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Asymptotic behaviour and symmetry of positive solutions to nonlinear elliptic equations in a half-space

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We consider the following equation:

$$-\Delta u = \lambda \frac{u}{d(x)^2} - d(x)^\theta f(u) \quad \text{in } \Omega,$$

where $d(x) = d(x, \partial\Omega)$, $\theta > -2$ and Ω is a half-space. The existence and non-existence of several kinds of positive solutions to this equation when $\lambda \leqslant \frac{1}{4}$, $f(u) = u^p (p > 1)$ and Ω is a bounded smooth domain were studied by Bandle, Moroz and Reichel in 2008. Here, we study exact the behaviour of positive solutions to this equation as $d(x) \to 0^+$ and $d(x) \to \infty$, respectively, and the symmetry of positive solutions when $\lambda > \frac{1}{4}$, Ω is a half-space and $f(u)$ is a more general nonlinearity term than u^p. Under suitable conditions for f, we show that the equation has a unique positive solution W, which is a function of x_1 only, and W satisfies

$$\lim_{x_1 \to 0^+} W(x) x_1^{(2+\theta)/(p-1)} = \left[\lambda + \frac{2+\theta}{p-1}\left(1 + \frac{2+\theta}{p-1}\right)\right]^{1/(p-1)}$$

and

$$\lim_{x_1 \to \infty} W(x) x_1^{(2+\theta)/(q-1)} = \left[\lambda + \frac{2+\theta}{q-1}\left(1 + \frac{2+\theta}{q-1}\right)\right]^{1/(q-1)}.$$

Keywords: Hardy potential; symmetry; uniqueness; asymptotic behaviour

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1. Introduction

We shall investigate the asymptotic behaviour and symmetry of positive solutions to the following equation:

$$-\Delta u = \lambda \frac{u}{d(x)^2} - d(x)^\theta f(u) \quad \text{in } T, \tag{1.1}$$

where $d(x) = d(x, \partial T)$, $\theta > -2$, $T \subset \mathbb{R}^N$ $(N \geqslant 2)$ and

$$T = \{x = (x_1, x_2, \ldots, x_N) \colon x_1 > 0\}.$$

Elliptic equations with singular potentials have been studied extensively for many years. When $\lambda = 0$, problem (1.1) with T replaced by a general bounded domain Ω becomes

$$-\Delta u = -d(x)^\theta f(u) \quad \text{in } \Omega,$$

where $d(x) = d(x, \partial\Omega)$ and $\theta > -2$. For such equations, the corresponding problems with boundary blow-up condition have attracted a great deal of attention, and some well-known results can be found in [14].

In [5], Bandle *et al*. consider

$$-\Delta u = \lambda \frac{u}{d(x)^2} - d(x)^\theta u^p \quad \text{in } \Omega, \tag{1.2}$$

where $\theta > -2$, $p > 1$, $d(x) = d(x, \partial\Omega)$ and $\Omega \subset \mathbb{R}^N$ $(N \geqslant 2)$ is a bounded smooth domain. They mainly establish the existence and non-existence of several kinds of positive solutions to (1.2) when $\lambda \leqslant \frac{1}{4}$, and pose some open questions. The third open question is 'what is the asymptotic behaviour near the boundary of the solutions for arbitrary $\lambda > \frac{1}{4}$?'. In order to answer the open question, in a recent paper, Du and Wei [20] prove that (1.2) has a unique positive solution u, and that u satisfies

$$\lim_{d(x)\to 0^+} \mathrm{d}(x)^{(2+\theta)/(p-1)} u(x) = \left[\lambda + \frac{2+\theta}{p-1}\left(1 + \frac{2+\theta}{p-1}\right)\right]^{1/(p-1)}$$

when Ω is a ball and $\lambda > \frac{1}{4}$. Du and Wei also prove that when Ω is a bounded smooth domain there exists $\lambda_* > \frac{1}{4}$ such that for any $\lambda > \lambda_*$ (1.2) has a unique positive solution, which has similar asymptotic behaviour. The constant $\frac{1}{4}$ is a Hardy constant on a convex domain and plays an important role in [5, 20]. Some well-known information about the Hardy constant can be found in [8, 28, 29]. In contrast to [20], in this paper the domain becomes a half-space and the nonlinearity term is also more general. In other words, on a half-space and for the more general nonlinearity term, we shall give complete answers to the third open question in [5].

For elliptic equations with another classic Hardy potential, Cîrstea [9] considered

$$-\Delta u = \lambda \frac{u}{|x|^2} - |x|^\theta f(u) \quad \text{in } \Omega \setminus \{0\}, \tag{1.3}$$

where $\theta > -2$ and Ω is a bounded smooth domain $(0 \in \Omega)$. Cîrstea gives a complete classification of positive solutions of (1.3) when $\lambda \leqslant \frac{1}{4}(N-2)^2$. Cîrstea and Du [11] give exact behaviour of positive solutions of (1.3) near the singular point when $\lambda = 0$. For the corresponding p-Laplacian problem, Cîrstea and Du [12] also obtain the exact behaviour of positive solutions. In [31], Wei and Feng considered

$$\left.\begin{aligned} -\Delta u &= \lambda \frac{u}{|x|^2} - b(x) u^p, && x \in \Omega \setminus \{0\}, \\ u &= 0, && x \in \partial\Omega, \end{aligned}\right\} \tag{1.4}$$

where $p > 1$ and $b(x)$ is a non-negative continuous function over $\bar{\Omega}$. Specially, when the parameter is supercritical, i.e. $\lambda > \frac{1}{4}(N-2)^2$ and $b(x)$ is a positive function, the existence of the minimal and maximal positive solutions was proved, and a rough estimate of positive solutions near the origin was established. This estimate shows that any positive solution of (1.4) must blow up at the origin. Let $\Omega_0 = \{x \in \Omega\colon b(x) = 0\}$ be C^2 and $\Omega_0 \subset \Omega$. If $0 \notin \Omega_0$ and $\frac{1}{4}(N-2)^2 < \lambda < \lambda_1[1/|x|^2, \Omega_0]$, (1.4) has a positive solution, and any positive solution of (1.4) must blow up at the origin. In recent work, Wei and Du [30] prove that (1.3) with the Dirichlet boundary condition has a unique positive solution and obtain that any positive solution of (1.3) has the same asymptotic behaviour near a singular point origin when $\lambda > \frac{1}{4}(N-2)^2$ and $f(u) = u^p$.

Problem (1.2) is a singularity problem. From our research in this paper, we see that it is related to boundary blow-up problems. Problem (1.3) is also a singularity problem. There are many important articles with respect to boundary singularity or isolated singularity problems, and the interested reader should refer to [3, 4, 6, 10, 18, 23–27] and the references therein.

When $\lambda = 0$ and $\theta = 0$ in (1.1) with the Dirichlet boundary condition, the corresponding problem becomes

$$\left.\begin{aligned} -\Delta u &= f(u) \quad \text{in } T, \\ u &= 0 \qquad\ \ \text{on } \partial T. \end{aligned}\right\} \tag{1.5}$$

This problem is well known and has some important results. Under suitable conditions for f, a well-known result of Angenent [2] says that any bounded positive solution of (1.5) is a function of x_1 only. There are some interesting conjectures and results that can be found in [1, 7]. Some results for (1.5) were extended to the p-Laplacian case in [16]. Moreover, these symmetry results remain valid if the boundary condition $u = 0$ is replaced by $u = \alpha$, where α is a positive constant or ∞ (see [13] for details). In this paper, we shall prove that such a symmetry result is valid under some light conditions for f, although problem (1.1) has no boundary condition and contains a Hardy potential.

Since T is a half-space, obviously, $d(x, \partial T) = x_1$. Therefore, problem (1.1) is equivalent to

$$-\Delta u = \lambda \frac{u}{x_1^2} - x_1^\theta f(u), \quad x_1 > 0. \tag{1.6}$$

Throughout this paper, we assume that $\lambda > \frac{1}{4}$ and that the following conditions hold:

(f_1) $f \in C^1([0, \infty))$ and $f(s)/s$ is increasing in $(0, \infty)$;

(f_2) $\lim_{s\to\infty} f(s)/s^p = a$, where $p > 1$ and $a > 0$;

(f_3) $\lim_{s\to 0^+} f(s)/s^q = b$, where $q > 1$ and $b > 0$.

For convenience, we define

$$\ell_p = \left[\frac{\lambda}{a} + \frac{2+\theta}{a(p-1)}\left(1 + \frac{2+\theta}{p-1}\right)\right]^{1/(p-1)}, \quad \ell_q = \left[\frac{\lambda}{b} + \frac{2+\theta}{b(q-1)}\left(1 + \frac{2+\theta}{q-1}\right)\right]^{1/(q-1)}.$$

In order to obtain some information on positive solutions of (1.1), we first need to establish an important result for the corresponding ordinary differential equation. In fact, the corresponding problem is just (1.1) with $N = 1$. The information can be given by the following theorem.

THEOREM 1.1. *Suppose that $\lambda > \frac{1}{4}$ and $\theta > -2$. Then the ordinary differential equation*

$$-u'' = \lambda \frac{u}{s^2} - s^\theta f(u) \quad \text{in } (0, \infty) \tag{1.7}$$

has a unique positive solution w. Moreover, w satisfies

$$\lim_{s \to 0^+} w(s) s^{(2+\theta)/(p-1)} = \ell_p, \qquad \lim_{s \to \infty} w(s) s^{(2+\theta)/(q-1)} = \ell_q. \tag{1.8}$$

For problem (1.1), our main conclusions can be given by the following theorem.

THEOREM 1.2. *Suppose that $\lambda > \frac{1}{4}$ and $\theta > -2$. Then problem (1.1) has a unique solution $W(x)$. Moreover, W is a function of x_1 only and satisfies*

$$\lim_{x_1 \to 0^+} W(x) x_1^{(2+\theta)/(p-1)} = \ell_p \quad \text{and} \quad \lim_{x_1 \to \infty} W(x) x_1^{(2+\theta)/(q-1)} = \ell_q. \tag{1.9}$$

This paper is organized as follows. In § 2, we give some preliminaries. In § 2.1, we recall an important comparison principle and establish the relationship between the Hardy constant and the first eigenvalue. In § 2.2, we prove the existence of the minimal and maximal positive solutions of (1.7), and show that any positive solution of (1.7) blows up at the origin and converges to 0 as $s \to \infty$. In § 3, we mainly prove theorem 1.1. In § 4, we give the proof of theorem 1.2.

2. Preliminaries

2.1. Several lemmas

The following comparison principle will be used frequently; it can be found in [14, 15, 19].

LEMMA 2.1. *Suppose that Ω is a bounded domain in $\mathbb{R}^N$, $\alpha(x)$ and $\beta(x)$ are continuous functions in Ω with $\|\alpha\|_\infty < \infty$ and $\beta(x)$ is non-negative and not identically zero. Let $u_1, u_2 \in C^1(\Omega)$ be positive in Ω and satisfy in the weak sense*

$$\Delta u_1 + \alpha(x) u_1 - \beta(x) g(u_1) \leqslant 0 \leqslant \Delta u_2 + \alpha(x) u_2 - \beta(x) g(u_2), \quad x \in \Omega,$$

and

$$\limsup_{x \to \partial\Omega} (u_2 - u_1) \leqslant 0,$$

where $g(u)$ is continuous and such that $g(u)/u$ is strictly increasing and non-negative for u in the range $\min\{u_1, u_2\} < u < \max\{u_1, u_2\}$. Then $u_2 \leqslant u_1$ in Ω.

In one-dimensional space, let $\lambda_1[(\delta, L), 1/s^2]$ define the first eigenvalue of

$$\begin{aligned} -u'' &= \lambda \frac{u}{s^2}, \quad s \in (\delta, L), \\ u &= 0, \qquad s = L \text{ or } \delta. \end{aligned}$$

For any bounded $\Omega \subset \mathbb{R}^N (N \geqslant 2)$, let $\lambda_1[\Omega, \alpha(x)]$ denote the first eigenvalue of

$$\begin{aligned} -\Delta u &= \lambda \alpha(x) u, \quad x \in \Omega, \\ u &= 0, \qquad\quad x \in \partial\Omega, \end{aligned}$$

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where $\alpha(x)$ is a positive continuous function over Ω. Define

$T_R := B_R(0) \cap \mathbb{R}^N_+ = \{x \in B_R(0)\colon x_1 > 0\}$ and $T_R^\delta := \{x \in T_R\colon d(x, \partial T_R) > \delta\}$.

For convenience, define $D_R(x) := d(x, \partial T_R)$. Obviously, we have $D_R(x) \leqslant x_1$ when $x \in T_R$.

Lemma 2.2. *With respect to the Hardy constant and the first eigenvalue, it follows that*

(i) *for any* $L > 0$,

$$\lim_{\delta \to 0^+} \lambda_1 \left[(\delta, L), \frac{1}{s^2} \right] = \frac{1}{4}; \tag{2.1}$$

(ii) *for any* $l_0 > 0$,

$$\lim_{L \to \infty} \lambda_1 \left[(l_0, L), \frac{1}{s^2} \right] = \frac{1}{4}; \tag{2.2}$$

(iii) $\lim_{R\to\infty} \lim_{\delta\to 0} \lambda_1[T_R^\delta, 1/x_1^2] = \frac{1}{4}$.

Proof. By [8, 28], we have

$$\frac{1}{4} = \inf_{\phi \in H_0^1(0,L)\setminus\{0\}} \left(\int_0^L \phi'(s)^2 \,\mathrm{d}s \right) \left(\int_0^L \frac{\phi(s)^2}{s^2} \,\mathrm{d}s \right)^{-1}.$$

By a similar method to that in [30, 31], we can obtain (i).

By (i), for any $\gamma \in (0, l_0)$,

$$\lim_{\delta \to 0^+} \lambda_1 \left[(\delta, \gamma), \frac{1}{s^2} \right] = \frac{1}{4}.$$

Since

$$\lambda_1 \left[(\delta, \gamma), \frac{1}{s^2} \right] = \lambda_1 \left[\left(l_0, \frac{l_0 \gamma}{\delta} \right), \frac{1}{s^2} \right],$$

we easily obtain (2.2).

Now we prove (iii). Since T_R is convex, from [28, 29] it follows that, for $R > 0$,

$$\inf_{\phi \in H_0^1(T_R)\setminus\{0\}} \left(\int_{T_R} |\nabla \phi|^2 \,\mathrm{d}x \right) \left(\int_{T_R} \frac{\phi(x)^2}{D_R(x)^2} \,\mathrm{d}x \right)^{-1} = \frac{1}{4}.$$

By [20], we have

$$\lambda_1 \left[T_R^\delta, \frac{1}{D_R(x)^2} \right] > \frac{1}{4} \quad \text{and} \quad \lim_{\delta \to 0^+} \lambda_1 \left[T_R^\delta, \frac{1}{D_R(x)^2} \right] = \frac{1}{4}.$$

By the monotone property of the first eigenvalue, $\lambda_1[T_R^\delta, 1/x_1^2]$ is decreasing when R increases or δ decreases. Therefore,

$$\lambda_R := \lim_{\delta \to 0^+} \lambda_1 \left[T_R^\delta, \frac{1}{x_1^2} \right] \quad \text{and} \quad \lambda_* := \lim_{R \to \infty} \lambda_R.$$

are well defined. Obviously, $D_R(x) \leqslant x_1$ implies that

$$\lambda_1\left[T_R^\delta, \frac{1}{x_1^2}\right] \geqslant \lambda_1\left[T_R^\delta, \frac{1}{D_R(x)^2}\right].$$

So we have $\lambda_R \geqslant \frac{1}{4}$. Further, we have $\lambda_* \geqslant \frac{1}{4}$. Suppose that $\lambda_* > \frac{1}{4}$. Then, there exist $\varepsilon_0 > 0$, R_n, δ_n with $R_n \to \infty$ and $\delta_n \to 0$ such that

$$\lambda_1\left[T_{R_n}^{\delta_n}, \frac{1}{x_1^2}\right] > \frac{1}{4} + \varepsilon_0. \tag{2.3}$$

From [28, 29], there exists $\phi \in C_0^\infty(T)$ such that

$$\left(\int_T |\nabla\phi|^2 \,\mathrm{d}x\right)\left(\int_T \frac{\phi^2}{x_1^2} \,\mathrm{d}x\right)^{-1} < \frac{1}{4} + \frac{\varepsilon_0}{2}.$$

Since $\phi \in C_0^\infty(T)$, for sufficiently large n, the support set of ϕ is a subset of $T_{R_n}^{\delta_n}$. Thus, it follows that, for sufficiently large n,

$$\phi \in C_0^\infty(T_{R_n}^{\delta_n}) \quad \text{and} \quad \left(\int_{T_{R_n}^{\delta_n}} |\nabla\phi|^2 \,\mathrm{d}x\right)\left(\int_{T_{R_n}^{\delta_n}} \frac{\phi^2}{x_1^2} \,\mathrm{d}x\right)^{-1} < \frac{1}{4} + \frac{\varepsilon_0}{2}.$$

By virtue of the variational form of the first eigenvalue, we can derive

$$\lambda_1\left[T_{R_n}^{\delta_n}, \frac{1}{x_1^2}\right] < \frac{1}{4} + \frac{\varepsilon_0}{2} \quad \text{for sufficiently large } n,$$

which contradicts (2.3). □

2.2. Minimal positive solution and maximal positive solution of (1.7)

In this subsection, we shall prove the existence of the minimal and maximal positive solutions of (1.7). We shall also approximately describe the behaviour of positive solutions of (1.7) near the origin and at infinity. Here and in the following sections, we need to use some arguments of elliptic equations with boundary blow-up conditions, which can be found in [14, 17, 21, 22].

Proposition 2.3. *For any $\lambda > \frac{1}{4}$, (1.7) has a minimal positive solution w_0 and a maximal positive solution w_∞.*

Proof. For positive integers n, m, consider the following two problems:

$$\left.\begin{aligned} -u'' &= \lambda\frac{u}{s^2} - s^\theta f(u), && s \in \left(\frac{1}{n}, m\right), \\ u &= 0, && s = m \text{ or } \frac{1}{n} \end{aligned}\right\} \tag{2.4}$$

and

$$\left.\begin{aligned} -u'' &= \lambda\frac{u}{s^2} - s^\theta f(u), && s \in \left(\frac{1}{n}, m\right), \\ u &= \infty, && s = m \text{ or } \frac{1}{n}. \end{aligned}\right\} \tag{2.5}$$

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By referring to standard results of boundary blow-up problems, (2.5) has a unique solution $u_{n,m}$. By lemma 2.1, $u_{n,m}$ is decreasing in m, and $u_{n,m}$ is decreasing in n. Therefore, $u_m(s) := \lim_{n\to\infty} u_{n,m}(s)$ is well defined in $(0, m)$, and $u_m(s) \geqslant u_{m+1}(s)$ for $s \in (0, m)$. Hence, $w_\infty(s) := \lim_{m\to\infty} u_m(s)$ is well defined in $(0, \infty)$. By regularity arguments of elliptic equations, w_∞ satisfies

$$-w_\infty'' = \lambda \frac{w_\infty}{s^2} - s^\theta f(w_\infty) \quad \text{in } (0, \infty).$$

Suppose that v is an arbitrary positive solution of (1.7). By lemma 2.1, $u_{n,m}(s) \geqslant v(s)$ in $(1/n, m)$. Letting $n \to \infty$ and $m \to \infty$ in turn, we see that $v(s) \leqslant w_\infty(s)$ in $(0, \infty)$. This implies that w_∞ is the maximal positive solution of (1.7).

Since $\lambda > \frac{1}{4}$, by lemma 2.2, there exist n_0 and m_0 such that

$$\lambda > \lambda_1\left[\left(\frac{1}{n}, m\right), \frac{1}{s^2}\right] \quad \text{for any } n \geqslant n_0 \text{ and } m \geqslant m_0.$$

By the standard arguments of logistic equations (see [14]), (2.4) has a unique positive solution $w_{n,m}$ when $n \geqslant n_0$ and $m \geqslant m_0$. By lemma 2.1, $w_{n,m}$ is increasing in m and $w_{n,m}$ is increasing in n. Therefore, $w_m(s) := \lim_{n\to\infty} w_{n,m}(s)$ is well defined in $(0, m)$, and $w_m(s) \leqslant w_{m+1}(s)$ for $s \in (0, m)$. Further, $w_0(s) := \lim_{m\to\infty} w_m(s)$ is well defined in $(0, \infty)$. By regularity arguments of elliptic equations, w_0 satisfies

$$-w_0'' = \lambda \frac{w_0}{s^2} - s^\theta f(w_0) \quad \text{in } (0, \infty).$$

Suppose that v is an arbitrary positive solution of (1.7). By lemma 2.1, $w_{n,m}(s) \leqslant v(s)$ in $(1/n, m)$. Letting $n \to \infty$ and $m \to \infty$ in turn, we see that $v(s) \geqslant w_0(s)$ in $(0, \infty)$. This implies that w_0 is the minimal positive solution of (1.7). □

Since it is not obvious whether positive solutions of (1.7) are bounded near the origin and at infinity, we need to determine approximate behaviours of positive solutions near the origin and at infinity. It suffices to determine behaviours of w_0 and w_∞ near the origin and at infinity.

PROPOSITION 2.4. *Let w_0 be the minimal positive solution of (1.7). Then $w_0'(s) \leqslant 0$ and w_0 blows up at the origin.*

Proof. As the proof of proposition 2.3, we see $w_0 = \lim_{m\to\infty} w_m$ and, by regularity arguments of elliptic equations, w_m satisfies

$$\left.\begin{aligned} -w_m'' = \lambda \frac{w_m}{s^2} - s^\theta f(w_m), \quad 0 < s < m, \\ w_m(m) = 0. \end{aligned}\right\} \tag{2.6}$$

If $w_m'(s) \leqslant 0$ for any m and all $s \in (0, m]$, then since $\{w_m\}$ has a subsequence converging to w_0 in $C^2_{\mathrm{loc}}(0, \infty)$, we derive $w_0'(s) \leqslant 0$ for all $s \in (0, \infty)$. Now, for any m, we prove that $w_m'(s) \leqslant 0$ for all $s \in (0, m]$. By Hopf's lemma, $w_m'(m) < 0$. By the continuity of $w_m'(s)$ with respect to s, $w_m'(s) < 0$ when $s < m$ and is close to m. Define

$$t_0 = \inf\{t\colon w_m'(s) \leqslant 0 \text{ for all } s \in (t, m]\}.$$

It suffices to show $t_0 = 0$. Assume to the contrary that $t_0 \in (0, m)$.

Case 1. Suppose that there exists $\tilde{t} \in (0, t_0)$ such that $w_m'(\tilde{t}) < 0$. In this case, we can always take $s_1 \in (\tilde{t}, t_0)$ such that

$$w_m''(s_1) \geqslant 0, \quad w_m'(s_1) = 0 \quad \text{and} \quad w_m(s_1) < w_m(t_0).$$

From the equation in (2.6), it follows that

$$\frac{f(w_m(s_1))}{w_m(s_1)} \geqslant \lambda s_1^{-\theta-2}. \tag{2.7}$$

From the definition of t_0, it follows that $w_m'(t_0) = 0$ and $w_m''(t_0) \leqslant 0$, which implies that

$$\frac{f(w_m(t_0))}{w_m(t_0)} \leqslant \lambda t_0^{-\theta-2}. \tag{2.8}$$

Since $(-\theta - 2) < 0$ and $t_0 > s_1$, (2.7) and (2.8) imply

$$\frac{f(w_m(t_0))}{w_m(t_0)} < \frac{f(w_m(s_1))}{w_m(s_1)}. \tag{2.9}$$

Since $f(u)/u$ is increasing in u, (2.9) is a contradiction to $w_m(s_1) < w_m(t_0)$.

Case 2. Suppose that $w_m'(s) \geqslant 0$ holds for all $s \in (0, t_0)$. In this case, there must be

$$\lim_{s \to 0^+} w_m(s) \in [0, \infty).$$

Since $\theta > -2$, we have

$$-s^2 w_m'' = \lambda w_m - s^{2+\theta} f(w_m) = (\lambda + o(1)) w_m \quad \text{as } s \to 0^+.$$

So, by virtue of $\lambda > \frac{1}{4}$, there exists $\tau \in (0, m)$ such that

$$-s^2 w_m'' > \frac{4\lambda + 1}{8} w_m \quad \text{when } s \in (0, \tau). \tag{2.10}$$

From (2.10), for any $\varepsilon \in (0, \tau)$, w_m satisfies

$$\left.\begin{aligned} -w_m'' > \frac{4\lambda+1}{8}\frac{w_m}{s^2}, \quad s \in (\varepsilon, \tau), \\ w_m(\varepsilon) > 0, \qquad w_m(\tau) > 0. \end{aligned}\right\} \tag{2.11}$$

By (2.11), w_m is a strictly positive supersolution of

$$-u'' = \frac{4\lambda+1}{8}\frac{u}{s^2} \quad \text{in } (\varepsilon, \tau), \qquad u(\varepsilon) = u(\tau) = 0.$$

So, we have $\lambda_1[(\varepsilon, \tau), 1/s^2] > \frac{1}{8}(4\lambda + 1) > \frac{1}{4}$ for all $\varepsilon \in (0, \tau)$, which is in contradiction with $\lambda_1[(\varepsilon, \tau), 1/s^2] \to \frac{1}{4}$ as $\varepsilon \to 0^+$.

We claim that w_0 blows up at $s = 0$. Otherwise, by $w_0'(s) \leqslant 0$ for all $s > 0$, $\lim_{s \to 0} w_m(s) \in (0, \infty)$ is well defined. Similar to case 2, we can derive a contradiction. □

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PROPOSITION 2.5. *Let w_∞ be the maximal positive solution of (1.7). Then*

$$\lim_{s\to\infty} w_\infty(s) = 0.$$

Proof. By proposition 2.4, we have $w_\infty(s) \to \infty$ as $s \to 0^+$. We suppose that $\lim_{s\to\infty} w_\infty(s) \neq 0$.

CASE 1. Suppose that $\lim_{s\to\infty} w_\infty(s)$ does not exist. Then, there are $s_2 > s_1 > 0$ such that

$$w_\infty(s_1) < w_\infty(s_2), \quad w''_\infty(s_1) \geqslant 0 \quad \text{and} \quad w''_\infty(s_2) \leqslant 0.$$

By the equation for w_∞, we can obtain

$$\frac{f(w_\infty(s_1))}{w_\infty(s_1)} \geqslant \lambda s_1^{-\theta-2} \quad \text{and} \quad \frac{f(w_\infty(s_2))}{w_\infty(s_2)} \leqslant \lambda s_2^{-\theta-2},$$

which imply

$$\frac{f(w_\infty(s_2))}{w_\infty(s_2)} < \frac{f(w_\infty(s_1))}{w_\infty(s_1)}.$$

Since $f(u)/u$ is increasing in u and $w_\infty(s_2) > w_\infty(s_1)$, we derive a contradiction immediately.

CASE 2. Suppose that $\lim_{s\to\infty} w_\infty(s) \in (0,\infty]$. By condition (f_2), there exist $r_0 > 0$ and $a_0 \in (0,a)$ such that

$$f(w_\infty(s)) \geqslant a_0 w_\infty(s)^p \quad \text{when } s \geqslant r_0.$$

So,

$$-w''_\infty \leqslant \lambda \frac{w_\infty}{s^2} - a_0 s^\theta w_\infty^p \quad \text{in } (r_0,\infty).$$

Let $r_1 > 2r_0$ and define

$$\phi(s) := r_1^{(2+\theta)/(p-1)} w_\infty(r_1 + \tfrac{1}{2} r_1 s) \quad \text{when } s \in (-1,1).$$

By calculation,

$$-\phi'' \leqslant \begin{cases} \lambda\phi - \dfrac{a_0}{2^{2+\theta}}\phi^p, & \theta \geqslant 0,\ s \in (-1,1), \\ \lambda\phi - \left(\dfrac{3}{2}\right)^\theta \dfrac{a_0}{4}\phi^p, & \theta < 0,\ s \in (-1,1). \end{cases}$$

Let $\phi_\infty(s)$ denote the unique positive solution (see [14]) of

$$-u'' = \lambda u - \frac{a_0}{2^{2+\theta}} u^p, \quad s \in (-1,1),$$
$$u(-1) = u(1) = \infty.$$

By the comparison principle, for $\theta \geqslant 0$, we have

$$\phi(s) \leqslant \phi_\infty(s) \quad \text{for } s \in (-1,1).$$

Let $s = 0$. Then we see that

$$w_\infty(r_1) \leqslant \phi_\infty(0) r_1^{-(2+\theta)/(p-1)}.$$

So, when $\theta \geqslant 0$, by the arbitrariness of r_1 we have $\lim_{s\to\infty} w_\infty(s) = 0$, which contradicts $\lim_{s\to\infty} w_\infty(s) \in (0,\infty]$. Similarly, when $\theta < 0$, we can also derive a contradiction.

From cases 1 and 2, we derive $\lim_{s\to\infty} w_\infty(s) = 0$. Thus, the proof is complete. □

Theorem 2.6. *Suppose that u is an arbitrary positive solution of (1.7). Then,*

$$\lim_{s\to 0^+} u(s) = \infty \quad \textit{and} \quad \lim_{s\to\infty} u(s) = 0. \tag{2.12}$$

Proof. For an arbitrary positive solution u of (1.7), we have

$$w_0(s) \leqslant u(s) \leqslant w_\infty(s) \quad \text{for } s \in (0,\infty).$$

By propositions 2.4 and 2.5, we obtain (2.12) directly. □

3. The exact behaviour of a positive solution to (1.7)

3.1. Some rough estimates and the uniqueness of positive solutions

In this subsection we shall give some estimates of positive solutions of (1.7). In fact, we only need to give some estimates for the maximal positive solution w_∞ and the minimal positive solution w_0 of (1.7).

Proposition 3.1. *There exist $C_1 > 0$, s_0 and $S_0 > 0$ such that*

$$C_1 s^{-(2+\theta)/(p-1)} \leqslant w_0(s) \quad \textit{for } s \in (0, s_0) \tag{3.1}$$

and

$$C_1 s^{-(2+\theta)/(q-1)} \leqslant w_0(s) \quad \textit{for } s \in (S_0, \infty). \tag{3.2}$$

Proof. By theorem 2.6 and (f_2), there exist $l > 0$ and $c > 0$ such that

$$f(w_0(s)) \leqslant c w_0(s)^p \quad \text{for } s \in (0,l).$$

So,

$$-w_0'' \geqslant \lambda \frac{w_0}{s^2} - c s^\theta w_0^p \quad \text{for } s \in (0,l). \tag{3.3}$$

Similarly to the proof of [20, proposition 3.2],

$$\begin{aligned} &-u'' = \lambda \frac{u}{s^2} - c s^\theta u^p, \quad s \in (0,l), \\ &u(l) = 0, \end{aligned}$$

has a unique positive solution u_l and

$$u_l(s) = \lim_{\tau\to 0^+} u_{\tau,l}(s), \quad s \in (0,l),$$

where $u_{\tau,l}$ is the unique positive solution of

$$\left.\begin{aligned} &-u'' = \lambda \frac{u}{s^2} - c s^\theta u^p, \quad s \in (\tau,l), \\ &u(\tau) = u(l) = 0. \end{aligned}\right\} \tag{3.4}$$

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By (3.3), w_0 is a supersolution of (3.4). The comparison principle implies

$$u_{\tau,l}(s) \leqslant w_0(s) \quad \text{in } (\tau, l).$$

Letting $\tau \to 0^+$, we have

$$u_l(s) \leqslant w_0(s) \quad \text{in } (0, l).$$

Similarly to the proof of [20, proposition 3.2], it follows that

$$\lim_{s\to 0^+} s^{(2+\theta)/(p-1)} u_l(s) = \left(\frac{\lambda}{c} + \frac{2+\theta}{c(p-1)}\left(\frac{2+\theta}{p-1} + 1\right)\right)^{1/(p-1)}.$$

So, there exist $s_0 > 0$ and $C_1 > 0$ such that (3.1) holds.

By theorem 2.6 and (f_3), there exist $L > 0$ and $c_1 > 0$ such that

$$f(w_0(s)) \leqslant c_1 w_0(s)^q \quad \text{in } (L, \infty).$$

Then,

$$-w_0'' \geqslant \lambda \frac{w_0}{s^2} - c_1 s^\theta w_0^q \quad \text{in } (L, \infty). \tag{3.5}$$

By lemma 2.2(i), for a given $\gamma_0 > 0$, there exists $\tau_0 \in (0, \frac{1}{2}\gamma_0)$ such that, for any $\tau \in (0, \tau_0]$, $\lambda > \lambda_1[(\tau, \gamma_0), 1/s^2]$. By the standard arguments of logistic equations,

$$\begin{aligned} -u'' &= \lambda \frac{u}{s^2} - c_1 s^\theta u^q, \quad s \in (\tau, \gamma_0), \\ u(\tau) &= u(\gamma_0) = 0, \end{aligned}$$

has a unique positive solution $V_{\gamma_0,\tau}$. Define $r_* = \gamma_0 L/\tau_0$ and, for $r > r_*$,

$$\Phi_r(s) = \left(\frac{\gamma_0}{r}\right)^{(2+\theta)/(q-1)} V_{\gamma_0,\gamma_0 L/r}\left(\frac{\gamma_0}{r}s\right) \quad \text{for } L \leqslant s \leqslant r.$$

Then, Φ_r solves

$$\left.\begin{aligned} -u'' &= \lambda \frac{u}{s^2} - c_1 s^\theta u^q, \quad s \in (L, r), \\ u(L) &= u(r) = 0. \end{aligned}\right\} \tag{3.6}$$

By (3.5), w_0 is a supersolution of (3.6). By the comparison principle, we have

$$w_0(s) \geqslant \Phi_r(s) \quad \text{for all } s \in (L, r). \tag{3.7}$$

Now, for arbitrary $r > r^* = \max\{2L, r_*\}$ and s satisfying $s = \frac{1}{2}r$, (3.7) implies

$$w_0\left(\frac{r}{2}\right) \geqslant \left(\frac{\gamma_0}{r}\right)^{(2+\theta)/(q-1)} V_{\gamma_0,\gamma_0 L/r}\left(\frac{\gamma_0}{2}\right).$$

Since $\tau_0 \geqslant \gamma_0 L/r$ and $V_{\gamma_0,\gamma_0 L/r}$ is non-decreasing in r, for the $s = \frac{1}{2}r$ above, we derive

$$w_0(s) \geqslant V_{\gamma_0,\tau_0}\left(\frac{\gamma_0}{2}\right)\left(\frac{\gamma_0}{2}\right)^{(2+\theta)/(q-1)} s^{-(2+\theta)/(q-1)}.$$

By the arbitrariness of r, letting $S_0 = \frac{1}{2}r_*$ and

$$C_1 = V_{\gamma_0,\tau_0}\left(\frac{\gamma_0}{2}\right)\left(\frac{\gamma_0}{2}\right)^{(2+\theta)/(q-1)},$$

we have (3.2). □

Proposition 3.2. *There exist $C_2 > 0$, s_0 and $S_0 > 0$ such that*

$$C_2 s^{-(2+\theta)/(p-1)} \geqslant w_\infty(s) \quad \text{for } s \in (0, s_0) \tag{3.8}$$

and

$$C_2 s^{-(2+\theta)/(q-1)} \geqslant w_\infty(s) \quad \text{for } s \in (S_0, \infty). \tag{3.9}$$

Proof. By theorem 2.6, there exist $c > 0$, $l > 0$ and $L > 0$ such that

$$f(w_\infty(s)) \geqslant c w_\infty(s)^p \quad \text{for all } s \in (0, l)$$

and

$$f(w_\infty(s)) \geqslant c w_\infty(s)^q \quad \text{for all } s \in (L, \infty).$$

So, we have

$$-w_\infty'' \leqslant \lambda \frac{w_\infty}{s^2} - c s^\theta w_\infty^p \quad \text{in } s \in (0, l) \tag{3.10}$$

and

$$-w_\infty'' \leqslant \lambda \frac{w_\infty}{s^2} - c s^\theta w_\infty^q \quad \text{in } s \in (L, \infty). \tag{3.11}$$

For arbitrary $t_0 \in (0, l/2)$, define

$$D(t_0) = \{s \in (0, \infty) \colon |s - t_0| < \tfrac{1}{2}t_0\}.$$

Obviously, $D(t_0) \subset (0, l)$. Define

$$U(s) := t_0^{(2+\theta)/(p-1)} w_\infty(t_0 + \tfrac{1}{2}t_0 s) \quad \text{when } s \in (-1, 1).$$

Suppose that $\theta \geqslant 0$. Then, by (3.10), we have

$$-U'' \leqslant \lambda U - \frac{c}{2^{2+\theta}} U^p \quad \text{in } (-1, 1).$$

From [14],

$$\left.\begin{aligned} -u'' = \lambda u - \frac{c}{2^{2+\theta}} u^p, \quad s \in (-1, 1),\\ u(-1) = u(1) = \infty, \end{aligned}\right\} \tag{3.12}$$

has a unique positive solution $U_{\infty,p}$. By the comparison principle, we have

$$U_{\infty,p}(s) \geqslant U(s) \quad \text{for all } s \in (-1, 1).$$

In particular, let $s = 0$. Then

$$U_{\infty,p}(0) \geqslant t_0^{(2+\theta)/(p-1)} w_\infty(t_0).$$

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By the arbitrariness of t_0, we let $s_0 = \frac{1}{2}l$, and then derive (3.8). When $-2 < \theta < 0$, we can similarly obtain (3.8).

Now, we prove (3.9). Choose $r > 2L$ and define

$$V(s) := r^{(2+\theta)/(q-1)} w_\infty(r + \tfrac{1}{2}rs) \quad \text{when } s \in (-1, 1).$$

Suppose $\theta \geqslant 0$. Then, by calculating, we see

$$-V'' \leqslant \lambda V - \frac{c}{2^{2+\theta}} V^q \quad \text{in } (-1, 1).$$

By the comparison principle and letting $s = 0$,

$$r^{(2+\theta)/(q-1)} w_\infty(r) \leqslant U_{\infty,q}(0),$$

where $U_{\infty,q}$ is the unique positive solution of (3.12) with p replaced by q. By the arbitrariness of r, let $S_0 = 2L$, and then we derive (3.9). When $-2 < \theta < 0$, we can similarly obtain (3.9). □

THEOREM 3.3. *Suppose that (f_1)–(f_3) hold. Then (1.7) has a unique positive solution w. Moreover, there are positive constants C_1, C_2, s_0, S_0 such that*

$$C_1 s^{-(2+\theta)/(p-1)} \leqslant w(s) \leqslant C_2 s^{-(2+\theta)/(p-1)} \quad \textit{when } s \in (0, s_0) \tag{3.13}$$

and

$$C_1 s^{-(2+\theta)/(q-1)} \leqslant w(s) \leqslant C_2 s^{-(2+\theta)/(q-1)} \quad \textit{when } s \in (S_0, \infty). \tag{3.14}$$

Proof. From propositions 3.1 and 3.2, it follows that the inequalities (3.13) and (3.14) hold for any positive solution of (1.7). So, there exists $C > 1$ such that

$$w_0(s) \leqslant w_\infty(s) \leqslant C w_0(s) \quad \text{for } s \in (0, \infty).$$

Condition (f_1) implies that $f(u)$ is a convex function in $(0, \infty)$. Therefore, the proof of uniqueness is standard, and thus we omit it. □

3.2. Exact behaviour near a singular point

In this subsection, we consider the exact behaviour of the unique positive solution w of (1.7). Obviously, $w = w_0 = w_\infty$ and

$$w(s) = \lim_{m\to\infty} \lim_{n\to\infty} w_{n,m}(s) = \lim_{m\to\infty} \lim_{n\to\infty} u_{n,m}(s) \quad \text{for } s \in (0, \infty).$$

THEOREM 3.4. *Let w be the unique positive solution of (1.7). Then*

$$\lim_{s\to 0^+} s^{(2+\theta)/(p-1)} w(s) = \ell_p. \tag{3.15}$$

Proof. We shall establish suitable local super- and subsolutions of (1.7). First, we establish a suitable local supersolution of (1.7). For any $\varepsilon > 0$, we choose large $M > 0$ such that $f(u) > (a - \varepsilon)u^p$ when $u \geqslant M$. Define

$$\tilde{\xi}_\varepsilon = \left(\frac{\lambda}{a - \varepsilon} + \frac{2 + \theta}{(p-1)(a - \varepsilon)} \left(1 + \frac{2 + \theta}{p - 1} \right) \right)^{1/(p-1)}$$

and $\bar{v}_\varepsilon = \tilde{\xi}_\varepsilon s^{-(2+\theta)/(p-1)} + M$. By calculation,

$$\begin{aligned}
&-\bar{v}_\varepsilon'' - \lambda\frac{\bar{v}_\varepsilon}{s^2} + s^\theta f(\bar{v}_\varepsilon)\\
&\quad= \lambda\frac{\tilde{\xi}_\varepsilon s^{-(2+\theta)/(p-1)}}{s^2} - (a-\varepsilon)s^\theta(\tilde{\xi}_\varepsilon s^{-(2+\theta)/(p-1)})^p - \lambda\frac{\bar{v}_\varepsilon}{s^2} + s^\theta f(\bar{v}_\varepsilon)\\
&\quad\geqslant (a-\varepsilon)s^\theta(M + \tilde{\xi}_\varepsilon s^{-(2+\theta)/(p-1)})^p - (a-\varepsilon)s^\theta(\tilde{\xi}_\varepsilon s^{-(2+\theta)/(p-1)})^p - \lambda\frac{M}{s^2}\\
&\quad= pM(a-\varepsilon)s^\theta\int_0^1(\tilde{\xi}_\varepsilon s^{-(2+\theta)/(p-1)} + M\tau)^{p-1}\,\mathrm{d}\tau - \lambda\frac{M}{s^2}\\
&\quad\geqslant pM(a-\varepsilon)s^{-2}\tilde{\xi}_\varepsilon^{p-1} - \lambda M s^{-2}\\
&\quad> 0.
\end{aligned}$$

By the comparison principle, for any large n, m,

$$\bar{v}_\varepsilon(s) \geqslant w_{n,m}(s) \quad \text{in } \left(\frac{1}{n}, m\right).$$

Letting $n \to \infty$ and $m \to \infty$ in turn, we derive

$$\bar{v}_\varepsilon(s) \geqslant w_0(s) \quad \text{for all } s \in (0,\infty).$$

By the uniqueness of positive solutions of (1.7), we see that

$$\limsup_{s\to 0^+} s^{(2+\theta)/(p-1)}w(s) \leqslant \tilde{\xi}_\varepsilon.$$

By the arbitrariness of ε, we obtain

$$\limsup_{s\to 0^+} s^{(2+\theta)/(p-1)}w(s) \leqslant \ell_p. \tag{3.16}$$

For arbitrary $\varepsilon > 0$, there exists $Z > 0$ such that

$$f(u) \leqslant (a+\varepsilon)u^p \quad \text{when } u > Z.$$

So, there exists $t_0 > 0$ such that, for sufficiently large n, m,

$$f(u_{n,m}(s)) \leqslant (a+\varepsilon)u_{n,m}(s)^p \quad \text{when } s \in \left(\frac{1}{n}, t_0\right),$$

where $u_{n,m}$ is given by proposition 2.3. Further, for sufficiently large n, m, we see

$$-u_{n,m}'' \geqslant \lambda\frac{u_{n,m}}{s^2} - (a+\varepsilon)u_{n,m}^p \quad \text{in } \left(\frac{1}{n}, t_0\right).$$

Define

$$\underline{\xi}_\varepsilon = \left(\frac{\lambda}{a+\varepsilon} + \frac{2+\theta}{(p-1)(a+\varepsilon)}\left(1 + \frac{2+\theta}{p-1}\right)\right)^{1/(p-1)}, \quad \beta = -\frac{2+\theta}{p-1},$$

and

$$\underline{v}_\varepsilon(s) = \underline{\xi}_\varepsilon s^\beta \eta(s), \quad \text{where } \eta(s) = \left(1 - \frac{s}{t_0}\right).$$

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Then,

$$\underline{v}_\varepsilon'' = \underline{\xi}_\varepsilon \beta(\beta-1)s^{\beta-2}\eta(s) + 2\underline{\xi}_\varepsilon \beta s^{\beta-1}\eta'(s) \quad \text{and} \quad \underline{v}_\varepsilon(t_0) = 0.$$

By calculating, for $s \in (0, t_0)$,

$$\begin{aligned}
&-\underline{v}_\varepsilon'' - \lambda\frac{\underline{v}_\varepsilon}{s^2} + (a+\varepsilon)s^\theta \underline{v}_\varepsilon^p \\
&\quad = \underline{\xi}_\varepsilon s^{\beta-2}[\eta(s)((a+\varepsilon)\underline{\xi}_\varepsilon^{p-1}\eta(s)^{p-1} - \lambda - \beta(\beta-1)) - 2\beta s\eta'(s)] \\
&\quad \leqslant \underline{\xi}_\varepsilon s^{\beta-2}\eta(s)[(a+\varepsilon)\underline{\xi}_\varepsilon^{p-1} - \lambda - \beta(\beta-1)] \\
&\quad = 0.
\end{aligned}$$

So, by the comparison principle, for sufficiently large m, n,

$$\underline{v}_\varepsilon(s) \leqslant u_{n,m}(s) \quad \text{for } s \in \left(\frac{1}{n}, t_0\right).$$

Letting $n \to \infty$ and $m \to \infty$ in turn,

$$\underline{v}_\varepsilon(s) \leqslant w_\infty(s) = w(s) \quad \text{for } s \in (0, t_0).$$

This implies

$$\liminf_{s\to 0^+} s^{(2+\theta)/(p-1)} w(s) \geqslant \underline{\xi}_\varepsilon.$$

By the arbitrariness of ε,

$$\liminf_{s\to 0^+} s^{(2+\theta)/(p-1)} w(s) \geqslant \ell_p. \tag{3.17}$$

Obviously, (3.16) and (3.17) imply (3.15). □

3.3. Exact asymptotic behaviour of positive solutions at ∞

In this subsection, we give the exact behaviour of the unique positive solution w when $s \to \infty$. It is possible that the exact behaviour of the unique positive solution at infinity cannot be derived by using the method of proving theorem 3.4, so we need give a different method to show the behaviour at infinity. Here, the method is inspired by the proof of [20, proposition 3.2].

THEOREM 3.5. *Let w be the unique positive solution of (1.7). Then*

$$\lim_{s\to\infty} s^{(2+\theta)/(q-1)} w(s) = \ell_q. \tag{3.18}$$

Proof. For any $\varepsilon > 0$, define

$$\begin{aligned}
\bar{\rho}_\varepsilon &= \left(\frac{\lambda}{b-\varepsilon} + \frac{2+\theta}{(b-\varepsilon)(q-1)}\left(1 + \frac{2+\theta}{q-1}\right)\right)^{1/(q-1)}, \\
\underline{\rho}_\varepsilon &= \left(\frac{\lambda}{b+\varepsilon} + \frac{2+\theta}{(b+\varepsilon)(q-1)}\left(1 + \frac{2+\theta}{q-1}\right)\right)^{1/(q-1)}.
\end{aligned}$$

By (f_3), there exists $l_0 > 0$ such that

$$-w'' \geqslant \lambda\frac{w}{s^2} - (b+\varepsilon)s^\theta w^q \quad \text{in } (l_0, \infty) \tag{3.19}$$

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and

$$-w'' \leqslant \lambda \frac{w}{s^2} - (b-\varepsilon)s^\theta w^q \quad \text{in } (l_0,\infty). \tag{3.20}$$

From (3.19) and (3.20), we shall prove that

$$\left.\begin{aligned} \liminf_{s\to\infty} s^{(2+\theta)/(q-1)} w(s) &\geqslant \underline{\rho}_\varepsilon, \\ \limsup_{s\to\infty} s^{(2+\theta)/(q-1)} w(s) &\leqslant \bar{\rho}_\varepsilon. \end{aligned}\right\} \tag{3.21}$$

If (3.21) holds, by the arbitrariness of ε, (3.18) holds. Here, we only prove the first inequality in (3.21). Since the second inequality can be proved by the similar method, we omit the proof.

Step 1. We claim that $w(s) \geqslant V(s)$ for all $s \in [l_0,\infty)$, where V is the unique positive solution of

$$\left.\begin{aligned} -u'' = \lambda \frac{u}{s^2} - (b+\varepsilon)s^\theta u^q, \quad s\in(l_0,\infty), \\ u(l_0) = 0. \end{aligned}\right\} \tag{3.22}$$

By lemma 2.2(ii) and $\lambda > \frac{1}{4}$, there exists $L_0 > l_0$ so that $\lambda > \lambda_1[(l_0,L),1/s^2]$ holds for any $L > L_0$. By the standard arguments of logistic equations,

$$\left.\begin{aligned} -u'' = \lambda \frac{u}{s^2} - (b+\varepsilon)s^\theta u^q, \quad s\in(l_0,L), \\ u(l_0) = u(L) = 0 \end{aligned}\right\} \tag{3.23}$$

has a unique positive solution V_L. From the standard arguments of boundary blow-up problems, it follows that

$$\begin{gathered} -u'' = \lambda \frac{u}{s^2} - (b+\varepsilon)s^\theta u^q, \quad s\in(l_0,L), \\ u(l_0) = 0, \qquad u(L) = \infty \end{gathered}$$

has a unique positive solution W_L. By the comparison principle,

$$V_L(s) \leqslant W_L(s) \quad \text{for } s\in(l_0,L),$$

V_L is non-decreasing in L and W_L is non-increasing in L. So,

$$V(s) := \lim_{L\to\infty} V_L(s) \quad \text{and} \quad W(s) := \lim_{L\to\infty} W_L(s) \text{ is well defined in } (l_0,\infty).$$

By regularity arguments of elliptic equations, V and W solve (3.22). By the comparison principle and (3.19), $V_L(s) \leqslant w(s)$ in $[l_0,L]$. Letting $L\to\infty$, we have $w(s) \geqslant V(s)$ in $[l_0,\infty)$.

To complete step 1, we now prove the uniqueness of positive solutions to (3.22). By the comparison principle, it can easily be seen that V is the minimal positive solution of (3.22) and W is the maximal positive solution of (3.22). Similar to the proof of propositions 3.1 and 3.2, there exist $S_0 > L_0$, $C_1 > 0$, $C_2 > 0$ such that

$$C_1 s^{-(2+\theta)/(q-1)} \leqslant V(s) \leqslant W(s) \leqslant C_2 s^{-(2+\theta)/(q-1)} \quad \text{when } s \geqslant S_0.$$

Further, by Hopf's lemma, without loss of generality, we have

$$C_1 s^{-(2+\theta)/(q-1)} \leqslant V(s) \leqslant W(s) \leqslant C_2 s^{-(2+\theta)/(q-1)} \quad \text{when } s \geqslant l_0.$$

By the standard method, we can obtain the uniqueness of positive solutions to (3.22).

Define $\phi(s) = \underline{\rho}_\varepsilon s^{-(2+\theta)/(q-1)}$. Then

$$-\phi'' = \lambda \frac{\phi}{s^2} - (b+\varepsilon)s^\theta \phi^q \quad \text{in } (0,\infty).$$

Similarly, from the comparison principle, it follows that $\phi(s) \geqslant V_L(s)$ for $s \in [l_0, L]$. Letting $L \to \infty$, we have $\phi(s) \geqslant V(s)$ in $[l_0,\infty)$. By the strong maximum principle, $\phi(s) > V(s)$ in $[l_0,\infty)$. Define

$$\alpha := \inf_{s\in(l_0,\infty)} \frac{\phi(s)}{V(s)}.$$

Then, $\alpha \geqslant 1$. If $\lim_{s\to\infty} \phi(s)/V(s) = 1$, then as $w(s) \geqslant V(s)$, for all $s \geqslant l_0$, we can obtain

$$\liminf_{s\to\infty} s^{(2+\theta)/(q-1)} w(s) \geqslant \underline{\rho}_\varepsilon.$$

STEP 2. We claim $\alpha = 1$. Suppose that this is not true. Then, there must be $\alpha > 1$. By the definition of α, we have

$$\phi(s) \geqslant \alpha V(s) \quad \text{for } s \in [l_0,\infty). \tag{3.24}$$

We claim that

$$\phi(s) > \alpha V(s) \quad \text{for } s \in [l_0,\infty). \tag{3.25}$$

If the inequality does not hold, then there exists $r_0 > l_0$ such that $\phi(r_0) = \alpha V(r_0)$. Since αV is a supersolution of

$$\left.\begin{aligned} -u'' = \lambda \frac{u}{s^2} - (b+\varepsilon)s^\theta u^q, \quad & s \in (r_0, L), \\ u(r_0) = \phi(r_0), \qquad & u(L) = 0, \end{aligned}\right\} \tag{3.26}$$

by the comparison principle, $\alpha V(s) \geqslant \phi_L(s)$ in (r_0, L), where ϕ_L is the unique positive solution of (3.26). Similarly to the proof of uniqueness, we can derive $\phi(s) = \lim_{L\to\infty} \phi_L(s)$ in $[r_0,\infty)$. So, we have $\alpha V(s) \geqslant \phi(s)$ in $[r_0,\infty)$. By the strong maximum principle,

$$\alpha V(s) > \phi(s) \quad \text{in } (r_0,\infty),$$

which is a contradiction to (3.24).

By (3.25) and the definition of α, we have $\liminf_{s\to\infty} \phi(s)/V(s) = \alpha$. We claim

$$\lim_{s\to\infty} \frac{\phi(s)}{V(s)} = \alpha. \tag{3.27}$$

Otherwise, there exists $r_0 > l_0$ such that r_0 is a minimal point of $\phi(s)/V(s)$ and $\phi(r_0)/V(r_0) > \alpha > 1$. Hence, we see that

$$\phi'(r_0) - \frac{\phi(r_0)}{V(r_0)} V'(r_0) = 0. \tag{3.28}$$

Since $\phi(r_0)V(s)/V(r_0)$ is a supersolution of (3.26), by the comparison principle,

$$\frac{\phi(r_0)}{V(r_0)}V(s) \geqslant \phi_L(s) \quad \text{for } s \in (r_0, L).$$

Letting $L \to \infty$, we see that

$$\frac{\phi(r_0)}{V(r_0)}V(s) \geqslant \phi(s) \quad \text{for } s \in (r_0, \infty).$$

By the strong maximum principle,

$$\frac{\phi(r_0)}{V(r_0)}V(s) > \phi(s) \quad \text{for } s \in (r_0, \infty).$$

By Hopf's lemma and (3.28), we obtain a contradiction.

Define $\gamma = \frac{1}{2}(1+\alpha)$ and $c \in (0,1)$ to be determined later. Let

$$\bar{w} = \gamma V - \frac{\phi - \gamma V}{c},$$

i.e.

$$\frac{c}{c+1}\bar{w} + \frac{1}{c+1}\phi = \gamma V.$$

By the property of convex functions and setting $\gamma > 1$, we have

$$-\bar{w}'' \geqslant \lambda\frac{\bar{w}}{s^2} - (b+\varepsilon)s^\theta \bar{w}^q \quad \text{in } (r_0, \infty). \tag{3.29}$$

Since $\lim_{s\to\infty} \phi(s)/V(s) = \alpha$ for sufficiently small $\mu > 0$, there exists $L_1 > r_0$ such that $\phi(s) < (\alpha+\mu)V(s)$ for all $s \geqslant L_1$. Now, we restrict c to satisfy $c \in ((\alpha+2\mu-1)/(\alpha+1), 1)$. Then we see that $\frac{1}{2}(c+1)(\alpha+1) > \alpha+\mu$. So, we derive

$$\bar{w} = \gamma V - \frac{\phi - \gamma V}{c} > 0 \quad \text{when } s \geqslant L_1. \tag{3.30}$$

By (3.29) and (3.30), $\bar{w}$ is a supersolution of

$$\left.\begin{aligned} -u'' = \lambda\frac{u}{s^2} - (b+\varepsilon)s^\theta u^q, \quad s \in (L_1, \infty),\\ u(L_1) = 0. \end{aligned}\right\} \tag{3.31}$$

Define

$$V(s; L_1) := \left(\frac{l_0}{L_1}\right)^{(2+\theta)/(q-1)} V\left(\frac{l_0}{L_1}s\right).$$

Then by calculation, $V(s;L_1)$ solves (3.31). For sufficiently large $L > \max\{L_1, L_0\}$, let $\tilde{V}_L(s;L_1)$ denote the unique positive solution of

$$\begin{gathered} -u'' = \lambda\frac{u}{s^2} - (b+\varepsilon)s^\theta u^q, \quad s \in (L_1, L), \\ u(L_1) = u(L) = 0. \end{gathered}$$

By the comparison principle, $\tilde{V}_L(s;L_1) \leqslant \bar{w}(s)$ for $s \in (L_1, L)$, and $\tilde{V}(s;L_1) := \lim_{L\to\infty} \tilde{V}_L(s;L_1)$ is well defined in $[L_1, \infty)$. By the regularity arguments, $\tilde{V}(s;L_1)$

solves (3.31). Similarly, we can show that (3.31) has a unique positive solution. Hence, there must exist $\tilde{V}(s;L_1) = V(s;L_1)$. Further, we see that

$$V(s;L_1) \leqslant \bar{w}(s) \quad \text{for } s \in (L_1, \infty).$$

Therefore, by the definition of $\bar{w}$,

$$\gamma \frac{V(s)}{V(s;L_1)} - \frac{1}{c}\left[\frac{\phi(s)}{V(s;L_1)} - \gamma \frac{V(s)}{V(s;L_1)}\right] \geqslant 1 \quad \text{when } s > L_1. \tag{3.32}$$

Obviously,

$$\phi(s;L_1) := \left(\frac{l_0}{L_1}\right)^{(2+\theta)/(q-1)} \phi\left(\frac{l_0}{L_1}s\right) \equiv \phi(s).$$

From (3.27), it follows that

$$\lim_{s\to\infty} \frac{\phi(s)}{V(s;L_1)} = \alpha, \qquad \lim_{s\to\infty} \frac{V(s)}{V(s;L_1)} = \lim_{s\to\infty} \frac{V(s)}{\phi(s)} \frac{\phi(s;L_1)}{V(s;L_1)} = 1.$$

Letting $s \to \infty$ in (3.32), we see that $\gamma - (\alpha - \gamma)/c \geqslant 1$, which is equivalent to $c \geqslant 1$. But this is a contradiction to $c \in ((\alpha + 2\mu - 1)/(\alpha + 1), 1)$.

STEP 3. We claim that

$$\lim_{s\to\infty} \frac{\phi(s)}{V(s)} = 1.$$

Since $\alpha = 1$ and $\phi(s) > V(s)$ for $s > l_0$, we deduce that $\liminf_{s\to\infty} \phi(s)/V(s) = 1$. It suffices to show that $\limsup_{s\to\infty} \phi(s)/V(s) \leqslant 1$. Assume on the contrary that

$$\limsup_{s\to\infty} \frac{\phi(s)}{V(s)} > 1.$$

Then there exists $\delta_0 > l_0$ such that δ_0 is a minimal point of $\phi(s)/V(s)$ and $\phi(\delta_0)/V(\delta_0) > 1$. Hence, we have

$$\phi'(\delta_0) - \frac{\phi(\delta_0)}{V(\delta_0)} V'(\delta_0) = 0.$$

Similarly to the proof in step 2, we obtain a contradiction. □

4. Proof of theorem 1.2

In a half-space, (1.1) is equivalent to (1.6). Consider the following problem:

$$\left.\begin{aligned} -\Delta u &= \lambda \frac{u}{x_1^2} - x_1^\theta f(u), & x &\in T_R^\delta, \\ u &= 0, & x &\in \partial T_R^\delta, \end{aligned}\right\} \tag{4.1}$$

and

$$\left.\begin{aligned} -\Delta u &= \lambda \frac{u}{x_1^2} - x_1^\theta f(u), & x &\in T_R^\delta, \\ u &= \infty, & x &\in \partial T_R^\delta. \end{aligned}\right\} \tag{4.2}$$

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For fixed $\lambda > \frac{1}{4}$, by lemma 2.2, there exist $R_0 > 0$ and δ_0 such that $\lambda_1[T_{R_0}^{\delta_0}, 1/x_1^2] < \lambda$. By the monotone property of the first eigenvalue with respect to domains, for arbitrary $\delta \in (0, \delta_0]$ and $R \geqslant R_0$, $\lambda_1[T_R^\delta, 1/x_1^2] < \lambda$. By the standard arguments of logistic equations and boundary blow-up problems, both (4.1) and (4.2) have unique positive solutions, and we denote these by $v_{R,\delta}$ and $V_{R,\delta}$, respectively. By the comparison principle, we have that

$$v_{R,\delta}(x) \leqslant V_{R,\delta}(x) \quad \text{for all } x \in T_R^\delta.$$

From the comparison principle, we also see that $v_{R,\delta}$ is non-decreasing when R increases or δ decreases, and $V_{R,\delta}$ is non-increasing when R increases or δ decreases. So,

$$v_*(x) := \lim_{R\to\infty} \lim_{\delta\to 0} v_{R,\delta}(x) \quad \text{and} \quad V_*(x) := \lim_{R\to\infty} \lim_{\delta\to 0} V_{R,\delta}(x)$$

are well-defined in T. As in the proof of proposition 2.3, it is easily proved that v_* is the minimal positive solution of (1.6) and V_* is the maximal positive solution of (1.6).

To complete the proof of theorem 1.2, it suffices to show that $v_* = V_*$ is a function of x_1 only, and satisfies (1.9). Since v_* and V_* are the minimal positive solution and the maximal positive solution of (1.6), respectively, the invariance-of-rotation transformation for the Laplacian operator implies that both v_* and V_* are functions of x_1 only. For convenience, we also define $v_*(x_1) = v_*(x)$ and $V_*(x_1) = V_*(x)$. Then, both $v_*(x_1)$ and $V_*(x_1)$ solve (1.7). The uniqueness in theorem 1.1 implies $v_* = V_*$. Defining $W(x) := v_*(x) = V_*(x)$, (1.8) in theorem 1.1 implies that W satisfies (1.9).

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References

1 G. Alberti, L. Ambrosio and X. Cabré. On a long-standing conjecture of E. DeGiorgi: symmetry in 3D for general nonlinearities and a local minimality property. *Acta Appl. Math.* **65** (2001), 9–33.

2 S. B. Angenent. Uniqueness of the solutions of a semilinear boundary value problem. *Math. Annalen* **272** (1985), 129–138.

3 C. Bandle and M. Marcus. Asymptotic behaviour of solutions and their derivatives, for semilinear elliptic problems with blowup on the boundary. *Annales Inst. H. Poincaré Analyse Non Linéaire* **12** (1995), 155–171.

4 C. Bandle and M. Marcus. Dependence of blowup rate of large solutions of semilinear elliptic equations, on the curvature of the boundary. *Complex Variables* **49** (2004), 555–570.

5 C. Bandle, V. Moroz and W. Reichel. 'Boundary blowup' type sub-solutions to semilinear elliptic equations with Hardy potential. *J. Lond. Math. Soc.* **77** (2008), 503–523.

6 C. Bandle, V. Moroz and W. Reichel. Large solutions to semilinear elliptic equations with Hardy potential and exponential nonlinearity. In *Around the research of Vladimir Maz′ya II*, International Mathematical Series, vol. 12, pp. 1–22 (Springer, 2012).

7 H. Berestycki, L. Caffarelli and L. Nirenberg. Further qualitative properties for elliptic equations in unbounded domains. *Annali Scuola Norm. Sup. Pisa IV* **25** (1997), 69–94.

8 H. Brézis and M. Marcus. Hardy's inequalities revisited. *Annali Scuola Norm. Sup. Pisa IV* **25** (1997), 217–237.

9 F. C. Cîrstea. *A complete classification of the isolated singularities for nonlinear elliptic equations with inverse square potentials.* Memoirs of the American Mathematical Society, vol. 227 (Providence, RI: American Mathematical Socity, 2014).

10 F. C. Cîrstea and Y. Du. General uniqueness results and variation speed for blow-up solutions of elliptic equations. *Proc. Lond. Math. Soc.* **91** (2005), 459–482.

11 F. C. Cîrstea and Y. Du. Asymptotic behavior of solutions of semilinear elliptic equations near an isolated singularity. *J. Funct. Analysis* **250** (2007), 317–346.

12 F. C. Cîrstea and Y. Du. Isolated singularities for weighted quasilinear elliptic equations. *J. Funct. Analysis* **259** (2010), 174–202.

13 W. Dong. Symmetry for boundary blow up solutions of elliptic equations in a half-space. *Nonlin. Analysis* **58** (2004), 159–173.

14 Y. Du. *Order structure and topological methods in nonlinear partial differential equations*, volume 1: *Maximum principle and applications* (Singapore, World Scientific Publishing, 2006).

15 Y. Du and Z. M. Guo. Boundary blow-up solutions and their applications in quasilinear elliptic problems. *J. Analyse Math.* **89** (2003), 277–302.

16 Y. Du and Z. M. Guo. Symmetry for elliptic equations in a half-space without strong maximum principle. *Proc. R. Soc. Edinb.* A **134** (2004), 259–269.

17 Y. Du and Q. Huang. Blow-up solutions for a class of semilinear elliptic and parabolic equations. *SIAM J. Math. Analysis* **31** (1999), 1–18.

18 Y. Du and L. S. Liu. Remarks on the uniqueness problem for the logistic equations on the entire space. *Bull. Austral. Math. Soc.* **73** (2006), 129–137.

19 Y. Du and L. Ma. Logistic type equations on $\mathbb{R}^N$ by a squeezing method involving boundary blow-up solutions. *J. Lond. Math. Soc.* **64** (2001), 107–124.

20 Y. Du and L. Wei. Boundary behavior of positive solutions to nonlinear elliptic equations with Hardy potential. *J. Lond. Math. Soc.* **91** (2015), 731–749.

21 J. García-Melián. Uniqueness for boundary blow-up problems with continuous weights. *Proc. Am. Math. Soc.* **135** (2007), 2785–2793.

22 J. García-Melián, R. Letelier Albornoz and J. Sabina de Lis. Uniqueness and asymptotic behaviour for solutions of semilinear problems with boundary blow-up. *Proc. Am. Math. Soc.* **129** (2001), 3593–3602.

23 B. Guerch and L. Véron. Local properties of stationary solutions of some nonlinear singular Schrödinger equations. *Rev. Mat. Ibero.* **7** (1991), 65–114.

24 Z. M. Guo and J. C. Wei. Symmetry of non-negative solutions of a semilinear elliptic equation with singular nonlinearity. *Proc. R. Soc. Edinb.* A **137** (2007), 963–994.

25 J. López-Gómez. The boundary blow-up rate of large solutions. *J. Diff. Eqns* **195** (2003), 25–45.

26 M. Marcus and L. Véron. Uniqueness and asymptotic behavior of solutions with boundary blow-up for a class of nonlinear elliptic equations. *Annales Inst. H. Poincaré Analyse Non Linéaire* **14** (1997), 237–274.

27 M. Marcus and L. Véron. Existence and uniqueness results for large solutions of general nonlinear elliptic equations. *J. Evol. Eqns* **3** (2003), 637–652.

28 M. Marcus, V. Mizel and Y. Pinchover. On the best constant for Hardy's inequality in $\mathbb{R}^n$. *Trans. Am. Math. Soc.* **350** (1998), 3237–3255.

29 J. Tidblom. A Hardy inequality in the half-space. *J. Funct. Analysis* **221** (2005), 482–495.

30 L. Wei and Y. Du. Exact singular behavior of positive solutions to nonlinear elliptic equations with a Hardy potential. Preprint, 2014. (Available at http://maths.xznu.edu.cn/_upload/article/5b/56/4435e2004c63a3cb129e34f2dba3/6a43fec1-5632-4c2d-8c65-cc25726c2669.pdf.)

31 L. Wei and Z. Feng. Isolated singularity for semilinear elliptic equations. *Discrete Contin. Dynam. Syst.* **35** (2015), 3239–3252.