A MAXIMUM PRINCIPLE FOR SUBHARMONIC AND PLURISUBHARMONIC FUNCTIONS

CHEN HUAIHUI AND P. M. GAUTHIER

ABSTRACT.   We give a simple description of boundary sets which may be ignored in calculating the maximum of subharmonic or plurisubharmonic functions.

Let $\bar{R}^n = R^n \cup \{\infty\}$ denote the one-point compactification of R^n and let Ω be an open subset of R^n. If s is a function defined on Ω and $p \in \partial\Omega$, we write

$$(1) \qquad\qquad \bar{s}(p) = \overline{\lim_{q \to p}} s(q).$$

In [4] we find the following characterization of those Ω for which the supremum of a subharmonic function is always equal to its supremum on the (finite) boundary of Ω.

THEOREM GGH.   *In order that*

$$\sup_{\Omega} s = \sup_{\partial\Omega \setminus \{\infty\}} \bar{s},$$

for each subharmonic function s on Ω, it is necessary and sufficient that ∞ not be accessible from Ω.

Sahakian [5] independently obtained a more precise result in the case of R^2 by showing that not only the point at infinity, but indeed, any closed inaccessible portion of the boundary, may be ignored when calculating the supremum.

THEOREM S.   *Let $\Omega \subset R^2$ and let E be a closed subset of $\partial\Omega$. In order that*

$$\sup_{\Omega} s = \sup_{\partial\Omega \setminus E} \bar{s},$$

for each subharmonic function s on Ω, it is necessary and sufficient that E is not accessible from Ω.

Actually, Theorem GGH was stated in a more general form on manifolds which, as was pointed out in [4], implies Theorem S. Our purpose in this note is to generalize Theorem S sufficiently to have the reverse implication. Moreover, our method will be very simple and different from that employed in [4].

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Let Ω be a manifold and $\bar{\Omega}$ a compactification of Ω. That is, $\bar{\Omega}$ is a compact Hausdorff space containing (a homeomorphic copy of) Ω as a dense open subset; the set $\partial\Omega = \bar{\Omega} \backslash \Omega$ is called the ideal boundary of Ω (relative to $\bar{\Omega}$). The compactification $\bar{\Omega}$ is said to be second countable if it has a countable basis. A subset E of the ideal boundary is said to be accessible (from Ω) if there is a path in Ω which is eventually in each neighbourhood of E, that is, there is a continuous mapping $\sigma \colon [0, \infty) \longrightarrow \Omega$ such that for each neighbourhood V of E, there is a $t' \in [0, \infty)$ such that $\sigma(t) \in V$, for each $t > t'$. A function s, defined on Ω and taking its values in the extended-reals is said to satisfy the local maximum principle if s is constant in a neighbourhood of each point at which s attains a local maximum. For $p \in \partial\Omega$, $\bar{s}(p)$ will be defined by formula (1). If s is upper semicontinuous on Ω, then the function defined as s on Ω and $\bar{s}$ on $\partial\Omega$ is the least upper semicontinuous extension of s to $\bar{\Omega}$.

THEOREM.   *Let $\bar{\Omega}$ be a second countable compactification of a manifold Ω. Suppose $E \subset \partial\Omega$ is not accessible from Ω. Then, for each upper semicontinuous function s on Ω, satisfying the local maximum principle, we have*

(*i*)
$$\sup_{\Omega} s = \sup_{\partial\Omega \backslash E} \bar{s}$$

and

(*ii*)
$$\sup_{E} \bar{s} = \overline{\lim_{\substack{p \longrightarrow E \\ p \in \partial\Omega \backslash E}}} \ \bar{s}(p)$$

COROLLARY.   *Let Ω be a Riemannian (respectively complex) manifold and let $\bar{\Omega}$ and E be as in the theorem. Then, (i) and (ii) hold for each function s subharmonic (respectively plurisubharmonic) on Ω.*

REMARK 1.   Maximum principles on certain unbounded domains also follow from Phragmén-Lindelöf-type theorems. In such theorems the functions under consideration are always assumed to be of restriced growth, however, in our theorem, there is no such restriction. In certain special situations, conclusions similar to the corollary can be easily obtained using arguments employing harmonic measure. Such arguments work only for bounded functions. Again, we do not assume that our functions are bounded. Rather, we infer that they are bounded, given that they are bounded along a certain portion of the boundary. Actually the maximum principle for upper semicontinuous functions satisfying the local maximum principle is equivalent to the maximum principle for **bounded** such functions. However, such is not the case for subharmonic or plurisubharmonic functions.

REMARK 2.   Our theorem has a converse. Namely, if E is closed and accessible from Ω, then it is trivial to construct an upper semicontinuous function satisfying the local maximum principle for which (i) and, consequently, (ii) fails. Thus, we have, in fact a complete characterization of closed boundary sets which can be disregarded in the maximum principle. If, moreover, Ω is a Riemannian manifold, then using recent results on extension and approximation [3], one can obtain such a function which is subharmonic and even [1] harmonic.

REMARK 3.   The hypothesis of our theorem is more general than in Theorem GGH, in which $\partial\Omega$, except ∞, is embedded in a manifold. Therefore, our theorem implies Theorem GGH, but the reverse implication is not valid and the method employed in [4] does not work for our case.

Proof of Theorem. Since (i) is an immediate consequence of (ii), we shall prove (ii) only. Set

$$M = \varlimsup_{\substack{p \to E \\ p \in \partial\Omega \setminus E}} \bar{s}(p)$$

We may assume that M is finite. Fix $\varepsilon > 0$. We shall show that

(2) $$\sup_{E} \bar{s} \leq M + \varepsilon.$$

We shall assume, at first, that E is closed. In order to study the behaviour of s near E, it is natural to construct a convenient neighbourhood system of E.

LEMMA 1.   *The compact set E has a countable neighbourhood system $V_1 \supset V_2 \supset \cdots$ such that $V_n \supset \bar{V}_{n+1}$, for $n = 1, 2, \ldots$, and there is a locally finite family β of open sets in Ω such that for each $n = 1, 2, \ldots$, $\Omega \cap V_n$ is a union of sets in β.*

Let A be a countable basis for the topology of $\bar{\Omega}$. Let A' be the family of sets $\bigcup_{i \in I} a_i$ which cover E, where $\{ a_i : i \in I \}$ are finite families in A. Then A' is a countable neighbourhood system of E, from which we can easily construct a nested countable neighbourhood system $G_1 \supset G_2 \supset \cdots$. Since $\bar{\Omega}$ is compact, we may further assume that $G_n \supset \bar{G}_{n+1}$, for $n = 1, 2, \ldots$.

Since $\bar{\Omega}$ is second countable, so is Ω and since, moreover, Ω is locally Euclidean, it is σ-compact and hence paracompact (see [2, 425X]). Thus, for $n = 2, 3, \ldots$, $\bar{G}_n \setminus \partial\Omega$ can be covered by a locally finite family β_n of parametric balls B such that $B \subset G_{n-1}$. By a parametric ball, we mean an open subset of Ω whose closure (in Ω) is homemorphic to a closed ball. Let $\beta = \bigcup_n \beta_n$ and for each $n = 1, 2, \ldots$, set

$$V_n = G_{n+1} \cup \bigcup\{ B : B \in \beta_{n+1} \}.$$

Then,

$$\Omega \cap V_n = \bigcup\{ B : B \in \beta_{n+1} \}$$

and the sets $V_1, V_2, \ldots$ have the properties affirmed in Lemma 1.

LEMMA 2. *If V is an arbitrary neighbourhood of E, then there is a smaller neighbourhood W of E, $W \subset V$, such that*

$$\sup_{\Omega \cap \partial W} s \leq M + \varepsilon.$$

Let $\{V_n\}$ be a neighbourhood system of E as in Lemma 1. We may assume that $V \supset V_1$ and that

$$(3) \qquad\qquad \sup_{\partial\Omega \cap \bar{V}_1 \setminus E} \bar{s} \leq M + \varepsilon/2$$

Set $H_1 = \Omega \cap V_1$ and denote by $X_{1,i}$, $i = 1, 2, \ldots$, those components of H_1 such that

$$\sup_{\Omega \cap \partial X_{1,i}} s \leq M + \varepsilon,$$

and by $Y_{1,j}$, $j = 1, 2, \ldots$, the other components. Thus, there exists $p_j \in \Omega \cap \partial Y_{1,j}$ with $s(p_j) > M + \varepsilon$. We claim that there are only finitely many $Y_{1,j}$. Indeed, if not, then there exists a sequence p_j which we may assume converges to some point $p \in \partial V_1$. Thus $p \notin E$ since $E \subset V_j$. Moreover, $p \notin \partial\Omega \setminus E$ by (3). Hence $p \in \Omega$. From Lemma 1, it follows that there is a neighbourhood of p which meets only finitely many components of H_1. This contradicts the statement that the sequence $\{p_j\}$ converges to p. Hence, there are only finitely many $Y_{1,j}$ as claimed. Set

$$H_2 = H_1 \cap V_2 \setminus \bigcup_i X_{1,i} = \Omega \cap V_2 \setminus \bigcup_i \bar{X}_{1,i}.$$

We redo the same procedure for H_2. That is, we divide the components into two classes $X_{2,i}$, $1, 2, \ldots$ and $Y_{2,j}$, $j = 1, 2, \ldots$ as above. We set

$$H_3 = H_2 \cap V_3 \setminus \bigcup_i X_{2,i} = \Omega \cap V_3 \setminus \bigcup_i \bar{X}_{2,i}.$$

We do this for each $n = 2, 3, \ldots$. Each component of H_n is also a component of $\Omega \cap V_n$ and so the components of H_n are a locally finite family. Thus, the same argument as at the first stage shows that there are at most finitely many $Y_{n,j}$ for each n.

Each $Y_{n+1,j}$ is contained in some component of H_n. But the $X_{n,i}$ have been removed. Thus, each $Y_{n+1,j}$ is contained in some $Y_{n,k}$. We say that $Y_{n+1,j}$ is a descendant of $Y_{n,k}$ in this case. Let us agree that a descendant of a descendant is also a descendant and so on to any number of generations.

If there are $Y_{n,j}$ for each stage n, then since there are only finitely many at the first stage, we may pick some $Y_{1,j}$, denote it by Y_1 such that Y_1 has infinitely many descendants. Now Y_1 has only finitely many descendants $Y_{2,j}$ at the second stage, and so we may choose one, denote it by Y_2, such that Y_2 has infinitely many descendants. Continuing in this manner, since at each stage there are at most finitely many $Y_{n,j}$, we construct a sequence $Y_1 \supset Y_2 \supset \cdots$, where each Y_n is some $Y_{n,j}$.

Now for each n, choose a point $q_n \in Y_n$ and construct a path which starts from q_1 and goes to q_2 within Y_1, then goes to q_3 within Y_2 etc. This path is eventually in each Y_n and, ipso facto, in each V_n. This contradicts the inaccessibility of E. Thus, at some stage m, there are no $Y_{m,j}$. Hence

$$H_m = H_{m-1} \cap V_m \setminus \bigcup_i \bar{X}_{m-1,i} = \bigcup_i X_{m,i}.$$

Set

$$W = V_m \cup \bigcup_{n<m} \bigcup_i X_{n,i}.$$

Then, W is a neighbourhood of E contained in V,

$$\Omega \cap W = \bigcup_{n\leq m} \bigcup_i X_{n,i},$$

and since $X_{n,i}$ form a locally finite family,

$$\Omega \cap \partial W = \bigcup_{n\leq m} \bigcup_i [\Omega \cap \partial X_{n,i}].$$

From the definition of the $X_{n,i}$, Lemma 2 follows.

By combining Lemma 1, Lemma 2, and the definition of M, we may construct a neighbourhood system of $E : W_1 \supset W_2 \supset \cdots$ such that

(4)
$$\sup_{(\bar{W}_1 \cap \partial\Omega) \setminus E} s \leq M + \varepsilon,$$

and for $n = 1, 2, \ldots$

$$W_n \supset \bar{W}_{n+1'}$$

and

(5)
$$\sup_{\Omega \cap \partial W_n} s \leq M + \varepsilon.$$

Since s satisfies the local maximum principle, it follows from (4) and (5) that

$$\sup_{\Omega \cap [W_n \setminus \bar{W}_{n+1}]} s \leq M + \varepsilon,$$

for $n = 1, 2, \ldots$. This proves (2).

Suppose now that E is not closed. Let W be an open neighbourhood of E such that

(6)
$$\sup_{(W \cap \partial\Omega) \setminus E} \bar{s} < M + \varepsilon,$$

and set

(7)
$$E(\varepsilon) = \left\{ p \in W \cap \partial\Omega : \bar{s}(p) \geq M + \varepsilon \right\}.$$

Then $E(\varepsilon)$ is a closed subset of E. By the result proved above for a closed set,

$$\sup_{E(\varepsilon)} \bar{s} \leq M + \varepsilon.$$

By (6), (7), and (8), (2) is also proved.

Let $\varepsilon \to 0$ in (2); we obtain (ii). This completes the proof of the theorem.

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Mathematics Department
Nanjing Normal University
Nanjing, China

Département de mathématiques et de statistique
Université de Montréal, C.P. 6128
Montréal, Quebec
H3C 3J7