal{C}$ is closed these sets are closed too. A set of $\mathcal{C}$-names $\{\omega_i \in A^{\mathbb{N}} : i \in I\}$ is said to *cover* X if $\bigcup_{i \in I} J^*(\omega_i) = X$ (beware! here subscripts *do not* denote coordinates).

When T is a homeomorphism it is often convenient to consider names in $A^{\mathbb{Z}}$. They are defined in the same way, with i varying from $-\infty$ to $+\infty$. In this article we sometimes use finite names: a word $w \in A^n$ is a $\mathcal{C}$-name of x if $x \in \bigcap_{i=0}^{n} T^{-i}(C_{w(i)})$, and for $\omega \in A^{\mathbb{N}}$ we call $J_n^*(\omega)$ the set of all points having the finite name $\omega(0, n-1)$; note that $J^*(\omega) = \lim_{n \to \infty} J_n^*(\omega)$.

2. *Topological complexity and equicontinuity*

Given a dynamical system (X, T) and a cover $\mathcal{C}$, put $\mathcal{C}_0^n = \bigvee_{i=0}^{n} T^{-i}\mathcal{C}$.

Definition. The *topological complexity function* of the finite cover $\mathcal{C}$ of (X, T) is the non-decreasing function

$$c(\mathcal{C}, n) = r(\mathcal{C}_0^n).$$

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We often call it *complexity* for short. Remark that $c(\mathcal{C}, n) = k$ means that there is a family of k $\mathcal{C}$-names of length n covering X.

If $\mathcal{C} \prec \mathcal{C}'$ then $c(\mathcal{C}, n) \leq c(\mathcal{C}', n)$, and $c(\mathcal{C} \vee \mathcal{D}, n) \leq c(\mathcal{C}, n) \times c(\mathcal{D}, n)$.

The cover $\mathcal{C}$ is said to have *bounded complexity* when $c(\mathcal{C}, n)$ is bounded, which means ultimately constant because $c(\mathcal{C}, n)$ is non-decreasing. Otherwise $\mathcal{C}$ is said to have *unbounded* or *infinite complexity*.

When $\mathcal{C}$ is the canonical clopen partition of a subshift (X, σ) according to the value of the zeroth coordinate, $c(\mathcal{C}, n)$ is just what is usually called the complexity function of (X, σ) **[F2]**. However, even in the case of subshifts we shall consider the complexity of various covers, not only that of the canonical partition.

When a finite cover $\mathcal{C}$ has bounded complexity there exists a finite covering collection of infinite $\mathcal{C}$-names, as expressed in the following lemma.

LEMMA 2.1. *Let (X, T) be a dynamical system (respectively a homeomorphism). A finite cover $\mathcal{C} = (U_1, \ldots, U_k)$ has complexity bounded by m if and only if there is a collection of m infinite (respectively bi-infinite) $\mathcal{C}$-names covering X.*

Proof. Let $\mathcal{C}$ have complexity bounded by m. Put $A = \{1, 2, \ldots, k\}$. To $\omega \in A^{\mathbb{N}}$ associate for given $n \in \mathbb{N}$ the set $J_n^*(\omega)$ of points of X for which $\omega(0, n-1)$ is a $\mathcal{C}$-name.

Call $H(n)$ the set of m-tuples $\{\omega_1, \ldots, \omega_m\}$ of elements of $A^{\mathbb{N}}$ such that $(J_n^*(\omega_1), \ldots, J_n^*(\omega_m))$ covers X. The set $H(n)$ is non-empty by definition of m; it is a closed subset of $(A^{\mathbb{N}})^m$. If $(J_n^*(\omega_1), \ldots, J_n^*(\omega_m))$ covers X then $(J_{n-1}^*(\omega_1), \ldots, J_{n-1}^*(\omega_m))$ covers X too, and therefore $H(n) \subseteq H(n-1)$. The intersection of decreasing closed sets $H = \bigcap_{n=0}^{\infty} H(n)$ contains at least one m-tuple $\underline{\omega}$; obviously $\bigcup_{i=1}^{m} J^*(\underline{\omega}_i) = \lim_{n\to\infty} \bigcup_{i=1}^{m} J_n^*(\underline{\omega}_i) = X$, which means that the set of $\mathcal{C}$-names $\underline{\omega}$ covers X.

The reverse is obvious. The proof for bi-infinite names is the same. □

Now we characterize equicontinuity in terms of the combinatorics of open covers.

PROPOSITION 2.2. *Let (X, T) be a dynamical system. The two following statements are equivalent:*

(1) *(X, T) is equicontinuous;*

(2) *for any finite open cover $\mathcal{C}$ of X, $c(\mathcal{C}, n)$ is bounded.*

Proof. (1) $\Rightarrow$ (2). Let $\epsilon > 0$ be a Lebesgue number of the finite open cover $\mathcal{C}$: this means any open ball B with radius ϵ is contained in at least one element of $\mathcal{C}$.

By the equicontinuity assumption there is $\eta > 0$, $\eta \leq \epsilon$, such that if $d(x, y) < \eta$ then for any $n \in \mathbb{N}$ one has $d(T^n x, T^n y) < \epsilon$. Let $x_1, x_2, \ldots, x_k$ be such that the open balls $B(x_i, \eta)$, $1 \leq i \leq k$, cover X. By equicontinuity for $j \in \mathbb{N}$ one has $T^j B(x_i, \eta) \subset B(T^j x_i, \epsilon)$, and since ϵ is a Lebesgue number of $\mathcal{C}$ there is an element $U_{i,j}$ of $\mathcal{C}$ containing completely $B(T^j x, \epsilon)$. Therefore $T^j B(x_i, \eta) \subset U_{i,j}$, and for any n

$$B(x_i, \eta) \subset \bigcap_{0 \leq j \leq n} T^{-j} U_{i,j}.$$

The last formula means that $(\bigcap_{0 \leq j \leq n} T^{-j} U_{i,j})$, $0 < i \leq k$, is a sub-cover of $\mathcal{C}_0^n$ with cardinality k independent of n. Therefore $c(\mathcal{C}, n) \leq k$.

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(2) $\Rightarrow$ (1). Suppose T is not equicontinuous, then there are $\epsilon > 0$ and a point $x \in X$ such that for any $\eta > 0$ one can find $y \in X$, $d(x, y) < \eta$, and $n \in \mathbb{N}$ such that $d(T^n x, T^n y) > \epsilon$. If the assumption (2) is true for (X, T), consider a finite cover $\mathcal{C}$ by open balls with radius $\epsilon/4$, and let $\overline{\mathcal{C}} = (U_1, \ldots, U_k)$ be the cover made up of the closures of the elements of $\mathcal{C}$: since $\overline{\mathcal{C}} \prec \mathcal{C}$ its complexity is also bounded. By Lemma 2.1 this means there is a closed cover $(X_1, X_2, \ldots, X_c)$ of X with

$$X_i = \bigcap_{j=0}^{\infty} T^{-j} U_{i,j}, \quad U_{i,j} \in \overline{\mathcal{C}}.$$

By definition of $\overline{\mathcal{C}}$ if y and z both belong to X_i the distance $d(T^j y, T^j z)$ is bounded by $\epsilon/2$ for any $j > 0$.

Let $\eta_n > 0$ go to zero; choose $y_n \in X$ such that $d(x, y_n) < \eta_n$ and there are integers k_n with $d(T^{k_n} x, T^{k_n} y) > \epsilon$. By taking a subsequence, the y_n can be assumed to belong all to the same set X_i, and since X_i is closed x belongs to it too. This implies $d(T^{k_n} x, T^{k_n} y_n) \leq \epsilon/2$, which contradicts the previous assumption. □

Remark 2.3. This is where the difference lies between an irrational rotation $(\mathbb{T}, \alpha)$ of the torus $\mathbb{T}$ and its Sturmian symbolic extension (X, σ): $(\mathbb{T}, \alpha)$ is equicontinuous and any open cover has bounded complexity; on the other hand the canonical clopen partition of (X, σ) has complexity $n + 1$; its image, the classical interval cover of $\mathbb{T}$, is closed but it is not the closure of an open cover.

3. *Complexity, mixing and transitivity*

Here we examine the relations between the complexity of covers and some classical topological properties.

Remark 3.1. The existence of a finite cover of (X, T) by non-dense open sets with bounded complexity, and the existence of a finite non-trivial closed cover with the same cardinality and bounded complexity, are equivalent properties: if $\mathcal{C}$ is open non-dense and has bounded complexity so has the non-trivial closed cover made up of the closures of its elements; and if a closed non-trivial cover has bounded complexity any coarser non-dense open cover has bounded complexity too. On the other hand all open covers may have bounded complexity while there exist closed covers with unbounded complexity, as was remarked for irrational rotations.

Call a dynamical system (X, T) *scattering*, respectively *2-scattering*, if any finite cover by non-dense open sets, respectively any standard cover, has unbounded complexity. In view of Remark 3.1 above one can replace open covers by non-trivial closed covers in these definitions. One could also define n-scattering for $n > 2$ in similar manner.

Question 3.2. Does 2-scattering imply scattering? This question is given its whole significance by the results of §4 and §5.

Relations between topological mixing and complexity are the matter of the next two propositions. We require one lemma for the proof of each.

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The first is purely combinatorial. Suppose the minimal cardinality of a sub-cover of $\mathcal{C}$ is $r(\mathcal{C}) = k$, let $I(\mathcal{C})$ be the set of intersections of $k-1$ complements of elements of $\mathcal{C}$; by definition of $r(\mathcal{C})$ none of these intersections is empty.

LEMMA 3.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be two finite non-trivial closed covers of X, $r(\mathcal{C}) = k$, $r(\mathcal{D}) = \ell$. Assume that for any $A \in I(\mathcal{C})$, any $B \in I(\mathcal{D})$, the set $A \cap B$ is non-empty. Then $r(\mathcal{C} \bigvee \mathcal{D}) \geq k + \ell - 1$.*

Proof. Consider $k + \ell - 2$ elements of $\mathcal{C} \bigvee \mathcal{D}$, each of them of the form $C_i \cap D_j$. The subset of X they do not cover is the complement of their union; it is a union of intersections of $k + \ell - 2$ sets C_i^c and D_j^c; some of these are intersections of an element of $I(\mathcal{C})$ and an element of $I(\mathcal{D})$. By hypothesis these are not empty, therefore no subset of size $k + \ell - 2$ of $\mathcal{C} \bigvee \mathcal{D}$ can be a sub-cover. □

PROPOSITION 3.4. *Weak topological mixing implies scattering.*

Proof. In this proof we use the definition of scattering by closed covers. Let (X, T) be weakly mixing and let $\mathcal{C} = (C_1, \ldots, C_k)$, with $k > 1$, be any non-trivial cover by closed sets.

Observe that since $\mathcal{C}$ is non-trivial the minimal cardinality $r_0 = r(\mathcal{C})$ of a sub-cover is at least two. We define inductively an increasing sequence of integers $(i_n, n \in \mathbb{N})$ such that $r(\mathcal{C}_0^{i_n})$ goes to infinity, starting from $i_0 = 0$ for which $r(\mathcal{C}) \geq 2$. Put $c(\mathcal{C}, i_n) = m_n$. As $\mathcal{C}$ is closed, the two families $I(\mathcal{C}_0^{i_n})$ and $I(\mathcal{C})$ are made up of non-empty open sets. By weak mixing of all orders, which we know is equivalent to weak mixing, there is an integer $j_n > 0$ such that whatever $A \in I(\mathcal{C}_0^{i_n})$, $B \in I(\mathcal{C})$, the set $A \cap T^{-j_n} B$ is not empty. We can thus apply Lemma 3.3 to the two covers $\mathcal{C}_0^{i_n}$ and $T^{-j_n}\mathcal{C}$: their refinement has minimal cardinality at least $m_n + m_0 - 1 > m_n$.

Putting $i_{n+1} = i_n + j_n$, observe that $\mathcal{C}_0^{i_n} \bigvee T^{-j_n}\mathcal{C} \prec \mathcal{C}_0^{i_{n+1}}$, thus $m_{i_n} \geq n + 1$. This proves that $\mathcal{C}$ has unbounded complexity. □

The next lemma, due to Banks, characterizes weak mixing for invertible as well as non-invertible systems; a similar Lemma was proved long ago in [**P**] for group actions.

LEMMA 3.5. *A dynamical system (X, T) is not weakly mixing if and only if there exist two non-empty open sets U and V such that*

$$\forall n > 0, \quad \textit{either } U \cap T^{-n} U = \emptyset \quad \textit{or} \quad U \cap T^{-n} V = \emptyset. \tag{1}$$

PROPOSITION 3.6. *Suppose each standard cover $\mathcal{C}$ of (X, T) is such that $c(\mathcal{C}, n) > n + 1$ for some n; then (X, T) is topologically weakly mixing.*

Proof. Suppose (X, T) is not weakly mixing; by Lemma 3.5 there are non-empty open sets U and V with property (1); without loss of generality assume their intersection is empty. Let $\mathcal{R}' = \{U', V'\}$ be a standard cover of X such that $V^c \subset U'$ and $U^c \subset V'$. Given a positive integer i, either $U \cap T^{-i} U \neq \emptyset$, in which case $U \subset (T^{-i} V)^c \subset T^{-i} U'$; or $U \cap T^{-i} U = \emptyset$ and $U \subset T^{-i} V'$. This means there is a sequence $(W_i, i > 0)$ with $W_i = U'$ or V', such that

$$U \subset U' \cap T^{-1} W_1 \cap \cdots \cap T^{-n} W_n \cap \cdots .$$

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Fixing n and choosing for each $x \in X$ the smallest integer i, $0 \le i \le n$, such that $T^i x \in U$ when there exists one, otherwise putting $i = n + 1$, we observe that the family of $n + 1$ sets

$$(V' \cap T^{-1}V' \cap \cdots \cap T^{-i+1}V' \cap T^{-i}U' \cap T^{-i-1}W_1 \cap \cdots \cap T^{-n}W_{n-i} : 0 \le i \le n+1)$$

is a sub-cover of $\mathcal{R}'^n_0$. Hence $c(\mathcal{R}', n) \le n + 1$ for all $n > 0$. □

Non-standard covers may have uncharacteristically small complexity. When (X, T) is weakly mixing and contains periodic points it is easy to find an open cover with bounded complexity. On the other hand it is a significant fact that some non-standard cover has unbounded complexity (see Proposition 5.1(2)).

Remark 3.7. (X, T) is said to be an *F-system* if it has a dense set of periodic points and if all powers of T act transitively on X [**Fu**]. F-systems are weakly mixing [**B1**] and therefore scattering.

PROPOSITION 3.8. *For a minimal dynamical system 2-scattering, scattering and weak mixing are equivalent.*

Proof. Any minimal system that is not weakly mixing has a non-trivial equicontinuous factor ([**KR**], see [**Au**] for a simple proof due to McMahon), and therefore possesses standard covers with bounded complexity. It cannot thus be scattering or 2-scattering. The converse results from Proposition 3.4. □

Question 3.9. Except for minimal systems Propositions 3.4 and 3.6 do not add up to a characterization of weak topological mixing. Scattering does not imply weak mixing [**AG**]. Nevertheless, is it possible to characterize weak mixing in terms of complexity?

PROPOSITION 3.10. *If (X, T) is 2-scattering, then (X, T^n) is 2-scattering and transitive for any $n > 0$.*

Proof. Let us show first that when (X, T) is 2-scattering so is (X, T^k). Let subscripts denote the transformations with respect to which a complexity is computed, and let $\mathcal{C}$ be a finite cover such that $c_T(\mathcal{C}, n) \to \infty$ when n goes to infinity. One has

$$\begin{aligned} c_T(\mathcal{C}, kn + k - 1) &= r\left(\bigvee_{i=0}^{kn+k-1} T^{-i}\mathcal{C}\right) = r\left(\bigvee_{i=0}^{n} T^{-ki}\mathcal{C} \vee \cdots \vee \bigvee_{i=0}^{n} T^{-ki-k+1}\mathcal{C}\right) \\ &\le r\left(\bigvee_{i=0}^{n} T^{-ki}\mathcal{C}\right) \times \cdots \times r\left(\bigvee_{i=0}^{n} T^{-ki-k+1}\mathcal{C}\right); \end{aligned}$$

but the k terms of the last expression are equal, because $r(\mathcal{C}) = r(T^{-1}\mathcal{C})$, and the first is equal to $c_{T^k}(\mathcal{C}, n)$. Thus $c_T(\mathcal{C}, kn + k - 1) \le c_{T^k}(\mathcal{C}, n)^k$ and if $\mathcal{C}$ has unbounded complexity under T the same is true under T^k. In particular, this applies to all standard covers, so that 2-scattering for T implies 2-scattering for T^k.

There remains to prove that any 2-scattering system is transitive. If (X, T) is not transitive let A, B be two non-empty open sets such that $A \cap T^{-n}B = \emptyset$ holds for any $n > 0$.

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1st case. Suppose $A \cap B \neq \emptyset$. Replacing A and B by their intersection we can assume $A = B$. Then $\mathcal{C} = (A^c, T^{-1}A^c)$ is a non-trivial closed cover: if the orbit of some point $x \in X$ hits $T^{-1}A$ at time n it hits A at time $n+1$, but it can belong to either of these two sets at any other time. The two names $010101\ldots$ and $101010\ldots$ thus cover X.

2nd case. Suppose now $A \cap B = \emptyset$.

(a) If there is $n > 0$ with $B \cap T^{-n}A \neq \emptyset$ let m be the minimum of such values. Then if $E = B \cap T^{-m}A$ one has $E \cap T^{-n}E = \emptyset$ for $n > 0$ so that we are in the first case again.

(b) If for any $n > 0$ one has $B \cap T^{-n}A = A \cap T^{-n}B = \emptyset$, putting $A^c = C_0$, $B^c = C_1$, $\mathcal{C} = (A^c, B^c)$ the two $\mathcal{C}$-names $000\ldots$ and $111\ldots$ cover X. □

The converse of Proposition 3.10 is false: irrational rotations of the 1-torus are totally transitive and not 2-scattering.

An equicontinuous system is one for which all points are equicontinuity points. Naturally enough the existence of an equicontinuity point in a system that is not equicontinuous has some consequences in terms of complexity, but these are not as clear as in the global case. Existence of an equicontinuity point contradicts weak mixing, so that the following statement is a consequence of Proposition 3.6.

PROPOSITION 3.11. *Suppose x is an equicontinuity point of (X, T). Then X is not weakly mixing and for any $y \in X$, $y \neq x$, there is a standard cover $\mathcal{C}$ separating x and y and with complexity $c(\mathcal{C}, n) \leq n + 1$.*

Proof. Let $x \in X$ be an equicontinuity point. Fix $y \neq x$, $\epsilon > 0$ such that $d(x, y) > 4\epsilon$ and the corresponding $\eta > 0$; without loss of generality assume $\eta \leq \epsilon$.

Call A the closed ball $B(x, \eta)$ and B the closed ball $B(y, \epsilon)$. One has $d(A, B) > 2\epsilon$ and $\mathcal{R} = (A^c, B^c)$ forms a standard cover of X, separating x and y. An essential feature of A and B is that for any $n > 0$, at least one of the sets $A \cap T^{-n}B$ and $A \cap T^{-n}A$ must be empty: for instance if $A \cap T^{-n}B \neq \emptyset$, by equicontinuity $d(T^n x, B) < \epsilon$; again by equicontinuity of x and because $d(A, B) > 2\epsilon$ one has $A \cap T^{-n}A = \emptyset$.

A sufficient condition for non-weak mixing is thus satisfied. Then just as in the proof of Proposition 3.6, one can find a sub-cover of $\mathcal{R}_0^{n-1}$ of cardinality at most $n + 1$. □

An easy consequence is that weak mixing implies sensitivity to initial conditions: recall that a transitive, non-sensitive system has equicontinuity points [**AAB**].

If there exists a cover with bounded complexity, this does not imply that there exists an equicontinuity point. A Sturmian symbolic system is expansive and cannot have equicontinuity points; but it also has infinitely many standard covers with bounded complexity (all those inherited from the rotation it has as a factor).

In the setting of Proposition 3.11, if one wishes to have a cover with bounded complexity separating x from some other point it is not enough to suppose that x is an equicontinuity point, i.e. that any point near enough x shadows its orbit; there must also be some assumption about what happens at some distance from x. The following example illustrates this point.

Example 3.12. Consider the homeomorphism $f : f(x) = x^2$ of the unit interval. The map f is equicontinuous everywhere except at one. Choose any cover $\mathcal{C} = (A, B)$, where

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$A^c = I_1 = [0, h)$ and $B^c = I_2$ is any open interval disjoint from I_1. Suppose the orbit of x_1 enters I_1 at time i_1, necessarily staying in B hereafter; there exists x_2, the orbit of which enters I_2 at time $i'_2 > i_1$ and I_1 at time i_2; any of its $\mathcal{C}$-names must have an A at time i'_2 and Bs after i_2, which implies that it is different from any $\mathcal{C}$-name of x_1. An easy induction based on this remark permits one to construct an infinite sequence of points $(x_n), n \in \mathbb{N}$, none of which may have the same $\mathcal{C}$-names. Therefore $\mathcal{C}$ has infinite complexity. However, this system is not transitive, which by Proposition 3.10 implies it is not 2-scattering.

Scattering does not imply that there exist no equicontinuity points [**AG**]. The same example shows that scattering systems are not necessarily sensitive to initial conditions (see §6): a transitive sensitive system has no equicontinuity points [**AAB**].

A class of transitive, non-equicontinuous systems with equicontinuity points is described in [**GW1, GW2**, Proposition 1.5]. Systems with this set of properties cannot be minimal, and no invariant measure with full support can exist on them [**GW1**, Theorem 1.3], and there exists a subsequence of (T^n) converging uniformly to the identity [**GW1**, Lemma 1.2]. Any system belonging specifically to the aforementioned class of examples has a surprising property: its cartesian product with some minimal weakly mixing system is not transitive.

4. *Covers with bounded complexity and a disjointness theorem*

Scattering and other definitions based on complexity may look rather abstract; they can also be hard to check for particular systems. Can one find a characterization which would be at once more familiar to dynamists and easier to manipulate? The next proposition answers this question: a system is scattering if and only if its cartesian product with *any* minimal system is transitive.

PROPOSITION 4.1. *For a transitive system (X, T) the following properties are equivalent:*

(1) *(X, T) is not scattering;*

(2) *there exists a minimal system (Y, S) such that $(X \times Y, T \times S)$ is not transitive. When such a system exists there is a minimal homeomorphism with the same property;*

(3) *there exists a minimal system (Y, S), a closed $T \times S$-invariant proper subset J of $X \times Y$ and an integer $N > 0$ such that $\bigcup_{0 \le n < N} (\mathrm{Id} \times S^n) J = X \times Y$. When such a system (Y, S) exists there is a minimal subshift with the same property.*

Proof. (1) $\Rightarrow$ (3). Assume $\mathcal{C} = \{C_1, \dots, C_k\}$ is a non-trivial closed cover of k elements with bounded complexity. Let $\Omega = \{1, \dots, k\}^{\mathbb{N}}$ be endowed with the shift σ. For $\omega \in \Omega$ let $J^*(\omega)$ be the closed set of all points $x \in X$ such that ω is a $\mathcal{C}$-name of x.

Since $\mathcal{C}$ has bounded complexity, by Lemma 2.1 there exist $m \ge 1$ and $\omega_1, \dots, \omega_m \in \Omega$ such that

$$\bigcup_{j=1}^{m} J^*(\omega_j) = X. \tag{2}$$

Call H the set of m-tuples $\underline{\omega} = (\omega_1, \dots, \omega_m)$ for which (2) holds. It is closed: suppose the sequence $(\underline{\omega}^{(n)}), n \in \mathbb{N}$, converges to $\underline{\omega}$ and each of its elements satisfies (2). As in §1 call $J^*_n(\omega)$ the set of points of X for which $\omega(0, n-1)$ is a $\mathcal{C}$-name; fix $j > 0$: when

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n is large enough ω_i and $\omega_i^{(n)}$ have the same first j coordinates, whatever $i \in \{1, \ldots, m\}$, and for the same values of n one has $J_j^*(\omega_i) = J_j^*(\omega_i^{(n)})$; hence $\bigcup_{i=1}^m J_j^*(\omega_i) = X$. This is true for all $j > 0$; passing to the limit (2) is seen to hold for $\underline{\omega}$. Let S be the joint shift $\sigma \times \cdots \times \sigma$: one easily checks that $SH \subset H$, so that the intersection $H' = \bigcap_{n=0}^\infty S^n H$ is a closed, invariant (i.e. $SH' = H'$) subset of H.

Therefore there exists $\underline{\omega} = (\omega_1, \ldots, \omega_m) \in H'$ having a minimal closed orbit Σ under S; (2) holds for any $\underline{\alpha} \in \Sigma$ because H' is closed and invariant. Denote by Y_i the invariant minimal closed orbit of ω_i under σ; the set Σ is a closed S-invariant subset of $\prod_{1 \le i \le m} Y_i$. For $1 \le i \le m$ put

$$K_i = \{(\underline{\alpha}, x) \in \Sigma \times X : x \in J^*(\alpha_i)\}.$$

Each K_i is closed in $\Sigma \times X$: this relies on the fact that $\mathcal{C}$ is a closed cover; as (2) holds for $\underline{\alpha} \in \Sigma$, one has $\bigcup_{i=1}^m K_i = \Sigma \times X$. For these reasons one of the K_i has non-empty interior. Assume it is K_1: this means there is a non-empty open set U of X, and for each i a non-empty open set $A_i \subset Y_i$ such that

$$Z = \Sigma \cap \left(\prod_{i=1}^m A_i \right) \neq \emptyset \quad \text{and} \quad \forall \underline{\alpha} = (\alpha_1, \ldots, \alpha_m) \in Z, \quad U \subset J^*(\alpha_1).$$

Z is defined as a non-empty open subset of Σ. Since (Σ, S) is minimal, there exists an integer N such that

$$\forall \underline{\alpha} \in \Sigma, \quad \exists k \in \{0, \ldots, N-1\}, \quad \text{with } S^k \underline{\alpha} \in Z,$$

which means by definition of Z that

$$\forall \omega \in Y_1, \quad U \subset \bigcup_{k=0}^{N-1} J^*(\sigma^k \omega);$$

remarking that Y_1 is invariant and that $TJ^*(\omega) \subset J^*(\sigma\omega)$, one obtains

$$\forall \omega \in Y_1, \quad \forall n \in \mathbb{N}, \quad T^n U \subset \bigcup_{k=0}^{N-1} J^*(\sigma^k \omega).$$

The open set U contains a transitive point, therefore $\bigcup_{n \in \mathbb{N}} T^n U$ is dense in X.

Put $Y = Y_1$. The set $\bigcup_{k=0}^{N-1} J^*(\sigma^k \omega)$ is equal to X: it is closed because $\mathcal{C}$ is a closed cover, and it contains the dense set $\bigcup_{n \in \mathbb{N}} T^n U$. The set $J' = \{(x, \alpha) \in X \times Y : x \in J^*(\alpha)\}$ is closed and $(T \times \sigma)$-invariant; suppose x belongs to the non-empty set C_i^c, where $C_i \in \mathcal{C}$: then $(x, \alpha) \notin J'$ whenever $\alpha(0) = i$, and consequently J' is a proper subset of $X \times Y$.

$(3) \Rightarrow (2)$. By Baire's theorem the interior of J is non-empty; it is an open non-dense invariant set. Taking the natural extension of (Y, S) we get a minimal homeomorphism with the same property.

$(2) \Rightarrow (3)$. Let $U \neq \emptyset$ be invariant, open and non-dense in $X \times Y$, and put $J = \overline{U}$. The projection of U to X is a non-empty open set; it contains a transitive point x_0. $J(x_0)$ is closed with non-empty interior in the minimal set Y, so that there is an integer N such that $\bigcup_{0 \le n < N} S^n J(x_0) = Y$. Let K be the closed invariant set $\bigcup_{0 \le n < N} (\mathrm{Id} \times S^n) J$: then $K(x_0) = Y$; since x_0 is transitive $K(x) = Y$ for all $x \in X$ and $K = X \times Y$.

(3) $\Rightarrow$ (1). Fix $y_0 \in Y$. Since $J \neq X \times Y$ and Y is minimal one has $J^*(y_0) \neq X$. This implies there exists a closed neighbourhood A of y_0 such that $J^*(A) \neq X$; put $F = J^*(A)$. By hypothesis one has $\bigcup_{0 \leq n < N} T^n J^*(y_0) = X$, thus $\mathcal{C} = (F, TF, \ldots, T^{N-1}F)$ is a non-trivial closed cover of X.

Choose M big enough for $\mathcal{A} = (A, SA, \ldots, S^{M-1}A)$ to cover Y, and let ω be an $\mathcal{A}$-name of y_0. We can suppose $N = M$. Let $x \in X$; there is n, $0 \leq n \leq N$, such that $x \in J^*(S^n y_0)$. For any $k \in \mathbb{N}$ one has $S^{n+k} y_0 \in S^{\omega_{n+k}} A$, whence

$$T^k x \in J^*(S^{n+k} y_0) \subset J^*(S^{\omega_{n+k}} A) = T^{\omega_{n+k}} F;$$

this means that $\sigma^n \omega$ is a $\mathcal{C}$-name of x, and the complexity of the cover $\mathcal{C}$ is bounded by N. □

It is worth noting that in fact the sets J and J' are joinings.

Furstenberg proved that weakly mixing systems are disjoint from minimal distal systems [**Fu**, Theorem II.3]. In his proof he only uses weak mixing through the fact that the cartesian product of a weakly mixing system with any minimal system is transitive [**Fu**, Proposition II.11]. Now Proposition 4.1 tells us that (X, T) is scattering if and only if its cartesian product with any minimal system is transitive (which by the way proves again that weak mixing implies scattering). A natural consequence is the following result.

PROPOSITION 4.2. *Scattering systems are disjoint from all minimal distal systems.*

Proof. Minimal distal systems form the smallest class containing the trivial system and stable under factor maps, minimal isometric extensions and inverse limits. The trivial system is disjoint from all others; disjointness from (X, T) is preserved under factor maps and inverse limits. We only have to prove that the disjointness property passes on to minimal isometric extensions: this is done in Lemmas 4.3 and 4.4. □

The two following statements are implicit in Furstenberg's proof. Our proof of Lemma 4.3 is based on the isometric extension property, whereas the one he gives relies on the properties of group extensions; the two are of course equivalent.

LEMMA 4.3. *Let $\pi : (Y, S) \to (X, T)$ be an isometric extension. If there is an invariant closed set $F \subset Y$ with $F \neq Y$ and $\pi(F) = X$, then (Y, S) is not transitive.*

Proof. For $\epsilon > 0$ put

$$F_\epsilon = \{y \in Y : \exists z \in F, \pi(z) = \pi(y) \text{ and } d(y, z) \leq \epsilon\}.$$

F_ϵ is closed in Y, and invariant because the extension is isometric; as $F \neq Y$ one can choose ϵ small enough for $F_\epsilon \neq Y$. Then its interior is a non-dense open invariant set; we must show it is not empty.

For $x \in X$ put $F(x) = \pi^{-1}(x) \cap F$. Recall that an ϵ-separated subset in $F(x)$ is a set $\{y_1, \ldots, y_k\} \subset F(x)$ with $d(y_i, y_j) \geq \epsilon$ whenever $1 \leq i \neq j \leq k$. Call $N(x)$ the maximum cardinality of an ϵ-separated subset in $F(x)$. For any x one has $F(x) \neq \emptyset$ and $1 \leq N(x) \leq \infty$; moreover since F is closed the set $U_k = \{x \in X : N(x) \leq k\}$ is open for any integer k. Choose the smallest value k such that $U_k \neq \emptyset$ and put $U_k = U$; thus U is a non-empty open set and $N(x) = k$ for $x \in U$.

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CLAIM. *Given* $\epsilon > 0$*, for any* $x \in U$ *there is a neighbourhood* V *of* x*,* $V \subset U$*, such that* $\forall y \in F(x), \forall x' \in V, \exists z \in F(x')$ *with* $d(y, z) < \epsilon$*.*

Assume the claim is not true: there are $x \in U$, a sequence (x_n) in U converging to x, and for any n a point $y_n \in F(x)$ with $d(y_n, z) \geq \epsilon$ for all $z \in F(x_n)$. For each n let $\{z_{n,1}, \ldots, z_{n,k}\}$ be an ϵ-separated subset of $F(x_n)$. By taking a subsequence one may suppose that for each $i \in \{1, \ldots, k\}$ the sequence $(z_{n,i})$ converges to some $z_i \in F(x)$, and that (y_n) converges to $y \in F(x)$. We assumed that $d(y_n, z_{n,i}) \geq \epsilon$ for all i, n, thus $d(y, z_i) \geq \epsilon$; we also assumed that $d(z_{n,i}, z_{n,j}) \geq \epsilon$ for $1 \leq i \neq j \leq k$, this implies $d(z_i, z_j) \geq \epsilon$. Thus $\{y, z_1, \ldots, z_k\}$ is an ϵ-separated subset of $F(x)$, which contradicts the definition of U: this proves the claim.

Let $y \in F$ with $\pi(y) \in U$, and let V be the neighbourhood associated to $\pi(y)$ for the value $\epsilon/2$ by the claim. There is a neighbourhood $W \subset B(y, \epsilon/2)$ of y such that $\pi(W) \subset V$. By the claim for $y' \in W$ there is a point $z \in F(\pi(y'))$ with $d(y, z) < \epsilon/2$, whence $d(y', z) < \epsilon$ and $y' \in F_\epsilon$. Thus $W \subset F_\epsilon$, so that the interior of F_ϵ is not empty and (Y, S) is not transitive. □

LEMMA 4.4. *Suppose* (X, T) *is scattering and let* $\pi : (Y, S) \to (Z, R)$ *be an isometric extension with* (Y, S) *minimal. If* (X, T) *and* (Z, R) *are disjoint so are* (X, T) *and* (Y, S)*.*

Proof. Let $J \subset X \times Y$ be any joining. $(\mathrm{Id} \times \pi)(J)$ is a joining of (X, T) and (Z, R) so by disjointness it is equal to $X \times Z$. On the other hand $\mathrm{Id} \times \pi : (X \times Y, T \times S) \to (X \times Z, T \times R)$ is an isometric extension. By Proposition 4.1 $(X \times Y, T \times S)$ is transitive; applying Lemma 4.3 one gets $J = X \times Y$. □

Remark 4.5. With the help of Proposition 4.1 above and the proof of [**GW2**, Lemma 2.2] it is not too hard to obtain the following result: any scattering system with an equicontinuity point contains only one minimal system which is a fixed point.

5. *Complexity pairs and the regionally proximal relation*

Let us define a relation between points of X. A couple of points (x, y) in $X^2 \setminus \Delta_X$ is said to be a *complexity pair* if for every standard cover $\mathcal{C}$ that separates them, $c(\mathcal{C}, n)$ tends to infinity. We denote by $\mathrm{Com}(X, T)$ the set of complexity pairs of (X, T). This notion is analogous to that of entropy pair introduced in [**Bl2**]. One easily sees that (X, T) is 2-scattering if and only if all couples of different points in X are complexity pairs. An entropy pair (x, y), i.e. one such that any standard cover separating x and y has exponentially growing complexity, is a complexity pair.

In the next proposition we gather all of the relevant facts about covers with unbounded complexity and complexity pairs. The proofs are left to the reader: they are practically identical to those of analogous properties of entropy pairs in [**Bl2**].

PROPOSITION 5.1. *Let* (X, T) *be a dynamical system.*

(1) *Let* $\mathcal{C}$ *be an open cover with unbounded complexity. Then there exists a standard cover with unbounded complexity.*
(2) *Let* $\mathcal{C} = (U, V)$ *be an open cover (not necessarily standard). If* $c(\mathcal{C}, n)$ *is not bounded there is a complexity pair in* $U^c \times V^c$*.*

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(3) *The set* $\mathrm{Com}(X,T)\cup\Delta_X$ *is closed.*

(4) *Let* $\pi : (X,T)\to(Y,S)$ *be a factor map. If* $(x,x')\in\mathrm{Com}(X,T)$ *and* $\pi(x)\neq\pi(x')$*, then* $(\pi(x),\pi(x'))\in\mathrm{Com}(Y,S)$*. If* $(y,y')\in\mathrm{Com}(Y,S)$*, then there is* $(x,x')\in\mathrm{Com}(X,T)$ *such that* $\pi(x)=y$*,* $\pi(x')=y'$*. In particular,* $\mathrm{Com}(X,T)\cup\Delta_X$ *is* $T\times T$*-invariant.*

In view of Proposition 5.1(2) observe that many systems have non-standard covers with unbounded complexity.

Example 5.2. Suppose X has no isolated points and (X,T) is transitive but not minimal with T invertible; let x be a transitive point. Using the invertibility assumption it is not hard to prove that if all preimages of x entered any of its neighbourhoods in bounded time then (X,T) would be minimal; so there exists some neighbourhood V of x such that the entrance time for preimages of x is unbounded: more precisely for any $n>0$ there are infinitely many points y in the negative orbit of x such that $T^j y\notin V$, $j\leq n$. Put $U=\{x\}^c$ and $\mathcal{C}=(U,V)$. Assume that for some positive n_k there are already k points in $\{T^{-i}x, i\leq n_k\}$ such that two of them can never be in the same open set of the cover $\mathcal{C}_0^{n_k}$; there exists $j>n_k$ such that $T^i(T^{-j}x)\notin V$ for $0\leq i\leq n_k$. Setting $n_{k+1}=j$ and adding $T^{-j}x$ to our collection we now have $k+1$ points in $\{T^{-i}x, i\leq n_{k+1}\}$ such that two of them can never be in the same element of the cover $\mathcal{C}_0^{n_{k+1}}$. Therefore the non-standard open cover $\mathcal{C}$ has unbounded complexity.

Remark 5.3. The relation $\mathrm{Com}(X,T)\cup\Delta_X$ is closed and invariant but not, in general, an equivalence relation: see Example 5.9 below.

A first consequence of the properties of complexity pairs is a disjointness theorem; it is not contained in Proposition 4.2, under the condition that 2-scattering and scattering are distinct properties.

PROPOSITION 5.4. *2-scattering dynamical systems are disjoint from minimal equicontinuous dynamical systems.*

Proof. Again the proof is close to that of [**Bl2**, Proposition 6]. We show that if (X,T) is 2-scattering and (Y,S) is minimal, and there is a non-trivial joining J between the two, there exists a complexity pair in (Y,S): then by Proposition 2.2 (Y,S) cannot be equicontinuous.

Call $J_*(x)$ the closed set $\{y\in Y : (x,y)\in J\}$. Suppose there are $x\neq x'\in X$ such that $J_*(x)\cap J_*(x')=\emptyset$. Let π and π' be the projections of J on X and Y. As $(x,x')\in\mathrm{Com}(X,T)$ by Proposition 5.1(4) applied to π, there exist $y,y'\in Y$ such that $((x,y),(x',y'))\in\mathrm{Com}(J,T\times S)$: our assumption that $J_*(x)\cap J_*(x')=\emptyset$ ensures that $y\neq y'$. Then, by Proposition 5.1(4) applied to π' in the reverse direction (y,y') is a complexity pair.

So we must only show that there exist $x,x'\in X$ with $J_*(x)\cap J_*(x')=\emptyset$. We can assume J is minimal as a joining, i.e. no proper subset of J is a joining of X and Y. Now suppose $J_*(x)\cap J_*(x')\neq\emptyset$ for $x,x'\in X$; consider the subset of J:

$$J'=\bigcup_{x\in X}\{x\}\times(J_*(x)\cap J_*(Tx))=\bigcup_{x\in X}\{x\}\times(J_*(x)\cap S(J_*(x)))=J\cap(\mathrm{Id}\times S)(J).$$

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One easily checks that it is closed and invariant, and by our assumption $J'_*(x) \neq \emptyset$ for $x \in X$, so $\pi(J') = X$; by minimality of (Y, S), $\pi'(J') = Y$. If one had $J' = J$ this would mean that for every x $J_*(x) = J_*(Tx)$, and as a non-empty closed invariant subset of the minimal set Y, the set $J_*(x)$ would be Y itself, hence $J = X \times Y$. Thus J' is a proper subjoining of J, so J cannot be minimal as a joining. So there has to be $x \in X$ with $J_*(x) \cap J_*(x') = \emptyset$, which finishes the proof. □

A transitive dynamical system on which there exists an invariant measure with full support is called an *E-system*.

PROPOSITION 5.5. *For an E-system 2-scattering is equivalent to scattering and also to disjointness from all minimal equicontinuous systems.*

Proof. Let (X, T) be an E-system. By Proposition 4.1 it is scattering if and only if its product with any minimal homeomorphism (Y, S) is transitive. Following the proof of [**Au**, Theorem 7, Chapter 11] such a cartesian product is transitive whenever the maximal equicontinuous factor of (Y, S) is disjoint from (X, T) (see the Appendix). This obviously holds when (X, T) is disjoint from all minimal equicontinuous systems, and this is true for 2-scattering systems by Proposition 5.4. □

Question 5.6. Here are, in fact, two in a series of natural questions, some of which are distinct only if scattering and 2-scattering are different properties. Is 2-scattering also a *necessary* condition for a transitive system to be disjoint from all minimal isometries? Is scattering a necessary condition for a transitive system to be disjoint from all minimal distal systems?

Let us now examine the connection between $\mathrm{Com}(X, T)$ and the well-known maximal equicontinuous factor of a dynamical system.

PROPOSITION 5.7. *Let* $\mathrm{Com}_0(X, T)$ *be the smallest closed invariant equivalence relation containing* $\mathrm{Com}(X, T) \cup \Delta_X$*, and* $\pi_0 : (X, T) \to (X_0, T_0)$ *be the factor map obtained by identifying points in the same equivalence class of this relation. Then* $\pi_0 : (X, T) \to (X_0, T_0)$ *is the maximal equicontinuous factor of* (X, T).

Proof. First (X_0, T_0) is equicontinuous. Suppose it is not. By Proposition 2.2 it has an open cover with unbounded complexity, then by Proposition 5.1(1) and (2), there exists a complexity pair for (X_0, T_0), i.e. by Proposition 5.1(4), two distinct points such that there is a complexity pair in their preimage for π_0. This is a contradiction: π_0 collapses all complexity pairs of X^2.

Now assume that $\pi : (X, T) \to (Y, S)$ is another equicontinuous factor, then π must collapse all complexity pairs of (X, T) and the corresponding equivalence relation must contain the one associated to π_0, which means π factors through π_0. □

For a homeomorphism (X, T) the *regionally proximal* relation is defined as

$$\mathrm{RP}(X, T) = \{(x, y) \in X^2 : (\forall U_x)(\forall U_y)(\forall \epsilon > 0) \\ (\exists x' \in U_x)(\exists y' \in U_y)(\exists n \in \mathbb{Z}) d(T^n(x'), T^n(y')) \leq \epsilon\}$$

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such that a pair (x, y) is regionally proximal if given $\epsilon > 0$ one can find in any neighbourhood of (x, y) a pair (x', y') such that the orbits of x' and y' get within ϵ of each other for some $n \in \mathbb{Z}$. This relation is closed and invariant, and in the minimal case it is an equivalence relation; the maximal equicontinous factor is the factor induced by the smallest closed invariant equivalence relation containing $\mathrm{RP}(X, T)$ [**Au**].

There are strong links between complexity pairs and regionally proximal pairs. When a pair is not regionally proximal, one can find a standard cover with bounded complexity separating it.

PROPOSITION 5.8. *Let (X, T) be a homeomorphism. Then* $\mathrm{Com}(X, T) \subseteq \mathrm{RP}(X, T)$.

Proof. Suppose that $(x, y) \notin \mathrm{RP}(X, T)$. Then there are neighbourhoods U_x and U_y of x and y, respectively, and $\epsilon > 0$ such that for every $x' \in U_x$, $y' \in U_y$ and $n \in \mathbb{Z}$, $d(T^n(x'), T^n(y')) > \epsilon$. By compactness, there is a cover $(C_1, \dots, C_k)$ with the following properties:

(i) C_i, $i = 1, \dots, k$, is closed with interior and $\mathrm{diam}(C_i) \le \epsilon/2$;
(ii) $x \in \mathrm{Int}(C_1)$ and $y \in \mathrm{Int}(C_k)$;
(iii) $C_1 \subset \overline{U}_x$ and $C_k \subset \overline{U}_y$.

We claim that for $i \in \{1, \dots, k\}$ and $n \in \mathbb{N}$, either $T^n(C_i) \cap C_1 \neq \emptyset$ or $T^n(C_i) \cap C_k \neq \emptyset$ but not both: if for some $i \in \{1, \dots, k\}$ there are two points $z, z' \in C_i$ and $n \in \mathbb{N}$ such that $T^n(z) \in C_1$ and $T^n(z') \in C_k$ then $d(T^{-n}(T^n(z)), T^{-n}(T^n(z'))) \ge \epsilon$; this contradicts the fact that $\mathrm{diam}(C_i) \le \epsilon/2$. Put $\mathcal{C} = (C_1^c, C_k^c)$: one can thus find a unique infinite $\mathcal{C}$-name for each set C_i. Then the standard cover $\mathcal{C}$, which by (ii) separates x from y, has complexity bounded by k and (x, y) cannot be a complexity pair. □

The converse inclusion is not, in general, true. The following property helps us prove this claim.

LEMMA 5.9. *Let (X, T) be a transitive homeomorphism containing a fixed point a. Then* $\mathrm{RP}(X, T) = X^2$.

Proof. Let x_0 and y_0 be two distinct points of X, and let U be a neighbourhood of x_0 and V be a neighbourhood of y_0. There is a transitive point x in U, and there is $k \in \mathbb{Z}$ with $y = T^k x \in V$. There is a sequence (n_j) of integers such that $T^{n_j} x \to a$. Thus $T^{n_j} y \to T^{n_j} a = a$ and $d(T^{n_j} x, T^{n_j} y) \to 0$. □

Here is a very simple symbolic system for which $\mathrm{Com}(X, T)$ is not an equivalence relation and $\mathrm{Com}(X, T) \neq \mathrm{RP}(X, T)$.

Example 5.10. Consider the transitive sofic subshift $S \subset \{0, 1\}^{\mathbb{Z}}$ defined by the condition that there is an odd number of zeros between any two occurrences of one. Put $A = \{\omega \in S / \omega_{2n} = 1 \text{ for some } n \in \mathbb{Z}\}$, $B = \{\omega \in S / \omega_{2n+1} = 1 \text{ for some } n \in \mathbb{Z}\}$ and let $\mathcal{C}$ be the closed cover (A^c, B^c). One easily checks that $c_n(\mathcal{C}) = 2$ for all n, and deduces from this that one point in A and one point in B can never form a complexity pair. On the other hand S is a factor of the cartesian product $(\{0, 1\}^{\mathbb{Z}}, \sigma) \times (\mathbb{Z}/2\mathbb{Z}, +1)$; applying Proposition 4.1(4) it is easy to check that any standard cover separating the unique fixed point on the letter zero from any other point has infinite complexity. These two facts prove that $\mathrm{Com}(S, \sigma) \cup \Delta_S$

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is not transitive. Lemma 5.9 shows that $\mathrm{RP}(S, \sigma) = X^2$. Thus $\mathrm{Com}(S, \sigma)$ is a strict subset of $\mathrm{RP}(S, \sigma)$.

This example also shows that systems with $\mathrm{RP}(X, T) = X^2$ are not necessarily disjoint from group rotations: (S, σ) has an obvious joining with the exchange of two points.

In the minimal case no system with these properties exists.

PROPOSITION 5.11. *If (X, T) is a minimal homeomorphism,* $\mathrm{Com}(X, T) = \mathrm{RP}(X, T)$.

Proof. By Proposition 5.7 it is enough to prove that $\mathrm{RP}(X, T) \subset \mathrm{Com}(X, T)$. Let $x_1 \neq x_2$ be in X and assume that $(x_1, x_2) \notin \mathrm{Com}(X, T)$. Let (U_1, U_2) be a standard cover of bounded complexity separating these points and put $F_1 = \overline{U_1}$, $F_2 = \overline{U_2}$; the cover $\mathcal{R} = (F_1, F_2)$ also has bounded complexity. Put $\Omega = \{1, 2\}^{\mathbb{Z}}$ and let J (the joining associated to $\mathcal{R}$-names), $J^*(\omega)$ (the sets of points in X having the name ω) and $J_*(x)$ (the set of names of x) be as defined in §1.

The cover $\mathcal{R}$ has bounded complexity, which by Lemma 2.1 implies that X is covered by a finite union of sets $J^*(\omega)$. These sets are closed and at least one of them must have non-empty interior. Thus there are $\alpha \in \Omega$ and a non-empty open set $W \subset X$ with $W \subset J^*(\alpha)$. Put

$$K = \{(x, y) \in X^2 : J_*(x) \cap J_*(y) \neq \emptyset\} = \bigcap_{n \in \mathbb{Z}} T^n \times T^n((F_1 \times F_1) \cup (F_2 \times F_2)).$$

K is a closed $T \times T$-invariant subset of X^2. If x and y are in W they have a common name α; thus $W \times W \subset K$, which means that K has a non-empty interior. Finally, by invariance one also has $T^n W \times T^n W \subset K$ for any n; this fact and minimality imply that K is a neighbourhood of the diagonal of X^2: there is $\epsilon > 0$ such that

$$d(x, y) < \epsilon \Longrightarrow (x, y) \in K. \tag{3}$$

Choose arbitrary points $y_1 \in F_2^c$ and $y_2 \in F_1^c$. Then $J_*(y_1) \subset [1]$ and $J_*(y_2) \subset [2]$, therefore $J_*(y_1) \cap J_*(y_2) = \emptyset$ and $(y_1, y_2) \notin K$. Since K is invariant, for every $n \in \mathbb{Z}$ one has $(T^n y_1, T^n y_2) \notin K$, hence by (3) $d(T^n y_1, T^n y_2) \geq \epsilon$; but F_2^c is a neighbourhood of x_1 and F_1^c is a neighbourhood of x_2: this implies that $(x_1, x_2) \notin \mathrm{RP}(X, T)$. □

Remark 5.12. In the minimal case, complexity pairs are exactly those that are collapsed by projection to the maximal equicontinuous factor. In the case of a Sturmian symbolic system this permits one to describe easily all complexity pairs, and to tell which standard covers have unbounded complexity; this would be a hard job by straight computation.

Remark 5.13. Suppose there is an ergodic measure μ with full support on (X, T), such that its Kronecker factor (its maximal factor with completely discrete spectrum) is trivial. Does this imply that (X, T) is scattering? The answer is almost obviously yes: a trivial Kronecker factor is equivalent to measure-theoretic weak mixing; this and full support imply weak topological mixing, which in turn implies that (X, T) is scattering by Proposition 3.4. On the contrary, it requires some real work to prove that any system on which there exists a K-measure with support has uniform positive entropy [**GW3**].

6. *Some remarks about determinism and chaos*

Here we very briefly address the most heuristic motivation of this article. One can consider equicontinuity as a good candidate for a mathematical definition of complete topological determinism: in an equicontinuous system when one knows precisely enough where a point lies, every point of its orbit is determined up to ϵ. This is especially patent for a symbolic system S. Suppose $p_S(n)$ is bounded; then when one knows a long enough block of coordinates of $x \in S$ this point is completely determined; when $p_S(n)$ is unbounded there are always blocks of coordinates with arbitrary length that can be prolonged by two different letters, hence containing at least two points. Granting that chaos is the opposite of determinism, by Proposition 2.2 it must have some relation with the complexity of covers.

In 1989 Devaney proposed to call chaotic any topological dynamical system that is transitive, sensitive to initial conditions, and contains a dense set of periodic points [**D**]; previously Auslander and Yorke had called chaotic all transitive sensitive systems and determined some properties of such systems [**AY**]. Several other tentative definitions of topological chaos are reviewed in [**KS**, §7]; although they are quite interesting we shall not examine them here.

It was later observed that transitivity and dense periodic points imply sensitivity [**BB, GW1**]. Sensitivity, nevertheless, remains a very interesting notion because it globally contradicts equicontinuity; it even does so locally in the transitive case: any transitive system that is not sensitive has equicontinuity points (actually coinciding with its transitive points) [**AAB**].

The scattering and 2-scattering properties also globally contradict equicontinuity. They do so in a way that is more closely connected with the point of view of Information Theory. By Proposition 3.10 both imply transitivity of all powers. Their rôle in the theory of chaos cannot be compared with that of sensitivity: scattering does not imply sensitivity [**AG**]; on the other hand sensitivity and transitivity do not even imply total transitivity.

Example 6.1. There exist non-scattering systems that are chaotic in the sense of Devaney. The cartesian product of a full shift and a 2-point set endowed with the exchange of points is obviously chaotic in this sense. It also has a clopen partition (C_0, C_1) with $TC_0 = C_1$ and $TC_1 = C_0$.

Another instance is the system (S, σ) of Example 5.10: it is also chaotic; the closed cover $\mathcal{C}$ is a partition except for the fixed point, and its elements also have period two. The following interval map f has the same property:

$$f(x) = 2x + \tfrac{1}{2} \quad \text{on } [0, \tfrac{1}{4}],$$
$$f(x) = -2(x - \tfrac{1}{2}) + \tfrac{1}{2} \quad \text{on } [\tfrac{1}{4}, \tfrac{3}{4}],$$
$$f(x) = 2(x - \tfrac{3}{4}) \quad \text{on } [\tfrac{3}{4}, 1].$$

In the three examples there exist almost continuous eigenfunctions—a not very chaotic feature that Devaney's definition does not exclude in general.

Let us now conclude with a very short discussion, partly inspired by the results in this article.

Actually, rather than competing definitions of chaos what one needs is a theory of

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chaoticity, inside which various degrees might be accepted. The really important point is not the definitions but their relations and fields of application.

Some definitions are *fitted to particular classes of transformations*. The existence of a dense set of periodic points certainly looks a real feature of chaos when dealing with interval maps (see for instance [**KS**]), and Devaney's definition is satisfactory in this class, if one does not mind its allowing the existence of eigenfunctions. It is possible that it is also valid for cellular automata. On the other hand, in the symbolic setting the assumption of dense periodic points excludes some systems that are highly chaotic in a sense, while accepting others that can be considered chaotic in a very weak sense only. For a less sketchy discussion of this hypothesis see [**GW1**].

There are indeed *several possible degrees of chaoticity* in the field of topological dynamical systems. One may suggest a set of weak definitions: for instance scattering (because it implies total transitivity and contradicts equicontinuity), or sensitivity and total transitivity (for similar reasons); these definitions are not equivalent. Weak mixing is strictly stronger than both previous definitions and may thus be considered as a mild degree of chaoticity. An example of a very strong definition is uniform positive entropy [**Bl1**]; there may be many others.

One should also distinguish between *requirements of complete or partial chaoticity*. An illustration of what we mean by partial chaos is to be found in [**GW1**], where the authors suggest that a chaotic system might be an E-system with positive entropy; these requirements do not exclude deterministic features but they ensure that there exist covers with exponentially growing complexity, and no equicontinuity points. A natural further step is to introduce various degrees of partial chaos.

Finally, one question: does the complexity of covers provide a scale of chaoticity? Presently the answer is far from obvious.

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Appendix

The following result generalizes [**Au**, Theorem 7, Chapter 11] to the transitive case. Auslander's proof is for the minimal case only.

PROPOSITION A.1. *Let (X, T) be a transitive system and μ be an invariant measure with full support. Let (Y, S) be a minimal homeomorphism. If (X, T) is disjoint from the maximal equicontinuous factor of (Y, S), then $(X \times Y, T \times S)$ is transitive.*

Proof. Assuming that $(X \times Y, T \times S)$ is not transitive we construct an equicontinuous factor (Z, τ) of (Y, S) and a non-trivial joining $(K, T \times \tau)$ of (X, T) and (Z, τ). It follows that (X, T) is not disjoint from (Z, τ), and *a fortiori* not disjoint from the maximal equicontinuous factor of (Y, S).

Step 1. (Construction of (Z, τ).) There exists an open invariant subset U of $X \times Y$, such that $F = \overline{U} \neq (X \times Y)$. By transitivity the projections of F to X and Y are onto: $(F, T \times S)$ is a non-trivial joining of (X, T) and (Y, S). For every $y \in Y$ put $F(y) = \{x \in X : (x, y) \in F\}$, and let $f_y = 1_{F(y)}$ be the characteristic function of $F(y)$, considered as an element of $L^1(\mu)$.

We show that the map $y \mapsto f_y$ is continuous from Y to $L^1(\mu)$. Fix $a \in Y$ and $\epsilon > 0$. There exists an open set V with $V \supset F(a)$ and $\mu(V \setminus F(a)) < \epsilon$. As F is closed, there exists a neighbourhood W of a such that $F(y) \subset V$ for every $y \in W$. Thus for $y \in W$ one has $\mu(F(y) \setminus F(a)) < \epsilon$. In particular, $\mu(F(y)) \leq \mu(F(a)) + \epsilon$; it follows that the map $y \mapsto \mu(F(y))$ is upper semicontinuous. A simple computation shows that it is S^{-1}-invariant, hence constant by minimality.

Let a, ϵ, V, W be as above. For every $y \in W$, $\mu(F(y) \setminus F(a)) < \epsilon$ and $\mu(F(y)) = \mu(F(a))$, thus $\mu(F(y) \Delta F(a)) < 2\epsilon$, i.e. $\|f_y - f_a\|_1 < 2\epsilon$: this proves our claim that f_y is continuous.

Denote by $\pi : Y \to L^1(\mu)$ the map $y \mapsto f_y$, and consider $Z = \pi(Y) \subset L^1(\mu)$, endowed with the topology of $L^1(\mu)$. As π is continuous, Z is compact. As F is invariant, we have $f_{S^{-1}y} = f_y \circ T$ and Z is invariant under the map $\tau : g \mapsto g \circ T$. Consider (Z, τ) as a dynamical system: then π is a factor map, and (Z, τ) is minimal. Moreover, this system is clearly isometric.

Step 2. Write $p = (\mathrm{Id} \times \pi) : (X \times Y) \to (X \times Z)$ and $K = p(F)$: p is a factor map and K is a joining of X and Z. We only have to prove that this joining is not trivial.

Let $(x, y) \in p^{-1}(p(U))$. There exists $y' \in Y$ with $(x, y') \in U$ and $\pi(y) = \pi(y')$ which implies that $f_y = f_{y'}$, i.e. $\mu(F(y) \Delta F(y')) = 0$. As μ has full support, $F(y)$ and $F(y')$ have the same interior, but x belongs to the interior of $F(y')$, hence to $F(y)$, and finally we get that $(x, y) \in F$: we have shown that

$$p^{-1}(p(U)) \subset F.$$

As $F \neq (X \times Y)$, there exists a non-empty open subset V of X and a non-empty open subset W of Y with $(V \times \overline{W}) \cap F = \emptyset$. By minimality,

$$\bigcup_{n \geq 0} S^n \overline{W} = Y, \quad \text{thus} \quad \bigcup_{n \geq 0} \tau^n \pi(\overline{W}) = Z$$

and by Baire's theorem the interior Ω of $\pi(\overline{W})$ is not empty. As $(V \times \overline{W}) \cap p^{-1}(p(U)) = \emptyset$, we have $(V \times \Omega) \cap p(U) = \emptyset$. Since $p(F) = \overline{p(U)}$, we get $(V \times \Omega) \cap p(F) = \emptyset$, and the joining $K = p(F)$ is not equal to $(X \times Z)$. □

With a few technical changes one can prove the same result without assuming that S is a homeomorphism.

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