Bull. Aust. Math. Soc. **101** (2020), 299–310
doi:10.1017/S0004972719000868

ON APPROXIMATELY ADDITIVE MAPPINGS IN 2-BANACH SPACES

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(Received 27 April 2019; accepted 19 June 2019; first published online 15 August 2019)

Abstract

We show how some Ulam stability issues can be approached for functions taking values in 2-Banach spaces. We use the example of the well-known Cauchy equation $f(x + y) = f(x) + f(y)$, but we believe that this method can be applied for many other equations. In particular we provide an extension of an earlier stability result that has been motivated by a problem of Th. M. Rassias. The main tool is a recent fixed point theorem in some spaces of functions with values in 2-Banach spaces.

2010 *Mathematics subject classification*: primary 39B82; secondary 39B52, 47H10.

Keywords and phrases: Cauchy equation, Ulam stability, 2-Banach space, 2-norm.

1. Introduction

The question of how much a function satisfying an equation approximately (for example, a difference, differential, functional or integral equation) may differ from a solution to the equation arises naturally in applications of mathematics. The theory of Ulam (also called Ulam–Hyers or Hyers–Ulam) stability provides some efficient tools to evaluate such errors (see [11, 21, 22] for further details and references).

Ulam pioneered these investigations when he posed a problem in 1940 in his talk at the University of Wisconsin (see [20, 21]). Roughly speaking, a functional equation is said to be stable in a class of functions if any function from that class, satisfying the equation approximately (in some sense), is near (in a given way) to an exact solution of the equation [1, 11, 21, 25]. The following definition corresponds to our considerations in this paper and makes this notion a bit more precise for a metric space (Y, d) and an equation in two variables. (Here, A^B means a family of all functions mapping a set $B \neq \emptyset$ into a set $A \neq \emptyset$ and $\mathbb{R}$ stands for the set of reals.)

Definition 1.1. Let $S \neq \emptyset$ be a set, $\mathcal{D}_0 \subset \mathcal{D} \subset Y^S$ and $E \subset \mathbb{R}^{S\times S}$ be nonempty, and consider maps $\mathcal{S} : E \to \mathbb{R}^S$, $H : Y \times Y \to Y$ and $\mathcal{T} : \mathcal{D} \to Y^{S\times S}$. The equation

$$(\mathcal{T}\psi)(s, t) = H(\psi(s), \psi(t)) \tag{1.1}$$

is said to be $\mathcal{S}$-stable in $\mathcal{D}_0$ if, for any $\psi \in \mathcal{D}_0$ and $\delta \in \mathcal{S}$ with

$$d((\mathcal{T}\psi)(s,t), H(\psi(s), \psi(t))) \leq \delta(s,t), \quad t, s \in S,$$

there is a solution $\phi \in \mathcal{D}$ of the equation such that $d(\phi(t), \psi(t)) \leq (\mathcal{S}\delta)(t)$ for $t \in \mathcal{S}$.

Such stability problems have been investigated mainly in classical spaces [9, 11, 21–24] and one of the most classical results is the following (see [10] and also [5, 7]).

Theorem 1.2. *Suppose that $p \in \mathbb{R} \setminus \{1\}$, V and W are normed spaces, $\emptyset \neq D \subset W \setminus \{0\}$ and consider the following three conditions (where $\mathbb{N}$ stands for the positive integers):*

(i) *if $p > 1$, then $D \subset 2D := \{2x : x \in D\}$;*
(ii) *if $p \in [0, 1)$, then $2D \subset D$;*
(iii) *if $p < 0$, then $-x, nx \in D$ for $x \in D$ and $n \in \mathbb{N}$, $n \geq n_0$, with some $n_0 \in \mathbb{N}$.*

Assume that $c \in \mathbb{R}_+ := [0, \infty)$ and $g \in V^D$ are such that

$$\|g(x_1 + x_2) - g(x_1) - g(x_2)\| \leq c(\|x_1\|^p + \|x_2\|^p) \tag{1.2}$$

for $x_1, x_2 \in D$ with $x_1 + x_2 \in D$. Then the following two statements hold.

(a) *If V is complete and (i) or (ii) holds, then there is a unique $h \in V^D$ that is additive on D (that is, $h(x + y) = h(x) + h(y)$ for $x, y \in D$ with $x + y \in D$) and such that*

$$\|g(x) - h(x)\| \leq c|1 - 2^{p-1}|^{-1}\|x_1\|^p, \quad x \in D. \tag{1.3}$$

(b) *If (iii) holds, then g is additive on D.*

However, it is clear that the concept of an approximate solution and the idea of nearness of two functions can be understood in various, nonstandard ways, depending on a particular situation. One such nonclassical way of measuring a distance can be introduced by the notion of 2-norm, proposed by Gähler in [18] (more information is provided in the next section). To the best of our knowledge, the first paper on the Ulam stability of functional equations in 2-Banach spaces is [19] (see also [8, 12–16, 26] for some later related results). We show how to deal with some Ulam stability issues (analogous to Theorem 1.2) in such spaces. We believe that those ideas can be applied in many other similar problems.

We also suggest an open problem concerning optimality of estimates (4.7) and (5.1) (see Remark 4.4).

2. Preliminaries

By a 2-*normed space* (see [17]) we mean a pair $(X, \|\cdot, \cdot\|)$ such that X is a real linear space of dimension not smaller than 2 and $\|\cdot, \cdot\| : X \times X \to \mathbb{R}_+$ ($\mathbb{R}_+$ denotes the set of nonnegative reals) is a function satisfying the following conditions:

(a) $\|x, y\| = 0$ if and only if x and y are linearly dependent;
(b) $\|x, y\| = \|y, x\|$ for $x, y \in X$;

https://doi.org/10.1017/S0004972719000868 Published online by Cambridge University Press

(c) $\|x, y+z\| \le \|x, y\| + \|x, z\|$ for $x, y, z \in X$;
(d) $\|\alpha x, y\| = |\alpha|\, \|x, y\|$ for $\alpha \in \mathbb{R}$ and $x, y \in X$.

A sequence $(x_n)_{n\in\mathbb{N}}$ ($\mathbb{N}$ denotes the set of positive integers) of elements of a 2-normed space X is a *Cauchy sequence* if there exist linearly independent $y, z \in X$ with

$$\lim_{n,m\to\infty} \|x_n - x_m, y\| = 0 = \lim_{n,m\to\infty} \|x_n - x_m, z\|;$$

whereas $(x_n)_{n\in\mathbb{N}}$ is said to be *convergent* if there exists $x \in X$ (called a *limit* of this sequence and denoted by $\lim_{n\to\infty} x_n$) such that $\lim_{n\to\infty} \|x_n - x, y\| = 0$ for $y \in X$.

In a 2-normed space a sequence has at most one limit and the standard property of the limit of a linear combination of two sequences is valid. A 2-normed space, in which every Cauchy sequence is convergent, is called a 2-*Banach space*. We recall a property from [26] and formulate an obvious, but useful, remark.

Lemma 2.1 [26]. *If X is a 2-normed space and $(x_n)_{n\in\mathbb{N}}$ is a convergent sequence of elements of X, then* $\lim_{n\to\infty} \|x_n, y\| = \|\lim_{n\to\infty} x_n, y\|$ *for each $y \in X$.*

Remark 2.2. Let X be a 2-normed space and $y, z \in X$. According to the condition (a) of the definition of a 2-norm, $\|z, y\| \neq 0$ if and only if the vectors z, y are linearly independent.

Note that (in view of the Cauchy–Schwarz inequality), if $\langle \cdot\,, \cdot \rangle$ is an inner product in a real linear space X of dimension at least 2 and

$$\|x, y\| := \sqrt{\|x\|^2 \|y\|^2 - \langle x, y\rangle^2}, \quad x, y \in X, \tag{2.1}$$

then conditions (a)–(d) are fulfilled. Moreover (see [8, Proposition 2.3]), if $(X, \langle \cdot\,, \cdot \rangle)$ is a real Hilbert space, then X is a 2-Banach space with respect to the 2-norm given by (2.1).

If we take in $\mathbb{R}^2$ the classical inner product: $\langle (x_1, x_2), (y_1, y_2)\rangle = x_1 y_1 + x_2 y_2$ for $(x_1, x_2), (y_1, y_2) \in \mathbb{R}^2$, then the corresponding 2-norm, given by (2.1), takes the form

$$\|(x_1, x_2), (y_1, y_2)\| := |x_1 y_2 - x_2 y_1|, \quad (x_1, x_2), (y_1, y_2) \in \mathbb{R}^2.$$

Clearly, if $\|\cdot\,, \cdot\|_1$ and $\|\cdot\,, \cdot\|_2$ are 2-norms in a real linear space X and $\alpha, \beta \in \mathbb{R}_+$, $\alpha^2 + \beta^2 > 0$, then $\|\cdot\,, \cdot\| = \alpha\|\cdot\,, \cdot\|_1 + \beta\|\cdot\,, \cdot\|_2$ is also a 2-norm in X.

An analogue of Definition 1.1 for 2-normed spaces could be formulated as follows.

Definition 2.3. Let $(Y, \|\cdot\,, \cdot\|)$ be a 2-normed space and $S \neq \emptyset$ be a set. Let $E \subset \mathbb{R}^{S^2 \times Y}$ and $\mathcal{D}_0 \subset \mathcal{D} \subset Y^S$ be nonempty, and $\mathcal{S} : E \to \mathbb{R}^{S\times Y}$, $H : Y \times Y \to Y$ and $\mathcal{T} : \mathcal{D} \to Y^{S^2}$. Then (1.1) is said to be $\mathcal{S}$-stable in $\mathcal{D}_0$ if, for any $\psi \in \mathcal{D}_0$ and $\delta \in E$ such that $\|(\mathcal{T}\psi)(s, t) - H(\psi(s), \psi(t)), y\| \le \delta(s, t, y)$ for every $s, t \in \mathcal{S}$ and $y \in Y$, there is a solution $\phi \in \mathcal{D}$ of (1.1) with $\|\phi(t) - \psi(t), y\| \le (\mathcal{S}\delta)(t, y)$ for $t \in \mathcal{S}$, $y \in Y$.

3. Fixed point theorem

Now, we present a fixed point theorem from [8, Theorem 1] that is the basic tool in the proof of our main result. To this end let us introduce three hypotheses.

(H1) $E \neq \emptyset$ is a set, $(Y, \|\cdot,\cdot\|)$ is a 2-Banach space, $Y_0 \subset Y$ contains two linearly independent vectors, $j \in \mathbb{N}$, $f_i \in E^E$, $g_i \in Y_0^{Y_0}$ and $L_i \in \mathbb{R}_+^{E\times Y}$ for $i = 1, \ldots, j$.

(H2) $\mathcal{T}\colon Y^E \to Y^E$ is an operator satisfying the inequality

$$\|(\mathcal{T}\xi)(x) - (\mathcal{T}\mu)(x), y\| \le \sum_{i=1}^{j} L_i(x,y)\|\xi(f_i(x)) - \mu(f_i(x)), g_i(y)\|,$$
$$\xi,\ \mu \in Y^E,\ x \in E,\ y \in Y_0.$$

(H3) $\Lambda\colon {\mathbb{R}_+}^{E\times Y_0} \to {\mathbb{R}_+}^{E\times Y_0}$ is an operator defined by

$$(\Lambda\delta)(x,y) := \sum_{i=1}^{j} L_i(x,y)\delta(f_i(x), g_i(y)), \quad \delta \in {\mathbb{R}_+}^{E\times Y_0},\ x \in E,\ y \in Y_0.$$

Now, we are in a position to present the fixed point theorem mentioned above.

Theorem 3.1. *Suppose that the hypotheses (H1)–(H3) hold and that the functions* $\varepsilon\colon E \times Y_0 \to \mathbb{R}_+$ *and* $\varphi : E \to Y$ *satisfy the two inequalities*

$$\|(\mathcal{T}\varphi)(x) - \varphi(x), y\| \le \varepsilon(x,y), \quad \varepsilon^*(x,y) := \sum_{l=0}^{\infty}(\Lambda^l\varepsilon)(x,y) < \infty \tag{3.1}$$

for $x \in E$ *and* $y \in Y_0$. *Then the operator* $\mathcal{T}$ *has a unique fixed point* $\psi \in Y^E$ *with* $\|\varphi(x) - \psi(x), y\| \le \varepsilon^*(x,y)$ *for* $x \in E$, $y \in Y_0$*; and* $\psi(x) = \lim_{l\to\infty}(\mathcal{T}^l\varphi)(x)$ *for* $x \in E$.

4. Stability

In this section we show that an analogue of the following theorem, proved for the classical normed spaces [6, Theorem 1.3], can be obtained also for functions taking values in 2-Banach spaces (actually, we prove it in a somewhat more general form).

Theorem 4.1. *Let* $p < 0$, V *be a Banach space,* W *be a normed space and* $D \subset W \setminus \{0\}$ *be nonempty. Assume that there is a positive integer* n_0 *with* $nx \in D$ *for* $x \in D$ *and* $n \in \mathbb{N}$, $n \ge n_0$. *If* $c \in \mathbb{R}_+$ *and* $g \in V^D$ *satisfies (1.2), then there is a unique* $h \in V^D$ *that is additive on* D *and such that*

$$\|g(x) - h(x)\| \le c\|x_1\|^p, \quad x \in D. \tag{4.1}$$

This result was motivated by an open problem raised by Th. M. Rassias and complements Theorem 1.2 by showing that the situation changes (because (4.1) is optimal) if we drop the assumption in (iii) that $-x \in D$ for $x \in D$. For some further comments and several other related outcomes, we refer to [2–5, 7, 10, 27–29].

In what follows, Y is a 2-Banach space, $(S,+)$ is an abelian semigroup and $\emptyset \neq X \subset S$. We assume that there exists a positive integer $k_0 > 1$ with

$$k_0 x \in X, \quad x \in X, \tag{4.2}$$

where $1x = x$ and $(n+1)x = nx + x$ for $x \in S$ and $n \in \mathbb{N}$. We write $N_0 := \{k_0^n : n \in \mathbb{N}\}$.

Let $A_1, A_2 : S \to Y$ be additive (that is, $A_i(x+z) = A_i(x) + A_i(z)$ for every $x, z \in S$), $C, D : Y \to Y$, $c, d \in \mathbb{R}_+$, $p, q \in (-\infty, 0)$ and $\psi : X^2 \times Y \to \mathbb{R}$ be a function such that

$$\psi^*_{k,i}(x,y) = \sum_{n=0}^{\infty} (\Lambda_k^n \psi_{k,i})(x,y) < \infty, \quad x \in X,\ y \in Y,\ i = 1,2,\ k \in N_0,\ k > \kappa, \tag{4.3}$$

with some $\kappa \in \mathbb{N}$ (see Remark 5.1), where we write $\psi_{k,1}(x,y) := \psi(x, kx, y)$, $\psi_{k,2}(x,y) := \psi(kx, x, y)$, $\Lambda_k : \mathbb{R}_+^{X \times Y} \to \mathbb{R}_+^{X \times Y}$ is given by

$$(\Lambda_k \delta)(x,y) := \delta((k+1)x, y) + \delta(kx, y), \quad x \in X,\ y \in Y,\ \delta \in \mathbb{R}_+^{X \times Y}, \tag{4.4}$$

and $\Lambda^0 \delta = \delta$, $\Lambda_k^n = \Lambda_k \circ \Lambda_k^{n-1}$ for $\delta \in \mathbb{R}_+^{X \times Y}$, $n \in \mathbb{N}$.

Moreover, we assume that the images of C and D, $C(Y)$ and $D(Y)$, are not 'too small', that is, the following hypothesis is valid ($\mathbb{R}y := \{ay : a \in \mathbb{R}\}$ for $y \in Y$).

($\mathcal{H}$) The set $D^{-1}(Y \setminus \mathbb{R}u) \cap C^{-1}(Y \setminus \mathbb{R}v)$ contains two linearly independent vectors for every $u, v \in Y$.

Define $\Psi : X^2 \times Y \to \mathbb{R}$ by the formula

$$\Psi(x_1, x_2, y) := c\|A_1(x_1), C(y)\|^p + d\|A_2(x_2), D(y)\|^q$$

when

$$\|A_1(x_1), C(y)\| \cdot \|A_2(x_2), D(y)\| \neq 0 \tag{4.5}$$

and $\Psi(x_1, x_2, y) := \psi(x_1, x_2, y)$ otherwise.

Our main result reads as follows.

THEOREM 4.2. *Let $f : X \to Y$ satisfy the inequality*

$$\|f(x_1 + x_2) - f(x_1) - f(x_2), y\| \leq \Psi(x_1, x_2, y) \tag{4.6}$$

for every $y \in Y$ and $x_1, x_2 \in X$ such that $x_1 + x_2 \in X$. Then there exists a unique $h : X \to Y$ that is additive on X and such that

$$\|f(x) - h(x), y\| \leq \min\{c\|A_1(x), C(y)\|^p, d\|A_2(x), D(y)\|^q\} \tag{4.7}$$

for every $x \in X$ and $y \in Y$ with $\|A_1(x), C(y)\| \cdot \|A_2(x), D(y)\| \neq 0$.

Moreover, for all $x \in X$ and $y \in Y$ with $\|A_1(x), C(y)\| \cdot \|A_2(x), D(y)\| = 0$,

$$\|f(x) - h(x), y\| \leq \inf_{k \in N_0} \psi^*_{k,0}(x,y),$$

where $\psi^*_{k,0}(x,y) := \min\{\psi^*_{k,1}(x,y), \psi^*_{k,2}(x,y)\}$ and $\psi^*_{k,i}$ is given by (4.3).

Proof. Fix $k \in N_0$, $k > \kappa$, with $k^p + (1+k)^p < 1$ and $k^q + (1+k)^q < 1$. It is easy to see that $x_2 = kx_1$ in (4.6) gives

$$\|f((k+1)x_1) - f(x_1) - f(kx_1), y\| \le c\|A_1(x_1), C(y)\|^p + k^q d\|A_2(x_1), D(y)\|^q \tag{4.8}$$

for $x_1 \in X$ and $y \in Y$ such that

$$\|A_1(x_1), C(y)\| \cdot \|A_2(x_1), D(y)\| \neq 0 \tag{4.9}$$

and

$$\|f((k+1)x_1) - f(x_1) - f(kx_1), y\| \le \psi(x_1, kx_1, y) \tag{4.10}$$

for $x_1 \in X$ and $y \in Y$ with $\|A_1(x_1), C(y)\| \cdot \|A_2(x_1), D(y)\| = 0$. Let $\varepsilon_0, \varepsilon_k : X \times Y \to \mathbb{R}$ for $k \in \mathbb{N}$ be defined as follows:

$$\varepsilon_0(x_1, y) = c\|A_1(x_1), C(y)\|^p, \quad \varepsilon_k(x_1, y) = k^q d\|A_2(x_1), D(y)\|^q$$

for $x_1 \in X$ and $y \in Y$ satisfying (4.9) and $\varepsilon_0(x_1, y) = \varepsilon_k(x_1, y) = \frac{1}{2}\psi(x_1, kx_1, y)$ otherwise. Let $\varepsilon = \varepsilon_0 + \varepsilon_k$ and $\Lambda_k : \mathbb{R}_+^X \to \mathbb{R}_+^X$ be given by (4.4).

Define the operator $\mathcal{T}_k : Y^X \to Y^X$ by

$$(\mathcal{T}_k\xi)(x) := \xi((k+1)x) - \xi(kx), \quad x \in X, \xi \in Y^X.$$

Then Λ_k has the same form as Λ in (H3) (with $E = S$, $j = 2$, $f_1(x) = (k+1)x$, $f_2(x) = kx$ and $L_i(x, y) = 1$, $g_i(y) = y$ for $x \in X$, $y \in Y_0 := Y$, $i = 1, 2$) and (H2) holds with $\mathcal{T} = \mathcal{T}_k$; moreover, (4.8) and (4.10) can be written jointly as

$$\|(\mathcal{T}_k f)(x_1) - f(x_1), y\| \le \varepsilon(x_1, y), \quad x_1 \in X, y \in Y.$$

Note that

$$\varepsilon_0(mx_1, y) = m^p\varepsilon_0(x_1, y), \quad \varepsilon_k(mx_1, y) = m^q\varepsilon_k(x_1, y), \quad m \in \mathbb{N}. \tag{4.11}$$

So, it is easy to prove by induction that, for every $x_1 \in X$ and $y \in Y$ satisfying (4.9),

$$(\Lambda_k^n\varepsilon_0)(x_1, y) = (k^p + (1+k)^p)^n\varepsilon_0(x_1, y), \quad (\Lambda_k^n\varepsilon_k)(x_1, y) = (k^q + (1+k)^q)^n\varepsilon_k(x_1, y)$$

for every $n \in \mathbb{N}$ and, consequently,

$$\begin{aligned} \sum_{n=0}^{\infty}(\Lambda_k^n\varepsilon)(x_1, y) &= \varepsilon_0(x_1, y)\sum_{n=0}^{\infty}(k^p + (1+k)^p)^n + \varepsilon_k(x_1, y)\sum_{n=0}^{\infty}(k^q + (1+k)^q)^n \\ &= \frac{\varepsilon_0(x_1, y)}{1 - k^p - (1+k)^p} + \frac{\varepsilon_k(x_1, y)}{1 - k^q - (1+k)^q}. \end{aligned} \tag{4.12}$$

If $x_1 \in X$ and $y \in Y$ are such that (4.9) does not hold, then, by (4.3),

$$\sum_{n=0}^{\infty}(\Lambda_k^n\varepsilon)(x_1, y) = \sum_{n=0}^{\infty}(\Lambda_k^n\psi_{k,1})(x_1, y) = \psi_{k,1}^*(x, y) < \infty.$$

Thus, we have shown that (3.1) is valid with $E = S$, $Y_0 = Y$, $\mathcal{T} = \mathcal{T}_k$, $\varphi = f$, $\Lambda = \Lambda_k$ and $\varepsilon = \psi_{k,1}$. Hence, in view of Theorem 3.1, there is a solution $T_k : X \to Y$ of the equation

$$T(x) = T((k+1)x) - T(kx) \tag{4.13}$$

such that

$$\|f(x_1) - T_k(x_1), y\| \le \varepsilon^*(x_1, y) = \sum_{n=0}^{\infty} (\Lambda_k^n \varepsilon)(x_1, y), \quad x_1 \in X,\ y \in Y, \tag{4.14}$$

and it is given by the formula

$$T_k(x_1) := \lim_{n\to\infty} \mathcal{T}_k^n f(x_1), \quad x_1 \in X. \tag{4.15}$$

Next, we prove by induction that, for every $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $y \in Y$ and $x_1, x_2 \in X$ such that $x_1 + x_2 \in X$ and (4.5) holds,

$$\begin{aligned} \|\mathcal{T}_k^n f(x_1 + x_2) - \mathcal{T}_k^n f(x_1) - \mathcal{T}_k^n f(x_2), y\| &\le (k^p + (k+1)^p)^n \varepsilon_0(x_1, y) \\ &\quad + (k^q + (k+1)^q)^n \varepsilon_k(x_1, y). \end{aligned} \tag{4.16}$$

Clearly, the case $n = 0$ follows from (4.6). So, take $n \in \mathbb{N}_0$ and assume (4.16) for every $y \in Y$ and $x_1, x_2 \in X$ such that $x_1 + x_2 \in X$ and (4.5) is valid. Hence, by (4.11),

$$\begin{aligned} &\|\mathcal{T}_k^{n+1} f(x_1 + x_2) - \mathcal{T}_k^{n+1} f(x_1) - \mathcal{T}_k^{n+1} f(x_2), y\| \\ &\quad \le \|\mathcal{T}_k^n f((k+1)x_1 + (k+1)x_2) - \mathcal{T}_k^n f((k+1)x_1) - \mathcal{T}_k^n f((k+1)x_2), y\| \\ &\qquad + \|\mathcal{T}_k^n f(kx_1 + kx_2) - \mathcal{T}_k^n f(kx_1) - \mathcal{T}_k^n f(kx_2), y\| \\ &\quad \le (k^p + (k+1)^p)^n \varepsilon_0((k+1)x_1, y) + (k^q + (k+1)^q)^n \varepsilon_k((k+1)x_2, y) \\ &\qquad + (k^p + (k+1)^p)^n \varepsilon_0(kx_1, y) + (k^q + (k+1)^q)^n \varepsilon_k(kx_2, y) \\ &\quad = (k^p + (k+1)^p)^{n+1} \varepsilon_0(x_1, y) + (k^q + (k+1)^q)^{n+1} \varepsilon_k(x_2, y) \end{aligned}$$

for $y \in Y$ and $x_1, x_2 \in X$ such that $x_1 + x_2 \in X$ and (4.5) holds, which proves (4.16).

Fix $x_1, x_2 \in X$ with $x_1 + x_2 \in X$. Now, by hypothesis $(\mathcal{H})$ (with $u = A_1(x_1)$ and $v = A_2(x_2)$), there are linearly independent vectors $z_1, z_2 \in Y$ such that the pairs of vectors $A_1(x_1), C(z_i)$ and $A_2(x_2), D(z_i)$ are linearly independent for $i = 1, 2$, which means that (4.5) holds for $y = z_i$ (see Remark 2.2). Now, letting $n \to \infty$ in (4.16) with $y = z_i$ for $i = 1, 2$, we get (see Lemma 2.1) $\|T_k(x_1 + x_2) - T_k(x_1) - T_k(x_2), z_i\| = 0$, whence $T_k(x_1 + x_2) - T_k(x_1) - T_k(x_2) = 0$, because z_1, z_2 are linearly independent. Thus, we have shown that T_k is additive on X.

Next, we prove that T_k is the unique function from Y^X that is additive on X and

$$\|f(x_1) - T_k(x_1), y\| \le \frac{B\varepsilon_0(x_1, y)}{1 - k^p - (k+1)^p} + \frac{B\varepsilon_k(x_1, y)}{1 - k^q - (k+1)^q}$$

for some $B > 0$ and every $x_1 \in X$ and $y \in Y$ satisfying (4.9). So, suppose that $B_0 \in (0, \infty)$, $T_0 : X \to Y$ is additive on X and

$$\|f(x_1) - T_0(x_1), y\| \le \frac{B_0\varepsilon_0(x_1, y)}{1 - k^p - (k+1)^p} + \frac{B_0\varepsilon_k(x_1, y)}{1 - k^q - (k+1)^q}$$

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for all $x_1 \in X, y \in Y$ such that (4.9) holds. Then (see (4.12) and (4.14))

$$\begin{aligned}\|T_k(x_1) - T_0(x_1), y\| &\le \|T_k(x_1) - f(x_1), y\| + \|f(x) - T_0(x_1), y\| \\ &\le \frac{(B_0+1)\varepsilon_0(x_1,y)}{1-k^p-(k+1)^p} + \frac{(B_0+1)\varepsilon_k(x_1,y)}{1-k^q-(k+1)^q} \\ &= (B_0+1)\varepsilon_0(x_1,y)\sum_{n=0}^{\infty}(k^p+(k+1)^p)^n \\ &\quad + (B_0+1)\varepsilon_k(x_1,y)\sum_{n=0}^{\infty}(k^q+(k+1)^q)^n \end{aligned} \tag{4.17}$$

for $x_1 \in X, y \in Y$ such that (4.9) holds.

Now, we show that, for each $j \in \mathbb{N}_0$ and $x_1 \in X, y \in Y$ satisfying (4.9),

$$\begin{aligned}\|T_k(x_1) - T_0(x_1), y\| &\le (B_0+1)\varepsilon_0(x_1,y)\sum_{n=j}^{\infty}(k^p+(k+1)^p)^n \\ &\quad + (B_0+1)\varepsilon_k(x_1,y)\sum_{n=j}^{\infty}(k^q+(k+1)^q)^n. \end{aligned} \tag{4.18}$$

Clearly, the case $j = 0$ is exactly (4.17). So, fix $j \in \mathbb{N}_0$ and assume that (4.18) holds. Note that T_k and T_0 are solutions to (4.13). Hence, by (4.17),

$$\begin{aligned}\|T_k(x_1) - T_0(x_1), y\| &= \|T_k((k+1)x_1) - T_k(kx_1) - T_0((k+1)x_1) + T_0(kx_1), y\| \\ &\le \|T_k((k+1)x_1) - T_0((k+1)x_1), y\| + \|T_k(kx_1) - T_0(kx_1), y\| \\ &\le (B_0+1)\varepsilon_0((k+1)x_1,y)\sum_{n=j}^{\infty}(k^p+(k+1)^p)^n \\ &\quad + (B_0+1)\varepsilon_k((k+1)x_1,y)\sum_{n=j}^{\infty}(k^q+(k+1)^q)^n \\ &\quad + (B_0+1)\varepsilon_0(kx_1,y)\sum_{n=j}^{\infty}(k^p+(k+1)^p)^n \\ &\quad + (B_0+1)\varepsilon_k(kx_1,y)\sum_{n=j}^{\infty}(k^q+(k+1)^q)^n \end{aligned}$$

for $x_1 \in X, y \in Y$ with (4.9). Finally, by (4.11),

$$\begin{aligned}\|T_k(x_1) - T_0(x_1), y\| &\le (B_0+1)\varepsilon_0(x_1,y)\sum_{n=j+1}^{\infty}(k^p+(k+1)^p)^n \\ &\quad + (B_0+1)\varepsilon_k(x_1,y)\sum_{n=j+1}^{\infty}(k^q+(k+1)^q)^n \end{aligned}$$

for $x_1 \in X, y \in Y$ such that (4.9) holds, which completes the proof of (4.18).

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For each $x_1 \in X$, there exist linearly independent vectors $z_1, z_2 \in Y$ such that the pairs of vectors $A_1(x_1), C(z_i)$ and $A_2(x_1), D(z_i)$ are linearly independent for $i = 1, 2$ (see hypothesis ($\mathcal{H}$)), which means that (see Remark 2.2) (4.9) holds with $y = z_i$ for $i = 1, 2$. Hence, letting $j \to \infty$ in (4.18) (with $y = z_i$ for $i = 1, 2$), we get $T_k = T_0$.

Arguing analogously, for each $l \in N_0$, $l > k$, we obtain a unique $T_l : X \to Y$ that is additive on X and

$$\|f(x_1) - T_l(x_1), y\| \le \frac{\varepsilon_0(x_1, y)}{1 - l^p - (l+1)^p} + \frac{\varepsilon_l(x_1, y)}{1 - l^q - (l+1)^q}$$

for $x_1 \in X, y \in Y$ with (4.9). Since, for each $l \in N_0$, $l > k$, we have $\varepsilon_l < \varepsilon_k$, the uniqueness of T_k implies that $T_k = T_l$, whence

$$\begin{aligned} \|f(x_1) - T_k(x_1), y\| &\le \frac{\varepsilon_0(x_1, y)}{1 - l^p - (l+1)^p} + \frac{\varepsilon_l(x_1, y)}{1 - l^q - (l+1)^q} \\ &= \frac{c\|A_1(x_1), C(y)\|^p}{1 - l^p - (l+1)^p} + \frac{l^q d\|A_2(x_1), D(y)\|^q}{1 - l^q - (l+1)^q} \end{aligned} \tag{4.19}$$

for $x_1 \in X, y \in Y$ satisfying (4.9). Now, letting $l \to \infty$ in (4.19), for every $x_1 \in X, y \in Y$ satisfying (4.9),

$$\|f(x_1) - T_k(x_1), y\| \le c\|A_1(x_1), C(y)\|^p. \tag{4.20}$$

On the other hand, (4.6) with $x_1 = kx_2$ gives

$$\|f(k+1)x_1 - f(kx_1) - f(x_1), y\| \le ck^p\|A_1(x_1), C(y)\|^p + d\|A_2(x_1), D(y)\|^q$$

for $x_1 \in X$ and $y \in Y$ such that (4.9) holds. So, we can repeat the whole reasoning of the proof analogously to obtain that

$$\|f(x_1) - T'_k(x_1), y\| \le d\|A_2(x_1), D(y)\|^q \tag{4.21}$$

for $x_1 \in X, y \in Y$ with (4.9), where

$$T'_k(x_1) := \lim_{n\to\infty} (\mathcal{T}_k^n f)(x_1), \qquad x_1 \in X. \tag{4.22}$$

Clearly, formulas (4.15) and (4.22) define the same mapping, whence (4.20) and (4.21) yield (4.7) with $h = T_k = T'_k$. The uniqueness of h is a consequence of the uniqueness property proved for T_k. □

REMARK 4.3. It follows from the proof of Theorem 4.2 that $h = T_k$ for any $k \in N_0, k > \kappa$, where T_k is given by (4.15) (or (4.22) with $T'_k = T_k$).

REMARK 4.4. Estimates (1.3) in Theorem 1.2 and (4.1) in Theorem 4.1 are optimal (see [10, Remark 3.7] and [6, Remark 3.2]). Therefore, there arises a natural open problem if this is also the case for (4.7) and (5.1) (see the next section).

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5. Final comments

The next remark provides an example of a function $\psi : X^2 \times Y \to \mathbb{R}$ such that (4.3) holds (many other similar examples can be constructed analogously).

Remark 5.1. Fix linearly independent $u_1, u_2 \in Y$. Then $\|z, u_1\|_2 + \|z, u_2\|_2 > 0$ for each $z \in Y \setminus \{0\}$ (see Remark 2.2).

Let $\alpha_1, \alpha_2, \beta_1, \beta_2 \in (0, \infty)$, $p_1, p_2 \in (-\infty, 0)$ and $\chi_1, \chi_2 : Y \to Y$. If $0 \notin A_i(X)$ for $i = 1, 2$, then we may define $\eta_1, \eta_2 : X \to \mathbb{R}_+$ and $\psi : X^2 \times Y \to \mathbb{R}_+$ by

$$\eta_i(x) = (\alpha_i\|A_i(x), u_1\| + \beta_i\|A_i(x), u_2\|)^{p_i}, \quad i = 1, 2,\ x \in X,$$
$$\psi(x_1, x_2, y) = \chi_1(y)\eta_1(x_1) + \chi_2(y)\eta_2(x_2), \quad x_1, x_2 \in X,\ y \in Y.$$

Then $\eta_i(mx) = m^{p_i}\eta_i(x)$ for $m \in \mathbb{N}$ and $i = 1, 2$, whence

$$\begin{aligned}(\Lambda_k\psi_{k,1})(x, y) &= \chi_1(y)(\eta_1((k + 1)x) + \eta_1(kx)) + \chi_2(y)(\eta_2((k + 1)kx) + \eta_2(k^2x)) \\ &\le \chi_1(y)((k + 1)^{p_1}\eta_1(x) + k^{p_1}\eta_1(x)) + \chi_2(y)((k + 1)^{p_2}\eta_2(kx) + k^{p_2}\eta_2(kx)) \\ &\le (k^{p_1} + (k + 1)^{p_1} + k^{p_2} + (k + 1)^{p_2})\psi_{k,1}(x, y)\end{aligned}$$

and analogously $(\Lambda_k\psi_{k,2})(x, y) \le (k^{p_1} + (k + 1)^{p_1} + k^{p_2} + (k + 1)^{p_2})\psi_{k,2}(x, y)$ for every $k \in N_0$, $x \in X$ and $y \in Y$. Since, for $k \in N$, Λ_k is linear, by induction we easily get

$$(\Lambda_k^n\psi_{k,i})(x, y) \le (k^{p_1} + (k + 1)^{p_1} + k^{p_2} + (k + 1)^{p_2})^n\psi_{k,i}(x, y), \quad n \in N_0,\ i = 1, 2,$$

for $x \in X$, $y \in Y$. If $\kappa \in \mathbb{N}$ and $\kappa^{p_i} + (\kappa + 1)^{p_i} < 1/2$ for $i = 1, 2$, then, for $k \in N_0$, $k \ge \kappa$,

$$\sum_{n=0}^{\infty}(\Lambda_k^n\psi_{k,i})(x, y) \le \frac{\psi_{k,i}(x, y)}{1 - (k^{p_1} + (k + 1)^{p_1} + k^{p_2} + (k + 1)^{p_2})}, \quad x \in X,\ y \in Y,\ i = 1, 2.$$

Note that the methods, from the proof of Theorem 4.2, can be applied in classical Banach spaces yielding the following generalisation of Theorem 4.1 (Theorem 5.2 also complements the main results in [5, 7]). Here S, X, p, q, c and d are as before.

Theorem 5.2. *Let V be a Banach space and $\xi_1, \xi_2 \in (V \setminus \{0\})^X$ be additive on X. Assume that there is $k_0 \in \mathbb{N}$ such that (4.2) holds. If $g \in V^X$ satisfies*

$$\|g(x_1 + x_2) - g(x_1) - g(x_2)\| \le c\|\xi_1(x_1)\|^p + d\|\xi_2(x_2)\|^q$$

for $x_1, x_2 \in X$ with $x_1 + x_2 \in X$, then there is a unique, additive on X, $h \in Y^X$ with

$$\|g(x) - h(x)\| \le \min\{c\|\xi_1(x)\|^p, d\|\xi_2(x)\|^q\}, \quad x \in X. \tag{5.1}$$

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