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A q-SUPERCONGRUENCE MODULO THE THIRD POWER OF A CYCLOTOMIC POLYNOMIAL

CHUANAN WEI

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Abstract

We derive a q-supercongruence modulo the third power of a cyclotomic polynomial with the help of Guo and Zudilin's method of creative microscoping ['A q-microscope for supercongruences', *Adv. Math.* **346** (2019), 329–358] and the q-Dixon formula. As consequences, we give several supercongruences including

$$\sum_{k=0}^{(p-2)/3} \frac{(\frac{2}{3})_k^3}{(1)_k^3} \equiv \frac{p}{2} \frac{(1)_{(p-2)/3}}{(\frac{4}{3})_{(p-2)/3}} \pmod{p^3},$$

where p is a prime with $p \equiv 5 \pmod 6$.

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1. Introduction

For any complex variable x, define the shifted-factorial by

$$(x)_0 = 1 \quad \text{and} \quad (x)_n = x(x+1)\cdots(x+n-1) \quad \text{for } n \in \mathbb{Z}^+,$$

and let $\Gamma_p(x)$ denote the p-adic Gamma function. Throughout the paper, p denotes an odd prime. In 1997, Van Hamme [9, (H.2)] proved that

$$\sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^3}{(1)_k^3} \equiv \begin{cases} -\Gamma_p\Big(\dfrac{1}{4}\Big)^4 \pmod{p^2} & \text{if } p \equiv 1 \pmod 4, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{1.1}$$

In 2016, Long and Ramakrishna [6] gave an extension of (1.1):

$$\sum_{k=0}^{(p-1)/2} \frac{(\frac{1}{2})_k^3}{(1)_k^3} \equiv \begin{cases} -\Gamma_p\Big(\dfrac{1}{4}\Big)^4 \pmod{p^3} & \text{if } p \equiv 1 \pmod 4, \\ -\dfrac{p^2}{16}\Gamma_p\Big(\dfrac{1}{4}\Big)^4 \pmod{p^3} & \text{if } p \equiv 3 \pmod 4. \end{cases} \tag{1.2}$$

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In the same year, Deines *et al.* [1] discovered the nice supercongruence: for $p \equiv 1 \pmod 6$,

$$\sum_{k=0}^{p-1} \frac{(\frac{2}{3})_k^3}{(1)_k^3} \equiv -\Gamma_p\Big(\frac{1}{3}\Big)^3 \pmod{p^3}.$$

Several years later, Mao and Pan [7] (see also Sun [8, Theorem 1.3]) found a result similar to (1.1): for $p \equiv 1 \pmod 4$,

$$\sum_{k=0}^{(p+1)/2} \frac{(-\frac{1}{2})_k^3}{(1)_k^3} \equiv 0 \pmod{p^2}. \tag{1.3}$$

For any complex numbers x and q, define the q-shifted factorial by

$$(x;q)_0 = 1 \quad \text{and} \quad (x;q)_n = (1-x)(1-xq)\cdots(1-xq^{n-1}) \quad \text{for } n \in \mathbb{Z}^+.$$

For simplicity, we also adopt the compact notation

$$(x_1, x_2, \ldots, x_m; q)_n = (x_1;q)_n (x_2;q)_n \cdots (x_m;q)_n,$$

where $m \in \mathbb{Z}^+$ and $n \in \mathbb{Z}^+ \cup \{0, \infty\}$. Let $[n] = (1-q^n)/(1-q)$ be the q-integer and let $\Phi_n(q)$ stand for the nth cyclotomic polynomial in q:

$$\Phi_n(q) = \prod_{\substack{1 \leqslant k \leqslant n \\ \gcd(k,n)=1}} (q - \zeta^k),$$

where ζ is a primitive nth root of unity. Recently, Wei [11] and Wang [10] established q-analogues of (1.2) for the first case: if $n \equiv 1 \pmod 4$, then modulo $\Phi_n(q)^3$,

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2 (q^2;q^4)_k}{(q^2;q^2)_k^2 (q^4;q^4)_k} q^{2k} \equiv q^{(n-1)/2} \frac{(q^2;q^4)_{(n-1)/4}^2}{(q^4;q^4)_{(n-1)/4}^2} \Big\{ 1 + 2[n]^2 \sum_{i=1}^{(n-1)/4} \frac{q^{4i-2}}{[4i-2]^2} \Big\},$$

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2 (q^2;q^4)_k}{(q^2;q^2)_k^2 (q^4;q^4)_k} q^{2k} \equiv [n] \frac{(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}} + [n]^3 \sum_{k=0}^{(n-3)/2} \frac{(1+q^{2k+1})(q^3;q^4)_k}{[2k+1]^2 (q^5;q^4)_k} q^{2k+1}.$$

Guo and Zudilin [5] and Guo [3] gave q-analogues of (1.2) for the second case: if $n \equiv 3 \pmod 4$, then

$$\sum_{k=0}^{(n-1)/2} \frac{1+q^{1+4k}}{1+q} \frac{(q^2;q^4)_k^3}{(q^4;q^4)_k^3} q^k \equiv [n]_{q^2} \frac{(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}} q^{(1-n)/2} \pmod{\Phi_n(q)^3},$$

$$\sum_{k=0}^{(n-1)/2} \frac{(q;q^2)_k^2 (q^2;q^4)_k}{(q^2;q^2)_k^2 (q^4;q^4)_k} q^{2k} \equiv [n] \frac{(q^3;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}} \pmod{\Phi_n(q)^3}. \tag{1.4}$$

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Guo and Zudilin [5] also found the q-supercongruence: for any positive integer $n > 1$ with $n \equiv 1 \pmod 4$,

$$\sum_{k=0}^{(n+1)/2} \frac{1+q^{4k-1}}{1+q} \frac{(q^{-2};q^4)_k^3}{(q^4;q^4)_k^3} q^{7k} \equiv [n]_{q^2} \frac{(q;q^4)_{(n-1)/2}}{(q^5;q^4)_{(n-1)/2}} q^{(n-3)/2} \pmod{\Phi_n(q)^3}. \tag{1.5}$$

Setting $n = p$ and then letting $q \to 1$ in (1.5), they obtain the extension of (1.3): for $p \equiv 1 \pmod 4$,

$$\sum_{k=0}^{(p+1)/2} \frac{(-\frac{1}{2})_k^3}{k!^3} \equiv p \frac{(\frac{1}{4})_{(p-1)/2}}{(\frac{7}{4})_{(p-1)/2}} \pmod{p^3}.$$

Motivated by the results just mentioned, we shall establish the following theorem.

THEOREM 1.1. *Let n, d be positive integers such that $n - dn + 2d \le r \le n + d$, $\gcd(n, d) = 1$ and $n \equiv r \pmod{2d}$. Then*

$$\sum_{k=0}^{(n-r+d)/d} \frac{1+q^{r-d+2dk}}{1+q^{r-d}} \frac{(q^{2r-2d};q^{2d})_k^3}{(q^{2d};q^{2d})_k^3} q^{(5d-3r)k}$$
$$\equiv \frac{1-q^{2n}}{1-q^{2r-2d}} \frac{(q^{3r-3d};q^{2d})_{(n-r+d)/d}}{(q^{r+d};q^{2d})_{(n-r+d)/d}} q^{(d-r)(n-r+d)/d} \pmod{\Phi_n(q)^3}.$$

Obviously, the $d = 2, r = 3$ case of Theorem 1.1 is exactly (1.4). When $d = 2, r = 1$, Theorem 1.1 reduces to (1.5). Taking $n = p$ and then letting $q \to 1$ in Theorem 1.1 gives the following supercongruence.

PROPOSITION 1.2. *Let p be an odd prime and let d be a positive integer such that $p - dp + 2d \le r \le p + d$ and $p \equiv r \pmod{2d}$. Then*

$$\sum_{k=0}^{(p-r+d)/d} \frac{(\frac{r-d}{d})_k^3}{(1)_k^3} \equiv \frac{p}{r-d} \frac{(\frac{3r-3d}{2d})_{(p-r+d)/d}}{(\frac{r+d}{2d})_{(p-r+d)/d}} \pmod{p^3}.$$

Specialising the parameters d and r, Proposition 1.2 can produce many concrete supercongruences. Six of these are given in the following corollaries.

COROLLARY 1.3. *Let p be a prime with $p \equiv 1 \pmod 6$. Then*

$$\sum_{k=0}^{(p+2)/3} \frac{(-\frac{2}{3})_k^3}{(1)_k^3} \equiv 0 \pmod{p^3}.$$

PROOF. Take $d = 3, r = 1$ in Proposition 1.2. □

COROLLARY 1.4. *Let p be a prime with $p \equiv 5 \pmod 6$. Then*

$$\sum_{k=0}^{(p-2)/3} \frac{(\frac{2}{3})_k^3}{(1)_k^3} \equiv \frac{p}{2} \frac{(1)_{(p-2)/3}}{(\frac{4}{3})_{(p-2)/3}} \pmod{p^3}.$$

PROOF. Take $d = 3, r = 5$ in Proposition 1.2. □

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COROLLARY 1.5. *Let p be a prime with $p \equiv 1 \pmod 8$. Then*

$$\sum_{k=0}^{(p+3)/4} \frac{(-\frac{3}{4})_k^3}{(1)_k^3} \equiv -\frac{p}{3}\frac{(-\frac{9}{8})_{(p+3)/4}}{(\frac{5}{8})_{(p+3)/4}} \pmod{p^3}.$$

PROOF. Take $d = 4, r = 1$ in Proposition 1.2. □

COROLLARY 1.6. *Let p be a prime with $p \equiv 3 \pmod 8$. Then*

$$\sum_{k=0}^{(p+1)/4} \frac{(-\frac{1}{4})_k^3}{(1)_k^3} \equiv -p\frac{(-\frac{3}{8})_{(p+1)/4}}{(\frac{7}{8})_{(p+1)/4}} \pmod{p^3}.$$

PROOF. Take $d = 4, r = 3$ in Proposition 1.2. □

COROLLARY 1.7. *Let p be a prime with $p \equiv 5 \pmod 8$. Then*

$$\sum_{k=0}^{(p-1)/4} \frac{(\frac{1}{4})_k^3}{(1)_k^3} \equiv p\frac{(\frac{3}{8})_{(p-1)/4}}{(\frac{9}{8})_{(p-1)/4}} \pmod{p^3}.$$

PROOF. Take $d = 4, r = 5$ in Proposition 1.2. □

COROLLARY 1.8. *Let p be a prime with $p \equiv 7 \pmod 8$. Then*

$$\sum_{k=0}^{(p-3)/4} \frac{(\frac{3}{4})_k^3}{(1)_k^3} \equiv \frac{p}{3}\frac{(\frac{9}{8})_{(p-3)/4}}{(\frac{11}{8})_{(p-3)/4}} \pmod{p^3}.$$

PROOF. Take $d = 4, r = 7$ in Proposition 1.2. □

We shall prove Theorem 1.1 in the next section by means of the creative microscoping method recently introduced by Guo and Zudilin [4].

2. Proof of Theorem 1.1

Following Gasper and Rahman [2], define the basic hypergeometric series by

$${}_{r+1}\phi_r\left[\begin{matrix} a_1, a_2, \dots, a_{r+1} \\ b_1, b_2, \dots, b_r \end{matrix}; q, z\right] = \sum_{k=0}^{\infty} \frac{(a_1, a_2, \dots, a_{r+1}; q)_k}{(q, b_1, b_2, \dots, b_r; q)_k} z^k.$$

To prove Theorem 1.1, we require the q-Dixon formula:

$${}_4\phi_3\left[\begin{matrix} a, -qa^{1/2}, b, c \\ -a^{1/2}, aq/b, aq/c \end{matrix}; q, \frac{qa^{1/2}}{bc}\right] = \frac{(aq, aq/bc, qa^{1/2}/b, qa^{1/2}/c; q)_\infty}{(aq/b, aq/c, qa^{1/2}, qa^{1/2}/bc; q)_\infty}, \tag{2.1}$$

where $|qa^{1/2}/bc| < 1$.

PROOF OF THEOREM 1.1. Setting $a = q^{-1-2n}$ in (2.1), we obtain

$${}_4\phi_3\left[\begin{matrix} q^{-1-2n}, -q^{1/2-n}, b, c \\ -q^{-1/2-n}, q^{-2n}/b, q^{-2n}/c \end{matrix}; q, \frac{q^{1/2-n}}{bc}\right] = 0.$$

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Performing the simultaneous replacements $n \mapsto (n-r)/2d$, $q \mapsto q^{2d}$, $b \mapsto bq^{2r-2d}$, $c \mapsto q^{2r-2d}/b$ in the above identity, we get

$$ {}_4\phi_3\left[\begin{matrix} q^{2r-2d-2n},\ -q^{r+d-n},\ bq^{2r-2d},\ q^{2r-2d}/b \\ -q^{r-d-n},\ q^{2d-2n}/b,\ bq^{2d-2n} \end{matrix} ; q^{2d},\ q^{5d-3r-n}\right] = 0. $$

This gives the following congruence: modulo $(a^2 - q^{2n})$,

$$ \sum_{k=0}^{(n-r+d)/d} \frac{1+q^{r-d+2dk}/a}{1+q^{r-d}/a} \frac{(q^{2r-2d}/a^2, bq^{2r-2d}, q^{2r-2d}/b; q^{2d})_k}{(q^{2d}, q^{2d}/a^2b, q^{2d}b/a^2; q^{2d})_k} \left(\frac{q^{5d-3r}}{a}\right)^k \equiv 0. \tag{2.2} $$

However, taking $c = q^{-1-2n}$ in (2.1), we obtain

$$ {}_4\phi_3\left[\begin{matrix} a,\ -qa^{1/2},\ b,\ q^{-1-2n} \\ -a^{1/2},\ aq/b,\ aq^{2+2n} \end{matrix} ; q,\ \frac{q^{2+2n}a^{1/2}}{b}\right] = \frac{(aq, qa^{1/2}/b; q)_{1+2n}}{(aq/b, qa^{1/2}; q)_{1+2n}}. $$

Employing the substitutions $n \mapsto (n-r)/2d$, $q \mapsto q^{2d}$, $a \mapsto q^{2r-2d}/a^2$, $b \mapsto q^{2r-2d+2n}$, we arrive at

$$ {}_4\phi_3\left[\begin{matrix} q^{2r-2d}/a^2,\ -q^{r+d}/a,\ q^{2r-2d+2n},\ q^{2r-2d-2n} \\ -q^{r-d}/a,\ q^{2d-2n}/a^2,\ q^{2d+2n}/a^2 \end{matrix} ; q^{2d},\ \frac{q^{5d-3r}}{a}\right] $$
$$ = (aq^{d-r})^{(n-r+d)/d} \frac{(q^{2r}/a^2, aq^{3r-3d}; q^{2d})_{(n-r+d)/d}}{(q^{r+d}/a, a^2q^{2r-2d}; q)_{(n-r+d)/d}}. $$

This gives the following congruence: modulo $(b - q^{2n})(1 - bq^{2n})$,

$$ \sum_{k=0}^{(n-r+d)/d} \frac{1+q^{r-d+2dk}/a}{1+q^{r-d}/a} \frac{(q^{2r-2d}/a^2, bq^{2r-2d}, q^{2r-2d}/b; q^{2d})_k}{(q^{2d}, q^{2d}/a^2b, q^{2d}b/a^2; q^{2d})_k} \left(\frac{q^{5d-3r}}{a}\right)^k $$
$$ \equiv (aq^{d-r})^{(n-r+d)/d} \frac{(q^{2r}/a^2, aq^{3r-3d}; q^{2d})_{(n-r+d)/d}}{(q^{r+d}/a, a^2q^{2r-2d}; q)_{(n-r+d)/d}}. \tag{2.3} $$

Note that the right-hand side of (2.2) can also be written as that of (2.3), since $(q^{2r}/a^2, aq^{3r-3d}; q^{2d})_{(n-r+d)/d}$ contains the factor $a^2 - q^{2n}$. It is clear that the polynomials $(a^2 - q^{2n})$ and $(b - q^{2n})(1 - bq^{2n})$ are relatively prime. Therefore, from (2.2) and (2.3) we deduce that, modulo $(a^2 - q^{2n})(b - q^{2n})(1 - bq^{2n})$,

$$ \sum_{k=0}^{(n-r+d)/d} \frac{1+q^{r-d+2dk}/a}{1+q^{r-d}/a} \frac{(q^{2r-2d}/a^2, bq^{2r-2d}, q^{2r-2d}/b; q^{2d})_k}{(q^{2d}, q^{2d}/a^2b, q^{2d}b/a^2; q^{2d})_k} \left(\frac{q^{5d-3r}}{a}\right)^k $$
$$ \equiv (aq^{d-r})^{(n-r+d)/d} \frac{(q^{2r}/a^2, aq^{3r-3d}; q^{2d})_{(n-r+d)/d}}{(q^{r+d}/a, a^2q^{2r-2d}; q)_{(n-r+d)/d}}. \tag{2.4} $$

Letting $a \to 1$, $b \to 1$ in (2.4), we are led to the q-supercongruence in Theorem 1.1. □

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3. Two open problems

Numerical calculations indicate the following two open problems related to Corollaries 1.3 and 1.5.

CONJECTURE 3.1. Let p be a prime with $p \equiv 1 \pmod 6$ and let s be a positive integer. Then

$$\sum_{k=0}^{(p^s+2)/3} \frac{(-\frac{2}{3})_k^3}{(1)_k^3} \equiv 0 \pmod{p^{3s}}.$$

CONJECTURE 3.2. Let p be a prime with $p \equiv 1 \pmod 8$ and let s be a positive integer. Then

$$\sum_{k=0}^{(p^s+3)/4} \frac{(-\frac{3}{4})_k^3}{(1)_k^3} \equiv -\frac{p^s}{3} \frac{(-\frac{9}{8})_{(p^s+3)/4}}{(\frac{5}{8})_{(p^s+3)/4}} \pmod{p^{s+2}}.$$

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CHUANAN WEI, School of Biomedical Information and Engineering,
Hainan Medical University, Haikou 571199, PR China
e-mail: weichuanan78@163.com

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