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FIRST PASSAGE PROBLEMS FOR UPWARDS SKIP-FREE RANDOM WALKS VIA THE SCALE FUNCTIONS PARADIGM

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Abstract

In this paper we develop the theory of the W and Z scale functions for right-continuous (upwards skip-free) discrete-time, discrete-space random walks, along the lines of the analogous theory for spectrally negative Lévy processes. Notably, we introduce for the first time in this context the one- and two-parameter scale functions Z, which appear for example in the joint deficit at ruin and time of ruin problems of actuarial science. Comparisons are made between the various theories of scale functions as one makes time and/or space continuous.

Keywords: Skip-free Markovian jump process; random walk; scale function; martingale; compound binomial risk model

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1. Introduction

First passage theory for random walks is a classic topic, excellently treated in, for example, [10], [15], [31], and [33], and this includes the right-continuous random walk, or what is the upwards skip-free compound binomial model of the actuarial literature. However, in light of recent developments in the parallel continuous-time theory of spectrally negative/upwards skip-free Lévy and Markov additive processes (see, for example, [2], [4], [5], [19], [20], and [35]) it seems worthwhile to revisit this topic.

Recall that in the Lévy case the scale functions $W^{(q)}$ and $Z^{(q)}$ have been known since [5] and [32], and that these functions intervene in important optimization problems. For example, $W^{(q)}$ provides the value function of the classic de Finetti problem of optimizing expected dividends until ruin with discount factor q [6], while $Z^{(q)}(\cdot, \theta)$ appears for instance in the moment generating function (with θ as argument) of the capital injections [20] and in the combined dividend payout-capital injections problem for a doubly reflected process [1], [6]. These are just two examples from an ever increasing list of problems [4], [23], [28], which can now be tackled by simple lookup in the list and using off-shelf packages computing the functions W and Z [19].

It was expected that the first passage theory developed in the world of spectrally negative Lévy processes, which we call the scale functions—or the Φ, W, Z—paradigm, should have

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parallels for other classes of spectrally negative/skip-free Markov processes. In particular, the three cases listed below, being precisely the processes with stationary independent increments that exhibit nonrandom overshoots [34] (modulo processes with monotone paths), were expected to be very similar:

(i) (discrete-time, discrete-space) right-continuous (i.e. skip-free to the right) random walks, also known in insurance as the compound binomial model;

(ii) (continuous-time, discrete-space) compound Poisson processes that live on a lattice $h\mathbb{Z}$, $h \in (0, \infty)$, jumping up only by h (what were called upwards skip-free Lévy chains in [35]);

(iii) (continuous-time, continuous-space) spectrally negative Lévy processes.

However, important steps were missing for the fully discrete setup. Notably, the second scale function $Z_v(\cdot, w)$ was absent from the previous literature and we provide below for the first time its generating function (z-transform) (14). Another contribution of our paper is spelling out the connections between the three types of first passage problems listed above. In particular, in Appendix B we provide a concise table featuring side-by-side some of the salient features of the Φ, W, Z theory for the three types of process (i)-(ii)-(iii) delineated above. It may serve as an inexhaustive summary and a quick reference; for the complete exposition, the main body of the text must be consulted.

Now, the doubly discrete (in time and space) random walk risk model [16], [30] is defined by

$$X_n = X_0 + cn - \sum_{i=1}^{n} C_i, \qquad n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\},$$

where X_0, taking values in $\mathbb{Z}$, is the initial capital, $c \in \mathbb{N}$ is the premium rate and the claims C_i, $i \in \mathbb{N}$, take values in $\mathbb{N}_0$ and are independent, identically distributed random variables with probability mass function $p_k = \mathbb{P}(C_1 = k)$ for $k \in \mathbb{N}_0$.

One advantage of the discrete setup over the more popular continuous-time models is the possibility to replace the Wiener–Hopf factorization by the conceptually simpler factorization of Laurent series (see, for example, [8] and [37]); another advantage is that one has access to Panjer recursions for computing compound distributions.

The results simplify considerably for the upwards skip-free compound binomial model obtained when $c = 1$ (see [11], [13, Section 4.1], [26], [29], [31], among others),

$$X_n = X_0 + n - \sum_{i=1}^{n} C_i, \qquad n \in \mathbb{N}_0, \tag{1}$$

that we now consider as having been fixed and to which we specialize all discussion henceforth. Ignoring its interpretation as an actuarial model, $X = (X_n)_{n \in \mathbb{N}_0}$ is nothing but a right-continuous random walk, and while our motivation for this investigation comes chiefly from risk models in the insurance context, and some actuarial-motivated vocabulary has (e.g. claims) and will (e.g. ruin probabilities) be used, the results presented are completely general and, hence, more widely applicable.

We insist throughout that $p_0 > 0$ and we let

$$\widetilde{p}(z) := \mathbb{E}[z^{C_1}] = \sum_{k=0}^{\infty} p_k z^k, \qquad z \in (0, 1],$$

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denote the probability generating function of the claims. Then, for $n \in \mathbb{N}$, (in the obvious notation) $\mathbb{E}[z^{\sum_{i=1}^{n} C_i}] = [\tilde{p}(z)]^n = (p_0 + (1 - p_0)\ \widetilde{p}_{C|C \geq 1}(z))^n$, which makes it manifest that $\sum_{i=1}^{n} C_i$, the total claims arising from n time periods, has a compound binomial distribution, explaining the name compound binomial model: at each instant in discrete time, a positive claim either occurs or not, with probability $1 - p_0$ and p_0, respectively, independently of the sizes of the positive claims.

Remark 1. By the independence of the claims we may also write, for $n \in \mathbb{N}_0$,

$$\mathbb{E}[z^{\sum_{i=1}^{n} (C_i - 1)}] = \left(\frac{\widetilde{p}(z)}{z}\right)^n \Longrightarrow \sum_{m=0}^{\infty} v^m \mathbb{E}[z^{\sum_{i=1}^{m} (C_i - 1)}] = \frac{1}{1 - v\widetilde{p}(z)/z}, \qquad v \in (0, z/\widetilde{p}(z)).$$

The last expression, called the 'unrestricted generating function' in [8, Equation (8)], identifies already potential singularities as the roots of the Lundberg equation [25] $\widetilde{p}(z)/z = v^{-1}$. The smallest (positive) root of this equation plays a central role in our story; see the next section.

Next, let

$$\tau_b^- = \inf\{t \geq 0 \colon X_t \leq b\} \quad \text{and} \quad \tau_b^+ = \inf\{t \geq 0 \colon X_t \geq b\} \tag{2}$$

respectively denote the first passage times below and above a level b (with $\inf \varnothing = \infty$).

Remark 2. Note that this differs slightly from the usual definition of these quantities for a spectrally negative Lévy process, say U. Here one replaces $t \geq 0$ by $t > 0$ and $\leq b$ ($\geq b$) by $< b$ ($> b$); and, of course, X by U. When considering $\tau_b^{\pm}$ for a spectrally negative Lévy process U, we shall mean these quantities with the latter replacements in effect.

Lastly, for convenience, we assume that a family of measures $(\mathbb{P}_x)_{x \in \mathbb{Z}}$ is given with corresponding expectation operators $(\mathbb{E}_x)_{x \in \mathbb{Z}}$, for which: (i) $\mathbb{P}_x[X_0 = x] = 1$ for all $x \in \mathbb{Z}$; and (ii) the C_i, $i \in \mathbb{N}$, have the same law under all the $\mathbb{P}_x$, $x \in \mathbb{Z}$, as they do under $\mathbb{P} = \mathbb{P}_0$.

Remark 3. The discrete-time, discrete-space compound binomial model is embedded into continuous time via subordination (time-change) by an independent homogeneous Poisson process N. In precise terms, allowing also a scaling of space, we have the following correspondence between the right-continuous random walk X of (1) and the upwards skip-free Lévy chain of [35, Section 2], which we denote by Y,

$$X \quad \rightsquigarrow \quad Y: \quad Y_t := hX_{N_t}, \qquad t \in [0, \infty),$$

where $h \in (0, \infty)$ is space scaling. In particular, denoting the intensity of N by γ, the Lévy measure λ of Y is given by $\lambda = \gamma \sum_{i \in \mathbb{N}_0 \setminus \{1\}} p_i \delta_{h(1-i)}$, and if we denote the Laplace exponent of Y by ψ (so $\psi(\beta) = \log \mathbb{E}[\mathrm{e}^{\beta Y_t}]/t$ for $\beta \in [0, \infty)$) then $\psi(\beta) = \gamma[\mathrm{e}^{\beta h}\widetilde{p}(\mathrm{e}^{-\beta h}) - 1]$. Note that the mass of the Lévy measure λ is $\gamma(1 - p_1)$, which may be strictly less than γ.

Remark 4. In the following, when the $\tau_b^{\pm}$ appear in the context of the upwards skip-free Lévy chain Y, they are to be interpreted in the sense of (2) with Y replacing X.

The remainder of the paper is organised as follows. In Sections 2, 3, and 4, we respectively review (with v indicating discounting) the smooth (i.e. upwards; hence, skip-free) one-sided first passage problem, which introduces the Lundberg root φ_v (analogue of $\Phi(q)$ from the Lévy theory); the nonsmooth (i.e. downwards) one-sided first passage problem, which involves the

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ruin and survival probabilities Ψ_v and $\bar{\Psi}_v$; the smooth (i.e. exiting at the upper boundary) two-sided first passage problem, where the fundamental scale function W_v first appears.

We then turn to original contributions in Section 5, computing the generating function (z-transform) of the second hero of the first passage theory: the $Z_v(\cdot, w)$ scale function. This is introduced via the problem of position on the nonsmooth two-sided exit. We provide the analogue (12) of the following two-sided exit identity for a spectrally negative Lévy process U (in standard notation, see below):

$$\mathbb{E}_x[\mathrm{e}^{-q\tau_0^- + \theta U(\tau_0^-)}; \tau_0^- < \tau_b^+] = Z^{(q)}(x, \theta) - \frac{W^{(q)}(x)}{W^{(q)}(b)} Z^{(q)}(b, \theta).$$

Ivanovs and Palmowski [20, Corollary 3] provide its beautiful probabilistic interpretation. We also determine the analogue (17) of the formula [23, Equation (8.9)] (again for a spectrally negative Lévy process, in standard notation, see below)

$$\mathbb{E}_x[\mathrm{e}^{-q\tau_0^-}; \tau_0^- < \infty] = Z^{(q)}(x) - \frac{q}{\Phi(q)} W^{(q)}(x), \qquad q > 0,$$

which is interesting, for example, since it reveals that the two protagonists of the 'reflected' and 'absorbed' smooth passage problems, $Z^{(q)}$ and $W^{(q)}$, have the same asymptotics at ∞, up to a constant. (For definiteness, let us mention that, in the preceding formulae, for $q \in [0, \infty)$: $\Phi(q)$ is the largest root of $\kappa - q$, where κ is the Laplace exponent of the underlying Lévy process U; $W^{(q)}$ is the unique function mapping $\mathbb{R} \to \mathbb{R}$, vanishing on $(-\infty, 0)$, continuous on $[0, \infty)$, and having Laplace transform $\int_0^\infty \mathrm{e}^{-\lambda x} W^{(q)}(x)\, \mathrm{d}x = (\kappa(\lambda) - q)^{-1}$ for $\lambda \in (\Phi(q), \infty)$; finally, $Z^{(q)}(x, \theta) = \mathrm{e}^{\theta x} + (q - \kappa(\theta)) \int_0^x \mathrm{e}^{\theta(x-y)} W^{(q)}(y)\, \mathrm{d}y$ for $x \in \mathbb{R}$.) A distinguishing element of the scale functions W_v and $Z_v(\cdot, w)$, in the present context, are explicit recursions available for their computation, see (9) and (13), respectively.

For applications of this theory to combined capital injections-dividend payout-penalty at ruin optimization problems, see an earlier version of this paper [3, Sections 6 and 7].

2. Smooth one-sided first passage problem: the Lundberg equation

The first key observation is that, for the first passage upwards, the stationary independent increments and skip-free properties imply a multiplicative structure. For integer $x \le b$ and $v \in (0, 1]$, we have

$$\mathbb{E}_x[v^{\tau_b^+}; \tau_b^+ < \infty] = \varphi_v^{b-x}, \tag{3}$$

where

$$\varphi_v := \mathbb{E}[v^{\tau_1^+}; \tau_1^+ < \infty] = \sum_{k=1}^{\infty} v^k \mathbb{P}[\tau_1^+ = k] \in (0, v].$$

Conditioning at time 1, we obtain

$$\varphi_v = v\, \mathbb{E}[\mathbb{E}_{1-C_1}[v^{\tau_1^+}; \tau_1^+ < \infty]] = v \sum_{k=0}^{\infty} p_k \varphi_v^k = v\widetilde{p}(\varphi_v),$$

which reveals that φ_v in (3) satisfies the Lundberg equation [12, Equation (3.3)], [18, Equation (6.8)]

$$\frac{\varphi_v}{\widetilde{p}(\varphi_v)} = v. \tag{4}$$

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Alternatively, this relation may be derived by looking for exponential martingales of the form $(v^t\xi^{-X_t})_{t\in\mathbb{N}_0}$, for fixed v, and ξ from $(0, 1]$: $(v^t\xi^{-X_t})_{t\in\mathbb{N}_0}$ is a martingale if and only if $\xi/\widetilde{p}(\xi)=v$; and then applying optional sampling.

Remark 5. The function $(0, 1]\ni\xi\mapsto\widetilde{p}(\xi)/\xi=\mathbb{E}[\xi^{C_1-1}]$ is strictly convex, equal to 1 at 1, and tending to ∞ at 0. It follows that the equation (in $\xi\in(0, 1]$) $\widetilde{p}(\xi)/\xi=v^{-1}$ has as its unique solution $\varphi_v\in(0, 1)$, when $v<1$ (furthermore, in this case, $\varphi_v<v$), whereas in the $v=1$ case, this equation has one or two solutions (one of which is always 1), according as to whether $\mathbb{E}[C_1]\leq 1$ or $\mathbb{E}[C_1]>1$. In the latter case X drifts to $-\infty$, and $\varphi_1\in(0, 1)$ is the smallest solution to $\xi=\widetilde{p}(\xi)$ (in $\xi\in(0, 1]$). Altogether, this gives a continuous strictly increasing bijection $\varphi\colon (0, 1]\to(0, \varphi_1]$.

Remark 6. If, for $q\in[0, \infty)$, we let $\Phi(q)$ be the largest zero of $\psi-q$, then we see from Remark 3 that $\varphi_v=\mathrm{e}^{-h\Phi(\gamma(v^{-1}-1))}$ for all $v\in(0, 1]$.

Remark 7. Note that (4) identifies τ_1^+ as a Lagrangian type distribution [14]. Indeed the distribution of τ_1^+ may be obtained using the Lagrange inversion formula [14, Paragraph 1.2.6]

$$\varphi_v=\sum_{n=1}^{\infty}\frac{v^n}{n!}\left[\left(\frac{\mathrm{d}}{\mathrm{d}w}\right)^{n-1}\widetilde{p}(w)^n\right]_{w=0}=\sum_{n=1}^{\infty}\frac{v^n}{n}p^{n*}(n-1),$$

where, for $n\in\mathbb{N}$, p^{n*} is the n-fold convolution of the distribution p with itself (the second equality follows from the fact that, for $n\in\mathbb{N}$, $\tilde{p}^n$ is the probability generating function of the probability mass function p^{n*}, viz. of the sum of n independent random variables, all distributed as C_1). More generally, for $b\in\mathbb{N}$,

$$\varphi_v^b=b\sum_{n=b}^{\infty}\frac{v^n}{n}p^{n*}(n-b),$$

yielding Kemperman's formula [21] for the distribution of τ_b^+,

$$\mathbb{P}[\tau_b^+=n]=\frac{b}{n}p^{n*}(n-b)=\frac{b}{n}\mathbb{P}[X_n=b],\qquad n\in\mathbb{N}_{\geq b}.$$

3. Nonsmooth one-sided first passage problem: ruin and survival probabilities; the Lundberg recurrence

For initial capital $x\in\mathbb{Z}$, the finite time and eventual ruin probabilities are defined by

$$\Psi(n; x):=\mathbb{P}_x[\tau_{-1}^-\leq n]\quad\text{for } n\in\mathbb{N}_0,\qquad \Psi(x):=\lim_{n\to\infty}\Psi(n; x)=\mathbb{P}_x[\tau_{-1}^-<\infty];$$

similarly, we introduce the finite time and perpetual survival probabilities

$$\bar{\Psi}(n; x):=\mathbb{P}_x[\tau_{-1}^->n]\quad\text{for } n\in\mathbb{N}_0,\qquad \bar{\Psi}(x):=\lim_{n\to\infty}\bar{\Psi}(n; x)=\mathbb{P}_x[\tau_{-1}^-=\infty].$$

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Of course, $\Psi(n; x) + \bar{\Psi}(n; x) = 1$, $\Psi(x) + \bar{\Psi}(x) = 1$, and we have the recursions, valid for all integers $x \geq 0$, $n \geq 1$,

$$\bar{\Psi}(n; x) = \sum_{i=0}^{x+1} p_i \bar{\Psi}(n-1; x+1-i), \qquad \bar{\Psi}(0; x) = 1, \tag{5}$$

$$\Psi(n; x) = \sum_{i=0}^{x+1} p_i \Psi(n-1; x+1-i) + \sum_{k=x+2}^{\infty} p_k, \qquad \Psi(0; x) = 0.$$

These two recurrences may, for a sequence of functions $f_n \colon \mathbb{Z} \to [0, 1]$, standing in lieu of $\Psi(n; \cdot\,)$ and $\bar{\Psi}(n; \cdot\,)$, be written symbolically as

$$f_n = K\,\widetilde{p}(K^{-1}) f_{n-1} \quad \text{on } \mathbb{N}_0,$$

which passes to the limit (as $n \to \infty$)

$$f = K\,\widetilde{p}(K^{-1}) f \quad \text{on } \mathbb{N}_0,$$

where K is the translation operator, $Kg(x) := g(x+1)$, and $f(x) := \lim_{n\to\infty} f_n(x)$. This limiting recurrence (satisfied by the eventual ruin and perpetual survival probabilities Ψ and $\bar{\Psi}$) may be called the 'Lundberg recurrence'. It constitutes a linear difference equation for f, whose characteristic equation is (in $x \neq 0$) $1 = x\widetilde{p}(1/x)$. The latter is (formally) just the Lundberg equation (4) with $v = 1$ upon substituting x^{-1} for φ_1. When the distribution p has a finite support, then from the theory of finite-order linear difference equations with constant coefficients, this implies that f, in particular the ultimate ruin and perpetual survival probabilities, may be expressed as combinations of powers of the roots of the characteristic equation (in $x \neq 0$)

$$1 = x\,\widetilde{p}\left(\frac{1}{x}\right).$$

Classical ruin theory proceeds by computing double (generating function) transforms, briefly reviewed in Appendix A. For example, one useful result, similar to the Pollaczek–Khinchine formula for the Cramér–Lundberg model, is

$$\widetilde{\bar{\Psi}}(z) := \sum_{x=0}^{\infty} z^x \bar{\Psi}(x) = \frac{(1 - \mathbb{E}[C_1]) \vee 0}{\widetilde{p}(z) - z}, \qquad z \in (0, 1); \tag{6}$$

see [36, Equation (3.5)]. Another is

$$\begin{aligned} \widetilde{\Psi}_v(z) &:= \sum_{x=0}^{\infty} \sum_{n=0}^{\infty} z^x v^n \Psi(n; x) \\ &= \frac{1}{z - v\widetilde{p}(z)} \left(\frac{v(z - \widetilde{p}(z))}{(1-v)(1-z)} + \frac{\varphi_v}{1 - \varphi_v} \right), \qquad v, z \in (0, 1),\ z \neq \varphi_v. \end{aligned} \tag{7}$$

We will follow next an alternate approach, which focuses on the two-sided exit problem from an interval.

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4. Smooth two-sided first passage problem: the W scale functions

In the context of Lévy processes, the $W^{(q)}$ scale function is often defined first for $q=0$, in the case when the underlying process drifts to ∞, by proportionality to the survival probability, and then in the remainder of the cases by an Esscher transform/approximation [9, Section VII.2], [23, Section 8.2], [35, Section 4.2].

In our setting of the right-continuous random walk X, we introduce, for $v \in (0, 1]$, the discrete-time analogue W_v of $W^{(q)}$, by setting $W_v(y) := (p_0 \mathbb{E}[v^{\tau_y^+}; \tau_y^+ < \tau_{-1}^-])^{-1}$ for $y \in \mathbb{N}_0$ and $W_v(y) = 0$ for $y \in -\mathbb{N}$. For integer $0 \le x \le N$, the Markov property at the time τ_x^+ and the skip-free property (yielding $X_{\tau_x^+} = x$ on $\{\tau_x^+ < \infty\}$) then imply that

$$\begin{aligned}\mathbb{E}[v^{\tau_N^+}; \tau_N^+ < \tau_{-1}^-] &= \mathbb{E}[v^{\tau_x^+} \mathbb{E}_{X_{\tau_x^+}}[v^{\tau_N^+}; \tau_N^+ < \tau_{-1}^-]; \tau_x^+ < \tau_{-1}^-] \\ &= \mathbb{E}_x[v^{\tau_N^+}; \tau_N^+ < \tau_{-1}^-]\mathbb{E}[v^{\tau_x^+}; \tau_x^+ < \tau_{-1}^-],\end{aligned}$$

i.e. the 'gambler's winning' relation [17], [26]

$$\mathbb{E}_x[v^{\tau_N^+}; \tau_N^+ < \tau_{-1}^-] = \frac{W_v(x)}{W_v(N)}, \tag{8}$$

which is valid for all integers $N \ge 0$ and $x \le N$ (it is trivial for $x < 0$ as both sides are then 0).

We call W_v the v-scale function and we simply write W for the 1-scale function W_1. (The choice of the normalization $W_v(0) = 1/p_0$ is somewhat arbitrary, though it is guided by obtaining the simplest possible form for the z-transform of W_v (see (10) below); by comparison to the W scale function of [35] (see Remark 10 below); and the simplicity of subsequent formulae in which W_v features.)

Remark 8. We use the subscript notation W_v for the scale functions of X, reserving the superscript version $W^{(q)}$ for the corresponding quantities from the Lévy setting. When only W appears, it will be clear from the context which of the two is meant. We will adhere to a similar convention with respect to the scale functions $Z^{(q)}(\cdot, \theta)$, $Z^{(q)} := Z^{(q)}(\cdot, 0)$ and (hence the notation) their discrete-time analogues $Z_v(\cdot, w)$, Z_v.

Conditioning on the first jump, (8) implies the harmonic recursion

$$W_v(x) = v \sum_{y=-1}^{x} W_v(x-y)p_{y+1}, \qquad x \in \mathbb{N}_0; \tag{9}$$

see [26, Equation (3.1)]

Taking the z-transform yields

$$\widetilde{W}_v(z) := \sum_{x=0}^{\infty} z^x W_v(x) = \frac{1}{\widetilde{p}(z) - z/v}, \qquad z \in (0, \varphi_v); \tag{10}$$

see [26, Equation (3.2)]

Since the z-transform (10) of W_v is known, the computation of the scale function W_v finally reduces to Taylor coefficient extraction of (10) expanded in a power series.

Remark 9. It is seen from (9), or directly from (8), that we have $W_v/v = {}^vW$, where ${}^vW$ is the 1-scale function of the process X geometrically killed with probability $1-v$, i.e. of the process which has, *ceteris paribus*, the sub-probability mass function $(vp_k)_{k \in \mathbb{N}_0}$ governing the sizes of the C_n, $n \in \mathbb{N}$.

Remark 10. For X embedded into continuous time as an upwards skip-free Lévy chain, i.e. for the process Y of Remark 3, (9) and (10) become, respectively, [35, Equations (20) and (16)]. This is seen through the identification $W^{(q)}(mh) = (1/\gamma h) W_{\gamma/(\gamma+q)}(m)$ for $m \in \mathbb{N}_0$, $q \in [0, \infty)$, where $W^{(q)}$ is the q-scale function of [35]. Note also that the normalization $W_v(0) = p_0^{-1}$ is consistent with $W^{(q)}(0) = 1/(h\lambda(\{h\})) = 1/(\gamma h p_0)$ of [35, Proposition 5]. On the other hand, in the spectrally negative case, there is no direct analogue of recursion (9), though one can consider the heuristic relation (it is rigorous in the upwards skip-free case [35, Remark 8]) $(L-q)W^{(q)} = 0$ on $(0, \infty)$ [22, p. 136], L being the infinitesimal generator of the underlying Lévy process, to be a close relative. (10) has the Laplace transform equivalent [22, Equation (8.8)] that formally differs from [35, Equation (16)] only by the factor $(\mathrm{e}^{\beta h} - 1)/(\beta h) \to 1$ as $h \downarrow 0$ (with β the argument of the Laplace transform).

Remark 11. An alternative form of recursion (9) is

$$W_v(n+1) = W_v(0) + \sum_{k=1}^{n+1} \frac{1/v - \sum_{l=0}^{k} p_l}{p_0} W_v(n+1-k), \qquad n \in \mathbb{N}_0;$$

see [35, Equation (23)]. In particular, we see via induction that, for each fixed integer x, the map $[1, \infty) \ni \xi \mapsto W_{1/\xi}(x)$ extends to a polynomial function defined on the whole complex plane.

Remark 12. When X drifts to ∞, i.e. when $\mathbb{E}[C_1] < 1$, then with $v = 1$, (10) coincides up to a multiplicative constant with the perpetual survival transform (6). We conclude that

$$\bar{\Psi}(x) = (1 - \mathbb{E}[C_1]) W(x).$$

Remark 13. It follows from (10), i.e. by inspecting the equality of generating functions, that ${}_vW(x) = (\varphi_v/v) W_v(x) \varphi_v^x$, where ${}_vW$ is the 1-scale function of the Esscher transformed process in which C_1 has the geometrically tilted probability mass function $\mathbb{N}_0 \ni k \mapsto (v/\varphi_v) p_k \varphi_v^k$ (and so the probability generating function $(0, 1] \ni z \mapsto (v/\varphi_v)\, \tilde{p}\, (z\varphi_v)$). Furthermore, we see that

$$\begin{aligned}
\lim_{x\to\infty} W_v(x)\varphi_v^{x+1} &= v \lim_{x\to\infty} {}_vW(x) \\
&= {}_vW(\infty) v \\
&= v \lim_{z\uparrow 1} \sum_{x=0}^{\infty} (1-z) z^x {}_vW(x) \\
&= v \lim_{z\uparrow 1} (1-z) {}_v\widetilde{W}(z) \\
&= v \lim_{z\uparrow 1} \frac{1-z}{(v/\varphi_v)\tilde{p}(z\varphi_v) - z} \\
&= \frac{v}{1 - v\widetilde{p}'(\varphi_v -)},
\end{aligned}$$

where we understand $1/0 = \infty$ (the equality ${}_vW(\infty) = \lim_{z\uparrow 1} \sum_{x=0}^{\infty} (1-z) z^x {}_vW(x)$ is seen to hold, for instance, by monotone convergence, because one can view $\sum_{x=0}^{\infty} (1-z) z^x {}_vW(x)$ as the expectation of ${}_vW(g_z)$, where g_z is a geometric random variable with success parameter $1-z$, and the g_z can be defined on a common probability space so as to be increasing to ∞ as $z \uparrow 1$). This confirms [35, Proposition 6(1)]. For a more detailed study of the behaviour of W_1 in the case when X oscillates, i.e. when $\widetilde{p}'(1-) = 1$ and $\varphi_1 = 1$, and so when the preceding does not yield the precise asymptotics of W at infinity, see [35, Proposition 6(2)].

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Remark 14. We note the following interesting observation of [26] that the scale function is essentially a determinant. For an arbitrary homogeneous Markov chain $(V_n)_{n\in\mathbb{N}_0}$ on a countable state space, let $(V'_n)_{n\in\mathbb{N}_0}$ denote the chain killed outside a finite nonempty set M, and let Q denote the corresponding restriction of the transition matrix to M. For $v \in (0, 1)$, denote by D_v the determinant of the matrix $I - vQ$. Then the killed resolvent expresses as

$$\sum_{n=0}^{\infty} \mathbb{P}_i[V'_n = j]v^n = ((I - vQ)^{-1})_{ij} = \frac{N_{ij}(v)}{D_v}, \qquad \{i, j\} \subset M,$$

where $N_{ij}(v)$ are the entries of the adjoint matrix $\mathrm{adj}(I - vQ)$ (see, for example, [26, Corollary 2.2]). Restricting now to the upwards skip-free case (while [26] considers the downwards skip-free case), for $v \in (0, 1]$, let $D_v(N)$, $N \in \mathbb{N}$, denote the determinant corresponding (in the above sense) to the restriction of X to $\{0, 1, 2, \ldots, N-1\}$, and set $D_v(0) := 1$. From [26, Proposition 3.3],

$$\mathbb{E}_i[v^{\tau_N^+}, \tau_N^+ < \tau_{-1}^-] = (p_0 v)^{N-i}\frac{D_v(i)}{D_v(N)}, \qquad \{i, N\} \subset \mathbb{N}_0,\ v \in (0, 1).$$

It follows that

$$W_v(i) = p_0^{-1}(p_0 v)^{-i}D_v(i) \quad \text{for all } i \in \mathbb{N}_0,\ v \in (0, 1].$$

Remark 15. For $N \in \mathbb{N}$, the resolvent of the process X killed on exiting $I_N := \{0, \ldots, N-1\}$, denoted by X', is given by

$$\sum_{n=0}^{\infty} \mathbb{P}_i[X'_n = j]v^n = v^{-1}\left(\frac{W_v(N-1-j)W_v(i)}{W_v(N)} - W_v(i-j-1)\right),$$

where $\{i, j\} \subset I_N$, $v \in (0, 1]$; see [26, Proposition 3.2]. For the analogue of the latter in the spectrally negative case, see, e.g. [23, Theorem 8.7].

We conclude this section with the following important observation.

Proposition 1. *For every $v \in (0, 1]$, $(v^{n\wedge\tau_{-1}^-}W_v(X_{n\wedge\tau_{-1}^-}))_{n\in\mathbb{N}_0}$ is a martingale under each $\mathbb{P}_x$, $x \in \mathbb{Z}$.*

Proof. This follows from the harmonic recurrence (9). $\square$

Remark 16. The analogue of Proposition 1 in the setting of upwards skip-free Lévy chains are the martingales, for $q \in [0, \infty)$, $(e^{-q(t\wedge\tau_{-h}^-)}W^{(q)}(Y_{t\wedge\tau_{-h}^-}))_{t\in[0,\infty)}$ [35, Corollary 2]. In the case of a spectrally negative Lévy process U, $(e^{-q(t\wedge\tau_0^-)}W^{(q)}(U_{t\wedge\tau_0^-}))_{t\in[0,\infty)}$ is a local martingale with localizing sequence $(\tau_n^+)_{n\in\mathbb{N}}$ [23, Example 8.12]. There are no issues with integrability in the discrete space case, because thanks to the skip-free property, $\mathbb{P}_x$-a.s. for any $x \in \mathbb{Z}$, by any deterministic time, the stopped process $X^{\tau_{-1}^-}$ is automatically bounded from above (and, for the upwards skip-free Lévy chain Y, the further subordination by the independent homogeneous Poisson process N does not ruin this).

Corollary 1. *For each $v \in (0, 1]$ and integer $x \le N$, $b \le N$,*

$$\mathbb{E}_x[W_v(X_{\tau_{b-1}^-})v^{\tau_{b-1}^-}; \tau_{b-1}^- < \tau_N^+] = W_v(x) - \frac{W_v(x-b)}{W_v(N-b)}W_v(N).$$

In particular,

$$\mathbb{E}_x[W_v(X_{\tau_{b-1}^-})v^{\tau_{b-1}^-}; \tau_{b-1}^- < \infty] = W_v(x) - W_v(x-b)\varphi_v^b.$$

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Proof. For any integer x, by optional sampling, the skip-free property and spatial homogeneity

$$\begin{aligned} W_v(x) &= \mathbb{E}_x[W_v(X(\tau_{b-1}^-))v^{\tau_{b-1}^-}; \tau_{b-1}^- < \tau_N^+] + \mathbb{E}_x[W_v(X(\tau_N^+))v^{\tau_N^+}; \tau_N^+ < \tau_{b-1}^-] \\ &= \mathbb{E}_x[W_v(X(\tau_{b-1}^-))v^{\tau_{b-1}^-}; \tau_{b-1}^- < \tau_N^+] + W_v(N)\mathbb{E}_x[v^{\tau_N^+}; \tau_N^+ < \tau_{b-1}^-] \\ &= \mathbb{E}_x[W_v(X(\tau_{b-1}^-))v^{\tau_{b-1}^-}; \tau_{b-1}^- < \tau_N^+] + W_v(N)\mathbb{E}_{x-b}[v^{\tau_{N-b}^+}; \tau_{N-b}^+ < \tau_{-1}^-]. \end{aligned}$$

The first identity then follows from (8). In particular, letting $N \uparrow \infty$ and using Remark 13, we obtain the second identity (for instance, first for $v < 1$ and then taking the limit $v \uparrow 1$). □

Remark 17. For the analogue of Corollary 1 in the spectrally negative Lévy setting, see [24, Lemma 2.1, Equation (19) and Lemma 2.2(i)].

5. Nonsmooth two-sided first passage problem: the Z scale functions

Let $v \in (0, 1]$, $w \in (0, 1]$. For integer $x \leq b$, $b \geq 0$, by the Markov property at time τ_b^+ and the skip-free property (yielding $X_{\tau_b^+} = b$ on $\{\tau_b^+ < \infty\}$),

$$\begin{aligned} &\mathbb{E}_x[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_{-1}^- < \tau_b^+] \\ &\quad = \mathbb{E}_x[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_{-1}^- < \infty] - \mathbb{E}_x[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_b^+ < \tau_{-1}^- < \infty] \\ &\quad = \mathbb{E}_x[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_{-1}^- < \infty] - \mathbb{E}_x[v^{\tau_b^+}; \tau_b^+ < \tau_{-1}^-]\mathbb{E}_b[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_{-1}^- < \infty]. \end{aligned}$$

Setting $\Psi_v(x, w) := \mathbb{E}_x[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_{-1}^- < \infty]$, we then have from the preceding and using (8), the neat identity $\mathbb{E}_x[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_{-1}^- < \tau_b^+] = \Psi_v(x, w) - (W_v(x)/W_v(b))\Psi_v(b, w)$. We introduce now, for some $\alpha_v(w) \in [0, \infty)$ that we shall specify later on,

$$Z_v(x, w) := \Psi_v(x, w) + \alpha_v(w) W_v(x), \tag{11}$$

a slightly modified $\Psi_v(\cdot, w)$, which also satisfies the identity

$$\mathbb{E}_x[v^{\tau_{-1}^-} w^{-X(\tau_{-1}^-)}; \tau_{-1}^- < \tau_b^+] = Z_v(x, w) - \frac{W_v(x)}{W_v(b)} Z_v(b, w) \tag{12}$$

(easy to check). The first motivation for preferring to use $Z_v(\cdot, w)$ with a suitable choice of $\alpha_v(w)$ instead of $\Psi_v(\cdot, w)$ appears below in (14); many other formulae where the analogue of $Z_v(\cdot, w)$ is preferable are known in the literature on spectrally negative Lévy processes, see, for example, [4] and [20].

Remark 18. Note that $Z_v(x, w) = \Psi_v(x, w) = w^{-x}$ for all integer $x \leq -1$.

We compute now the z-transform of Z. Conditioning on the first jump, we obtain from (11) and the definition of $\Psi_v(\cdot, w)$, via (9), the recurrence relation

$$\frac{Z_v(x, w)}{v} = \sum_{k=-1}^{x} p_{k+1} Z_v(x-k, w) + \sum_{k=x+1}^{\infty} w^{k-x} p_{k+1}, \qquad x \in \mathbb{N}_0. \tag{13}$$

Hence, the generating function

$$\widetilde{Z}_v(z, w) := \sum_{x=0}^{\infty} z^x Z_v(x, w)$$

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satisfies, for $z \in (0, \varphi_v)\backslash\{w\}$,

$$\begin{aligned}
&\frac{\widetilde{Z}_v(z, w)}{v} \\
&\quad = p_0 \frac{\widetilde{Z}_v(z, w) - Z_v(0, w)}{z} + \sum_{x=0}^{\infty} z^x \sum_{k=0}^{x} Z_v(x-k, w) p_{k+1} + \sum_{x=0}^{\infty} z^x \sum_{k=x+1}^{\infty} w^{k-x} p_{k+1} \\
&\quad = p_0 \frac{\widetilde{Z}_v(z, w) - Z_v(0, w)}{z} + \sum_{k=0}^{\infty} p_{k+1} z^k \sum_{x=k}^{\infty} z^{x-k} Z_v(x-k, w) \\
&\qquad + \sum_{k=1}^{\infty} p_{k+1} w^k \sum_{x=0}^{k-1} \left(\frac{z}{w}\right)^x \\
&\quad = p_0 \frac{\widetilde{Z}_v(z, w) - Z_v(0, w)}{z} + \widetilde{Z}_v(z, w) \sum_{k=0}^{\infty} p_{k+1} z^k + \sum_{k=1}^{\infty} p_{k+1} w^k \frac{1 - (z/w)^k}{1 - z/w} \\
&\quad = p_0 \frac{\widetilde{Z}_v(z, w) - Z_v(0, w)}{z} + \widetilde{Z}_v(z, w) \frac{\widetilde{p}(z) - p_0}{z} + \frac{(\widetilde{p}(w) - p_0)/w - (\widetilde{p}(z) - p_0)/z}{1 - z/w},
\end{aligned}$$

i.e. in view of (10),

$$\widetilde{Z}_v(z, w) = -p_0(1 - Z_v(0, w)) \widetilde{W}_v(z) + \frac{z\widetilde{p}(w) - w\widetilde{p}(z)}{(z-w)(\widetilde{p}(z) - z/v)}.$$

Recall now that in the Lévy case, $Z^{(q)}(0, \theta)$ is chosen so as to ensure a 'smooth fit' [7, Definition 5.8] to the boundary condition $e^{x\theta}$ for $x \in (-\infty, 0)$. The analogue in the discrete case is to insist on $Z_v(0, w) = 1$, which we may do by choosing (cf. (11)) $\alpha_v(w) = p_0(1 - \Psi_v(0, w))$. Furthermore, this choice (that we assume henceforth) leads to the simple expression

$$\widetilde{Z}_v(z, w) = \frac{1}{\widetilde{p}(z) - z/v} \frac{z\widetilde{p}(w) - w\widetilde{p}(z)}{z - w}, \qquad z \in (0, \varphi_v),\ v \in (0, 1],\ w \in (0, 1] \tag{14}$$

(where the quotient must be understood in the limiting sense when $z = w$).

Extracting the coefficients of the z-power series yields finally an expression similar to that of the Dickson–Hipp type representation in the Lévy case (see [20])

$$Z_v(x, w) = \left(\widetilde{p}(w) - \frac{w}{v}\right) \sum_{k=0}^{\infty} w^k W_v(x+k), \qquad w \in (0, \varphi_v),\ v \in (0, 1],\ x \in \mathbb{N}_0$$

(it is easy to check that this expression has z-transform (14)).

In the special case $w = 1$ we set $Z_v(x) := Z_v(x, 1)$, (14) simplifies to

$$\widetilde{Z}_v(z) := \sum_{x=0}^{\infty} z^x Z_v(x) = \frac{\widetilde{p}(z) - z}{(\widetilde{p}(z) - z/v)(1 - z)}, \qquad z \in (0, \varphi_v),\ v \in (0, 1], \tag{15}$$

and we have the representation

$$Z_v(x) = 1 + \left(\frac{1}{v} - 1\right) \sum_{y=0}^{x-1} W_v(y), \qquad v \in (0, 1],\ x \in \mathbb{N}_0. \tag{16}$$

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Remark 19. Using (7) in the form

$$\widetilde{\Psi}_v(z) = \frac{1}{z/v - \widetilde{p}(z)}\left(\frac{z-\widetilde{p}(z)}{(1-v)(1-z)} + \frac{\varphi_v}{v(1-\varphi_v)}\right), \qquad v, z \in (0,1),\ z \neq \varphi_v,$$

it follows from (10) and (15) that

$$\Psi_v(x) := \sum_{n=0}^{\infty} v^n \Psi(n;x) = \frac{1}{1-v} Z_v(x) - \frac{\varphi_v}{v(1-\varphi_v)} W_v(x),$$

i.e.

$$\begin{aligned}\mathbb{E}_x[v^{\tau_{-1}^-}; \tau_{-1}^- < \infty] &= Z_v(x) - \frac{\varphi_v(1-v)}{v(1-\varphi_v)} W_v(x) \\ &= Z_v(x) - \alpha_v W_v(x), \qquad x \in \mathbb{N}_0,\ v \in (0,1),\end{aligned} \tag{17}$$

where we have set $\alpha_v := \alpha_v(1)$ (recall that we have chosen $\alpha_v(1)$ so that $Z_v(0) = 1 = \mathbb{E}[v^{\tau_{-1}^-}; \tau_{-1}^- < \infty] + \alpha_v(1) W_v(0)$). Passing to the limit $v \uparrow 1$, we find that $\mathbb{P}_x(\tau_{-1}^- < \infty) = 1 - W(x)(1 - \widetilde{p}'(1-) \wedge 1)$.

Remark 20. It is seen from (16), Remark 10, and [35, Definition 4] that we have the identification $Z^{(q)}(mh) = Z_{\gamma/(\gamma+q)}(m)$ for $q \in [0,\infty)$, $m \in \mathbb{Z}$, where $Z^{(q)}$ is the Z q-scale function of [35]. Then (14), (12) and (13), with $w = 1$, become [35, Equation (19), Proposition 8, and Equation (21)], respectively; (17) becomes [35, Equation (4.8)]. For an alternative form of (13) (when $w = 1$), see [35, Equation (18)].

Proposition 2. *For all $v \in (0,1]$, $w \in (0,1]$, the process $(v^{n \wedge \tau_{-1}^-} Z_v(X_{n \wedge \tau_{-1}^-}, w))_{n \in \mathbb{N}_0}$ is a martingale.*

Proof. This follows, for instance, by linearity, from Proposition 1, and from the definition of $Z_v(\cdot, w)$ via the Markov property and the terminal time property of τ_{-1}^-. □

Remark 21. For the $w = 1$ case, the analogue of Proposition 2 in the setting of upwards skip-free Lévy chains are the martingales $(\mathrm{e}^{-q(t \wedge \tau_{-h}^-)} Z^{(q)}(Y_{t \wedge \tau_{-h}^-}))_{t \in [0,\infty)}$ for $q \in [0,\infty)$ [35, Corollary 2]. In the case of a spectrally negative Lévy processes U, $(\mathrm{e}^{-q(t \wedge \tau_0^-)} Z^{(q)}(U_{t \wedge \tau_0^-}))_{t \in [0,\infty)}$ is a local martingale with localizing sequence $(\tau_n^+)_{n \in \mathbb{N}}$ [23, Example 8.12]. See also [7], in which Gerber–Shiu functions are defined as solutions to martingale problems [7, Definition 5.1], and the $Z^{(q)}(\cdot, \theta)$ function is the Gerber-Shiu function with boundary condition $\mathrm{e}^{\theta x}$ for $x \in (-\infty, 0)$ [7, Definition 5.8].

Remark 22. Assume that $\mathbb{E}[C_1] < \infty$. Let $v \in (0,1]$, $x \in \mathbb{Z}$. We can obtain the expected undershoot at ruin by differentiating (12) with respect to w from the left at 1. Setting

$$Z_{1,v}(x) := -\left.\frac{\partial Z_v(x,w)}{\partial w}\right|_{w=1-},$$

we find that, for $b \in \mathbb{N}_0$,

$$\mathbb{E}_x[X(\tau_{-1}^-)\, v^{\tau_{-1}^-}; \tau_{-1}^- < \tau_b^+] = Z_{1,v}(x) - \frac{W_v(x)}{W_v(b)} Z_{1,v}(b), \qquad x \leq b.$$

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The generating function transform of $Z_{1,v}$ is given by

$$\widetilde{Z}_{1,v}(z) := \sum_{k=0}^{\infty} z^k Z_{1,v}(k) = \frac{z}{1-z}\frac{1}{\widetilde{p}(z)-z/v}\left(\frac{\widetilde{p}(z)-z}{1-z}-(1-\widetilde{p}'(1-))\right), \qquad z\in(0,\varphi_v).$$

Setting for $f\colon \mathbb{N}_0\to\mathbb{R}$ and $y\in\mathbb{N}_0$, $\bar{f}(y) := \sum_{z=0}^{y-1} f(z)$ (in particular, $\bar{f}(0)=0$), and using

$$\sum_{k=0}^{\infty} z^k \bar{f}(k) = \left(\frac{z}{1-z}\right)\sum_{k=0}^{\infty} z^k f(k) \quad \text{for } z\in(0,1],$$

we see that this coincides with the generating function of

$$\mathbb{N}_0 \ni x \mapsto \bar{Z}_v(x) - (1-\widetilde{p}'(1-))\bar{W}_v(x),$$

i.e.

$$Z_{1,v}(x) = \bar{Z}_v(x) - (1-\widetilde{p}'(1-))\bar{W}_v(x), \qquad x\in\mathbb{N}_0, \tag{18}$$

Note also that when $x<0$, $Z_{1,v}(x)=x$.

Appendix A. Double (generating function) transforms of ruin probabilities

Recall the notation of Section 3. From [36, Equations (2.7) and (2.13)], we can deduce the double transform

$$\begin{aligned}\widetilde{\bar{\Psi}}_v(z) &:= \sum_{n=0}^{\infty} v^n \bar{\Psi}_z(n)\\ &:= \sum_{n=0}^{\infty} v^n\left(\sum_{x=0}^{\infty} z^x \bar{\Psi}(n;x)\right)\\ &= \frac{z/(1-z)-\varphi_v/(1-\varphi_v)}{z-v\widetilde{p}(z)}, \qquad v,z\in(0,1),\ z\neq\varphi_v,\end{aligned} \tag{19}$$

where $\varphi_v\in(0,1)$ is the Lundberg root (4) (note that $z=\varphi_v$ is a removable singularity). Indeed, from (5), for all $n\in\mathbb{N}$,

$$z\bar{\Psi}_z(n) = \widetilde{p}(z)\bar{\Psi}_z(n-1) - p_0\bar{\Psi}(n-1;0)$$

(see [36, Equation (2.3)]), and summing over n after multiplication by v^n yields

$$\begin{aligned} z(\widetilde{\bar{\Psi}}_v(z)-(1-z)^{-1}) = v\widetilde{p}(z)\widetilde{\bar{\Psi}}_v(z) - p_0 v\sum_{n=0}^{\infty} v^n\bar{\Psi}(n;0)\\ \Rightarrow (z-v\widetilde{p}(z))\widetilde{\bar{\Psi}}_v(z)\\ = \frac{z}{1-z} - p_0 v\sum_{n=0}^{\infty} v^n\bar{\Psi}(n;0)\end{aligned}$$

(see [36, Equation (2.7)]), from where (19) is obtained by requiring that the root $z=\varphi_v$ on the left-hand side annihilates also the right-hand side.

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Equation (19) implies the transform (for $v, z \in (0, 1), z \neq \varphi_v$)

$$\begin{aligned}\widetilde{\Psi}_v(z) &:= \sum_{x=0}^{\infty}\sum_{n=0}^{\infty} z^x v^n \Psi(n; x) \\ &= \frac{1}{(1-z)(1-v)} - \widetilde{\bar{\Psi}}_v(z) \\ &= \frac{1}{z - v\widetilde{p}(z)}\left(\frac{v(z-\widetilde{p}(z))}{(1-v)(1-z)} + \frac{\varphi_v}{1-\varphi_v}\right).\end{aligned}$$

Remark 23. Note the single transforms

$$\widetilde{\bar{\Psi}}(z) := \sum_{x=0}^{\infty} z^x \bar{\Psi}(x) = \lim_{v\uparrow 1}(1-v)\widetilde{\bar{\Psi}}_v(z) = \frac{(1-\mathbb{E}[C_1]) \vee 0}{\widetilde{p}(z) - z}, \qquad z \in (0, 1),$$

$$\widetilde{\Psi}(z) := \sum_{x=0}^{\infty} z^x \Psi(x) = \frac{1}{1-z} - \frac{(1-\mathbb{E}[C_1]) \vee 0}{\widetilde{p}(z) - z}, \qquad z \in (0, 1)$$

(see [36, Equation (3.5)]), which are similar to the Pollaczek–Khinchine formulae of the Cramér–Lundberg model. We also have

$$\bar{\Psi}(0) = \lim_{z\downarrow 0}\widetilde{\bar{\Psi}}(z) = \frac{(1-\mathbb{E}[C_1]) \vee 0}{p_0};$$

see [30, Equation (2.14)].

Appendix B

See Table 1 for a summary of the features of the Φ, W, Z theory for the three types of processes (i)-(ii)-(iii) as discussed in the introduction.

Remark 24. Every spectrally negative Lévy process may be seen as a (weak) limit of a net Y^h of upwards skip-free Lévy chains, as $h \downarrow 0$ [27]. This means that a great many relations in the spectrally negative Lévy setting may be obtained (at least naively) by simply passing to the limit $h \downarrow 0$ (formally, one must of course pay attention to whether or not the relevant functional is continuous with respect to such a weak limit).

Remark 25. One of the important contributions of having a unified Φ, W, Z theory developed in all three settings featured in Table 1 is that whenever a result is available for one of them, it may often be simply 'guessed' in the others, by 'translating' one set of quantities into the other (though ultimately it still needs to be proved). We have seen this time and again in the results of this paper.

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TABLE 1: Summary table.

Right-continuous random walk	Upwards skip-free Lévy chain	Spectrally negative Lévy process
$\{n, b\} \subset \mathbb{N}_0$, $x \in \mathbb{Z}$, $x \le b$, $v \in (0, 1]$	$\{q, \beta\} \subset [0, \infty)$, $\{b, x\} \subset h\mathbb{Z}$, $x \le b$, $b \ge 0$	$\{q, \beta\} \subset [0, \infty)$, $\{b, x\} \subset \mathbb{R}$, $x \le b$, $b > 0$
$X_n = X_0 + n - \sum_{i=1}^{n} C_i$	$Y = hX_N$, N independent homogeneous Poisson processes of intensity γ, $\{\gamma, h\} \subset (0, \infty)$	Lévy process U having a.s. nonmonotone paths and no positive jumps
C_n i.i.d., $\mathbb{N}_0$-valued; PMF p, $p_0 \in (0, 1)$; PGF $\widetilde{p}$	Lévy measure $\lambda = \gamma \sum_{i \in \mathbb{Z} \setminus \{1\}} p_i \delta_{h(1-i)}$; Laplace exponent ψ, where $\psi(\beta) = \gamma[\mathrm{e}^{\beta h}\widetilde{p}(\mathrm{e}^{-\beta h}) - 1]$	Laplace exponent ψ; δ is the drift when X has bounded variation
$\tau_b^- = \inf\{m \in \mathbb{N}_0 : X_m \le b\}$; $\tau_b^+ = \inf\{m \in \mathbb{N}_0 : X_t \ge b\}$	$\tau_b^- = \inf\{t \in [0, \infty) : Y_t \le b\}$; $\tau_b^+ = \inf\{t \in [0, \infty) : Y_t \ge b\}$	$\tau_b^- = \inf\{t \in (0, \infty) : U_t < b\}$; $\tau_b^+ = \inf\{t \in (0, \infty) : U_t > b\}$
$\varphi_v =$ smallest root of $\widetilde{p}(\xi)/\xi = v^{-1}$ (in $\xi \in (0, 1]$)	$\Phi(q) =$ largest root of $\psi(\lambda) - q$ (in $\lambda \in [0, \infty)$); $\mathrm{e}^{-h\Phi(q)} = \varphi_{\gamma/(\gamma+q)}$	$\Phi(q) =$ largest root of $\psi(\lambda) - q$ (in $\lambda \in [0, \infty)$)
$\mathbb{E}_x[v^{\tau_b^+}; \tau_b^+ < \infty] = \varphi_v^{b-x}$	$\mathbb{E}_x[\mathrm{e}^{-q\tau_b^+}; \tau_b^+ < \infty] = \mathrm{e}^{-\Phi(q)(b-x)}$	
$\sum_{y=0}^{\infty} z^y W_v(y) = 1/(\widetilde{p}(z) - z/v),\ z \in (0, \varphi_v)$	$\int_0^\infty \mathrm{e}^{-\beta y} W^{(q)}(y)\,\mathrm{d}y = (\mathrm{e}^{\beta h} - 1)/(\beta h(\psi(\beta) - q))$, $\beta \in (\Phi(q), \infty)$; $W^{(q)}$ càdlàg & constant on each interval $[x, x + h)$; $W^{(q)}(x) = W_{\gamma/(\gamma+q)}(x/h)/(\gamma h)$	$\int_0^\infty \mathrm{e}^{-\beta y} W^{(q)}(y)\,\mathrm{d}y = 1/(\psi(\beta) - q),\ \beta \in (\Phi(q), \infty)$; $W^{(q)}$ continuous on $[0, \infty)$
$\mathbb{E}_x[v^{\tau_b}; \tau_b^+ < \tau_{-1}^-] = W_v(x)/W_v(b)$	$\mathbb{E}_x[\mathrm{e}^{-q\tau_b^+}; \tau_b^+ < \tau_{-h}^-] = W^{(q)}(x)/W^{(q)}(b)$	$\mathbb{E}_x[\mathrm{e}^{-q\tau_b^+}; \tau_b^+ < \tau_0^-] = W^{(q)}(x)/W^{(q)}(b)$
$\mathbb{P}_x(\tau_{-1}^- < \infty) = 1 - W_1(x)(1 - \widetilde{p}'(1-) \wedge 1)$	$\mathbb{P}_x(\tau_{-h}^- < \infty) = 1 - W^{(0)}(x)(\psi'(0+) \vee 0)$	$\mathbb{P}_x(\tau_0^- < \infty) = 1 - W^{(0)}(x)(\psi'(0+) \vee 0)$
$W_v(x) = 0$, $x < 0$; $W_v(0) = 1/p_0$	$W^{(q)}(x) = 0$, $x < 0$; $W^{(q)}(0) = 1/(h\lambda(\{h\}))$	$W^{(q)}(x) = 0$, $x < 0$; $W^{(q)}(0) = 0$ if U has unbounded variation, $W^{(q)}(0) = 1/\delta$ otherwise
$\lim_{y\to\infty} W_v(y)\varphi_v^{y+1} = v/(1 - v\widetilde{p}'(\varphi_v -))$	$\lim_{y\to\infty} W^{(q)}(y)\mathrm{e}^{-\Phi(q)(y+h)} = 1/\psi'(\Phi(q)+)$	$\lim_{y\to\infty} W^{(q)}(y)\mathrm{e}^{-\Phi(q)(y)} = 1/\psi'(\Phi(q)+)$
$Z_v(x) = 1 + (1/v - 1)\sum_{y=0}^{x-1} W_v(y),\ x \ge 0$	$Z^{(q)}(x) = 1 + q\sum_{k=0}^{x/h-1} W^{(q)}(kh),\ x \ge 0$; $Z^{(q)}(x) = Z_{\gamma/(\gamma+q)}(x/h)$	$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)\,\mathrm{d}y,\ x \ge 0$
$Z_v(x) = 1$, $x \le 0$	$Z^{(q)}(x) = 1,\ x \le 0$	
$\sum_{y=0}^{\infty} z^y Z_v(y) = (\widetilde{p}(z) - z)/((1 - z)(\widetilde{p}(z) - z/v)),\ z \in (0, \varphi_v)$	$\int_0^\infty Z^{(q)}(y)\mathrm{e}^{-\beta y}\,\mathrm{d}y = \psi(\beta)/(\beta(\psi(\beta) - q)),\ \beta \in (\Phi(q), \infty)$	
$\mathbb{E}_x[v^{\tau_{-1}^-}; \tau_{-1}^- < \tau_b^+] = Z_v(x) - (W_v(x)/W_v(b))Z_v(b)$	$\mathbb{E}_x[\mathrm{e}^{-q\tau_{-h}^-}; \tau_{-h}^- < \tau_b^+] = Z^{(q)}(x) - (W^{(q)}(x)/W^{(q)}(b))Z^{(q)}(b)$	$\mathbb{E}_x[\mathrm{e}^{-q\tau_0^-}; \tau_0^- < \tau_b^+] = Z^{(q)}(x) - (W^{(q)}(x)/W^{(q)}(b))Z^{(q)}(b)$
$\mathbb{E}_x[v^{\tau_{-1}^-}; \tau_{-1}^- < \infty]$ $= Z_v(x) - (\varphi_v(1 - v)/(v(1 - \varphi_v)))W_v(x),\ v < 1$	$\mathbb{E}_x[\mathrm{e}^{-q\tau_{-h}^-}; \tau_{-h}^- < \infty]$ $= Z^{(q)}(x) - (qh/(\mathrm{e}^{\Phi(q)h} - 1))W^{(q)}(x),\ q > 0$	$\mathbb{E}_x[\mathrm{e}^{-q\tau_0^-}; \tau_0^- < \infty] = Z^{(q)}(x) - qW^{(q)}(x)/\Phi(q),\ q > 0$
$v^{m\wedge\tau_{-1}^m} W_v(X_{m\wedge\tau_{-1}^-})$ is a martingale in $m \in \mathbb{N}_0$	$\mathrm{e}^{-q(t\wedge\tau_{-h}^-)} W^{(q)}(X_{t\wedge\tau_{-h}^-})$ is a martingale in $t \in [0, \infty)$	$\mathrm{e}^{-q(t\wedge\tau_0^-\wedge\tau_b^+)} W^{(q)}(X_{t\wedge\tau_0^-\wedge\tau_b^+})$ is a martingale in $t \in [0, \infty)$

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