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PARITY RESULTS FOR PARTITIONS WHEREIN EACH PART APPEARS AN ODD NUMBER OF TIMES

MICHAEL D. HIRSCHHORN and JAMES A. SELLERS✉

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Abstract

We consider the function $f(n)$ that enumerates partitions of weight n wherein each part appears an odd number of times. Chern ['Unlimited parity alternating partitions', *Quaest. Math.* (to appear)] noted that such partitions can be placed in one-to-one correspondence with the partitions of n which he calls unlimited parity alternating partitions with smallest part odd. Our goal is to study the parity of $f(n)$ in detail. In particular, we prove a characterisation of $f(2n)$ modulo 2 which implies that there are infinitely many Ramanujan-like congruences modulo 2 satisfied by the function f. The proof techniques are elementary and involve classical generating function dissection tools.

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1. Introduction

In a recent note, Chern [2] defined the function $pa_o(n)$ to be the number of unlimited parity alternating partitions of n with smallest part odd. Chern's work is motivated by work of Andrews [1] who defined a partition of n as 'parity alternating' if the parts of the partition in question alternate in parity.

Chern notes in passing that $pa_o(n)$ also counts the number of partitions of n in which each part appears an odd number of times. (Indeed, one can place the unlimited parity alternating partitions of n with smallest part odd and the partitions of n in which each part appears an odd number of times in one-to-one correspondence via conjugation.)

In order to simplify the notation, we let $f(n)$ be the number of partitions of n in which each part appears an odd number of times. Our primary goal in this note is to prove the following characterisation of $f(2n)$ modulo 2.

THEOREM 1.1. *For all $n \geq 0$,*

$$f(2n) \equiv \begin{cases} 1 \pmod 2 & \text{if } n = k^2 \text{ for some integer } k \text{ with } 3 \nmid k, \\ 0 \pmod 2 & \text{otherwise.} \end{cases}$$

At the conclusion of the note, we will highlight infinite families of Ramanujan-like congruences modulo 2 that are satisfied by f. We will also note how Theorem 1.1 implies a characterisation modulo 2 of $a_3(n)$, the number of 3-cores of n (see [4]).

2. An elementary generating function proof

In order to prove Theorem 1.1, we will utilise some well-known generating function results and elementary manipulations thereof. We describe this foundation here.

We begin by setting some standard notation. In particular, we define $(a;q)_\infty$, which is the usual Pochhammer symbol, to be

$$(a;q)_\infty = (1-a)(1-aq)(1-aq^2)(1-aq^3)\dots$$

Next, we provide three important lemmas.

Lemma 2.1.

$$\frac{(q;q)_\infty}{(q^3;q^3)_\infty} = \frac{(q^2;q^2)_\infty}{(q^6;q^6)_\infty^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n}.$$

Proof. Observe that

$$\begin{aligned}\sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n} &= (q;q^6)_\infty(q^5;q^6)_\infty(q^6;q^6)_\infty \\ &= \frac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty}.\end{aligned}$$

The result follows. □

Lemma 2.2.

$$\frac{(q^3;q^3)_\infty^3}{(q;q)_\infty} \equiv \sum_{n=-\infty}^{\infty} q^{3n^2-2n} \pmod 2.$$

Proof. Working modulo 2,

$$\begin{aligned}\sum_{n=-\infty}^{\infty} q^{3n^2-2n} &\equiv \sum_{n=-\infty}^{\infty}(-1)^n q^{3n^2-2n} \pmod 2 \\ &= \frac{(q;q)_\infty(q^6;q^6)_\infty^2}{(q^2;q^2)_\infty(q^3;q^3)_\infty} \\ &\equiv \frac{(q;q)_\infty(q^3;q^3)_\infty^4}{(q;q)_\infty^2(q^3;q^3)_\infty} \pmod 2 \\ &= \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty}.\end{aligned}$$

□

As an aside, we note that Lemma 2.2 yields a mod 2 characterisation for the number of 3-core partitions of n [4]. We will return to this observation at the end of this paper.

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Lemma 2.3. *If, as usual,*

$$\psi(q)=\sum_{n\geq 0}q^{(n^2+n)/2}=\frac{(q^2;q^2)_\infty^2}{(q;q)_\infty}\quad \textit{and}\quad \Pi(q)=\sum_{n=-\infty}^{\infty}q^{(3n^2-n)/2},$$

then

$$\psi(q)=\Pi(q)+q\psi(q^9).$$

Proof. See [3, Ch. 1]. □

We are now in a position to prove Theorem 1.1.

Proof of Theorem 1.1.

$$\begin{aligned}
\sum_{n\geq 0}f(n)q^n &= \prod_{n\geq 1}\left(1+\frac{q^n}{1-q^{2n}}\right)\\
&= \prod_{n\geq 1}\frac{1+q^n-q^{2n}}{1-q^{2n}}\\
&\equiv \prod_{n\geq 1}\frac{1+q^n+q^{2n}}{1-q^{2n}} \pmod 2\\
&= \prod_{n\geq 1}\frac{(1-q^{3n})}{(1-q^n)(1-q^{2n})}\\
&= \frac{(q^3;q^3)_\infty}{(q;q)_\infty(q^2;q^2)_\infty}\\
&= \frac{(q^3;q^3)_\infty^2}{(q;q)_\infty^2(q^2;q^2)_\infty}\cdot\frac{(q;q)_\infty}{(q^3;q^3)_\infty}\\
&\equiv \frac{(q^6;q^6)_\infty}{(q^2;q^2)_\infty^2}\cdot\frac{(q;q)_\infty}{(q^3;q^3)_\infty} \pmod 2\\
&= \frac{(q^6;q^6)_\infty}{(q^2;q^2)_\infty^2}\cdot\frac{(q^2;q^2)_\infty}{(q^6;q^6)_\infty^2}\sum_{n=-\infty}^{\infty}(-1)^nq^{3n^2-2n} \quad \text{by Lemma 2.1}\\
&= \frac{1}{(q^2;q^2)_\infty(q^6;q^6)_\infty}\sum_{n=-\infty}^{\infty}(-1)^nq^{3n^2-2n}\\
&= \frac{1}{(q^2;q^2)_\infty(q^6;q^6)_\infty}\left(\sum_{n=-\infty}^{\infty}q^{12n^2-4n}-q\sum_{n=-\infty}^{\infty}q^{12n^2-8n}\right).
\end{aligned}$$

It follows that

$$\begin{aligned}
\sum_{n\geq 0}f(2n)q^n &\equiv \frac{1}{(q;q)_\infty(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}q^{6n^2-2n} \pmod 2\\
&\equiv \frac{1}{(q;q)_\infty(q^3;q^3)_\infty}\sum_{n=-\infty}^{\infty}(-1)^nq^{6n^2-2n} \pmod 2
\end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{(q;q)_\infty (q^3;q^3)_\infty}(q^4;q^4)_\infty \\
&\equiv \frac{(q^2;q^2)_\infty^2}{(q;q)_\infty (q^3;q^3)_\infty} \pmod 2 \\
&= \frac{\psi(q)}{(q^3;q^3)_\infty} \\
&= \frac{\Pi(q^3) + q\psi(q^9)}{(q^3;q^3)_\infty} \quad \text{by Lemma 2.3} \\
&\equiv \frac{(q^3;q^3)_\infty + q\psi(q^9)}{(q^3;q^3)_\infty} \pmod 2 \\
&= 1 + q\frac{(q^{18};q^{18})_\infty^2}{(q^3;q^3)_\infty (q^9;q^9)_\infty} \\
&\equiv 1 + q\frac{(q^9;q^9)_\infty^4}{(q^3;q^3)_\infty (q^9;q^9)_\infty} \pmod 2 \\
&= 1 + q\frac{(q^9;q^9)_\infty^3}{(q^3;q^3)_\infty} \\
&\equiv 1 + q\sum_{n=-\infty}^{\infty} q^{9n^2-6n} \pmod 2 \quad \text{by Lemma 2.2} \\
&= 1 + \sum_{n=-\infty}^{\infty} q^{(3n-1)^2} \\
&= 1 + \sum_{n>0,\ 3\nmid n} q^{n^2}.
\end{aligned}
$$

The result follows. □

Several comments are in order as we close.

First, note that we can now prove a variety of corollaries which provide infinitely many Ramanujan-like congruences modulo 2 involving $f(2n)$. We simply need to make sure that we avoid arguments of the form $2n$ where n is square. So, although not exhaustive, we provide two such corollaries here.

Corollary 2.4. *Let $p \geq 3$ be prime and let r be a quadratic nonresidue modulo p. Then, for all $M \geq 1$ and $n \geq 0$,*

$$f(2M^2(pn + r)) \equiv 0 \pmod 2.$$

Proof. Thanks to Theorem 1.1, we need to see whether $pn + r$ can be written as $pn + r = k^2$ with $3 \nmid k$. However, note that $pn + r = k^2$ implies that $r \equiv k^2 \pmod p$. This contradicts the definition of r given in the corollary. We also know that $M^2(pn + r)$ cannot be square because it is the product of a square and a nonsquare. The result follows. □

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COROLLARY 2.5. *For all $M \geq 1$ and $n \geq 0$,*

$$f(2M^2(4n+2)) \equiv 0 \pmod{2}.$$

PROOF. Note that, for $M = 1$, the result follows because $4n + 2$ is never square. (All squares are congruent to either 0 or 1 modulo 4.) Next, we need to ask whether $M^2(4n+2)$ can ever be square. Clearly, this also cannot be the case given that $M^2(4n+2)$ is the product of a square with a nonsquare. □

Secondly, we highlight an unrelated observation about the parity of $a_3(n)$, the number of 3-core partitions of n [4]. Since the generating function for $a_3(n)$ is given by

$$\sum_{n\geq 0} a_3(n)q^n = \frac{(q^3;q^3)_\infty^3}{(q;q)_\infty},$$

it is clear that Lemma 2.2 yields the following result.

THEOREM 2.6. *For all $n \geq 0$,*

$$a_3(n) \equiv \begin{cases} 1 \pmod{2} & \text{if } n = 3m^2 + 2m \text{ for some integer } m, \\ 0 \pmod{2} & \text{otherwise.} \end{cases}$$

Finally, we note that a combinatorial proof of Theorem 1.1 would be very illuminating.

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MICHAEL D. HIRSCHHORN, School of Mathematics and Statistics,
UNSW, Sydney 2052, Australia
e-mail: m.hirschhorn@unsw.edu.au

JAMES A. SELLERS, Department of Mathematics,
Penn State University, University Park, PA 16802, USA
e-mail: sellersj@psu.edu

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