Journal of the Inst. of Math. Jussieu (2011) **10**(1), 59–118 59
doi:10.1017/S147474800900022X

ON DESCENT THEORY AND MAIN CONJECTURES IN NON-COMMUTATIVE IWASAWA THEORY

DAVID BURNS[1] AND OTMAR VENJAKOB[2]

[1] *Department of Mathematics, King's College London, Strand, London WC2R 2LS, UK* (david.burns@kcl.ac.uk)
[2] *Mathematisches Institut der Universität Heidelberg, 69120 Heidelberg, Germany* (venjakob@mathi.uni-heidelberg.de)

(Received 13 March 2008; revised 3 July 2009; accepted 24 July 2009)

Abstract We develop an explicit descent theory in the context of Whitehead groups of non-commutative Iwasawa algebras. We apply this theory to describe the precise connection between main conjectures of non-commutative Iwasawa theory (in the spirit of Coates, Fukaya, Kato, Sujatha and Venjakob) and the equivariant Tamagawa number conjecture. The latter result is both a converse to a theorem of Fukaya and Kato and also provides an important means of deriving explicit consequences of the main conjecture and proving special cases of the equivariant Tamagawa number conjecture.

Keywords: Iwasawa theory; L-functions; motives; K-theory; Whitehead groups

AMS 2010 *Mathematics subject classification:* Primary 11G40
Secondary 11R65; 19A31; 19B28

Introduction

There has been much interest in the study of non-commutative Iwasawa theory over the last few years. Nevertheless, there is still no satisfactory understanding of the explicit consequences for Hasse–Weil L-functions that are implied by a 'main conjecture' of the kind formulated by Coates, Fukaya, Kato, Sujatha and the second named author in [**15**]. Indeed, while explicit consequences of such a conjecture for the values (at $s = 1$) of twisted Hasse–Weil L-functions have been studied by Coates *et al*. in [**15**], by Kato in [**21**] and by Dokchister and Dokchister in [**17**], all of these consequences become trivial whenever the L-functions vanish at $s = 1$. Further, the conjecture of Birch and Swinnerton-Dyer implies that these L-functions should vanish whenever the relevant component of the Mordell–Weil group has strictly positive rank and by a recent result of Mazur and Rubin [**23**], which is itself equivalent to a special case of an earlier result of Nekovář [**25**, Theorem 10.7.17], this should often be the case. It is therefore of interest to understand what a main conjecture of the kind formulated in [**15**] predicts concerning the values of *derivatives* of Hasse–Weil L-functions at $s = 1$.

In this article we take the first step towards developing such a theory by describing a general formalism of descent in non-commutative Iwasawa theory. In a subsequent article it will be shown that the results proved here can indeed be combined with techniques developed by the first named author in [**7**] to derive from the main conjecture of non-commutative Iwasawa theory a variety of explicit (and highly non-trivial) congruence relations between values of derivatives of twisted Hasse–Weil L-functions. In other directions, the descent theory developed here has also played a key role in obtaining the first verification of the equivariant Tamagawa number conjecture (for certain Tate motives) for an interesting class of non-abelian extensions of number fields and in the proof of a long-standing conjecture of Chinburg in the setting of global function fields (for more details see [**9**] and [**8**] respectively).

However, as preparation for the above applications, we must first develop several aspects of the theory that appear themselves to be of some independent interest. These include proving a natural Weierstrass Preparation Theorem for Whitehead groups of Iwasawa algebras, defining a canonical 'characteristic series' for torsion modules over (localized) Iwasawa algebras, satisfactorily resolving the descent problem in non-commutative Iwasawa theory and formulating a main conjecture in the spirit of Coates *et al*. that deals with interpolation properties of the 'leading terms at Artin representations' (in the sense introduced in [**13**]) of analytic p-adic L-functions.

In a little more detail, the main contents of this article are as follows. In § 1 we recall some useful preliminaries concerning localization of Iwasawa algebras, K-theory, virtual objects and derived categories. In § 2 we state the main K-theoretical results that are proved in this article. In § 3 we define a suitable notion of μ-invariant and in § 4 we combine this notion with a result of Schneider and the second named author from [**28**] and the formalism developed by Fukaya and Kato in [**18**] to define canonical 'characteristic series' in non-commutative Iwasawa theory (this construction extends the notion of 'algebraic p-adic L-functions' introduced by the first named author in [**6**] and hence also refines the notion of 'Akashi series' introduced by Coates, Schneider and Sujatha in [**14**]). As a first application of these characteristic series we use them in § 5 to prove an explicit formula for the 'leading terms at Artin representations ' of elements of Whitehead groups of non-commutative Iwasawa algebras: this result provides a suitable 'descent formalism' in non-commutative Iwasawa theory and in particular plays a crucial role in proving the arithmetic results discussed in the remainder of the article. In § 6 we present a result of Kato that allows reduction to a convenient special class of extensions when formulating main conjectures and, in particular, shows that the main result of § 5 is indeed a satisfactory resolution of the descent problem in the context of non-commutative Iwasawa theory. In § 7 we formulate explicit main conjectures of non-commutative Iwasawa theory for both Tate motives and certain critical motives. The approach here is finer than that of [**15**] since we consider interpolation properties for leading terms of analytic p-adic L-functions. In § 8 we combine the descent formalism described in § 5 with the main results of our earlier article [**13**] to prove that, under suitable hypotheses, the main conjectures formulated in § 7 imply the relevant special cases of the equivariant Tamagawa number conjecture formulated by Flach and the first named author in [**10**, Conjecture 4(iv)]. These results are both a converse to the result of Fukaya and Kato in [**18**] which asserts that, under suitable hypotheses, the 'non-commutative Tamagawa number conjecture' of [**18**] implies the main conjecture of Coates *et al.* [**15**] and can also be used to derive explicit consequences of the main conjecture. Finally, in several appendices, we review relevant aspects of the algebraic formalism of localized K_1-groups and Bockstein homomorphisms and clarify certain normalizations used in [**13**].

Part I. K-theory

1. Preliminaries

1.1. Iwasawa algebras

We fix a prime p. For any compact p-adic Lie group G we write $\Lambda(G)$ and $\Omega(G)$ for the 'Iwasawa algebras' $\varprojlim_U \mathbb{Z}_p[G/U]$ and $\varprojlim_U \mathbb{F}_p[G/U]$ where U runs over all open normal subgroups of G and the limits are taken with respect to the natural projection maps $\mathbb{Z}_p[G/U] \to \mathbb{Z}_p[G/U']$ and $\mathbb{F}_p[G/U] \to \mathbb{F}_p[G/U']$ for $U \subseteq U'$. The rings $\Lambda(G)$ and $\Omega(G)$ are both noetherian and, if G has no element of order p, they are also regular in the sense that their (left and right) global dimensions are finite. We write $Q(G)$ for the total quotient ring of $\Lambda(G)$. If $\mathcal{O}$ is any subring of $\mathbb{Q}_p^c$ that contains $\mathbb{Z}_p$, then we set $\Lambda_{\mathcal{O}}(G) := \mathcal{O} \otimes_{\mathbb{Z}_p} \Lambda(G)$ and write $Q_{\mathcal{O}}(G)$ for its total quotient ring.

We assume throughout that the following condition is satisfied.

- G has a closed normal subgroup H for which the quotient group $\Gamma := G/H$ is isomorphic (topologically) to the additive group of $\mathbb{Z}_p$.

We write $\pi_\Gamma : G \to \Gamma$ for the natural projection and fix a topological generator γ of Γ. We use γ to identify $\Lambda(\Gamma)$ with the power series ring $\mathbb{Z}_p[\![T]\!]$ in an indeterminate T (via the identification $T = \gamma - 1$).

We recall from [**15**, §§ 2–3] that there are canonical left and right denominator sets $S_{G,H}$ and $S^*_{G,H}$ of $\Lambda(G)$ where

$$S_{G,H} := \{\lambda \in \Lambda(G) : \Lambda(G)/(\Lambda(G) \cdot \lambda) \text{ is a finitely generated } \Lambda(H)\text{-module}\}$$

and $S^*_{G,H} := \bigcup_{i \geqslant 0} p^i S_{G,H}$. When G and H are clear from context we usually abbreviate $S_{G,H}$ to S. We also write $\mathfrak{M}_S(G)$ and $\mathfrak{M}_{S^*}(G)$ for the categories of finitely generated $\Lambda(G)$-modules M with $\Lambda(G)_S \otimes_{\Lambda(G)} M = 0$ and $\Lambda(G)_{S^*} \otimes_{\Lambda(G)} M = 0$ respectively.

For any $\mathbb{Z}_p$-module M we write M_{tor} for its $\mathbb{Z}_p$-torsion submodule and set $M_{\mathrm{tf}} := M/M_{\mathrm{tor}}$. We recall from [**15**, Proposition 2.3] that a finitely generated $\Lambda(G)$-module M belongs to $\mathfrak{M}_S(G)$, respectively $\mathfrak{M}_{S^*}(G)$, if and only if it is a finitely generated $\Lambda(H)$-module (by restriction), respectively when M_{tf} belongs to $\mathfrak{M}_S(G)$. This means in particular that $\mathfrak{M}_{S^*}(G)$ coincides with the category $\mathfrak{M}_H(G)$ introduced in [**15**].

We write $M \hat{\otimes}_{\Lambda(G)} N$ for the completed tensor product of compact $\Lambda(G)$-modules M and N and we recall that if either M or N is a finitely generated $\Lambda(G)$-module, then $M \hat{\otimes}_{\Lambda(G)} N$ identifies with the usual tensor product $M \otimes_{\Lambda(G)} N$.

1.2. *K*-groups

For any ring homomorphism $R \to R'$ we write $K_0(R, R')$ for the associated relative algebraic K_0-group. We recall that this group is generated by symbols of the form (P, λ, Q) where P and Q are finitely generated projective (left) R-modules and λ is an isomorphism of R'-modules $R' \otimes_R P \to R' \otimes_R Q$ (for more details see [**31**, p. 215]). For any ring homomorphisms $R \to R' \to R''$ there is a natural commutative diagram of long exact sequences

$$\begin{array}{ccccccc} K_1(R) & \longrightarrow & K_1(R') & \xrightarrow{\partial_{R,R'}} & K_0(R,R') & \longrightarrow & K_0(R) \\ \| & & \downarrow & & \downarrow & & \| \\ K_1(R) & \longrightarrow & K_1(R'') & \xrightarrow{\partial_{R,R''}} & K_0(R,R'') & \longrightarrow & K_0(R) \end{array} \tag{1.1}$$

If G has no element of order p and Σ denotes either S or S^*, then $\Lambda(G)$ is a noetherian regular ring and $K_0(\Lambda(G), \Lambda(G)_\Sigma)$ can be identified with the Grothendieck group $K_0(\mathfrak{M}_\Sigma(G))$ of the category $\mathfrak{M}_\Sigma(G)$. To be precise we normalize this isomorphism as follows: if $g = s^{-1}h$ with $s \in \Sigma$ and $h \in \Lambda(G) \cap \Lambda(G)_\Sigma^\times$, then the element $(\Lambda(G), \mathrm{r}_g, \Lambda(G))$ of $K_0(\Lambda(G), \Lambda(G)_\Sigma)$ corresponds to $[\mathrm{cok}(\mathrm{r}_h)] - [\mathrm{cok}(\mathrm{r}_s)]$ in $K_0(\mathfrak{M}_\Sigma(G))$ where $[X]$ is the element of $K_0(\mathfrak{M}_\Sigma(G))$ associated to an object X of $\mathfrak{M}_\Sigma(G)$ and r_g, r_h and r_s denote the automorphisms of $\Lambda(G)_\Sigma$ that are induced by right multiplication by g, h and s respectively. In particular, with respect to this isomorphism, the upper row of (1.1) with $R = \Lambda(G)$ and $R' = \Lambda(G)_{S^*}$ identifies with the exact sequence of [**15**, (24)].

If $R = \Lambda(G)$ and $R' = \Lambda(G)_{S^*}$, respectively $R = \mathbb{Z}_p[\mathcal{G}]$ and $R' = \mathbb{Q}_p^c[\mathcal{G}]$ for a finite group $\mathcal{G}$, then we often abbreviate the connecting homomorphism $\partial_{R,R'}$ in diagram (1.1) to ∂_G, respectively $\partial_{\bar{G}}$.

1.3. Virtual objects

We let $\mathcal{P}_0$ denote the Picard category with unique object $\mathbf{1}_{\mathcal{P}_0}$ and $\mathrm{Aut}_{\mathcal{P}_0}(\mathbf{1}_{\mathcal{P}_0}) = 0$. For any associative unital ring R we also write $V(R)$ for the Picard category of virtual objects associated to the category of finitely generated projective R-modules and we fix a unit object $\mathbf{1}_R$ in $V(R)$. For any homomorphism of such rings $R \to R'$ we then define $V(R, R')$ to be the fibre product category in the diagram

$$\begin{array}{ccc} V(R,R') & \longrightarrow & \mathcal{P}_0 \\ \downarrow & & \downarrow{\scriptstyle F_2} \\ V(R) & \xrightarrow{F_1} & V(R') \end{array}$$

where F_2 is the (monoidal) functor sending $\mathbf{1}_{\mathcal{P}_0}$ to $\mathbf{1}_{R'}$ and $F_1(L) = R' \otimes_R L$ for each object L of $V(R)$. We regard the canonical isomorphism

$$\pi_0(V(R, R')) \cong K_0(R, R') \tag{1.2}$$

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of [**3**, Lemma 5.1] (and [**10**, Proposition 2.5]) as an identification. In particular, for each object L of $V(R)$ and each morphism $\mu : F_1(L) \to \mathbf{1}_{R'}$ in $V(R')$ we write $[L,\mu]$ for the associated element of $K_0(R,R')$.

1.4. Euler characteristics

For any ring R we write $D(R)$ for the derived category of R-modules. If R is noetherian, then we also write $D^{\mathrm{fg},-}(R)$, respectively $D^{\mathrm{fg}}(R)$ for the full triangulated subcategory of $D(R)$ comprising complexes that are isomorphic to a bounded above, respectively bounded, complex of finitely generated R-modules and we let $D^{\mathrm{p}}(R)$ denote the full subcategory of $D^{\mathrm{fg}}(R)$ comprising complexes that are isomorphic to an object of the category $C^{\mathrm{p}}(R)$ of bounded complexes of finitely generated projective R-modules.

If Σ denotes either S or S^*, then we write $D^{\mathrm{p}}_{\Sigma}(\Lambda(G))$ for the full triangulated subcategory of $D^{\mathrm{p}}(\Lambda(G))$ comprising those complexes C such that $\Lambda(G)_\Sigma \hat{\otimes}^{\mathbb{L}}_{\Lambda(G)} C$ is acyclic. For each such C we write $\chi(C)$ for the *inverse* of the element of $K_0(\Lambda(G),\Lambda(G)_{S^*})$ that corresponds under (1.2) (with $R=\Lambda(G)$ and $R'=\Lambda(G)_{S^*}$) to the pair $([P^\bullet],\iota_{P^\bullet})$ with $[P^\bullet]$ the object of $\mathcal{V}(\Lambda(G))$ associated to any $P^\bullet$ in $C^{\mathrm{p}}(\Lambda(G))$ that is isomorphic in $D^{\mathrm{p}}(\Lambda(G))$ to C and $\iota_{P^\bullet}$ the morphism in $\mathcal{V}(\Lambda(G)_{S^*})$ associated to the isomorphism $\Lambda(G)_{S^*}\otimes_{\Lambda(G)} P^\bullet \cong \Lambda(G)_{S^*}\hat{\otimes}^{\mathbb{L}}_{\Lambda(G)} C \cong 0$ in $D^{\mathrm{p}}(\Lambda(G)_{S^*})$. This element $\chi(C)$ is the inverse of the Euler characteristic $\chi_{\Lambda(G),\Lambda(G)_{S^*}}(C,t)$ that is defined in [**3**, Definition 5.5] with t equal to the isomorphism

$$\bigoplus_{i\in\mathbb{Z}} H^{2i}(\Lambda(G)_{S^*}\hat{\otimes}^{\mathbb{L}}_{\Lambda(G)} C) \cong 0 \cong \bigoplus_{i\in\mathbb{Z}} H^{2i+1}(\Lambda(G)_{S^*}\hat{\otimes}^{\mathbb{L}}_{\Lambda(G)} C).$$

(We prefer to define $\chi(C)$ in terms of the inverse in order to ensure that if G has no element of order p, then the isomorphism $K_0(\Lambda(G),\Lambda(G)_{S^*}) \cong K_0(\mathfrak{M}_{S^*}(G))$ discussed in § 1.2 sends $\chi(C)$ to $\sum_{i\in\mathbb{Z}}(-1)^i[H^i(C)]$.)

1.5. Wedderburn decompositions

We fix an algebraic closure $\mathbb{Q}_p^c$ of $\mathbb{Q}_p$. For any finite group $\mathcal{G}$ we write $\mathrm{Irr}(\mathcal{G})$ for the set of irreducible finite-dimensional $\mathbb{Q}_p^c$-valued characters of $\mathcal{G}$. Then the Wedderburn decomposition of the (finite-dimensional semisimple) $\mathbb{Q}_p^c$-algebra $\mathbb{Q}_p^c[\mathcal{G}]$ induces a decomposition of its centre

$$\zeta(\mathbb{Q}_p^c[\mathcal{G}]) \cong \prod_{\mathrm{Irr}(\mathcal{G})} \mathbb{Q}_p^c. \tag{1.3}$$

The natural reduced norm map $\mathrm{Nrd}_{\mathbb{Q}_p^c[\mathcal{G}]} : K_1(\mathbb{Q}_p^c[\mathcal{G}]) \to \zeta(\mathbb{Q}_p^c[\mathcal{G}])^\times$ is bijective and we often (and without explicit comment) combine this map with (1.3) to regard elements of $\prod_{\mathrm{Irr}(\mathcal{G})}\mathbb{Q}_p^{c,\times}$ as elements of the Whitehead group $K_1(\mathbb{Q}_p^c[\mathcal{G}])$. In particular, we write $\partial_{\mathcal{G}} : \prod_{\mathrm{Irr}(\mathcal{G})}\mathbb{Q}_p^{c,\times} \to K_0(\mathbb{Z}_p[\mathcal{G}],\mathbb{Q}_p^c[\mathcal{G}])$ for the connecting homomorphism of relative K-theory (normalized as per the discussion following diagram (1.1)).

For each $\rho\in\mathrm{Irr}(\mathcal{G})$ we fix a minimal idempotent e_ρ in $\mathbb{Q}_p^c[\mathcal{G}]$ for which the left action of $\mathcal{G}$ on $\mathbb{Q}_p^c[\mathcal{G}]$ given by $x\mapsto xg^{-1}$ for $g\in\mathcal{G}$ induces upon restriction an isomorphism of (left) $\mathbb{Q}_p^c[\mathcal{G}]$-modules $e_\rho\mathbb{Q}_p^c[\mathcal{G}] \cong V_{\rho^*}$ where $V_{\rho^*}\cong(\mathbb{Q}_p^c)^{n_\rho}$ is the representation space of

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the contragredient ρ^* of ρ over $\mathbb{Q}_p^c$. Then for each complex D in $D^{\mathrm{p}}(\mathbb{Z}_p[\mathcal{G}])$ the theory of Morita equivalence induces an identification of morphism groups

$$\begin{aligned}\mathrm{Mor}_{V(\mathbb{Q}_p^c[\mathcal{G}])}(\mathbf{d}_{\mathbb{Q}_p^c[\mathcal{G}]}(\mathbb{Q}_p^c[\mathcal{G}]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[\mathcal{G}]} D), \mathbf{1}_{\mathbb{Q}_p^c[\mathcal{G}]})\\ \cong \prod_{\mathrm{Irr}(\mathcal{G})} \mathrm{Mor}_{V(\mathbb{Q}^c)}(\mathbf{d}_{\mathbb{Q}_p^c}(e_\rho\mathbb{Q}_p^c[\mathcal{G}]\otimes^{\mathbb{L}}_{\mathbb{Z}_p[\mathcal{G}]} D), \mathbf{1}_{\mathbb{Q}_p^c}).\end{aligned} \tag{1.4}$$

Details of the 'non-commutative determinants' $\mathbf{d}_{\mathbb{Q}_p^c[\mathcal{G}]}(-)$ and $\mathbf{d}_{\mathbb{Q}_p^c}(-)$ that are used here are recalled in Appendix A.

2. Statement of the main results in Part I

The first main result we prove in Part I is the following decomposition theorem for Whitehead groups.

Theorem 2.1. *If G has no element of order p, then there is a natural isomorphism of abelian groups*

$$K_0(\Omega(G)) \oplus K_0(\mathfrak{M}_S(G)) \oplus \mathrm{im}(K_1(\Lambda(G)) \to K_1(\Lambda(G)_{S^*})) \cong K_1(\Lambda(G)_{S^*}).$$

Our proof of Theorem 2.1 will show that if $G = \Gamma$, then the above isomorphism reduces to the assertion that every element of $Q(\Gamma)^\times$ can be written uniquely in the form $p^m du$ where m is an integer, d is a quotient of distinguished polynomials and u a unit in $\Lambda(\Gamma)$ (see Remark 4.2 and § 4.3). Theorem 2.1 is therefore a natural generalization of the classical Weierstrass Preparation Theorem. For an alternative approach to generalizing the latter result see [**34**].

In the remainder of Part I we apply the decomposition in Theorem 2.1 to help resolve the 'descent problem' in non-commutative Iwasawa theory. Before stating our main result in this regard we recall that for each Artin representation $\rho : G \to \mathrm{GL}_n(\mathcal{O})$ the ring homomorphism $\Lambda(G)_{S^*} \to M_n(Q_{\mathcal{O}}(\Gamma))$ that sends each element g of G to $\rho(g)\pi_\Gamma(g)$ induces a homomorphism of groups

$$\Phi_\rho : K_1(\Lambda(G)_{S^*}) \to K_1(M_n(Q_{\mathcal{O}}(\Gamma))) \cong K_1(Q_{\mathcal{O}}(\Gamma)) \cong Q_{\mathcal{O}}(\Gamma)^\times \cong Q(\mathcal{O}[\![T]\!])^\times, \tag{2.1}$$

where the first isomorphism is induced by the theory of Morita equivalence, the second by taking determinants (over $Q_{\mathcal{O}}(\Gamma)$) and the third by the identification $\gamma - 1 = T$. The 'leading term' $\xi^*(\rho)$ at ρ of an element ξ of $K_1(\Lambda(G)_{S^*})$ is then defined to be the leading term $(\Phi_\rho(\xi))^*(0)$ at $T = 0$ of the power series $\Phi_\rho(\xi)$ (this definition can also be interpreted as a leading term at zero of a p-adic meromorphic function; see [**13**, Lemma 3.17]).

The problem of descent in (non-commutative) Iwasawa theory is then the following: given an element ξ of $K_1(\Lambda(G)_{S^*})$ and a finite quotient $\bar{G}$ of G, can one use knowledge of the image of ξ under the connecting homomorphism ∂_G to give an explicit formula for the image of $(\xi^*(\rho))_{\rho\in\mathrm{Irr}(\bar{G})} \in K_1(\mathbb{Q}_p^c[\mathcal{G}])$ under the connecting homomorphism $\partial_{\bar{G}}$? This has been known for some time to be an important and delicate problem.

Before stating our main result in this regard we quickly recall some terminology from [**13**]. For a complex C in $D^p(\Lambda(G))$ and representation ρ as above we define a complex in $D^p(\Lambda_{\mathcal{O}}(\Gamma))$

$$C_\rho := (\Lambda_{\mathcal{O}}(\Gamma) \otimes_{\mathcal{O}} \mathcal{O}^n) \otimes^{\mathbb{L}}_{\Lambda(G)} C \cong \Lambda_{\mathcal{O}}(\Gamma) \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(G)} C(\rho^*) \tag{2.2}$$

(see § 3.1 for details about the G-actions) and an associated Euler characteristic

$$r_G(C)(\rho) := \sum_{i\in\mathbb{Z}} (-1)^{i+1} \dim_{\mathbb{Q}_p^c}(H^i(\mathbb{Q}_p^c \otimes_{\mathcal{O}} C_\rho)^\Gamma).$$

(In our arithmetic applications the integer $r_G(C)(\rho)$ corresponds to the 'algebraic Mordell–Weil rank of the ρ-component' of the motive under consideration.) As a natural generalization of the classical notion of 'semisimplicity at zero' of $\mathbb{Z}_p[\![T]\!]$-modules we also say that C is 'semisimple at ρ' if a certain associated complex is acyclic and in such a case we obtain a canonical morphism in $V(\mathbb{Q}_p^c)$ of the form

$$t(C_\rho) : \mathbf{d}_{\mathbb{Q}_p^c}(e_\rho \mathbb{Q}_p^c[\bar{G}] \hat{\otimes}^{\mathbb{L}}_{\Lambda(G)} C) \to \mathbf{1}_{\mathbb{Q}_p^c}.$$

Finally, we define an Ore set

$$\tilde{S} := \begin{cases} S, & \text{if } G \text{ has an element of order } p, \\ S^*, & \text{otherwise.} \end{cases} \tag{2.3}$$

Theorem 2.2. *Let $\bar{G}$ be a finite quotient of G. Let ξ be an element of $K_1(\Lambda(G)_{S^*})$ with $\partial_G(\xi) = \chi(C)$ where C is a complex that belongs to $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G))$ and is semisimple at each representation ρ in $\mathrm{Irr}(\bar{G})$. Then in $K_0(\mathbb{Z}_p[\bar{G}], \mathbb{Q}_p^c[\bar{G}])$ one has*

$$\partial_{\bar{G}}((\xi^*(\rho))_{\rho\in\mathrm{Irr}(\bar{G})}) = -[\mathbf{d}_{\mathbb{Z}_p[\bar{G}]}(\mathbb{Z}_p[\bar{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} C), t(C)_{\bar{G}}],$$

where $t(C)_{\bar{G}}$ denotes the morphism $\mathbf{d}_{\mathbb{Q}_p^c[\bar{G}]}(\mathbb{Q}_p^c[\bar{G}]\hat{\otimes}^{\mathbb{L}}_{\Lambda(G)} C) \to \mathbf{1}_{\mathbb{Q}_p^c[\bar{G}]}$ in $V(\mathbb{Q}_p^c[\bar{G}])$ that corresponds via (1.4) (with $\mathcal{G} = \bar{G}$ and $D = \mathbb{Z}_p[\bar{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} C$) to the tuple $((-1)^{r_G(C)(\rho)} t(C_\rho))_{\rho\in\mathrm{Irr}(\bar{G})}$, and on the right-hand side of the displayed equality we use the notation introduced after (1.2).

Remark 2.3. The homomorphism Φ_ρ, leading term $\xi^*(\rho)$ and morphisms $t(C_\rho)$ and $t(C)_{\bar{G}}$ all depend upon the precise choice of γ but for typographic simplicity we do not indicate this dependence. For more details about the invariants $r_G(C)(\rho)$, the morphisms $t(C_\rho)$ and the concept of semisimplicity at ρ see § 5.2.4.

Remark 2.4. In the case that the complex $e_\rho \mathbb{Q}_p^c[\bar{G}]\hat{\otimes}^{\mathbb{L}}_{\Lambda(G)} C$ is acyclic one knows that C is automatically semisimple at ρ, that $r_G(C)(\rho) = 0$ and that $t(C_\rho)$ is the inverse of the canonical morphism that is induced by property A(e) in Appendix A of the determinant functor described in Appendix A. In particular, if $G = \Gamma$, $C = M[0]$ for a finitely generated torsion $\Lambda(\Gamma)$-module M for which both M^Γ and M_Γ are finite and ρ is the trivial character, then the equality of Theorem 2.2 is equivalent to the classical descent formula

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discussed in [**36**, p. 318, Example 13.12]. Upon appropriate specialization, Theorem 2.2 also recovers the descent formalism proved in certain commutative cases by Greither and the first named author in [**11**, § 8] and is therefore related to the earlier (commutative) work of Nekovář in [**25**, § 11].

In § 6 we will prove that it suffices to consider main conjectures of non-commutative Iwasawa theory in the case that G has no element of order p. Theorem 2.2 therefore represents a satisfactory resolution of the descent problem in this context. Indeed, in Part II (§§ 6–8) of this article we shall combine Theorem 2.2 with the main results of [**13**] to describe the precise connection between main conjectures of non-commutative Iwasawa theory (in the spirit of Coates *et al.* [**15**]) and the appropriate cases of the equivariant Tamagawa number conjecture. Other interesting arithmetic applications of Theorem 2.2 are described in [**8**] and [**9**].

3. Generalized μ-invariants

The key ingredient in our proof of Theorem 2.1 is the construction of canonical 'characteristic series' in non-commutative Iwasawa theory. In this section we prepare for this construction by generalizing the classical notion of μ-invariant.

3.1. The definition

Below we write $\mu_\Gamma(M)$ for the 'μ-invariant' of a finitely generated $\Lambda(\Gamma)$-module M. For each complex C in $D^{\mathrm{p}}(\Lambda(\Gamma))$ we also set

$$\mu_\Gamma(C) := \sum_{i\in\mathbb{Z}} (-1)^i \mu_\Gamma(H^i(C)).$$

For modules or complexes over $\Lambda_{\mathcal{O}}(G)$ we similarly write $\mu_{\Gamma,\mathcal{O}}$ for the corresponding μ-invariant.

Let $\rho : G \to GL_n(\mathcal{O})$ be a continuous representation of G and write $E_\rho \cong \mathcal{O}^n$ for the associated representation module, where $\mathcal{O} = \mathcal{O}_L$ denotes the ring of integers of a finite extension L of $\mathbb{Q}_p$. We denote the corresponding L-linear representation $L \otimes_{\mathcal{O}} E_\rho$ by V_ρ. We fix a uniformizing parameter π of $\mathcal{O}$ and denote the residue class field of $\mathcal{O}$ by κ. We write $\bar{\rho}$ for the reduction of ρ modulo π and denote the associated representation space by $\overline{E_\rho}$.

For each C in $D^{\mathrm{p}}(\Lambda(G))$ we set $C(\rho^*) := \mathcal{O}^n \otimes_{\mathbb{Z}_p} C$, regarded as an object of $D^{\mathrm{p}}(\Lambda_{\mathcal{O}}(G))$ via the action $g(x \otimes_{\mathbb{Z}_p} c^i) = \rho^*(g)(x) \otimes_{\mathbb{Z}_p} g(c^i)$ for each g in G, x in $\mathcal{O}^n$ and c^i in C^i. We then set

$$C_\rho := (\Lambda_{\mathcal{O}}(\Gamma) \otimes_{\mathcal{O}} \mathcal{O}^n) \otimes^{\mathbb{L}}_{\Lambda(G)} C \cong \Lambda_{\mathcal{O}}(\Gamma) \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(G)} C(\rho^*) \tag{3.1}$$

and also

$$\mu(C,\rho) := \mu_{\Gamma,\mathcal{O}}(C_\rho) \in \mathbb{Z}. \tag{3.2}$$

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3.2. Basic properties

For each complex C we write $\mathrm{H}(C)$ for the complex which has $\mathrm{H}(C)^i = H^i(C)$ in each degree i and for which all differentials are zero. If $\mathrm{H}(C)$ belongs to $D^{\mathrm{p}}(\Lambda(G))$, then we shall say that C is 'cohomologically perfect'.

Lemma 3.1. *Fix a continuous representation* $\rho : G \to GL_n(\mathcal{O})$.

(i) *If* $C_1 \to C_2 \to C_3 \to C_1[1]$ *is an exact triangle in* $D^{\mathrm{p}}_{S^*}(\Lambda(G))$, *then*

$$\mu(C_2, \rho) = \mu(C_1, \rho) + \mu(C_3, \rho).$$

(ii) *If* $C \in D^{\mathrm{p}}_{S^*}(\Lambda(G))$ *is cohomologically perfect, then* $\mu(C, \rho) = \mu(\mathrm{H}(C), \rho)$.

(iii) *If* $C \in D^{\mathrm{p}}_{S}(\Lambda(G))$ *is cohomologically perfect, then* $\mu(C, \rho) = 0$.

(iv) *If* U *is any closed normal subgroup of* G *such that* $U \subseteq H \cap \ker(\rho)$, *then for any* C *in* $D^{\mathrm{p}}(\Lambda(G))$ *we have*

$$\mu(C, \rho) = \mu(\Lambda(G/U) \otimes^{\mathbb{L}}_{\Lambda(G)} C, \rho).$$

Here the first μ*-invariant is formed with respect to the group* G *and the second with respect to* G/U.

(v) *If* U *is any open subgroup of* G, *then for any continuous representation* $\psi : U \to GL_n(\mathcal{O})$ *and any* $C \in D^{\mathrm{p}}(\Lambda(G))$ *one has*

$$\mu(C, \mathrm{Ind}^G_U \psi) = \mu(\mathrm{Res}^G_U C, \psi)$$

where the first μ*-invariant is formed with respect to* G *and the second with respect to* U. *Here* Res^G_U *denotes the restriction functor from* $\Lambda(G)$*-modules to* $\Lambda(U)$*-modules.*

Proof. For each D in $D^{\mathrm{p}}_{S^*}(\Lambda(G))$ all of the $\Lambda_{\mathcal{O}}(\Gamma)$-modules $H^i(D_\rho)$ are both finitely generated and torsion. Since $\mu_{\Gamma,\mathcal{O}}(-)$ is additive on exact sequences of finitely generated torsion $\Lambda_{\mathcal{O}}(\Gamma)$-modules, claim (i) therefore follows from the long exact sequence of cohomology of the exact triangle $C_{1,\rho} \to C_{2,\rho} \to C_{3,\rho} \to C_{1,\rho}[1]$ in $D^{\mathrm{p}}(\Lambda_{\mathcal{O}}(\Gamma))$ that is induced by the given triangle.

If $C \cong H^i(C)[i]$ for some i, then claim (ii) is clear. The general case can then be proved by induction with respect to the cohomological length: indeed, one need only combine claim (i) together with the exact triangles given by the (non-naive) truncation functor.

In order to prove claim (iii) it is sufficient by claim (ii) to consider the case $C = M[0]$ with M a $\Lambda(G)$-module that is finitely generated over $\Lambda(H)$. But then $H^i(C_\rho)$ is a finitely generated $\mathbb{Z}_p$-module in each degree i and so it is clear that $\mu(C, \rho) = 0$.

In the situation of claim (iv) there is a canonical isomorphism in $D^{\mathrm{p}}(\Lambda_{\mathcal{O}}(G/U))$ of the form

$$\Lambda_{\mathcal{O}}(G/U) \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(G)} C(\rho^*) \cong (\Lambda(G/U) \otimes^{\mathbb{L}}_{\Lambda(G)} C)(\rho^*),$$

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from which the claim follows immediately. Similarly, in the situation of claim (v) we have a canonical isomorphism $\mathrm{Ind}_U^G((\mathrm{Res}_U^G C)(\psi^*)) \cong C(\mathrm{Ind}_U^G \psi^*)$ in $D^{\mathrm{p}}(\Lambda_{\mathcal{O}}(G))$ which corresponds to

$$\Lambda_{\mathcal{O}}(G) \otimes_{\Lambda_{\mathcal{O}}(U)} (\mathcal{O}^n \otimes_{\mathbb{Z}_p} \mathrm{Res}_U^G C) \cong (\Lambda(G) \otimes_{\Lambda(U)} \mathcal{O}^n) \otimes_{\mathbb{Z}_p} C,$$
$$g \otimes (a \otimes c) \mapsto (g \otimes a) \otimes g(c).$$

Now we write Γ_U for the image of U in Γ under the natural projection and obtain

$$\begin{aligned}\mu(C, \mathrm{Ind}_U^G \psi) &= \mu_\Gamma(\Lambda_{\mathcal{O}}(\Gamma) \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(G)} C(\mathrm{Ind}_U^G \psi^*)) \\ &= \mu_\Gamma(\Lambda_{\mathcal{O}}(\Gamma) \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(G)} \mathrm{Ind}_U^G((\mathrm{Res}_U^G C)(\psi^*))) \\ &= \mu_\Gamma(\Lambda_{\mathcal{O}}(\Gamma) \otimes_{\Lambda_{\mathcal{O}}(\Gamma_U)} (\Lambda_{\mathcal{O}}(\Gamma_U) \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(U)} (\mathrm{Res}_U^G C)(\psi^*))) \\ &= \mu_{\Gamma_U}(\Lambda_{\mathcal{O}}(\Gamma_U) \otimes^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(U)} (\mathrm{Res}_U^G C)(\psi^*)) \\ &= \mu(\mathrm{Res}_U^G C, \psi),\end{aligned}$$

as had to be shown. □

3.3. Modules and idempotents

In order to make a closer examination of the μ-invariant defined in (3.2) we recall some standard module theory. This section is inspired by [**1**], where further details can also be found.

We write $\mathrm{Jac}(\Lambda(G))$ for the Jacobson radical of $\Lambda(G)$ and $\prod_{i\in I} A_i$ for the Wedderburn decomposition of the finite-dimensional semisimple $\mathbb{F}_p$-algebra $A := A(G) := \Lambda(G)/\mathrm{Jac}(\Lambda(G))$. The set I is therefore finite and it is convenient to regard I as the set of integers $\{m : 1 \leqslant m \leqslant |I|\}$. We then fix a set of mutually orthogonal primitive idempotents $\{a_i : i \in I\}$ of A in such a way that for each $i \in I$ the module $R_i = Aa_i$ is a representative of the unique isomorphism class of simple A_i-modules. In making such a choice we assume that $A_1 = \mathbb{F}_p = R_1$. For each $i \in I$ we denote the corresponding representation by $\psi_i : G \to GL(R_i)$. We also fix a set of mutually orthogonal primitive idempotents $\{e_i : i \in I\}$ of $\Lambda(G)$ such that, for each $i \in I$, the image of e_i in $A(G)$ is equal to a_i.

For each $i \in I$ we define a projective $\Lambda(G)$-module $X_i := \Lambda(G)e_i$ and a projective $\Omega(G)$-module $Y_i := X_i/pX_i$. These modules are projective hulls of R_i since $A(G) \otimes_{\Lambda(G)} X_i = A(G) \otimes_{\Omega(G)} Y_i = R_i$. Further, every finitely generated projective $\Lambda(G)$-module X, respectively $\Omega(G)$-module Y, decomposes in a unique way as a direct sum

$$X = \bigoplus_{i\in I} X_i^{\langle X, X_i\rangle}, \quad \text{respectively } Y = \bigoplus_{i\in I} Y_i^{\langle Y, Y_i\rangle},$$

for suitable natural numbers $\langle X, X_i\rangle$, respectively $\langle Y, Y_i\rangle$. For each $i \in I$ we write $l_{R_i}(\psi)$ for the multiplicity with which R_i occurs in an $\mathbb{F}_p$-linear representation ψ. For a $\Lambda(G)$-module M we also write

$$\chi(G, M) := \prod_{j\in\mathbb{Z}} |\mathrm{H}_j(G, M)|^{(-1)^j},$$

if this is finite, for the Euler–Poincaré characteristic of M.

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Finally, we note that similar constructions to the above can be made, mutatis mutandis, if $\Lambda(G)$ and $\Omega(G)$ are replaced by $\Lambda_{\mathcal{O}}(G)$ and $\Omega_\kappa(G) := \Lambda_{\mathcal{O}}(G)/\Lambda_{\mathcal{O}}(G)\pi$ respectively.

Lemma 3.2. *Let Y be a finitely generated projective $\Omega_\kappa(G)$-module.*

(i) *$\Lambda_{\mathcal{O}}(\Gamma) \otimes_{\Lambda_{\mathcal{O}}(G)} Y$ is naturally isomorphic to $\Omega_\kappa(\Gamma) \otimes_{\Omega_\kappa(G)} Y = \Omega_\kappa(\Gamma)^{\langle Y,Y_1\rangle}$ and thus $\chi(G,Y) = p^{[\kappa:\mathbb{F}_p]\langle Y,Y_1\rangle}$.*

(ii) *$\langle Y(\psi^*), Y_1\rangle = \sum_{i\in I} l_{R_i}(\psi)\dim_\kappa(\mathrm{End}_{\Omega_\kappa(G)}(R_i))\langle Y, Y_i\rangle$.*

Proof. For each index i the finitely generated projective $\Omega_\kappa(\Gamma)$-module $\Omega_\kappa(\Gamma) \otimes_{\Omega_\kappa(G)} Y_i$ is isomorphic to $\Omega_\kappa(\Gamma)^{n_i}$ for some natural number n_i. Since then $\kappa^{n_i} \cong \kappa \otimes_{\Omega_\kappa(G)} Y_i \cong A_1 \otimes_{\Omega_\kappa(G)} Y_i = a_1 R_i$ is isomorphic to $R_1 \cong \kappa$ if $i = 1$ and is zero if $i \neq 1$, the first claim follows.

Claim (ii) is true because $\dim_\kappa(\mathrm{End}_{\Omega_\kappa(G)}(R_i))\langle Y, Y_i\rangle = \dim_\kappa(\mathrm{Hom}_{\Omega_\kappa(G)}(Y, R_i))$ and $\mathrm{Hom}_{\Omega_\kappa(G)}(Y, R_i)$ is isomorphic to $\mathrm{Hom}_{\Omega_\kappa(G)}(Y(\psi_i^*), R_1)$ (cf. [**1**, Proposition 4.1 and Lemma 4.4]). □

3.4. The regular case

In this section we study the μ-invariants of § 3.1 in the case that G has no element of order p.

3.4.1. *Pairings*

For a field K we write $R_K(G)$ for the Grothendieck group of the category of finite-dimensional continuous K-linear representations of G which have finite image. The tensor product induces a structure of rings on both $R_L(G)$ and $R_\kappa(G)$ and there exists a canonical surjective homomorphisms of rings $R_L(G) \twoheadrightarrow R_\kappa(G)$ that is induced by reducing modulo π any G-stable $\mathcal{O}$-lattice of a representation of the above type (cf. [**30**]).

Proposition 3.3. *Assume that G has no element of order p.*

(i) *If $C \in D^{\mathrm{p}}_{S^*}(\Lambda(G))$, then for each continuous representation $\rho : G \to GL_n(\mathcal{O})$ one has*

$$\mu(C,\rho) = \sum_{i\in\mathbb{Z}} (-1)^i \mu_\Gamma(\Lambda(\Gamma) \otimes^{\mathbb{L}}_{\Lambda(G)} (\overline{E_\rho}^{\,*} \otimes_{\mathbb{F}_p} \mathrm{gr}(H^i(C)_{\mathrm{tor}})[0])),$$

where $\overline{E_\rho}^{\,*}$ denotes the contragredient module $\mathrm{Hom}_\kappa(\bar{E}_\rho, \kappa)$ while for a $\mathbb{Z}_p$-module M endowed with the p-adic filtration we denote by $\mathrm{gr}(M)$ the associated graded $\mathbb{F}_p$-module.

(ii) *The μ-invariant induces a $\mathbb{Z}$-bilinear pairing*

$$\mu(-,-) : K_0(D^{\mathrm{p}}_{S^*}(\Lambda(G))) \times R_L(G) \to \mathbb{Z}.$$

This pairing induces a $\mathbb{Z}$-bilinear pairing of the form

$$\mu(-,-) : K_0(D^{\mathrm{p}}_{S^*}(\Lambda(G))) \times R_\kappa(G) \to \mathbb{Z}.$$

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(iii) *If* $C \in D^{\mathrm{p}}_{S^*}(\Lambda(G))$ *and* $i \in I$*, then the integer* $\mu(C,\psi_i)$ *defined by claim (ii) is divisible by* $\dim_{\mathbb{F}_p}(\mathrm{End}_{\Omega(G)}(R_i))$*.*

(iv) *If* $C \in D^{\mathrm{p}}_{S}(\Lambda(G))$*, then* $\mu(C,\psi_i) = 0$ *for all* $i \in I$*.*

Proof. To prove claim (i), we write $\mu'(C,\bar\rho)$ for the term on the right-hand side of the claimed equality. Then $\mu'(C,\bar\rho) = \mu'(\mathrm{H}(C),\bar\rho)$ by definition and $\mu(C,\rho) = \mu(\mathrm{H}(C),\rho)$ by Lemma 3.1 (ii) and so we need only consider the case where $C \cong M[0]$ with M in $\mathfrak{M}_{S^*}(G)$. Further, since both μ-invariants are additive on exact triangles (cf. Lemma 3.1 (i)), it is actually sufficient to prove the following two special cases (recall that M/M_{tor} belongs to $\mathfrak{M}_S(G)$ for all M in $\mathfrak{M}_{S^*}(G)$):

(1) If M is in $\mathfrak{M}_S(G)$, then both $H^i(\Lambda(\Gamma) \otimes^{\mathbb{L}}_{\Lambda(G)} (\overline{E_\rho} \otimes_{\mathbb{F}_p} \mathrm{gr}(M_{\mathrm{tor}})[0]))$ and $H^i(C_\rho)$ are finitely generated $\mathbb{Z}_p$-modules (in all degrees i) and so $\mu(C,\rho) = 0 = \mu'(C,\bar\rho)$.

(2) If $p^n M = 0$ for some n, we argue by induction on n. For $n = 1$ the isomorphism $\overline{E_\rho}^* \otimes_{\mathbb{F}_p} \mathrm{gr}(M_{\mathrm{tor}}) \cong \overline{E_\rho}^* \otimes_{\mathbb{F}_p} M \cong E^*_\rho \otimes_{\mathbb{Z}_p} M$ implies the equality of the μ-invariants. For $n > 1$ one uses dévissage and again the additivity of both μ-invariants.

This proves claim (i).

To prove the existence of the first pairing in claim (ii) it suffices to show that $\mu(C,\rho)$ depends only on the space V_ρ. To this end we assume that $E_{\rho'}$ is another G-stable lattice in V_ρ and we have to show that $\mu(C,\rho) = \mu(C,\rho')$. By Lemma 3.1 (ii) and dévissage we may assume that $C \cong M[0]$ with $pM = 0$ and similarly that $E^*_{\rho'} \subseteq E^*_\rho$ with $\pi T = 0$ for $T := E^*_\rho/E^*_{\rho'}$. In this situation there is an exact sequence

$$0 \to M \otimes_{\mathbb{F}_p} T \to M \otimes_{\mathbb{F}_p} \overline{E^*_\rho} \to M \otimes_{\mathbb{F}_p} \overline{E^*_\rho} \to M \otimes_{\mathbb{F}_p} T \to 0$$

of $\Lambda(G)$-modules. The required claim now follows from the known additivity of μ-invariants and the fact that the $\Lambda(G)$-modules $M \otimes_{\mathbb{F}_p} \overline{E^*_\rho}$ and $M \otimes_{\mathbb{F}_p} \overline{E^*_\rho}$ are isomorphic to $M(\rho^*)$ and $M((\rho')^*)$ respectively. The second assertion of claim (ii) then follows from claim (i).

To prove claim (iii) we may assume that $C \cong M[0]$ with M a finitely generated $\Omega(G)$-module. After choosing a finite resolution P of M by finitely generated projective $\Omega(G)$-modules and using the additivity of $\mu(-,\psi_i)$ on short exact sequences the proof is immediately reduced to the case of a projective $\Omega(G)$-module because $\mu(P,\psi_i) = \sum_{j\in\mathbb{Z}}(-1)^j\mu(P^j,\psi_i)$. But for every projective $\Omega(G)$-module Y, considered also as a $\Lambda(G)$-module, and for each $i \in I$ we have

$$\begin{aligned}\mu(Y,\psi_i) &= \mu_{\Gamma,\mathcal{O}}(\Omega_\kappa(\Gamma) \otimes_{\Omega_\kappa(G)} Y(\psi_i^*)) = \mu_{\Gamma,\mathcal{O}}(\Omega_\kappa(\Gamma)^{\langle Y(\psi_i^*),Y_1\rangle}) \\ &= \langle Y(\psi_i^*), Y_1\rangle = \dim_{\mathbb{F}_p}(\mathrm{End}_{\Omega(G)}(R_i))\langle Y, Y_i\rangle\end{aligned}$$

by Lemma 3.2.

Claim (iv) follows from Lemma 3.1 (iii). □

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If G has no element of order p, then Proposition 3.3 (iii) allows us to define an integer $\mu^i_{\Lambda(G)}(C)$ for each complex C in $D^{\mathrm{p}}_{S^*}(\Lambda(G))$ and each index i in I by setting

$$\mu^i_{\Lambda(G)}(C) := \mu(C, \psi_i) \cdot \dim_{\mathbb{F}_p}(\mathrm{End}_{\Omega(G)}(R_i))^{-1}.$$

We note in particular that for each index a in I this definition ensures that

$$\mu^i_{\Lambda(G)}(Y_a[0]) = \begin{cases} 1, & \text{if } i = a, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

3.4.2. *K-groups*

We continue to assume that G has no element of order p and write $\mathfrak{D}(G)$ for the category of finitely generated $\Lambda(G)$-modules that are annihilated by a power of p. Then, by dévissage and lifting of idempotents, one obtains the following isomorphisms

$$K_0(\mathfrak{D}(G)) \cong K_0(\Omega(G)) \cong K_0(\Lambda(G)) \cong \mathbb{Z}^I, \tag{3.4}$$

where the ith basis vector of the free $\mathbb{Z}$-module on the right corresponds to the classes of Y_i in $K_0(\mathfrak{D}(G))$ and $K_0(\Omega(G))$. Lemma 3.2 (ii) implies that if M belongs to $\mathfrak{D}(G)$, then the map in (3.4) sends the class of M to the vector

$$\mu(M) := (\mu^i_{\Lambda(G)}(M[0]))_{i \in I}. \tag{3.5}$$

The proof of the following result is a natural generalization of that given by Kato in [**21**, Proposition 8.6].

Proposition 3.4. *If G has no element of order p, then there are isomorphisms*

$$K_0(\mathfrak{M}_{S^*}(G)) \cong K_0(\mathfrak{M}_S(G)) \oplus K_0(\Omega(G)),$$
$$K_1(\Lambda(G)_{S^*}) \cong K_1(\Lambda(G)_S) \oplus K_0(\Omega(G)).$$

The first of these isomorphisms is induced by the embeddings of categories $\mathfrak{M}_S(G) \subset \mathfrak{M}_{S^*}(G)$ and $\mathfrak{D}(G) \subset \mathfrak{M}_{S^*}(G)$ combined with the first isomorphism in (3.4). The second isomorphism depends on the choice of a splitting of $K_1(\Lambda(G)_{S^*}) \twoheadrightarrow K_0(\mathfrak{D}(G)) \cong \mathbb{Z}^I$; once we have fixed an idempotent e_i for each $i \in I$ a 'natural' choice is induced by sending the ith basis vector of $\mathbb{Z}^I$ to the class of the element $1 + (p-1)e_i$ in $K_1(\Lambda(G)_{S^*})$.

Proof. For a closed normal subgroup N of G we denote by $\Omega(G/N)_S$ the localization of $\Omega(G/N)$ with respect to the image of S under the natural projection map (which coincides with the localization of $\Omega(G/N)$ considered as $\Lambda(G)$-module, whence the notation!). We first prove the surjectivity of the homomorphism ∂_2 in the long exact localization sequence of K-theory

$$K_2(\Lambda(G)_{S^*}) \xrightarrow{\partial_2} K_1(\Omega(G)_S) \to K_1(\Lambda(G)_S) \to K_1(\Lambda(G)_{S^*}) \xrightarrow{\partial_1} K_0(\Omega(G)_S).$$

But, since $\Omega(G)_S$ is semi-local by [**15**, Proposition 4.2], the natural homomorphism $\Omega(G)^*_S \twoheadrightarrow K_1(\Omega(G)_S)$ is surjective and hence $K_1(\Omega(G)_S)$ is generated by the image of S. The surjectivity of ∂_2 thus follows from the fact that for each $f \in S$ one has $\partial_2(\{f,p\}) = [f] \in K_1(\Omega(G)_S)$, where $\{f,p\}$ denotes the symbol of f and p in $K_2(\Lambda(G)_{S^*})$ (indeed, the latter equality is proved by the argument of [**19**, Proposition 5]). From the above exact sequence we therefore obtain an exact sequence

$$0 \to K_1(\Lambda(G)_S) \to K_1(\Lambda(G)_{S^*}) \xrightarrow{\partial_1} K_0(\Omega(G)_S). \tag{3.6}$$

We next consider the composite map

$$\mathbb{Z}^I \to K_1(\Lambda(G)_{S^*}) \xrightarrow{\partial_1} K_0(\Omega(G)_S) \xrightarrow{\alpha} K_0(B(G)) \cong \mathbb{Z}^J. \tag{3.7}$$

Here the first map is given by sending the ith basis vector of $\mathbb{Z}^I$ to the class of the element $f_i := 1 + (p-1)e_i$ (note that $\Lambda(G)/\Lambda(G)f_i$ is isomorphic to Y_i), we set $B(G) := \Omega(G)_S/\operatorname{Jac}(\Omega(G)_S)$, the canonical map α is injective by [**2**, Chapter IX, Proposition 1.3] and the index set J parametrizes the isomorphism classes of simple modules over the semisimple Artinian ring $B(G)$. Let N be any closed normal subgroup of G which is both pro-p and open in H. Then it is straightforward to check that (3.7) factorizes through the surjective composite map

$$\mathbb{Z}^I \cong K_0(\Omega(G/N)) \xrightarrow{\beta} K_0(\Omega(G/N)_S) \xrightarrow{\gamma} K_0(B(G)). \tag{3.8}$$

Here the (surjective) map β comes from the exact localization sequence and γ is induced from the fact that $\Omega(G)_S \twoheadrightarrow B(G)$ factors as the composite $\Omega(G)_S \twoheadrightarrow \Omega(G/N)_S \twoheadrightarrow B(G)$ by the proof of [**15**, Lemma 4.3]. By [**15**], $\Omega(G/N)_S$ is an Artinian ring and thus $\operatorname{Jac}(\Omega(G/N)_S)$ is a nilpotent ideal with $\Omega(G/N)_S/\operatorname{Jac}(\Omega(G/N)_S) = B(G)$. Hence, by [**15**, Lemma 4.1], the map γ is an isomorphism. It follows that the map in (3.7) is surjective and hence that α is bijective and ∂_1 is surjective.

If $\mathfrak{D}_S(G)$ denotes the category of finitely generated $\Lambda(G)_S$-modules which are $\mathbb{Z}_p$-torsion, then we have shown that the composite map

$$K_0(\mathfrak{D}_S(G)) \cong K_0(\Omega(G)_S) \cong K_0(B(G)) = \mathbb{Z}^J \tag{3.9}$$

is bijective and that $|J| \leqslant |I|$.

By combining (3.6) with the surjectivity of ∂_1, the bijectivity of (3.9) and the assertion of Lemma 3.5 (i) below we obtain an exact sequence

$$0 \to K_1(\Lambda(G)_S) \to K_1(\Lambda(G)_{S^*}) \xrightarrow{\partial_1'} K_0(\mathfrak{D}(G)) \to 0. \tag{3.10}$$

Further, it is straightforward to show that, with respect to the isomorphism $K_0(\mathfrak{D}(G)) \cong \mathbb{Z}^I$ of (3.4), this sequence is split by the map which sends the ith basis vector of $\mathbb{Z}^I$ to the class of f_i in $K_1(\Lambda(G)_{S^*})$. This proves the final assertion of Proposition 3.4.

We next consider the following diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{im}(\iota_{S^*}) & \longrightarrow & K_1(\Lambda(G)_S) & \longrightarrow & K_0(\mathfrak{M}_S(G)) & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle \delta} & & \\ 0 & \longrightarrow & \operatorname{im}(\iota_{S^*}) & \longrightarrow & K_1(\Lambda(G)_{S^*}) & \longrightarrow & K_0(\mathfrak{M}_{S^*}(G)) & \longrightarrow & 0 \end{array} \tag{3.11}$$

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where ι_{S^*} is the natural map $K_1(\Lambda(G)) \to K_1(\Lambda(G)_{S^*})$, δ is induced by the embedding of categories $\mathfrak{M}_S(G) \subset \mathfrak{M}_{S^*}(G)$ and [**15**, Proposition 3.4] implies that each row is indeed exact. By applying the snake lemma to this diagram and comparing with the sequence (3.10) we obtain an exact sequence of the form

$$0 \to K_0(\mathfrak{M}_S(G)) \xrightarrow{\delta} K_0(\mathfrak{M}_{S^*}(G)) \to K_0(\mathfrak{D}(G)) \to 0.$$

The first assertion of Proposition 3.4 now follows because this sequence is split by the homomorphism $K_0(\mathfrak{D}(G)) \to K_0(\mathfrak{M}_{S^*}(G))$ that is induced by the embedding of categories $\mathfrak{D}(G) \subset \mathfrak{M}_{S^*}(G)$. □

Lemma 3.5.

(i) *The exact scalar extension functor from $\Lambda(G)$-mod to $\Lambda(G)_S$-mod identifies $\mathfrak{D}(G)$ with a full subcategory of $\mathfrak{D}_S(G)$ and induces an isomorphism $K_0(\mathfrak{D}(G)) \cong K_0(\mathfrak{D}_S(G))$.*

(ii) *The natural map $\iota : K_0(\mathfrak{D}(G)) \to K_0(\mathfrak{M}_{S^*}(G))$ is injective.*

(iii) *The natural map $K_0(\Omega(G/N)) \to K_0(\Omega(G/N)_S)$ is bijective.*

Proof. The assignment $M \mapsto (\mu^i_{\Lambda(G)}(M[0]))_{i\in I}$ induces a homomorphism

$$\mu : K_0(\mathfrak{M}_{S^*}(G)) \to \mathbb{Z}^I.$$

Now from (3.4) and (3.5) we know that $\mu \circ \iota$ is bijective while from Lemma 3.1 (iii) we know $\delta(K_0(\mathfrak{M}_S(G))) \subseteq \ker(\mu)$ where δ is the homomorphism in diagram (3.11). This implies that ι is injective (so proving claim (ii)), that μ is surjective and that $|I| \leqslant |J|$. But $|J| \leqslant |I|$ (see just after (3.9)) and so $|I| = |J|$.

Since $|I| = |J|$ the isomorphisms of (3.4) and (3.8) combine to imply that the natural map $K_0(\mathfrak{D}(G)) \to K_0(\mathfrak{D}_S(G))$ is bijective (proving claim (i)).

In a similar way, claim (iii) follows by combining the equality $|I| = |J|$ together with the surjectivity of the map β in (3.8) and the definition of the index set J (in (3.7)). □

4. Characteristic series

In this section we associate a canonical 'characteristic series' to each complex in $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G))$. This construction extends the notion of 'algebraic p-adic L-functions' introduced by the first named author in [**6**] and hence refines the 'Akashi series' introduced by Coates, Schneider and Sujatha in [**14**]. It will also play a key role in our proof of Theorem 2.1 (see in particular the proof of Lemma 5.7).

4.1. The definition

If M is any compact (left) $\Lambda(G)$-module, then the completed tensor product

$$\mathrm{I}^G_H(M) := \Lambda(G)\hat{\otimes}_{\Lambda(H)}\mathrm{Res}^G_H(M)$$

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has a natural structure as a compact $\Lambda(G)$-module via multiplication on the left. With respect to this action, one obtains a (well-defined) endomorphism Δ_γ of $\mathrm{I}_H^G(M)$ by setting

$$\Delta_\gamma(x \otimes_{\Lambda(H)} y) := x\tilde{\gamma}^{-1} \otimes_{\Lambda(H)} \tilde{\gamma}(y)$$

for each $x \in \Lambda(G)$ and $y \in M$, where $\tilde{\gamma}$ is any lift of γ through the natural projection $G \to \Gamma$. It is easily checked that Δ_γ is independent of the precise choice of $\tilde{\gamma}$. Further, if M belongs to $\mathfrak{M}_{S^*}(G)$, then [**6**, Lemma 2.1] implies that

$$\delta_\gamma := \mathrm{id}_{\mathrm{I}_H^G(M)} - \Delta_\gamma$$

induces an automorphism of the (finitely generated) $\Lambda(G)_{S^*}$-module $\mathrm{I}_H^G(M)_{S^*}$.

Now the ring $\Lambda(G)_{S^*}$ is both noetherian and regular [**18**, Proposition 4.3.4] and so $K_1(\Lambda(G)_{S^*})$ is naturally isomorphic to the group $G_1(\Lambda(G)_{S^*})$ that is generated (multiplicatively) by symbols $\langle \alpha \mid M \rangle$ where α is an automorphism of a finitely generated $\Lambda(G)_{S^*}$-module M (cf. [**31**, Theorem 16.11]). For each complex C in $D^{\mathrm{p}}_{S^*}(\Lambda(G))$ we may therefore define an element of $K_1(\Lambda(G)_{S^*})$ by setting

$$\mathrm{char}^*_{G,\gamma}(C) := \prod_{i \in \mathbb{Z}} \langle \delta_\gamma \mid \mathrm{I}_H^G(H^i(C))_{S^*} \rangle^{(-1)^i}.$$

For each C in $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G))$ we also define an 'equivariant multiplicative μ-invariant' in the image of the natural map

$$K_1\left(\Lambda(G)\left[\frac{1}{p}\right]\right) \to K_1(\Lambda(G)_{S^*})$$

by setting

$$\chi^\mu_G(C) := \begin{cases} \left\langle 1 + \sum_{i \in I} (p^{\mu^i_{\Lambda(G)}(C)} - 1)e_i \;\middle|\; \Lambda(G)_{S^*} \right\rangle, & \text{if } G \text{ has no element of order } p, \\ 1, & \text{if } C \text{ belongs to } D^{\mathrm{p}}_S(\Lambda(G)), \end{cases}$$

where the integer $\mu^i_{\Lambda(G)}(C)$ is as defined at the end of § 3.4.1. We note this definition is both consistent (because if G has no element of order p and C belongs to $D^{\mathrm{p}}_S(\Lambda(G))$, then Proposition 3.3 (iv) implies $\mu^i_{\Lambda(G)}(C) = 0$ for all $i \in I$) and also applies to all C in $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G))$ (because $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G)) = D^{\mathrm{p}}_S(\Lambda(G))$ if G has an element of order p).

Definition 4.1. For each C in $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G))$ the *characteristic series of* C is the element

$$\mathrm{char}_{G,\gamma}(C) := \chi^\mu_G(C) \cdot \mathrm{char}^*_{G,\gamma}(C)$$

of $K_1(\Lambda(G)_{S^*})$.

Remark 4.2. If $G = \Gamma$, then $\Lambda(G)_{S^*} = Q(\Gamma)$ and so there is a natural isomorphism $\iota : K_1(\Lambda(G)_{S^*}) \cong Q(\Gamma)^\times$. Further, if M is any finitely generated torsion $\Lambda(\Gamma)$-module, then $\iota(\mathrm{char}_{G,\gamma}(M[0])) = (1+T)^{-\lambda(M)}\mathrm{char}_T(M)$ where $\lambda(M)$ is the Iwasawa λ-invariant of M and $\mathrm{char}_T(M)$ is the characteristic polynomial of M with respect to the variable $T = \gamma - 1$. (For a proof of this fact see [**6**, Lemma 2.3].)

Remark 4.3. In Proposition 4.7 (i) we will prove that if G has no element of order p, then $\mathrm{char}_{G,\gamma}(C)$ is a 'characteristic element for C' in the sense of [**15**, (33)] (and see also Remark 6.2 in this regard). In [**6**, Theorem 4.1] it is proved that this is also true if G has rank one as a p-adic Lie group. In these cases it therefore seems reasonable to regard $\mathrm{char}_{G,\gamma}(C)$ as a canonical 'algebraic p-adic L-function' associated to C.

4.2. Basic properties

Lemma 4.4. *If $C_1 \to C_2 \to C_3 \to C_1[1]$ is an exact triangle in $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G))$, then* $\mathrm{char}_{G,\gamma}(C_2) = \mathrm{char}_{G,\gamma}(C_1)\mathrm{char}_{G,\gamma}(C_3)$.

Proof. We claim first that $\chi^{\mu}_G(C_2) = \chi^{\mu}_G(C_1)\chi^{\mu}_G(C_3)$. This is obvious if G has an element of order p since then each complex C_j belongs to $D^{\mathrm{p}}_S(\Lambda(G))$ and so $\chi^{\mu}_G(C_j) = 1$. If on the other hand G has no element of order p, then the claimed equality is a consequence of Lemma 3.1 (i). Indeed, if for each $j \in \{1,2,3\}$ we set $x_j := 1 + \sum_{i\in I}(p^{\mu^i_{\Lambda(G)}(C_j)} - 1)e_i$, then $\chi^{\mu}_G(C_j) = \langle x_j \mid \Lambda(G)_{S^*}\rangle$ and, since

$$x_j = \left(1 - \sum_{i\in I} e_i\right) + \sum_{i\in I} p^{\mu^i_{\Lambda(G)}(C_j)}e_i$$

and the idempotents e_i are mutually orthogonal, Lemma 3.1 (i) implies an equality $x_2 = x_1x_3$.

Finally, we note that the equality $\mathrm{char}^*_{G,\gamma}(C_2) = \mathrm{char}^*_{G,\gamma}(C_1)\mathrm{char}^*_{G,\gamma}(C_3)$ is equivalent to that of [**6**, Proposition 3.1]. □

Let U be a closed subgroup of H that is normal in G and set $G_1 := G/U$, $H_1 := H/U$ and $S_1 := S_{G_1,H_1}$. Then there exists a natural ring homomorphism $\pi_{G_1} : \Lambda(G)_{S^*} \to \Lambda(G_1)_{S_1^*}$ and hence an induced homomorphism of groups

$$\pi_{G_1,*} : K_1(\Lambda(G)_{S^*}) \to K_1(\Lambda(G_1)_{S_1^*}).$$

Lemma 4.5. *Let G_1, H_1 and S_1 be as above. Fix C in $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G))$ and assume that either C belongs to $D^{\mathrm{p}}_S(\Lambda(G))$ or that G_1 has no element of order p. Then $C_1 := \Lambda(G_1) \otimes^{\mathbb{L}}_{\Lambda(G)} C$ belongs to $D^{\mathrm{p}}_{\tilde{S}_1}(\Lambda(G_1))$ and $\pi_{G_1,*}(\mathrm{char}_{G,\gamma}(C)) = \mathrm{char}_{G_1,\gamma}(C_1)$.*

Proof. We claim first that $\pi_{G_1,*}(\chi^{\mu}_G(C)) = \chi^{\mu}_{G_1}(C_1)$. If C belongs to $D^{\mathrm{p}}_S(\Lambda(G))$, then C_1 belongs to $D^{\mathrm{p}}_{S_1}(\Lambda(G_1))$ and so $\pi_{G_1,*}(\chi^{\mu}_G(C)) = \pi_{G_1,*}(1) = 1 = \chi^{\mu}_{G_1}(C_1)$. If on the other hand C does not belong to $D^{\mathrm{p}}_S(\Lambda(G))$, then, by assumption, neither G or G_1 has an element of order p and the claimed equality follows from Lemma 3.1 (iv). To show this we may regard the index set I_1 corresponding to the irreducible representations of G_1 as a subset of I by considering them as representations of G via the natural projection: then, for each $i \in I_1$ Lemma 3.1 (iv) implies $\mu^i_{\Lambda(G)}(C) = \mu^i_{\Lambda(G_1)}(C_1)$. Furthermore, we can assume that the image under the natural projection of e_i for each $i \in I_1$ coincides with the corresponding idempotent chosen with respect to G_1 while the image of e_i in $\Lambda(G_1)$ is zero for $i \in I \setminus I_1$. The claimed equality $\pi_{G_1,*}(\chi^{\mu}_G(C)) = \chi^{\mu}_{G_1}(C_1)$ is therefore clear.

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Finally, we note that the equality $\pi_{G_1,*}(\mathrm{char}^*_{G,\gamma}(C)) = \mathrm{char}^*_{G_1,\gamma}(C_1)$ is equivalent to that of [**6**, Proposition 3.2]. □

In the next result we fix an open subgroup U of G and set $H_U := H \cap U$ and $\Gamma_U := U/H_U$. We use the natural isomorphism $\Gamma_U \cong HU/H$ to regard Γ_U as an open subgroup of Γ, we set $d_U := [\Gamma : \Gamma_U]$ and write γ_U for the topological generator γ^{d_U} of Γ_U. We set $S_U := S_{U,H_U}$ and note that $\Lambda(G)$, respectively $\Lambda(G)_S$, respectively $\Lambda(G)_{S^*}$ is a free $\Lambda(U)$-module, respectively $\Lambda(U)_{S_U}$-module, respectively $\Lambda(U)_{S_U^*}$-module, of rank $[G : U]$. In particular, restriction of scalars gives natural functors $D^{\mathrm{p}}_S(\Lambda(G)) \to D^{\mathrm{p}}_{S_U}(\Lambda(U))$ and $D^{\mathrm{p}}_{S^*}(\Lambda(G)) \to D^{\mathrm{p}}_{S_U^*}(\Lambda(U))$ and a natural homomorphism

$$\mathrm{res}_{U,*} : K_1(\Lambda(G)_{S^*}) \to K_1(\Lambda(U)_{S_U^*}).$$

Lemma 4.6. *Let G and U be as above and fix C in $D^{\mathrm{p}}_{\tilde{S}}(\Lambda(G))$. Then $C_1 := \mathrm{res}^G_U C$ belongs to $D^{\mathrm{p}}_{\tilde{S}_U}(\Lambda(U))$ and one has $\mathrm{res}_{U,*}(\mathrm{char}_{G,\gamma}(C)) = \mathrm{char}_{U,\gamma_U}(C_1)$.*

Proof. We prove first that $\mathrm{res}_{U,*}(\chi^\mu_G(C)) = \chi^\mu_U(C_1)$. The complex C belongs to $D^{\mathrm{p}}_S(\Lambda(G))$ if and only if C_1 belongs to $D^{\mathrm{p}}_{S_U}(\Lambda(U))$ and in this case one has

$$\mathrm{res}_{U,*}(\chi^\mu_G(C)) = \mathrm{res}_{U,*}(1) = 1 = \chi^\mu_U(C_1).$$

We may thus assume that C belongs to $D^{\mathrm{p}}_{S^*}(\Lambda(G)) \setminus D^{\mathrm{p}}_S(\Lambda(G))$ and hence that G (and therefore also U) has no element of order p. By the same argument as used in the proof of Proposition 3.3 (iii), we can also assume that $C = Y_a[0]$ for some index a in I. Then for each index i in I one has $\mu^i_{\Lambda(G)}(C) = \langle Y_a, Y_i\rangle = \delta_{ai}$ and so

$$\mathrm{res}_{U,*}(\chi^\mu_G(C)) = \mathrm{res}_{U,*}(\langle(1-e_a) + pe_a \mid \Lambda(G)_{S^*}\rangle) = \mathrm{res}_{U,*}(\langle p \mid \Lambda(G)_{S^*}e_a\rangle).$$

We write $\{\hat{e}_j : j \in J\}$ for the idempotents of $\Lambda(U)$ that are analogous to the idempotents e_i of $\Lambda(G)$ defined in § 3.3 and $\{X(U)_j : j \in J\}$, respectively $\{Y(U)_j : j \in J\}$, for the submodules of $\Lambda(U)$, respectively $\Omega(U)$, that are analogous to the modules X_i, respectively Y_i, defined in § 3.3. For each j in J we set $m_j := \langle \Lambda(G)e_a, X(U)_j\rangle = \langle Y_a, Y(U)_j\rangle$. Then the $\Lambda(U)_{S_U^*}$-module $\Lambda(G)_{S^*}e_a$ is isomorphic to $\bigoplus_{j\in J}(\Lambda(U)_{S_U^*}\hat{e}_j)^{m_j}$ and so the last displayed expression is equal to

$$\begin{aligned}\prod_{j\in J}\langle p \mid \Lambda(U)_{S_U^*}\hat{e}_j\rangle^{m_j} &= \prod_{j\in J}\langle(1-\hat{e}_j) + p^{m_j}\hat{e}_j \mid \Lambda(U)_{S_U^*}\rangle \\ &= \left\langle \prod_{j\in J}((1-\hat{e}_j) + p^{m_j}\hat{e}_j) \,\middle|\, \Lambda(U)_{S_U^*}\right\rangle \\ &= \left\langle \left(1 - \sum_{j\in J}\hat{e}_j\right) + \sum_{j\in J}p^{m_j}\hat{e}_j \,\middle|\, \Lambda(U)_{S_U^*}\right\rangle \\ &= \chi^\mu_U(\mathrm{res}^G_U Y_a[0]),\end{aligned}$$

where the third equality is valid because

$$\prod_{j\in J}((1-\hat{e}_j) + p^{m_j}\hat{e}_j) = (1 - \sum_{j\in J}\hat{e}_j) + \sum_{j\in J}p^{m_j}\hat{e}_j.$$

This completes the proof that $\mathrm{res}_{U,*}(\chi^\mu_G(C)) = \chi^\mu_U(C_1)$.

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It remains to prove that $\mathrm{res}_{U,*}(\mathrm{char}^*_{G,\gamma}(C)) = \mathrm{char}^*_{U,\gamma_U}(C_1)$. To do this we may assume that $C = M[0]$ for a module M in $\mathfrak{M}_{S^*}(G)$ so that $\mathrm{char}^*_{G,\gamma}(C)$ is equal to $\langle \delta_\gamma \mid \mathrm{I}^G_H(M)_{S^*}\rangle$. Now the $\Lambda(U)_{S^*_U}$-module $\mathrm{I}^G_H(M)_{S^*}$ is equal to the direct sum

$$\bigoplus_{i=0}^{i=d_U-1} \Delta_{\gamma^i}(\mathrm{I}^U_{H_U}(M)_{S^*_U}) \cong \bigoplus_{i=0}^{i=d_U-1} \mathrm{I}^U_{H_U}(M)_{S^*_U},$$

where the isomorphism identifies each translate $\Delta_{\gamma^i}(\mathrm{I}^U_{H_U}(M)_{S^*_U})$ with $\mathrm{I}^U_{H_U}(M)_{S^*_U}$ in the natural way. With respect to this decomposition δ_γ is the automorphism given by the $d_U \times d_U$ matrix

$$\begin{pmatrix} \mathrm{id} & -\mathrm{id} & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \mathrm{id} & -\mathrm{id} & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \mathrm{id} & -\mathrm{id} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & \mathrm{id} & -\mathrm{id} \\ -\Delta_{\gamma^{d_U}} & 0 & \cdots & \cdots & \cdots & 0 & \mathrm{id} \end{pmatrix}.$$

Elementary row and column operations show that this automorphism represents the same element of $K_1(\Lambda(U)_{S^*_U})$ as does the automorphism α of $\bigoplus_{i=0}^{i=d_U-1} \mathrm{I}^U_{H_U}(M)_{S^*_U}$ that acts as $\mathrm{id} - \Delta_{\gamma^{d_U}}$ on the last direct summand and as the identity on all other summands. The required result thus follows because, since $\delta_{\gamma_U} := \mathrm{id} - \Delta_{\gamma^{d_U}}$, the class of α in $K_1(\Lambda(U)_{S^*_U})$ is equal to $\langle \delta_{\gamma_U} \mid \mathrm{I}^U_{H_U}(M)_{S^*_U}\rangle =: \mathrm{char}^*_{U,\gamma_U}(C_1)$. □

4.3. The proof of Theorem 2.1

We deduce Theorem 2.1 as a consequence of the following result.

Proposition 4.7. *Assume that G has no element of order p.*

(i) *For each C in $D^{\mathrm{p}}_{S^*}(\Lambda(G))$ one has $\partial_G(\mathrm{char}_{G,\gamma}(C)) = \sum_{i\in\mathbb{Z}}(-1)^{i+1}[H^i(C)]$.*

(ii) *There exists a (unique) homomorphism $\chi_{G,\gamma}$ from $K_0(\mathfrak{M}_{S^*}(G))$ to $K_1(\Lambda(G)_{S^*})$ that simultaneously satisfies the following conditions.*

  (a) *For each M in $\mathfrak{M}_{S^*}(G)$ one has $\chi_{G,\gamma}([M]) = \mathrm{char}_{G,\gamma}(M[1])$.*

  (b) *$\chi_{G,\gamma}$ is right inverse to ∂_G.*

  (c) *$\chi_{G,\gamma}$ respects the isomorphisms of Proposition 3.4.*

  (d) *Let U be a closed subgroup of H that is normal in G and such that $\bar{G} := G/U$ has no element of order p. Set $\bar{H} := H/U$ and $\bar{S} := S_{\bar{G},\bar{H}}$. Then there is a commutative diagram*

$$\begin{array}{ccc} K_0(\mathfrak{M}_{S^*}(G)) & \xrightarrow{\chi_{G,\gamma}} & K_1(\Lambda(G)_{S^*}) \\ \downarrow & & \downarrow \\ K_0(\mathfrak{M}_{\bar{S}^*}(\bar{G})) & \xrightarrow{\chi_{\bar{G},\gamma}} & K_1(\Lambda(\bar{G})_{\bar{S}^*}) \end{array}$$

  where the vertical arrows are the natural projection homomorphisms.

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Remark 4.8. Proposition 4.7 (i) shows that $\mathrm{char}_{G,\gamma}(C)$ is a 'characteristic element for C' in the sense of [**15**, (33)]. The surjectivity of ∂_G (which follows directly from Proposition 4.7 (ii)(b)) was first proved in [**15**, Proposition 3.4].

The proof of Proposition 4.7 will be the subject of § 4.4. However, we now show that it implies Theorem 2.1. To do this we consider the map

$$\iota_G : K_0(\Omega(G)) \oplus K_0(\mathfrak{M}_S(G)) \oplus \mathrm{im}(K_1(\Lambda(G)) \to K_1(\Lambda(G)_{S^*})) \to K_1(\Lambda(G)_{S^*}),$$

which for each M in $\Omega(G)$, N in $\mathfrak{M}_S(G)$ and u in $\mathrm{im}(K_1(\Lambda(G)) \to K_1(\Lambda(G)_{S^*}))$ satisfies $\iota_G(([M],[N],u)) = \mathrm{char}_{G,\gamma}(M[1])\mathrm{char}_{G,\gamma}(N[1])u$. Then Proposition 4.7 (ii) implies that ι_G is a well-defined homomorphism which, upon restriction to the summand $K_0(\Omega(G)) \oplus K_0(\mathfrak{M}_S(G))$, gives a right inverse to the composite $K_1(\Lambda(G)_{S^*}) \to K_0(\mathfrak{M}_{S^*}(G)) \to K_0(\Omega(G)) \oplus K_0(\mathfrak{M}_S(G))$ where the first arrow is ∂_G and the second is the isomorphism of Proposition 3.4. The exactness of the lower row of (3.11) thus implies that ι_G is bijective. This completes the proof of Theorem 2.1.

Remark 4.9. The characteristic series of a module M in $\mathfrak{M}_{S^*}(G)$ and hence also the splitting $\chi_{G,\gamma}$ of ∂_G in Proposition 4.7 can be defined just in terms of modules instead of using derived categories. One can therefore avoid the use of derived categories in the proof of Theorem 2.1. However, since our applications involve complexes we prefer to use this language from the outset.

4.4. The proof of Proposition 4.7

In addition to proving Proposition 4.7 we shall also now translate Definition 4.1 into the language of localized K_1-groups introduced by Fukaya and Kato in [**18**]. We therefore use the notation of Appendix A.

4.4.1. *S-acyclic complexes*

In [**28**, Proposition 2.2, Remark 2.3] Schneider and the second named author have proved that for each bounded complex P of projective $\Lambda(G)$-modules in $D^{\mathrm{p}}_S(\Lambda(G))$ there exists an exact sequence of complexes in $D^{\mathrm{p}}(\Lambda(G))$

$$0 \to \mathrm{I}^G_H(P) \xrightarrow{\delta_\gamma(P)} \mathrm{I}^G_H(P) \xrightarrow{\pi(P)} P \to 0, \tag{4.1}$$

where in each degree i the morphisms $\delta_\gamma(P)^i$ and $\pi(P)^i$ are equal to $\delta_\gamma : \mathrm{I}^G_H(P^i) \to \mathrm{I}^G_H(P^i)$ and the natural projection $\mathrm{I}^G_H(P^i) \to P^i$ respectively. We may therefore define a morphism

$$\mathbf{1}_{\Lambda(G)} \to \mathbf{d}_{\Lambda(G)}(\mathrm{I}^G_H(P))\mathbf{d}_{\Lambda(G)}(\mathrm{I}^G_H(P))^{-1} \to \mathbf{d}_{\Lambda(G)}(P),$$

where the first arrow is induced by the identity map on $\mathrm{I}^G_H(P)$ and the second by applying property A(d) in Appendix A to (4.1). By using property A(g) in Appendix A of the functor $\mathbf{d}_{\Lambda(G)}$ we then extend this definition to obtain for any object C of $D^{\mathrm{p}}_S(\Lambda(G))$ a canonical morphism $\delta_\gamma(C) : I^G_H(C) \to I^G_H(C)$ in $D^{\mathrm{p}}(\Lambda(G))$ and an induced morphism in $V(\Lambda(G))$ of the form

$$t_S(C) : \mathbf{1}_{\Lambda(G)} \to \mathbf{d}_{\Lambda(G)}(\mathrm{I}^G_H(C))\mathbf{d}_{\Lambda(G)}(\mathrm{I}^G_H(C))^{-1} \to \mathbf{d}_{\Lambda(G)}(C).$$

This morphism is closely analogous to those that arise naturally in the context of varieties over finite fields (cf. [**20**, Lemma 3.5.8] and [**5**, § 3.2]).

In the following result we use the homomorphism $\mathrm{ch}_{\Lambda(G),\Sigma_C}$ and notation $[-,-]_{\mathrm{FK}}$ defined in Appendix A.

Lemma 4.10. *If G has no element of order p, then for each C in $D^{\mathrm{p}}_S(\Lambda(G))$ one has $\mathrm{ch}_{\Lambda(G),\Sigma_C}([C,t_S(C)]_{\mathrm{FK}}) = \mathrm{char}^*_{G,\gamma}(C)$.*

Proof. We first recall from Appendix A that $\mathrm{ch}_{\Lambda(G),\Sigma_C}([C,t_S(C)]_{\mathrm{FK}}) = \theta_{C,t_S(C)}$ where the latter element is as defined in (A.1). To compute $\theta_{C,t_S(C)}$ explicitly we set $R := \Lambda(G)$, $Q := R_{S^*}$, $C_Q := Q \otimes_R C$, $\mathrm{H}(C)_Q := Q \otimes_R \mathrm{H}(C)$, $\mathrm{H}(C)_{H,Q} := Q \otimes_{\Lambda(H)} \mathrm{H}(C)$ and $H^i(C)_{H,Q} := Q \otimes_{\Lambda(H)} H^i(C)$. We consider the following diagram in $V(Q)$:

$$\begin{array}{ccccccc}
\mathbf{1}_Q & \xrightarrow{\alpha_1} & \mathbf{d}_Q(\mathrm{H}(C)_{H,Q})\mathbf{d}_Q(\mathrm{H}(C)_{H,Q})^{-1} & \xrightarrow{\alpha_2} & \mathbf{d}_Q(\mathrm{H}(C)_Q) & \xrightarrow{\alpha_3} & \mathbf{d}_Q(C_Q) \\
{\scriptstyle\alpha_5}\downarrow & & \| & & & & \| \\
\mathbf{1}_Q & \xrightarrow{\alpha_4} & \mathbf{d}_Q(\mathrm{H}(C)_{H,Q})\mathbf{d}_Q(\mathrm{H}(C)_{H,Q})^{-1} & \xrightarrow{\alpha_2} & \mathbf{d}_Q(\mathrm{H}(C)_Q) & \xrightarrow{\alpha_3} & \mathbf{d}_Q(C_Q)
\end{array}$$

In this diagram α_1 is induced by the identity map on $\mathrm{H}(C)_{H,Q}$, α_2 is induced by the second arrow in the definition of $t_S(\mathrm{H}(C))$, α_3 is property A(h) in Appendix A, α_4 is induced by the automorphism $Q \otimes_R H^i(\delta_\gamma(C))$ of each term $H^i(C)_{H,Q} = \mathrm{H}(C)^i_{H,Q}$ and α_5 is defined so that the first square commutes. Now the upper row of the diagram is equal to the morphism $Q \otimes_R t_S(C)$ while the lower row agrees with the morphism $\mathbf{1}_Q \to \mathbf{d}_Q(C_Q)$ induced by the acyclicity of C_Q. From the commutativity of the diagram we thus deduce that the element $\theta_{C,t_S(C)}$ of $K_1(Q)$ is represented by α_5. On the other hand, an explicit comparison of the maps α_1 and α_4 shows that α_5 represents the same element of $K_1(Q)$ as does the morphism

$$\mathbf{d}_Q(\mathrm{H}(C)_{H,Q}) \cong \prod_{i\in\mathbb{Z}} \mathbf{d}_Q(H^i(C)_{H,Q})^{(-1)^i} \to \prod_{i\in\mathbb{Z}} \mathbf{d}_Q(H^i(C)_{H,Q})^{(-1)^i} \cong \mathbf{d}_Q(\mathrm{H}(C)_{H,Q}),$$

where the first and third maps use property A(h) in Appendix A for the complex $\mathrm{H}(C)_{H,Q}$ and the second map is $\prod_{i\in\mathbb{Z}} \mathbf{d}_Q(Q \otimes_R H^i(\delta_\gamma(C)))^{(-1)^i}$. From here we deduce the required equality in $K_1(Q)$:

$$\theta_{C,t_S(C)} = \prod_{i\in\mathbb{Z}} \langle \delta_\gamma \mid H^i(C)_{H,Q}\rangle^{(-1)^i} =: \mathrm{char}^*_{G,\gamma}(C).$$

□

4.4.2. *p-torsion complexes*

In this subsection we assume that G has no element of order p (so that $D^{\mathrm{p}}(\Lambda(G))$ identifies with $D^{\mathrm{fg}}(\Lambda(G))$).

If T is a bounded complex of finitely generated $\Omega(G)$-modules, then there is a bounded complex of finitely generated projective $\Omega(G)$-modules $\bar{P}$ that is isomorphic in $D(\Omega(G))$

(and hence also in $D^{\mathrm{p}}(\Lambda(G))$) to T. Also, following the discussion of § 3.3, in each degree i there is a finitely generated projective $\Lambda(G)$-module P^i and an exact sequence of $\Lambda(G)$-modules

$$0 \to P^i \xrightarrow{p} P^i \to \bar{P}^i \to 0. \tag{4.2}$$

We may therefore define a morphism

$$t(T) : \mathbf{1}_{\Lambda(G)} \to \prod_{i\in\mathbb{Z}} (\mathbf{d}_{\Lambda(G)}(P^i)\mathbf{d}_{\Lambda(G)}(P^i)^{-1})^{(-1)^i} \\ \to \prod_{i\in\mathbb{Z}} \mathbf{d}_{\Lambda(G)}(\bar{P}^i)^{(-1)^i} = \mathbf{d}_{\Lambda(G)}(\bar{P}) \to \mathbf{d}_{\Lambda(G)}(T),$$

where the first arrow is induced by the identity map on each module P^i, the second by applying property A(d) in Appendix A to each of the sequences (4.2) and the last by the given quasi-isomorphism $\bar{P} \cong T$. If now C is any bounded complex of modules in $\mathfrak{D}(G)$, then there exists a finite length filtration of C by complexes

$$0 = C_d \subset C_{d-1} \subset \cdots \subset C_1 \subset C_0 = C \tag{4.3}$$

so that each quotient complex $T_i := C_i/C_{i+1}$ belongs to $D^{\mathrm{p}}(\Omega(G))$. This gives associated exact triangles in $D^{\mathrm{p}}_{S^*}(\Lambda(G))$ of the form

$$C_{a+1} \to C_a \to T_a \to C_{a+1}[1], \quad 0 \leqslant a < d, \tag{4.4}$$

which combine to induce an identification $\mathbf{d}_{\Lambda(G)}(C) = \prod_{0\leqslant a<d} \mathbf{d}_{\Lambda(G)}(T_a)$. With respect to this identification, we set

$$t(C) := \prod_{0\leqslant a<d} t(T_a).$$

This definition is easily checked to be independent of the choice of filtration (4.3) and, for each a, of isomorphism $\bar{P} \cong T_a$ and resolution (4.2) used to define $t(T_a)$.

Lemma 4.11. *If G has no element of order p and C is any bounded complex of modules in $\mathfrak{D}(G)$, then in $K_1(\Lambda(G)_{S^*})$ one has $\mathrm{ch}_{\Lambda(G),\Sigma_{S^*}}([C, t(C)]_{\mathrm{FK}}) = \chi^{\mu}_G(C)$.*

Proof. Given the above definition of $t(C)$, a straightforward reduction argument combining the exact triangles (4.4) with both relation (2) in the definition of $K_1(\Lambda(G), \Sigma_{S^*})$ (cf. Definition A.1) and Lemma 4.4 shows that it is enough to consider the case that C is equal to a bounded complex of finitely generated $\Omega(G)$-modules T. A similar reduction argument then allows one to further assume that $T = \bar{P}[0]$ for a finitely generated projective $\Omega(G)$-module $\bar{P}$. Now, in terms of the notation introduced in § 3.3, one has a direct sum decomposition $\bar{P} \cong \bigoplus_{i\in I} Y_i^{m_i}$ with $m_i := \langle \bar{P}, Y_i\rangle$ for each $i \in I$. There is therefore a resolution (4.2) of the form $0 \to \Lambda(G)^n \xrightarrow{d} \Lambda(G)^n \to \bar{P} \to 0$ where $n := \sum_{i\in I} m_i$ and the homomorphism d is given with respect to the canonical $\Lambda(G)$-basis of $\Lambda(G)^n$ by the diagonal matrix with ath entry equal to $f_b := 1 + (p-1)e_b$ if $\sum_{i=1}^{i=b-1} m_i < a \leqslant \sum_{i=1}^{i=b} m_i$

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for any $b \in I$. The definition of $\mathrm{ch}_{\Lambda(G),\Sigma_{S^*}}$ then implies that

$$\begin{aligned}\mathrm{ch}_{\Lambda(G),\Sigma_{S^*}}([T,t(T)]_{\mathrm{FK}}) &= \langle d \mid \Lambda(G)^n_{S^*}\rangle\\ &= \prod_{i\in I}\langle f_i \mid \Lambda(G)^{m_i}_{S^*}\rangle\\ &= \prod_{i\in I}\langle f_i^{m_i} \mid \Lambda(G)_{S^*}\rangle\\ &= \prod_{i\in I}\langle 1+(p^{m_i}-1)e_i \mid \Lambda(G)_{S^*}\rangle\\ &= \left\langle 1+\sum_{i\in I}(p^{m_i}-1)e_i \;\middle|\; \Lambda(G)_{S^*}\right\rangle.\end{aligned}$$

Recalling the explicit definition of $\chi^{\mu}_G(\bar{P}[0])$ it is thus enough to show that for each $i \in I$ one has $m_i = \mu^i_{\Lambda(G)}(\bar{P}[0])$. But the decomposition $\bar{P} \cong \bigoplus_{a\in I} Y_a^{m_a}$ implies that $\mu^i_{\Lambda(G)}(\bar{P}[0]) = \sum_{a\in I} m_a \mu^i_{\Lambda(G)}(Y_a[0])$ and (3.3) implies that this sum is indeed equal to m_i. □

4.4.3. *The proof of Proposition 4.7*

For each complex C in Σ_{S^*} we write $\mathrm{H}(C)_{\mathrm{tor}}$ and $\mathrm{H}(C)_{\mathrm{tf}}$ for the complexes with $\mathrm{H}(C)^i_{\mathrm{tor}} = H^i(C)_{\mathrm{tor}}$ and $\mathrm{H}(C)^i_{\mathrm{tf}} = H^i(C)_{\mathrm{tf}}$ in each degree i and in which all differentials are zero. There is then a tautological exact sequence of complexes $0 \to \mathrm{H}(C)_{\mathrm{tor}} \to \mathrm{H}(C) \to \mathrm{H}(C)_{\mathrm{tf}} \to 0$ and hence an equality $\chi(C) = \chi(\mathrm{H}(C)) = \chi(\mathrm{H}(C)_{\mathrm{tor}}) + \chi(\mathrm{H}(C)_{\mathrm{tf}})$ in $K_0(\mathfrak{M}_{S^*}(G))$. From the definitions of $\chi^{\mu}_G(-)$ and $\mathrm{char}^*_{G,\gamma}(-)$ it is also clear that $\chi^{\mu}_G(C) = \chi^{\mu}_G(\mathrm{H}(C)_{\mathrm{tor}})$ and $\mathrm{char}^*_{G,\gamma}(C) = \mathrm{char}^*_{G,\gamma}(\mathrm{H}(C)_{\mathrm{tf}})$. Claim (i) therefore follows upon combining Lemmas 4.10 and 4.11 (with C replaced by $\mathrm{H}(C)_{\mathrm{tf}}$ and $\mathrm{H}(C)_{\mathrm{tor}}$ respectively) with the following fact: there is a commutative diagram of homomorphisms of abelian groups

$$\begin{array}{ccc} K_1(\Lambda(G),\Sigma_{S^*}) & \xrightarrow{\partial'} & K_0(\Sigma_{S^*})\\ \downarrow{\scriptstyle \mathrm{ch}_{\Lambda(G),\Sigma_{S^*}}} & & \downarrow{\scriptstyle\iota}\\ K_1(\Lambda(G)_{S^*}) & \xrightarrow{\partial_G} & K_0(\mathfrak{M}_{S^*}(G)) \end{array}$$

where ∂' sends each class $[C,a]_{\mathrm{FK}}$ to $-[\![C]\!]$ (cf. [**18**, Theorem 1.3.15]) and ι sends each class $[\![C]\!]$ to $\chi(C) = \sum_{i\in\mathbb{Z}}(-1)^i[H^i(C)]$ (cf. [**18**, § 4.3.3]).

Regarding claim (ii) we note first that if a homomorphism $\chi_{G,\gamma}$ exists satisfying property (a), then it is automatically unique. Next we note that Lemma 4.4 implies the assignment $M \mapsto \mathrm{char}_{G,\gamma}(M[1])$ for each M in $\mathfrak{M}_{S^*}(G)$ induces a well-defined homomorphism $\chi_{G,\gamma} : K_0(\mathfrak{M}_{S^*}(G)) \to K_1(\Lambda(G)_{S^*})$ and claim (i) implies that this homomorphism is a right inverse to ∂_G. Further, for each M in $\mathfrak{D}(G)$ and N in $\mathfrak{M}_S(G)$ one has $\mathrm{char}_{G,\gamma}(M[1]) = \chi^{\mu}_{G,\gamma}(M[1]) \in \iota(K_0(\Omega(G)))$ and $\mathrm{char}_{G,\gamma}(N[1]) = \mathrm{char}^*_{G,\gamma}(N[1]) \in \iota(K_0(\mathfrak{M}_S(G)))$ (where the latter equality follows from Lemma 3.1 (iii)) and so property (c) is satisfied. Finally, the commutativity of the diagram in (d) is a direct consequence of Lemma 4.5.

https://doi.org/10.1017/S147474800900022X Published online by Cambridge University Press

5. Descent theory

In this section we shall discuss leading terms of elements of $K_1(\Lambda(G)_{S^*})$ and in particular prove Theorem 2.2. The approach of this section was initially developed by the first named author in an unpublished early version of the article [**6**].

We deal first with the case that C is acyclic. In this case the complex $\mathbb{Z}_p[\bar{G}]\otimes^{\mathbb{L}}_{\Lambda(G)}C$ is also acyclic so $[\mathbf{d}_{\mathbb{Z}_p[\bar{G}]}(\mathbb{Z}_p[\bar{G}]\otimes^{\mathbb{L}}_{\Lambda(G)}C), t(C)_{\bar{G}}]$ is the zero element of $K_0(\mathbb{Z}_p[\bar{G}],\mathbb{Q}_p^c[\bar{G}])$ and in addition ξ belongs to the image of the natural map $K_1(\Lambda(G))\to K_1(\Lambda(G)_{S^*})$. The equality of Theorem 2.2 is therefore a consequence of the following result.

Lemma 5.1. *If u belongs to the image of the natural map $\lambda : K_1(\Lambda(G))\to K_1(\Lambda(G)_{S^*})$, then for each finite quotient $\bar{G}$ of G the element $(u^*(\rho))_{\rho\in\mathrm{Irr}(\bar{G})}$ belongs to $\ker(\partial_{\bar{G}})$.*

Proof. Let $\mathcal{O}$ be the valuation ring of a finite extension L of $\mathbb{Q}_p$ such that all representations of $\bar{G}$ can be realized over $\mathcal{O}$. If $v\in K_1(\Lambda(G))$ with $u=\lambda(v)$, then $u^*(\rho)=u(\rho)=\lambda(v)(\rho)\in\mathcal{O}^\times$ for all $\rho\in\mathrm{Irr}(\bar{G})$. Thus by functoriality of K-theory and the fact that the canonical map $K_1(\Lambda_{\mathcal{O}}(\Gamma))\to K_1(\mathcal{O})$ is equal to the 'evaluation at 0' homomorphism $\Lambda_{\mathcal{O}}(\Gamma)^\times\to\mathcal{O}^\times$, the image of v in $K_1(\mathbb{Z}_p[\bar{G}])$ under the natural projection is mapped to $(u^*(\rho))_{\rho\in\mathrm{Irr}(\bar{G})}\in K_1(L[\bar{G}])$. □

5.1. Reduction to S-acyclicity

We now reduce the general case of Theorem 2.2 to the case that C belongs to $D^{\mathrm{p}}_S(\Lambda(G))$. In this subsection we therefore assume that G has no element of order p.

Lemma 5.2. *If G has no element of order p, then it is enough to prove Theorem 2.2 in the case that C is acyclic outside at most one degree.*

Proof. We assume that the result of Theorem 2.2 is true for all complexes that are acyclic outside at most one degree. To deduce Theorem 2.2 in the general case we use induction on the number of non-zero cohomology groups of C (which we assume to be at least two). We let n denote the largest integer m for which $H^m(C)$ is non-zero. We write C_1 for the (non-naive!) truncation of C in degrees less than n (which has fewer non-zero cohomology groups than does C) and set $C_2:=H^n(C)[-n]$. Then there is an exact triangle in $D^{\mathrm{p}}_{S^*}(\Lambda(G))$ of the form

$$C_1\to C\to C_2\to C_1[1] \tag{5.1}$$

and the assumption that C is semisimple at ρ implies that both C_1 and C_2 are also semisimple at ρ (for any ρ in $\mathrm{Irr}(\bar{G})$). Let ξ be an element such that $\partial_G(\xi)=\chi(C)$. If ξ_1 is such that $\partial_G(\xi_1)=\chi(C_1)$, then $\xi_2:=\xi\xi_1^{-1}$ satisfies $\partial_G(\xi_2)=\partial_G(\xi)-\partial_G(\xi_1)=\chi(C)-\chi(C_1)=\chi(C_2)$, where the last equality follows from (5.1). Hence, by the inductive hypothesis, one has

$$\begin{aligned}\partial_{\bar{G}}((\xi^*(\rho))_\rho)&=\partial_{\bar{G}}((\xi_1^*(\rho))_\rho)+\partial_{\bar{G}}((\xi_2^*(\rho))_\rho)\\&=-[\mathbf{d}_{\mathbb{Z}_p[\bar{G}]}(C_{1,\bar{G}}),t(C_1)_{\bar{G}}]-[\mathbf{d}_{\mathbb{Z}_p[\bar{G}]}(C_{2,\bar{G}}),t(C_2)_{\bar{G}}],\end{aligned} \tag{5.2}$$

where $C_{i,\bar{G}} := \mathbb{Z}_p[\bar{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} C_i$ for $i = 1, 2$. Now if we set $C_{\bar{G}} := \mathbb{Z}_p[\bar{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} C$, then (5.1) induces an exact triangle in $D^{\mathrm{p}}(\mathbb{Z}_p[\bar{G}])$

$$C_{1,\bar{G}} \to C_{\bar{G}} \to C_{2,\bar{G}} \to C_{1,\bar{G}}[1] \tag{5.3}$$

and, with respect to this triangle, the trivializations $t(C_1)_{\bar{G}}$, $t(C)_{\bar{G}}$ and $t(C_2)_{\bar{G}}$ satisfy the 'additivity criterion' of [**3**, Corollary 6.6]. Indeed, for each ρ in $\mathrm{Irr}(\bar{G})$ the exact triangle (5.1) combines with the definition (3.1) of each of the complexes $C_{1,\rho}$, C_ρ and $C_{2,\rho}$ to induce an exact triangle

$$\mathbb{Q}_p^c \hat{\otimes}^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(\Gamma)} C_{1,\rho} \to \mathbb{Q}_p^c \hat{\otimes}^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(\Gamma)} C_{\rho} \to \mathbb{Q}_p^c \hat{\otimes}^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(\Gamma)} C_{2,\rho} \to \mathbb{Q}_p^c \hat{\otimes}^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(\Gamma)} C_{1,\rho}[1]$$

and the cohomology sequence of this triangle gives an equality $r_G(C)(\rho) = r_G(C_1)(\rho) + r_G(C_2)(\rho)$ and a short exact sequence of complexes

$$0 \to \mathrm{H}_\beta(\triangle(C_{1,\rho}, \gamma)) \to \mathrm{H}_\beta(\triangle(C_\rho, \gamma)) \to \mathrm{H}_\beta(\triangle(C_{2,\rho}, \gamma)) \to 0.$$

Here we use the notation of Appendix B and write $\triangle(C_\rho, \gamma)$ for the triangle

$$\mathbb{Q}_p^c \otimes^{\mathbb{L}}_{\mathcal{O}} C_\rho \xrightarrow{\theta_{\gamma,\rho}} \mathbb{Q}_p^c \otimes^{\mathbb{L}}_{\mathcal{O}} C_\rho \to \mathbb{Q}_p^c \hat{\otimes}^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(\Gamma)} C_\rho \to \mathbb{Q}_p^c \otimes^{\mathbb{L}}_{\mathcal{O}} C_\rho[1], \tag{5.4}$$

where $\theta_{\gamma,\rho}$ is induced by multiplication by $\gamma - 1$, and we use similar notation for C_1 and C_2. This means that the criterion of [**3**, Corollary 6.6] is satisfied if one takes (in the notation of [**3**]) Σ to be $\mathbb{Q}_p^c[\bar{G}]$,

$$P \xrightarrow{a} Q \xrightarrow{b} R \xrightarrow{c} P[1]$$

to be the exact triangle (5.3) (so $\ker(H^{\mathrm{ev}} a_\Sigma) = \ker(H^{\mathrm{od}} a_\Sigma) = 0$) and the trivializations t_P, t_Q and t_R to be induced by $(-1)^{r_G(C_1)(\rho)} t(C_{1,\rho})$, $(-1)^{r_G(C)(\rho)} t(C_\rho)$ and $(-1)^{r_G(C_2)(\rho)} t(C_{2,\rho})$ respectively. From [**3**, Corollary 6.6] we therefore deduce that the last element in (5.2) is indeed equal to $-[\mathbf{d}_{\mathbb{Z}_p[\bar{G}]}(C_{\bar{G}}), t(C)_{\bar{G}}]$, as required. □

Taking account of Lemmas 5.1 and 5.2 we now assume that C is acyclic outside precisely one degree. To be specific, we assume that $C = M[0]$ with M in $\mathfrak{M}_{S^*}(\Lambda(G))$. Then there is an exact triangle of the form

$$M_{\mathrm{tor}}[0] \to M[0] \to M_{\mathrm{tf}}[0] \to M_{\mathrm{tor}}[1], \tag{5.5}$$

where M_{tor} belongs to $\mathfrak{D}(G)$ and M_{tf} to $\mathfrak{M}_S(\Lambda(G))$. In this case one has $t(M[0])_{\bar{G}} = t(M_{\mathrm{tf}}[0])_{\bar{G}}$ and so (by another application of [**3**, Corollary 6.6])

$$\begin{aligned}&[\mathbf{d}_{\mathbb{Z}_p[\bar{G}]}(\mathbb{Z}_p[\bar{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} M[0]), t(M[0])_{\bar{G}}] \\ &\quad = [\mathbf{d}_{\mathbb{Z}_p[\bar{G}]}(\mathbb{Z}_p[\bar{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} M_{\mathrm{tor}}[0]), \mathrm{can}] + [\mathbf{d}_{\mathbb{Z}_p[\bar{G}]}(\mathbb{Z}_p[\bar{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} M_{\mathrm{tf}}[0]), t(M[0])_{\bar{G}}]\end{aligned} \tag{5.6}$$

with 'can' the canonical morphism

$$\mathbf{d}_{\mathbb{Q}_p[\bar{G}]}(\mathbb{Q}_p[\bar{G}] \hat{\otimes}^{\mathbb{L}}_{\Lambda(G)} M_{\mathrm{tor}}[0]) = \mathbf{d}_{\mathbb{Q}_p[\bar{G}]}(0) \to \mathbf{1}_{\mathbb{Q}_p[\bar{G}]}.$$

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Lemma 5.3. *Let N be an object of $\mathfrak{D}(G)$. If ξ is any element of $K_1(\Lambda(G)_{S^*})$ with $\partial_G(\xi)=\chi(N[0])$, then for any finite quotient $\bar{G}$ of G one has*

$$\partial_{\bar{G}}((\xi^*(\rho))_\rho) = -[\mathbf{d}_{\mathbb{Z}_p[\bar{G}]}(\mathbb{Z}_p[\bar{G}]\otimes^{\mathbb{L}}_{\Lambda(G)} N[0]), \mathrm{can}].$$

Proof. An easy reduction (using dévissage and the additivity of Euler characteristics on exact sequences in $\mathfrak{D}(G)$) allows us to assume that N is an object of $\mathfrak{D}(G)$ which lies in an exact sequence of $\Lambda(G)$-modules of the form

$$0 \to Q \xrightarrow{d} P \to N \to 0, \tag{5.7}$$

where Q and P are both finitely generated and projective. (Indeed, it is actually enough to consider the case that $Q=P=\Lambda(G)e_i$ for an idempotent e_i as in § 3.3 and with d equal to multiplication by p.)

To proceed we identify the subgroup $K_0(\mathfrak{D}(G))$ of $K_0(\mathfrak{M}_{S^*}(G))$ with the group $K_0(\Lambda(G),\Lambda(G)[1/p])$. To be compatible with the normalizations used in § 1.2 we must fix this isomorphism so that for every exact sequence (5.7) the element $[N]$ of $K_0(\mathfrak{D}(G))$ corresponds to the element (Q,d',P) of $K_0(\Lambda(G),\Lambda(G)[1/p])$ with $d' := \Lambda(G)[1/p]\otimes_{\Lambda(G)} d$.

Now since $\Lambda(G)[1/p]\otimes_{\Lambda(G)} N = 0$ the localization sequence of K-theory implies that any element ξ as above belongs to the image of $K_1(\Lambda(G)[1/p])$ in $K_1(\Lambda(G)_{S^*})$. This implies in particular that $\xi^*(\rho)=\xi(\rho)$ for all ρ in $\mathrm{Irr}(\bar{G})$. The natural commutative diagram of connecting homomorphisms

$$\begin{array}{ccc} K_1(\Lambda(G)[1/p]) & \longrightarrow & K_0(\Lambda(G),\Lambda(G)[1/p]) \\ \downarrow & & \downarrow \\ K_1(\mathbb{Q}_p[\bar{G}]) & \xrightarrow{\partial_{\bar{G}}} & K_0(\mathbb{Z}_p[\bar{G}],\mathbb{Q}_p[\bar{G}]) \end{array}$$

also then implies that $\partial_{\bar{G}}((\xi^*(\rho))_\rho) = (Q_{\bar{G}}, d'_{\bar{G}}, P_{\bar{G}})$ with $Q_{\bar{G}} := \mathbb{Z}_p[\bar{G}]\otimes_{\Lambda(G)} Q$, $P_{\bar{G}} := \mathbb{Z}_p[\bar{G}]\otimes_{\Lambda(G)} P$ and $d'_{\bar{G}} := \mathbb{Q}_p[\bar{G}]\otimes_{\Lambda(G)} d$. Hence, with respect to the isomorphism (1.2) (with $R=\mathbb{Z}_p[\bar{G}]$ and $R'=\mathbb{Q}_p[\bar{G}]$), one has

$$\partial_{\bar{G}}((\xi^*(\rho))_\rho) = [\mathbf{d}_{\mathbb{Z}_p[\bar{G}]}(Q_{\bar{G}})\mathbf{d}_{\mathbb{Z}_p[\bar{G}]}(P_{\bar{G}})^{-1}, \tau] \tag{5.8}$$

with τ equal to the composite morphism

$$\mathbf{d}_{\mathbb{Q}_p[\bar{G}]}(\mathbb{Q}_p\otimes_{\mathbb{Z}_p} Q_{\bar{G}})\mathbf{d}_{\mathbb{Q}_p[\bar{G}]}(\mathbb{Q}_p\otimes_{\mathbb{Z}_p} P_{\bar{G}})^{-1} \\ \to \mathbf{d}_{\mathbb{Q}_p[\bar{G}]}(\mathbb{Q}_p\otimes_{\mathbb{Z}_p} P_{\bar{G}})\mathbf{d}_{\mathbb{Q}_p[\bar{G}]}(\mathbb{Q}_p\otimes_{\mathbb{Z}_p} P_{\bar{G}})^{-1} = \mathbf{1}_{\mathbb{Q}_p[\bar{G}]},$$

where the first arrow is induced by $\mathbf{d}_{\mathbb{Q}_p[\bar{G}]}(d'_{\bar{G}})$. Finally, we note that the image under $\mathbb{Z}_p[\bar{G}]\otimes_{\Lambda(G)} -$ of the sequence (5.7) induces an isomorphism in $D^{\mathrm{p}}(\mathbb{Z}_p[\bar{G}])$ between $\mathbb{Z}_p[\bar{G}]\otimes^{\mathbb{L}}_{\Lambda(G)} N[0]$ and the complex $Q_{\bar{G}} \xrightarrow{d_{\bar{G}}} P_{\bar{G}}$ where the first term is placed in degree -1 and $d_{\bar{G}} := \mathbb{Z}_p[\bar{G}]\otimes_{\Lambda(G)} d$ and this implies that the element on the right-hand side of (5.8) is the inverse of $[\mathbf{d}_{\mathbb{Z}_p[\bar{G}]}(\mathbb{Z}_p[\bar{G}]\otimes^{\mathbb{L}}_{\Lambda(G)} N[0]), \mathrm{can}]$, as required. □

Lemmas 5.1, 5.2 and 5.3 combine with (5.5) and (5.6) to reduce the proof of Theorem 2.2 to consideration of complexes in $D^{\mathrm{p}}_S(\Lambda(G))$ (now in both cases, whether G has an element of order p or not!). In the remainder of § 5 we shall therefore assume that C belongs to $D^{\mathrm{p}}_S(\Lambda(G))$.

5.2. Equivariant twists

In this subsection we introduce the algebraic formalism that is key to a proper understanding of descent.

5.2.1. *The definition*

We fix an open normal subgroup U of G and set $\bar{G} := G/U$. We write

$$\Delta_{\bar{G}} : \Lambda(G) \to \Lambda(\bar{G} \times G) \cong \mathbb{Z}_p[\bar{G}] \otimes_{\mathbb{Z}_p} \Lambda(G)$$

for the (flat) ring homomorphism which sends each element σ of G to $\bar{\sigma} \otimes \sigma$ where $\bar{\sigma}$ is the image of σ in $\bar{G}$. Then for each $\Lambda(G)$-module M the induced $\Lambda(\bar{G} \times G)$-module $\Lambda(\bar{G} \times G) \otimes_{\Lambda(G),\Delta_{\bar{G}}} M$ can be identified with the module

$$\mathrm{tw}_{\bar{G}}(M) := \mathbb{Z}_p[\bar{G}] \otimes_{\mathbb{Z}_p} M$$

upon which $\bar{G}$ acts via left multiplication and each $\sigma \in G$ acts by sending $x \otimes y$ to $x\bar{\sigma}^{-1} \otimes \sigma(y)$. This construction extends to give an exact functor $C \mapsto \mathrm{tw}_{\bar{G}}(C)$ from $D^{\mathrm{p}}(\Lambda(G))$ to $D^{\mathrm{p}}(\Lambda(\bar{G} \times G))$ and for each such C we set

$$\mathrm{tw}_{\bar{G}}(C)_H := \Lambda(\bar{G} \times \Gamma) \otimes^{\mathbb{L}}_{\Lambda(\bar{G} \times G)} \mathrm{tw}_{\bar{G}}(C) \in D^{\mathrm{p}}(\Lambda(\bar{G} \times \Gamma)).$$

5.2.2. *Base change*

For each $s \in \Lambda(G)$ we write r_s and $\mathrm{r}_{\Delta_{\bar{G}}(s)}$ for the endomorphisms of $\Lambda(G)$ and $\Lambda(\bar{G} \times G)$ given by right multiplication by s and $\Delta_{\bar{G}}(s)$ respectively. Then $\mathrm{cok}(\mathrm{r}_{\Delta_{\bar{G}}(s)})$ is isomorphic as a $\Lambda(\bar{G} \times G)$-module to $\mathrm{tw}_{\bar{G}}(\mathrm{cok}(\mathrm{r}_s))$ and so is finitely generated over $\Lambda(\bar{G} \times H)$ if $\mathrm{cok}(\mathrm{r}_s)$ is finitely generated over $\Lambda(H)$. This implies that $\Delta_{\bar{G}}(S^*) \subseteq S_1^*$, where $S := S_{G,H}$ and $S_1 := S_{\bar{G} \times G, \bar{G} \times H}$ and so $\Delta_{\bar{G}}$ induces a ring homomorphism

$$\Lambda(G)_{S^*} \to \Lambda(\bar{G} \times G)_{S_1^*} \to \Lambda(\bar{G} \times \Gamma)_{S_2^*} = Q(\bar{G} \times \Gamma), \tag{5.9}$$

where $S_2 := S_{\bar{G} \times \Gamma, \bar{G}}$, the second arrow is the natural projection and the equality is because $\bar{G} \times \Gamma$ has rank one (as a p-adic Lie group). These maps induce a group homomorphism

$$\pi_{\bar{G} \times \Gamma} : K_1(\Lambda(G)_{S^*}) \to K_1(Q(\bar{G} \times \Gamma))$$

which forms the upper row of a natural commutative diagram of connecting homomorphisms:

$$\begin{array}{ccccc} K_1(\Lambda(G)_{S^*}) & \longrightarrow & K_1(\Lambda(\bar{G} \times G)_{S_1^*}) & \longrightarrow & K_1(\Lambda(\bar{G} \times \Gamma)_{S_2^*}) \\ \downarrow & & \downarrow & & \downarrow \\ K_0(\Lambda(G), \Lambda(G)_{S^*}) & \longrightarrow & K_0(\Lambda(\bar{G} \times G), S_1^*) & \longrightarrow & K_0(\Lambda(\bar{G} \times \Gamma), S_2^*) \end{array} \tag{5.10}$$

where we write $K_0(\Lambda(\bar{G} \times G), S_1^*)$ and $K_0(\Lambda(\bar{G} \times \Gamma), S_2^*)$ for $K_0(\Lambda(\bar{G} \times G), \Lambda(\bar{G} \times G)_{S_1^*})$ and $K_0(\Lambda(\bar{G} \times \Gamma), \Lambda(\bar{G} \times \Gamma)_{S_2^*})$ respectively.

5.2.3. *Reduced norms*

We set $R := \Lambda(\bar{G}\times\Gamma)$. Then the algebra $Q(R)$ identifies with the group ring $Q(\Gamma)[\bar{G}]$ and, with respect to this identification, one has

$$\zeta(Q(R)) \subset \zeta(Q^c(R)) = \prod_{\rho\in\mathrm{Irr}(\bar{G})} Q^c(\Gamma) \tag{5.11}$$

where $Q^c(R) := \mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} Q(R)$ and $Q^c(\Gamma) := \mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} Q(\Gamma)$. We write $x = (x_\rho)_\rho$ for the corresponding decomposition of each element x of $\zeta(Q^c(R))$.

In the next result we write $\mathrm{Nrd}_{Q(R)} : K_1(Q(R)) \to \zeta(Q^c(R))^\times$ for the reduced norm map of the semisimple algebra $Q(R)$ and use the homomorphism Φ_ρ and Ore set $\tilde{S}$ defined in (2.1) and (2.3) respectively.

Lemma 5.4. *For each ξ in $K_1(\Lambda(G)_{\tilde{S}})$ one has* $\mathrm{Nrd}_{Q(R)}(\pi_{\bar{G}\times\Gamma}(\xi)) = (\Phi_\rho(\xi))_{\rho\in\mathrm{Irr}(\bar{G})}$.

Proof. It suffices to prove that, with respect to the decomposition (5.11), one has $\Phi_\rho(\xi) = \mathrm{Nrd}_{Q(R)}(\pi_{\bar{G}\times\Gamma}(\xi))_\rho$ for each fixed ρ in $\mathrm{Irr}(\bar{G})$. Further, since [**15**, Proposition 4.2, Theorem 4.4] implies that the natural map $\Lambda(G)_{\tilde{S}}^\times \to K_1(\Lambda(G)_{\tilde{S}})$ is surjective, it is enough to verify this for all elements ξ of the form $\langle \mathrm{r}_s \mid \Lambda(G)_{\tilde{S}}\rangle$ with $s \in \Lambda(G)\cap\Lambda(G)_{\tilde{S}}^\times$.

To do this we fix a finite-dimensional $\mathbb{Q}_p^c$-space V_ρ that corresponds to ρ and write V_{ρ^*} for the space $\mathrm{Hom}_{\mathbb{Q}_p^c}(V_\rho, \mathbb{Q}_p^c)$ that corresponds to ρ^*. Then for each $x = \sum_{\delta\in\bar{G}} c_\delta\delta$ in $Q(\Gamma)[\bar{G}]^\times = Q(R)^\times$ the argument of Ritter and Weiss in [**27**, § 3] shows that

$$\mathrm{Nrd}_{Q(R)}(\langle \mathrm{r}_x \mid Q(R)\rangle)_\rho = \det_{Q^c(\Gamma)}(\alpha_x), \tag{5.12}$$

where α_x is the automorphism of $V_{\rho^*}\otimes_{\mathbb{Q}_p^c} Q^c(\Gamma)$ given by $\sum_{\delta\in\bar{G}}\delta^{-1}\otimes\mu(c_\delta)$ with $\mu(c_\delta)$ denoting multiplication by c_δ. Now the matrix of the action of δ^{-1} on V_{ρ^*} (with respect to a fixed $\mathbb{Q}_p^c$-basis) is the transpose of the matrix of the action of δ on V_ρ (with respect to the dual $\mathbb{Q}_p^c$-basis). Using this fact, and an explication of the role of Morita equivalence in (2.1), one finds that $\Phi_\rho(s) = \det_{Q^c(\Gamma)}(\alpha_{\Delta_{\bar{G}}(s)})$ for each $s\in\Lambda(G)\cap\Lambda(G)_{\tilde{S}}^\times$. Since $\pi_{\bar{G}\times\Gamma}(\langle \mathrm{r}_s \mid \Lambda(G)_{\tilde{S}}\rangle) = \langle \mathrm{r}_{\Delta_{\bar{G}}(s)} \mid Q(R)\rangle$ the claimed result is therefore a consequence of the description (5.12). □

5.2.4. *Semisimplicity*

There are natural isomorphisms in $D^{\mathrm{p}}(\mathbb{Z}_p[\bar{G}])$ of the form

$$\mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(\Gamma)} \mathrm{tw}_{\bar{G}}(C)_H \cong \mathbb{Z}_p \otimes^{\mathbb{L}}_{\Lambda(G)} \mathrm{tw}_{\bar{G}}(C) \cong \mathbb{Z}_p[\bar{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} C$$

and hence an exact triangle in $D(\Lambda(\bar{G}\times\Gamma))$ of the form

$$\triangle(\mathrm{tw}_{\bar{G}}(C),\gamma) : \mathrm{tw}_{\bar{G}}(C)_H \xrightarrow{\theta_\gamma} \mathrm{tw}_{\bar{G}}(C)_H \to \mathbb{Z}_p[\bar{G}]\otimes^{\mathbb{L}}_{\Lambda(G)} C \to \mathrm{tw}_{\bar{G}}(C)_H[1],$$

where θ_γ is induced by multiplication by $\gamma - \mathrm{id}\in\Lambda(\Gamma)$ on $\Lambda(\bar{G}\times\Gamma)$.

In the next result we use the terminology and notation of Appendix B. For each $\mathbb{Q}_p[\bar{G}]$-module M we also define a $\mathbb{Q}_p^c$-module $M^\rho := \mathrm{Hom}_{\mathbb{Q}_p^c[\bar{G}]}(V_\rho, \mathbb{Q}_p^c\otimes_{\mathbb{Z}_p} M)$.

Lemma 5.5.

(i) *The image of* $\triangle(\mathrm{tw}_{\bar{G}}(C), \gamma)$ *under the (exact) functor* $e_\rho \mathbb{Q}_p^c[\bar{G}] \otimes_{\mathbb{Z}_p[\bar{G}]} -$ *is naturally isomorphic to the exact triangle* $\triangle(C_\rho, \gamma)$ *defined in (5.4).*

(ii) *For each* ρ *in* $\mathrm{Irr}(\bar{G})$ *one has*

$$r_G(C)(\rho) = \sum_{i \in \mathbb{Z}} (-1)^{i+1} \dim_{\mathbb{Q}_p^c}(H^i(\mathrm{tw}_{\bar{G}}(C)_H)^{\Gamma,\rho}).$$

(iii) *The morphism* θ_γ *is semisimple if and only if* C *is semisimple at* ρ *(in the sense of [**13**, Definition 3.11]) for every* ρ *in* $\mathrm{Irr}(\bar{G})$.

(iv) *If* θ_γ *is semisimple, then*

$$r_G(C)(\rho) = \sum_{i \in \mathbb{Z}} (-1)^{i+1} i \cdot \dim_{\mathbb{Q}_p^c}(H^i(\mathbb{Z}_p[\bar{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} C)^\rho)$$

for every ρ *in* $\mathrm{Irr}(\bar{G})$ *and, with respect to the decomposition (1.4), one has for the Bockstein morphism (as defined in Appendix B)*

$$t(\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \triangle(\mathrm{tw}_{\bar{G}}(C), \gamma)) = (t(C_\rho))_{\rho \in \mathrm{Irr}(\bar{G})},$$

where $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \triangle(\mathrm{tw}_{\bar{G}}(C), \gamma)$ *denotes the exact triangle in* $D^{\mathrm{p}}(\Lambda(\bar{G} \times \Gamma)[1/p])$ *that is obtained from* $\triangle(\mathrm{tw}_{\bar{G}}(C), \gamma)$ *by scalar extension.*

Proof. For every $\Lambda(G)$-module P there is a natural isomorphism of $\Lambda(\bar{G} \times \Gamma)$-modules

$$\Lambda(\bar{G} \times \Gamma) \otimes_{\Lambda(\bar{G} \times G)} (\mathbb{Z}_p[\bar{G}] \otimes_{\mathbb{Z}_p} P) \cong \Lambda(\Gamma) \otimes_{\Lambda(G)} (\mathbb{Z}_p[\bar{G}] \otimes_{\mathbb{Z}_p} P)$$

where the action of $\bar{G}$ on the second module is just on $\mathbb{Z}_p[\bar{G}]$ (from the left). This fact gives rise to natural isomorphisms in $D(\Lambda_{\mathbb{Q}_p^c}(\Gamma))$ of the form

$$\begin{aligned} e_\rho \mathbb{Q}_p^c[\bar{G}] \otimes_{\mathbb{Z}_p[\bar{G}]} \mathrm{tw}_{\bar{G}}(C)_H &\cong \Lambda_{\mathbb{Q}_p^c}(\Gamma) \otimes^{\mathbb{L}}_{\Lambda_{\mathbb{Q}_p^c}(G)} (e_\rho \mathbb{Q}_p^c[\bar{G}] \otimes_{\mathbb{Z}_p} C) \\ &\cong \Lambda_{\mathbb{Q}_p^c}(\Gamma) \otimes^{\mathbb{L}}_{\Lambda_{\mathbb{Q}_p^c}(G)} (V_{\rho*} \otimes_{\mathbb{Z}_p} C) \\ &\cong \mathbb{Q}_p^c \otimes_{\mathcal{O}} C_\rho, \end{aligned} \tag{5.13}$$

where e_ρ denotes a primitive idempotent of $\mathbb{Q}_p^c[\bar{G}]$ for which $e_\rho \mathbb{Q}_p^c[\bar{G}]$ is isomorphic to $V_{\rho*}$. We now set $C_{\bar{G}} := \mathbb{Z}_p[\bar{G}] \otimes^{\mathbb{L}}_{\Lambda(G)} C$. Then claim (i) follows upon combining the isomorphism (5.13) together with the following natural isomorphism in $D(\Lambda_{\mathbb{Q}_p^c}(\Gamma))$:

$$e_\rho \mathbb{Q}_p^c[\bar{G}] \otimes_{\mathbb{Z}_p[\bar{G}]} C_{\bar{G}} \cong V_{\rho^*} \hat{\otimes}^{\mathbb{L}}_{\Lambda(G)} C \cong \mathbb{Q}_p^c \hat{\otimes}^{\mathbb{L}}_{\Lambda_{\mathcal{O}}(\Gamma)} C_\rho.$$

Claim (ii) follows by combining the equality

$$r_G(C)(\rho) = \sum_{i \in \mathbb{Z}} (-1)^{i+1} \dim_{\mathbb{Q}_p^c}(H^i(\mathbb{Q}_p^c \otimes_{\mathcal{O}} C_\rho)^\Gamma)$$

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with the isomorphisms of $\Lambda_{\mathbb{Q}_p^c}(\Gamma)$-modules

$$H^i(\mathbb{Q}_p^c \otimes_{\mathcal{O}} C_\rho) \cong H^i(e_\rho \mathbb{Q}_p^c[\bar{G}] \otimes_{\mathbb{Z}_p[\bar{G}]} \mathrm{tw}_{\bar{G}}(C)_H) \cong H^i(\mathrm{tw}_{\bar{G}}(C)_H)^\rho$$

that are induced by (5.13).

Next we note that claim (i) implies that θ_γ is semisimple if and only if for every ρ in $\mathrm{Irr}(\bar{G})$ the morphism $\theta_{\gamma,\rho}$ which occurs in (5.4) is semisimple. Claim (iii) thus follows immediately from the very definition of 'semisimplicity at ρ' (in terms of $\theta_{\gamma,\rho}$).

In each degree i the exact triangle $\triangle(\mathrm{tw}_{\bar{G}}(C), \gamma)$ induces a short exact sequence

$$0 \to H^i(\mathrm{tw}_{\bar{G}}(C)_H)_\Gamma \to H^i(C_{\bar{G}}) \to H^{i+1}(\mathrm{tw}_{\bar{G}}(C)_H)^\Gamma \to 0.$$

So by applying the exact functor $M \mapsto M^\rho$ to this sequence one finds that

$$\dim_{\mathbb{Q}_p^c}(H^i(C_{\bar{G}})^\rho) = \dim_{\mathbb{Q}_p^c}(H^i(\mathrm{tw}_{\bar{G}}(C)_H)^{\Gamma,\rho}) + \dim_{\mathbb{Q}_p^c}(H^{i+1}(\mathrm{tw}_{\bar{G}}(C)_H)^{\Gamma,\rho})$$

for each integer i and hence that $\sum_{i\in\mathbb{Z}}(-1)^{i+1} i \cdot \dim_{\mathbb{Q}_p^c}(H^i(C_{\bar{G}})^\rho)$ is equal to

$$\sum_{i\in\mathbb{Z}}(-1)^{i+1} i(\dim_{\mathbb{Q}_p^c}(H^i(\mathrm{tw}_{\bar{G}}(C)_H)^{\Gamma,\rho}) + \dim_{\mathbb{Q}_p^c}(H^{i+1}(\mathrm{tw}_{\bar{G}}(C)_H)^{\Gamma,\rho}))$$
$$= \sum_{i\in\mathbb{Z}}(-1)^{i+1} \dim_{\mathbb{Q}_p^c}(H^i(\mathrm{tw}_{\bar{G}}(C)_H)^{\Gamma,\rho}).$$

This proves the explicit formula for $r_G(C)(\rho)$ in claim (iv). The explicit description of $t(\mathbb{Q}_p \otimes \triangle(\mathrm{tw}_{\bar{G}}(C), \gamma))$ in claim (iv) follows from the identification in claim (i) and the fact that $t(C_\rho)$ is defined in [**13**] to be equal to the morphism $t(\triangle(C_\rho, \gamma))$. □

5.3. Leading terms

We now fix an element ξ and a complex C as in Theorem 2.2. Then Lemma 5.5 (iii) implies that the morphism $\theta_\gamma : \mathrm{tw}_{\bar{G}}(C)_H \to \mathrm{tw}_{\bar{G}}(C)_H$ is semisimple and so in each degree i there is a direct sum decomposition of $\Lambda(\bar{G} \times \Gamma)[1/p]$-modules

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i(\mathrm{tw}_{\bar{G}}(C)_H) = D_0^i \oplus D_1^i, \tag{5.14}$$

where

$$D_0^i := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \ker(H^i(\theta_\gamma)) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i(\mathrm{tw}_{\bar{G}}(C)_H)^\Gamma \quad \text{and} \quad D_1^i := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{im}(H^i(\theta_\gamma)).$$

By assumption, both D_0^i and D_1^i are finitely generated (projective) $\mathbb{Q}_p[\bar{G}]$-modules and $H^i(\theta_\gamma)$ induces an automorphism of D_1^i.

The proof of the following result will occupy the rest of this section.

Proposition 5.6. $\partial_{\bar{G}}(((-1)^{r_G(C)(\rho)}\xi^*(\rho))_{\rho\in\mathrm{Irr}(\bar{G})}) = \sum_{i\in\mathbb{Z}}(-1)^i \partial_{\bar{G}}(\langle H^i(\theta_\gamma) \mid D_1^i\rangle)$.

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5.3.1. *The descent to* $Q(R)$

We write Σ for the subset of R consisting of those elements of $\Lambda(\Gamma)$ with non-zero image under the projection $\Lambda(\Gamma) \to \mathbb{Z}_p$. This is a multiplicatively closed Ore set in R which consists of central regular elements. It can be shown that $R_\Sigma = R_{S_2} \subseteq Q(R)$ where S_2 denotes (as in § 5.2.2) the Ore set $S_{\bar{G}\times\Gamma,\bar{G}}$.

Lemma 5.7. *For each integer* i *we set* $M^i := (\mathrm{I}_{\bar{G}}^{\bar{G}\times\Gamma}(H^i(\mathrm{tw}_{\bar{G}}(C)_H)^\Gamma))_{S^*}$. *Then the element*

$$y_\xi := \mathrm{Nrd}_{Q(R)}\left(\pi_{\bar{G}\times\Gamma}(\xi)\prod_{i\in\mathbb{Z}}\langle\delta_\gamma \mid M^i\rangle^{(-1)^{i+1}}\right)$$

belongs to $\zeta(R_\Sigma)^\times \subseteq \zeta(Q(R))^\times$.

Proof. We set $X := \mathrm{tw}_{\bar{G}}(C)_H$. Then the commutative diagram (5.10) implies that $\partial_{\bar{G}\times\Gamma}(\pi_{\bar{G}\times\Gamma}(\xi)) = \chi(X)$ in $K_0(R, Q(R))$. But X belongs to $D^{\mathrm{p}}_S(R)$ and so [**6**, Theorem 4.1(ii)] also implies that $\partial_{\bar{G}\times\Gamma}(\mathrm{char}_{\bar{G}\times\Gamma,\gamma}(X)) = \chi(X)$. Hence the upper row of (1.1) with $R' = Q(R)$ implies that there exists an element u of $K_1(R)$ with

$$\pi_{\bar{G}\times\Gamma}(\xi) = \iota_1(u)\mathrm{char}_{\bar{G}\times\Gamma,\gamma}(X), \tag{5.15}$$

where ι_1 is the natural homomorphism $K_1(R) \to K_1(R_\Sigma) \to K_1(Q(R))$.

For each integer i we set $N^i := (\mathrm{I}_{\bar{G}}^{\bar{G}\times\Gamma}(H^i(\mathrm{tw}_{\bar{G}}(C)_H)))_{S^*}$. Then the term $\langle\delta_\gamma \mid N^i\rangle$ occurs in the definition of $\mathrm{char}_{\bar{G}\times\Gamma,\gamma}(X) = \mathrm{char}^*_{\bar{G}\times\Gamma,\gamma}(X)$. Also, from Lemma 5.8 below, the action of δ_γ on $N^i = Q(R) \otimes_{\mathbb{Q}_p[\bar{G}]} (\mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^i(\mathrm{tw}_{\bar{G}}(C)_H))$ restricts to give an automorphism of $R_\Sigma \otimes_{\mathbb{Q}_p[\bar{G}]} D^i_1$ and so (5.14) implies that $\langle\delta_\gamma \mid N^i\rangle$ is equal to

$$\begin{aligned}\langle\delta_\gamma \mid Q(R)\otimes_{\mathbb{Q}_p[\bar{G}]} D^i_0\rangle\langle\delta_\gamma \mid Q(R)\otimes_{\mathbb{Q}_p[\bar{G}]} D^i_1\rangle \\ = \langle\delta_\gamma \mid Q(R)\otimes_{\mathbb{Q}_p[\bar{G}]} D^i_0\rangle\iota_\Sigma(\langle\delta_\gamma \mid R_\Sigma\otimes_{\mathbb{Q}_p[\bar{G}]} D^i_1\rangle),\end{aligned}$$

where ι_Σ is the natural homomorphism $K_1(R_\Sigma) \to K_1(Q(R))$. Hence by combining (5.15) with the definition of $\mathrm{char}_{\bar{G}\times\Gamma,\gamma}(X)$ one finds that

$$\begin{aligned}\pi_{\bar{G}\times\Gamma}(\xi)\prod_{i\in\mathbb{Z}}\langle\delta_\gamma \mid M^i\rangle^{(-1)^{i+1}} &= \pi_{\bar{G}\times\Gamma}(\xi)\prod_{i\in\mathbb{Z}}\langle\delta_\gamma \mid Q(R)\otimes_{\mathbb{Q}_p[\bar{G}]} D^i_0\rangle^{(-1)^{i+1}} \\ &= \iota_1(u)\iota_\Sigma\left(\prod_{i\in\mathbb{Z}}\langle\delta_\gamma \mid R_\Sigma\otimes_{\mathbb{Q}_p[\bar{G}]} D^i_1\rangle^{(-1)^i}\right) \in \mathrm{im}(\iota_\Sigma). \quad (5.16)\end{aligned}$$

Now R_Σ is finitely generated as a module over the commutative local ring $\Lambda(\Gamma)_\Sigma$ and so is itself a semi-local ring (cf. [**16**, Proposition (5.28)(ii)]). The natural homomorphism $R^\times_\Sigma \to K_1(R_\Sigma)$ is thus surjective (by [**16**, Theorem (40.31)]) and so (5.16) implies that the element $\pi_{\bar{G}\times\Gamma}(\xi)\prod_{i\in\mathbb{Z}}\langle\delta_\gamma \mid M^i\rangle^{(-1)^{i+1}}$ is represented by a pair of the form $\langle \mathrm{r}_y \mid Q(R)\rangle$ with $y \in R^\times_\Sigma$. Now both y and y^{-1} are of the form $z\sigma^{-1}$ for suitable elements $z \in R \cap Q(R)^\times$ and $\sigma \in \Sigma$. Thus, to complete the proof of the lemma, it suffices to prove that for all such z and σ both $\mathrm{Nrd}_{Q(R)}(\langle \mathrm{r}_z \mid Q(R)\rangle)$ and $\mathrm{Nrd}_{Q(R)}(\langle \mathrm{r}_{\sigma^{-1}} \mid Q(R)\rangle)$ belong to $\zeta(R_\Sigma)$. But (5.12) implies $\mathrm{Nrd}_{Q(R)}(\langle \mathrm{r}_{\sigma^{-1}} \mid Q(R)\rangle) = (\sigma^{-d_\rho})_\rho \in \zeta(R_\Sigma)$ with $d_\rho := \dim_{\mathbb{Q}^c_p}(V_\rho)$.

Also, if $\mathbb{Z}_p^c$ is the integral closure of $\mathbb{Z}_p$ in $\mathbb{Q}_p^c$ and T_ρ is any full $\mathbb{Z}_p^c$-sublattice of V_ρ, then the action of $\bar{G}$ on T_ρ induces a homomorphism $\varrho : R = \Lambda(\Gamma)[\bar{G}] \to \mathrm{M}_{d_\rho}(\mathbb{Z}_p^c \otimes_{\mathbb{Z}_p} \Lambda(\Gamma))$ and (5.12) implies

$$\begin{aligned}\mathrm{Nrd}_{Q(R)}(\langle \mathrm{r}_x \mid Q(R)\rangle) &= (\det(\varrho(z)))_\rho \in (\mathbb{Q}_p^c \otimes_{\mathbb{Z}_p} \zeta(R)) \cap \zeta(Q(R)) \\ &= \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \zeta(R) \subseteq \zeta(R_\Sigma),\end{aligned}$$

as required. □

Lemma 5.8. *δ_γ induces an automorphism of $R_\Sigma \otimes_{\mathbb{Q}_p[\bar{G}]} D_1^i$.*

Proof. The argument of [**28**, Proposition 2.2, Remark 2.3] gives a short exact sequence

$$0 \to R \otimes_{\mathbb{Z}_p[\bar{G}]} D_1^i \xrightarrow{\delta_\gamma} R \otimes_{\mathbb{Z}_p[\bar{G}]} D_1^i \to D_1^i \to 0$$

and so it suffices to show that $(D_1^i)_\Sigma = 0$. But $D_1^i := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{im}(H^i(\theta_\gamma))$ and, regarding $\mathrm{im}(H^i(\theta_\gamma))$ as a (finitely generated) module over $\Lambda(\Gamma) \subseteq R$, the decomposition (5.14) implies that $\mathrm{im}(H^i(\theta_\gamma))_\Gamma$ is finite. This implies that $\mathrm{im}(H^i(\theta_\gamma))$ is a finitely generated torsion $\Lambda(\Gamma)$-module whose characteristic polynomial $f(T)$ is coprime to T. It follows that $f(T)$ is invertible in R_Σ and so $(D_1^i)_\Sigma = \mathrm{im}(H^i(\theta_\gamma))_\Sigma = 0$, as required. □

5.3.2. *The proof of Proposition 5.6*

We write $(-)^*(0) : \mathbb{Q}_p^c \otimes_{\mathbb{Q}_p} Q(\mathbb{Z}_p[\![T]\!])^\times \to \mathbb{Q}_p^{c,\times}$ for the 'leading term' homomorphism which assigns to each series of the form $T^r G(T)$ with $G(0) \neq 0$ the value $G(0)$. Then Lemma 5.4 implies that the 'leading term at ρ' homomorphism $(-)^*(\rho)$ from $K_1(\Lambda(G)_{\tilde{S}})$ to $\mathbb{Q}_p^{c,\times}$ coincides with the composite

$$\begin{aligned}&K_1(\Lambda(G)_{\tilde{S}}) \xrightarrow{\pi_{\bar{G}\times\Gamma}} K_1(Q(R)) \\ &\qquad \xrightarrow{\mathrm{Nrd}_{Q(R)}} \zeta(Q(R))^\times \subset \prod_{\rho \in \mathrm{Irr}(\bar{G})} Q^c(\Gamma)^\times \xrightarrow{\mathrm{pr}_\rho} Q^c(\Gamma)^\times \xrightarrow{(-)^*(0)} \mathbb{Q}_p^{c,\times},\end{aligned}$$

where pr_ρ denotes projection to the ρ-component. Thus from Lemma 5.7 we know that

$$\left(\xi^*(\rho) \prod_{i\in\mathbb{Z}} (\mathrm{Nrd}_{Q(R)}(\langle \delta_\gamma \mid M^i\rangle))_\rho^*(0)^{(-1)^{i+1}}\right)_{\rho\in\mathrm{Irr}(\bar{G})} = \pi(y_\xi), \tag{5.17}$$

where π is the natural projection $\zeta(R_\Sigma)^\times \to \zeta(\mathbb{Q}_p[\bar{G}])^\times$. But if x is in $R_\Sigma^\times$, then $\mathrm{Nrd}_{Q(R)}(\langle \mathrm{r}_x \mid Q(R)\rangle)$ belongs to $\zeta(R_\Sigma)^\times$ (see the proof of Lemma 5.7) and (5.12) implies $\pi(\mathrm{Nrd}_{Q(R)}(\langle \mathrm{r}_x \mid Q(R)\rangle)) = \mathrm{Nrd}_{\mathbb{Q}_p[\bar{G}]}(\langle \mathrm{r}_{\bar{x}} \mid \mathbb{Q}_p[\bar{G}]\rangle)$ with $\bar{x}$ the image of x in $\mathbb{Q}_p[\bar{G}]^\times$. Hence (5.16) implies

$$\pi(y_\xi) = \mathrm{Nrd}_{\mathbb{Q}_p[\bar{G}]}(\bar{u}) \prod_{i\in\mathbb{Z}} \mathrm{Nrd}_{\mathbb{Q}_p[\bar{G}]}(\langle H^i(\theta_\gamma) \mid D_1^i\rangle)^{(-1)^i},$$

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where $\bar{u}$ is the image of u under the natural composite homomorphism $K_1(R) \to K_1(\mathbb{Z}_p[\bar{G}]) \to K_1(\mathbb{Q}_p[\bar{G}])$. Since $\partial_{\bar{G}}(\bar{u}) = 0$ one therefore has

$$\partial_{\bar{G}}(\pi(y_\xi)) = \sum_{i\in\mathbb{Z}} (-1)^i \partial_{\bar{G}}(\langle H^i(\theta_\gamma) \mid D_1^i\rangle). \tag{5.18}$$

The equality of Proposition 5.6 now follows upon substituting (5.17) into (5.18) and then using both the explicit formula for $r_G(C)(\rho)$ given in Lemma 5.5 (ii) and the following result (with $M = H^i(\mathrm{tw}_{\bar{G}}(C)_H)$ for each integer i).

Lemma 5.9. *If M is any finitely generated R-module, then for every ρ in $\mathrm{Irr}(\bar{G})$ one has*

$$(\mathrm{Nrd}_{Q(R)}(\delta_\gamma \mid \mathrm{I}_{\bar{G}}^{\bar{G}\times\Gamma}(M^\Gamma)_{S^*}))^*_\rho(0) = (-1)^{\dim_{\mathbb{Q}_p^c}(M^{\Gamma,\rho})}.$$

Proof. There are natural isomorphisms of $Q^c(\Gamma)$-modules of the form

$$\mathrm{Hom}_{\mathbb{Q}_p^c[\bar{G}]}(V_\rho, \mathrm{I}_{\bar{G}}^{\bar{G}\times\Gamma}(M^\Gamma)_{S^*}) \cong \mathrm{Hom}_{\mathbb{Q}_p^c[\bar{G}]}(V_\rho, Q^c(\Gamma)\otimes_{\mathbb{Z}_p} M^\Gamma) \cong M^{\Gamma,\rho}\otimes_{\mathbb{Q}_p^c} Q^c(\Gamma)$$

under which the induced action of δ_γ on the first module corresponds to the endomorphism $\tilde{\delta}_\gamma$ of the third module that sends $m\otimes x$ to $m\otimes(\gamma^{-1}-1)x$. It follows that the reduced norm $\mathrm{Nrd}_{Q(R)}(\delta_\gamma \mid \mathrm{I}_{\bar{G}}^{\bar{G}\times\Gamma}(M^\Gamma)_{S^*})_\rho$ is equal to

$$\begin{aligned}\det_{Q^c(\Gamma)}(\tilde{\delta}_\gamma \mid M^{\Gamma,\rho}\otimes_{\mathbb{Q}_p^c} Q^c(\Gamma)) &= \det_{Q^c(\Gamma)}(\gamma^{-1}-1 \mid Q^c(\Gamma))^{\dim_{\mathbb{Q}_p^c}(M^{\Gamma,\rho})} \\ &= (-T/(1+T))^{\dim_{\mathbb{Q}_p^c}(M^{\Gamma,\rho})},\end{aligned}$$

where the last equality follows from the fact that $\gamma^{-1}-1 = (1-\gamma)/\gamma = -T/(1+T)$. From this explicit formula it is clear that the first non-zero coefficient of T in the series $\mathrm{Nrd}_{Q(R)}(\delta_\gamma \mid \mathrm{I}_{\bar{G}}^{\bar{G}\times\Gamma}(M^\Gamma)_{S^*})_\rho$ is equal to $(-1)^{\dim_{\mathbb{Q}_p^c}(M^{\Gamma,\rho})}$. □

5.4. Completion of the proof of Theorem 2.2

We set $N := U\cap H$. Then in each degree i there is a natural isomorphism of $\mathbb{Z}_p[\bar{G}]$-modules

$$H^i(\mathrm{tw}_{\bar{G}}(C)_H) \cong \mathbb{Z}_p[\bar{G}]\otimes_{\Lambda(H/N)} H^i(\Lambda(G/N)\otimes^{\mathbb{L}}_{\Lambda(G)} C).$$

But $\Lambda(G/N)\otimes^{\mathbb{L}}_{\Lambda(G)} C$ belongs to $D^{\mathrm{p}}_{S_{G/N,H/N}}(\Lambda(G/N))$ and so each module $H^i(\mathrm{tw}_{\bar{G}}(C)_H)$ is finitely generated over $\mathbb{Z}_p[\bar{G}]$. This implies that $\Delta(\mathrm{tw}_{\bar{G}}(C),\gamma)$ is an exact triangle in $D^{\mathrm{p}}(\mathbb{Z}_p[\bar{G}])$. In view of Lemma 5.5 (iv) and Proposition 5.6 we may thus deduce Theorem 2.2 by applying the following result with $\mathcal{G} = \bar{G}$, $R = \mathbb{Z}_p$ and $\Delta = \Delta(\mathrm{tw}_{\bar{G}}(C),\gamma)$.

For each finite group $\mathcal{G}$ and $R[\mathcal{G}]$-module M we write M_F for the associated $F[\mathcal{G}]$-module $F\otimes_R M$. We also use similar notation for both homomorphisms and complexes of $R[\mathcal{G}]$-modules.

Proposition 5.10. *Let $\mathcal{G}$ be a finite group, R an integral domain and F the field of fractions of R. Let $\Delta : C \xrightarrow{\theta} C \to D \to C[1]$ be an exact triangle in $D^{\mathrm{p}}(R[\mathcal{G}])$. We assume that θ is semisimple and in each degree i fix an $F[\mathcal{G}][H^i(\theta)]$-equivariant direct complement W^i to $\ker(H^i(\theta))_F$ in $H^i(C)_F$. Then $H^i(\theta)$ induces an automorphism of the*

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(finitely generated projective) $F[\mathcal{G}]$-module W^i, the element $\langle H^i(\theta)\rangle^* := \langle H^i(\theta) \mid W^i\rangle$ of $K_1(F[\mathcal{G}])$ is independent of the choice of W^i and in $K_0(R[\mathcal{G}], F[\mathcal{G}])$ one has

$$\sum_{i\in\mathbb{Z}}(-1)^i\partial_{\mathcal{G}}(\langle H^i(\theta)\rangle^*) = -[\mathbf{d}_{R[\mathcal{G}]}(D), t(\Delta)], \tag{5.19}$$

where $\partial_{\mathcal{G}}$ is the connecting homomorphism $K_1(F[\mathcal{G}]) \to K_0(R[\mathcal{G}], F[\mathcal{G}])$ and the morphism $t(\Delta)$ is as defined in (B.1).

Proof. It is clear that $H^i(\theta)$ induces an automorphism of W^i and straightforward to verify that $\langle H^i(\theta)\rangle^*$ is independent of the choice of W^i. However to prove (5.19) we shall give a very explicit argument in order to avoid any ambiguities of sign.

To do this we replace C by a complex P in $C^{\mathrm{p}}(R[\mathcal{G}])$ for which there exists an isomorphism $q : P \to C$ in $D^{\mathrm{p}}(R[\mathcal{G}])$ and we shall argue by induction on $|P| := \max\{i : P^i \neq 0\} - \min\{j : P^j \neq 0\}$. We fix a morphism of complexes $\phi : P \to P$ such that $q \circ \phi = \theta \circ q$ in $D^{\mathrm{p}}(R[\mathcal{G}])$.

If $|P| = 0$, then $P = P^m[-m] = H^m(P)$ (and $\phi = \phi^m = H^m(\phi)$) for some integer m. In this case D can be identified with the mapping cone $P^m \xrightarrow{\phi^m} P^m$ of ϕ (so the first term of this complex is placed in degree $m-1$) in such a way that the homomorphism $H^{m-1}(D) \to \ker(H^m(\theta)) \to \operatorname{cok}(H^m(\theta)) \to H^m(D)$ induced by Δ corresponds to the tautological map $\tau : \ker(\phi^m) \to \operatorname{cok}(\phi^m)$. Further, if W is a direct complement to $\ker(\phi^m)_F$ in P^m_F, then $\phi^m(W) = W$ (since ϕ is semisimple) and $[\mathbf{d}_{R[\mathcal{G}]}(D), t(\Delta)] = (P^m, \iota^{(-1)^{m-1}}, P^m)$ with ι the composite isomorphism

$$P^m_F = \ker(\phi^m)_F \oplus W \xrightarrow{(F\otimes_R\tau, H^m(\phi))} \operatorname{cok}(\phi^m)_F \oplus W \cong P^m_F,$$

where the isomorphism is induced by a choice of splitting of the tautological exact sequence $0 \to W \to P^m_F \to \operatorname{cok}(\phi^m)_F \to 0$ (this explicit description of $[\mathbf{d}_{R[\mathcal{G}]}(D), t(\Delta)]$ follows, for example, from [**3**, Theorem 6.2]). The equality (5.19) is therefore valid because one has

$$-(P^m, \iota^{(-1)^{m-1}}, P^m) = (-1)^m\partial_{\mathcal{G}}(\langle \iota \mid P^m_F\rangle) = (-1)^m\partial_{\mathcal{G}}(\langle H^m(\phi) \mid W\rangle).$$

We now assume that $|P| = n > 0$ and, to fix notation, that $\min\{j : P^j \neq 0\} = 0$. We set $C_2 := P$ and $\phi_2 = \phi$. We also write C_3 for the naive truncation in degree $n-1$ of P and ϕ_3 for the morphism $C_3 \to C_3$ obtained by restricting ϕ and set $C_1 := P^n[-n]$ and write ϕ_1 for the morphism $C_1 \to C_1$ given by ϕ^n. Then one has a tautological short exact sequence of complexes in $C^{\mathrm{p}}(R[\mathcal{G}])$

$$C_1 \to C_2 \to C_3. \tag{5.20}$$

From the long exact sequence of cohomology of this short exact sequence we deduce that $H^i(C_2) = H^i(C_3)$ if $i < n-1$ and that there are commutative diagrams of short exact

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sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^{n-1}(C_2) & \longrightarrow & H^{n-1}(C_3) & \longrightarrow & B^n(C_2) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle H^{n-1}(\phi_2)} & & \Big\downarrow{\scriptstyle H^{n-1}(\phi_3)} & & \Big\downarrow{\scriptstyle \phi^n} & & \\ 0 & \longrightarrow & H^{n-1}(C_2) & \longrightarrow & H^{n-1}(C_3) & \longrightarrow & B^n(C_2) & \longrightarrow & 0 \end{array} \tag{5.21}$$

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B^n(C_2) & \longrightarrow & H^n(C_1) & \longrightarrow & H^n(C_2) & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \phi^n} & & \Big\downarrow{\scriptstyle H^n(\phi_1)} & & \Big\downarrow{\scriptstyle H^n(\phi_2)} & & \\ 0 & \longrightarrow & B^n(C_2) & \longrightarrow & H^n(C_1) & \longrightarrow & H^n(C_2) & \longrightarrow & 0 \end{array} \tag{5.22}$$

where $B^n(C_2)$ denotes the coboundaries of C_2 in degree n. Further, by mimicking the argument of [**6**, Lemma 4.4] we may change ϕ by a homotopy in order to assume that, in each degree i, the restriction of ϕ^i to $B^i(C_2)$ induces an automorphism of $B^i(C_2)_F$. This assumption has two important consequences. Firstly, the above diagrams then imply that the morphisms ϕ_1 and ϕ_3 of C_1 and C_3 that are induced by ϕ are both semisimple. Secondly, if we write D_i for the mapping cone of ϕ_i for $i = 1, 2, 3$, then (5.20) induces a short exact sequence of complexes in $C^{\mathrm{p}}(R[\mathcal{G}])$ of the form

$$D_1 \to D_2 \to D_3 \tag{5.23}$$

and there are also commutative diagrams of short exact sequences of complexes in $C^{\mathrm{p}}(F[\mathcal{G}])$ of the form

$$\left.\begin{array}{c} \begin{array}{ccccc} \mathrm{Z}(D_{1,F}) & \longrightarrow & \mathrm{Z}(D_{2,F}) & \longrightarrow & \mathrm{Z}(D_{3,F}) \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ D_{1,F} & \longrightarrow & D_{2,F} & \longrightarrow & D_{3,F} \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ \mathrm{B}(D_{1,F})[1] & \longrightarrow & \mathrm{B}(D_{2,F})[1] & \longrightarrow & \mathrm{B}(D_{3,F})[1] \end{array} \\ \\ \begin{array}{ccccc} \mathrm{B}(D_{1,F}) & \longrightarrow & \mathrm{B}(D_{2,F}) & \longrightarrow & \mathrm{B}(D_{3,F}) \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ \mathrm{Z}(D_{1,F}) & \longrightarrow & \mathrm{Z}(D_{2,F}) & \longrightarrow & \mathrm{Z}(D_{3,F}) \\ \Big\downarrow & & \Big\downarrow & & \Big\downarrow \\ \mathrm{H}(D_{1,F}) & \longrightarrow & \mathrm{H}(D_{2,F}) & \longrightarrow & \mathrm{H}(D_{3,F}) \end{array} \end{array}\right\} \tag{5.24}$$

We refer to the left- and right-hand diagrams here as $(5.24)_{\mathrm{l}}$ and $(5.24)_{\mathrm{r}}$. In these diagrams we write $\mathrm{B}(D_{i,F})$ and $\mathrm{Z}(D_{i,F})$ for the complexes of coboundaries and cocycles of $D_{i,F}$

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(each with zero differentials), all horizontal morphisms are induced by those in (5.23) and all vertical morphisms are tautological. It is thus clear that all columns in (5.24) are short exact sequences and that (5.23) implies the central row of $(5.24)_{\mathrm{l}}$ is a short exact sequence. The fact that the lower row of $(5.24)_{\mathrm{r}}$ is a short exact sequence is proved as follows: the exact sequences obtained by applying the snake lemma to (5.21) and (5.22) combine with the equalities $H^i(C_2) = H^i(C_3)$ for $i < n-1$ and our assumption that (the restriction of) ϕ^n induces an automorphism of $B^n(C_2)_F$ to imply that the exact sequence of cohomology of (5.23) induces an isomorphism $H^i(D_2)_F \cong H^i(D_3)_F$ for each $i < n-1$, a short exact sequence $0 \to H^{n-1}(D_1)_F \to H^{n-1}(D_2)_F \to H^{n-1}(D_3)_F \to 0$ and an isomorphism $H^n(D_1)_F \cong H^n(D_2)_F$; to deduce that the lower row of $(5.24)_{\mathrm{r}}$ is a short exact sequence one then need only note that $H^i(D_1)_F$ vanishes for each $i < n-1$ and that $H^n(D_3)$ vanishes. Given that the central row of $(5.24)_{\mathrm{l}}$ and lower row of $(5.24)_{\mathrm{r}}$ are now known to be short exact sequences it is a straightforward exercise to show that all of the remaining rows in (5.24) are also short exact sequences.

We next consider the following diagram in $V(F[\mathcal{G}])$:

$$\begin{array}{ccccccc}
\mathbf{d}(D_{2,F}) & \xrightarrow{\alpha_1} & \mathbf{d}(\mathrm{H}(D_{2,F})) & = & \mathbf{d}(\mathrm{H}_\beta(D_{2,F})) & \xrightarrow{\alpha_2} & \mathbf{1} \\
\downarrow{\scriptstyle\alpha_3} & & \downarrow{\scriptstyle\alpha_4} & & \downarrow{\scriptstyle\alpha_5} & & \| \\
\mathbf{d}(D_{1,F})\mathbf{d}(D_{3,F}) & \xrightarrow{\alpha_6} & \mathbf{d}(\mathrm{H}(D_{1,F}))\mathbf{d}(\mathrm{H}(D_{3,F})) & = & \mathbf{d}(\mathrm{H}_\beta(D_{1,F}))\mathbf{d}(\mathrm{H}_\beta(D_{3,F})) & \xrightarrow{\alpha_7} & \mathbf{1}\cdot\mathbf{1}
\end{array}$$

Here we have abbreviated $\mathbf{d}_{F[\mathcal{G}]}(-)$ and $\mathbf{1}_{V(F[\mathcal{G}])}$ to $\mathbf{d}(-)$ and $\mathbf{1}$ and used the following morphisms: α_1 and α_6 are induced by property A(h) in Appendix A; α_2 and α_7 are induced by property A(e) and the acyclicity of $\mathrm{H}_\beta(D_{i,F})$; α_3 and α_4 are induced by applying property A(d) to the central row of $(5.24)_{\mathrm{l}}$ and to the bottom row of $(5.24)_{\mathrm{r}}$; if we write Δ_i for the tautological exact triangle

$$C_i \xrightarrow{\phi_i} C_i \to D_i \to C_i[1],$$

then α_5 is induced by applying property A(d) to the short exact sequence $\mathrm{H}_\beta(\Delta_{1,F}) \to \mathrm{H}_\beta(\Delta_{2,F}) \to \mathrm{H}_\beta(\Delta_{3,F})$ which is induced by the bottom row of $(5.24)_{\mathrm{r}}$. Now by applying property A(d) to the diagrams (5.24) one finds that the first square in the above diagram commutes and it is clear that the other squares in this diagram also commute. But the upper and lower rows of the diagram are equal to the morphisms $t(\Delta_2)$ and $t(\Delta_1)t(\Delta_3)$ respectively. Hence, by combining the commutativity of this diagram with the equality $\mathbf{d}_{R[\mathcal{G}]}(D_2) = \mathbf{d}_{R[\mathcal{G}]}(D_1)\mathbf{d}_{R[\mathcal{G}]}(D_3)$ induced by applying property A(d) to (5.23), and recalling the group structure of $V(R[\mathcal{G}], F[\mathcal{G}])$, one deduces that in $\pi_0(V(R[\mathcal{G}], F[\mathcal{G}])) \cong K_0(R[\mathcal{G}], F[\mathcal{G}])$ there is an equality

$$[\mathbf{d}_{R[\mathcal{G}]}(D_2), t(\Delta_2)] = [\mathbf{d}_{R[\mathcal{G}]}(D_1), t(\Delta_1)] + [\mathbf{d}_{R[\mathcal{G}]}(D_3), t(\Delta_3)].$$

But the inductive hypothesis implies

$$-[\mathbf{d}_{R[\mathcal{G}]}(D_3), t(\Delta_3)] = \sum_{i=0}^{i=n-1} (-1)^i \partial_{\mathcal{G}}(\langle H^i(\phi_3)\rangle^*)$$

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and, since $|C_1| = 0$, our earlier argument proves

$$-[\mathbf{d}_{R[\mathcal{G}]}(D_1), t(\Delta_1)] = (-1)^n \partial_{\mathcal{G}}(\langle H^n(\phi_1)\rangle^*).$$

One also has $\langle H^i(\phi_3)\rangle^* = \langle H^i(\phi_2)\rangle^*$ for $i < n-1$, while (5.21) and (5.22) imply

$$\langle H^{n-1}(\phi_3)\rangle^* = \langle H^{n-1}(\phi_2)\rangle^* \langle \phi^n \mid B^n(C_2)_F\rangle,$$
$$\langle H^n(\phi_1)\rangle^* = \langle \phi^n \mid B^n(C_2)_F\rangle \langle H^n(\phi_2)\rangle^*.$$

The claimed description of $-[\mathbf{d}_{R[\mathcal{G}]}(D), t(\Delta)] = -[\mathbf{d}_{R[\mathcal{G}]}(D_2), t(\Delta_2)]$ thus follows upon combining the last five displayed equations. □

Remark 5.11. If $\mathcal{G}$ is abelian, then Proposition 5.10 can be reinterpreted in terms of graded determinants and in this case has been proved to within a 'sign ambiguity' by Kato in [**20**, Lemma 3.5.8]. (This ambiguity arises because Kato uses ungraded determinants; for more details in this regard see [**20**, Remark 3.2.3(3) and 3.2.6(3),(5)] and [**10**, Remark 9].)

Part II. Arithmetic

For any Galois extension of fields F/E we set $G_{F/E} := \mathrm{Gal}(F/E)$. For any field E we also fix an algebraic closure E^c and abbreviate $G_{E^c/E}$ to G_E.

6. Field-theoretic preliminaries

We first introduce the class of fields for which the techniques of [**15**] allow one to formulate a main conjecture of non-commutative Iwasawa theory.

We fix an odd prime p and for each number field k we write $\mathcal{F}_k$ for the set of Galois extensions L of k inside $\mathbb{Q}^c$ which satisfy the following conditions:

(i) L contains the cyclotomic $\mathbb{Z}_p$-extension k^{cyc} of k;

(ii) L/k is unramified outside a finite set of places;

(iii) $G_{L/k}$ is a compact p-adic Lie group.

If k is totally real, then we also let $\mathcal{F}_k^+$ denote the subset of $\mathcal{F}_k$ comprising those fields that are totally real.

The following result was explained to us by Kazuya Kato. It provides an important general reduction step and also shows that Theorem 2.2 constitutes a satisfactory resolution of the descent problem in the setting of non-commutative Iwasawa theory.

Lemma 6.1. *For any number field k and any F in $\mathcal{F}_k$ there exists a field F' in $\mathcal{F}_k$ with $F \subseteq F'$ and such that $G_{F'/k}$ has no element of order p. If k is totally real and F belongs to $\mathcal{F}_k^+$, then one can also choose F' in $\mathcal{F}_k^+$.*

Proof. For any extension E of k we write $E(\zeta_{p^\infty})$ for the extension of E generated by all p-power roots of unity (in $\mathbb{Q}^c$).

We set $\tilde{F} := F(\zeta_{p^\infty})$ and choose a p-torsion free open normal subgroup U of $V := G_{\tilde{F}/k(\zeta_{p^\infty})}$. We let L be the extension of k in $\tilde{F}$ that corresponds to U and for each non-trivial p-torsion element σ_i of V/U we write L_i for the fixed subfield of L by σ_i. Then $L = L_i(a_i^{1/p^n})$ for some $a_i \in L_i^\times$. Let a_{ij}, $1 \leqslant j \leqslant s(i)$, be all conjugates of a_i over F and let L_i' denote the field generated over L_i by the set $\{a_{ij}^{1/p^n} : 1 \leqslant j \leqslant s(i),\ n \geqslant 1\}$. Then $\tilde{F}L_i'$ is a Galois extension of F that contains L. Furthermore, G_{L_i'/L_i} is isomorphic to a subgroup of $\mathbb{Z}_p^{s(i)}$ by $\tau \mapsto (r(j))_j$ with $\tau(a_j^{1/p^n})/a_j^{1/p^n} = \zeta_{p^n}^{r(j)}$. Let F' be the composite field of $\tilde{F}$ and L_i' for all i.

The group $G_{F'/k}$ is a compact p-adic Lie group and we now prove that it has no element of order p. We note first that $G_{k(\zeta_{p^\infty})/k}$ is isomorphic to a subgroup of $\mathbb{Z}_p^\times$ and hence is p-torsion free by the assumption $p \neq 2$. Thus if $\sigma \in G_{F'/k}$ has order p, then the image of σ in $G_{\tilde{F}/k}$ is contained in V and so the image of σ in V/U coincides with σ_i for some i. Thus σ fixes all elements of L_i. But then the image of σ in G_{L_i'/L_i} is both p-torsion and also non-trivial (for its restriction to G_{L/L_i} is non-trivial). This contradicts the fact that G_{L_i'/L_i} has no element of order p. Hence $G_{F'/k}$ has no element of order p, as claimed.

Lastly we assume that F (and hence k) is totally real. Then $\tilde{F}$ is a CM field with maximal real subfield $\tilde{F}^+$ equal to the compositum of F and the maximal totally real subfield of $k(\zeta_{p^\infty})$. Also, by the above construction, the extension $F'/\tilde{F}$ is pro-p. Since p is odd, the group $G_{F'/\tilde{F}^+}$ therefore contains a unique element of order 2 and the fixed field $(F')^+$ of F' by this element is totally real, contains F, is Galois over k and such that $G_{(F')^+/k}$ has no element of order p. □

In the remainder of this article we set $\Gamma_k := G_{k^{\mathrm{cyc}}/k}$, $H_{L/k} := G_{L/k^{\mathrm{cyc}}}$, $\Lambda(L/k) := \Lambda(G_{L/k})$ and $\Omega(L/k) := \Omega(G_{L/k})$ for each L in $\mathcal{F}_k$. We also fix a topological generator $\gamma_{\mathbb{Q}}$ of $\Gamma_{\mathbb{Q}}$, set $d_k := [k \cap \mathbb{Q}^{\mathrm{cyc}} : \mathbb{Q}]$ and write γ_k for the topological generator $\gamma_{\mathbb{Q}}^{d_k}$ of Γ_k.

Remark 6.2. If $\mathcal{C}$ denotes either $\mathcal{F}_k$ or $\mathcal{F}_k^+$, then it is an ordered set (by inclusion). Lemma 6.1 implies that the subset $\mathcal{C}'$ of $\mathcal{C}$ comprising those fields F for which $G_{F/k}$ has no element of order p is cofinal. Taking account of the functorial properties of the isomorphism in Theorem 2.1 and of the results in Proposition 4.7 (ii) we may therefore deduce the following extensions of these results.

- There is a natural isomorphism of abelian groups

$$\varprojlim_{F\in\mathcal{C}} K_1(\Lambda(F/k)_{S^*}) \cong \varprojlim_{F\in\mathcal{C}} K_0(\Omega(F/k)) \oplus \varprojlim_{F\in\mathcal{C}} K_0(\Lambda(F/k), \Lambda(F/k)_S) \oplus \varprojlim_{F\in\mathcal{C}} \operatorname{im}(\lambda_{F/k}),$$

  where $\lambda_{F/k}$ is the natural homomorphism $K_1(\Lambda(F/k)) \to K_1(\Lambda(F/k)_{S^*})$ and in each inverse limit the transition maps are induced by the homomorphism $\Lambda(F/k) \to \Lambda(F'/k)$ for each $F' \subseteq F$.

- Let $(x_F)_F$ be an element of $\varprojlim_{F\in\mathcal{C}} K_0(\Lambda(F/k), \Lambda(F/k)_{S^*})$. Then for each F in $\mathcal{C}$ we may define an element $\mathrm{char}_{G_{F/k},\gamma_k}(x_F)$ of $K_1(\Lambda(F/k)_{S^*})$ in the following way:

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we choose F' in $\mathcal{C}'$ with $F \subseteq F'$ and let $\mathrm{char}_{G_{F/k},\gamma_k}(x_F)$ denote the image of $\mathrm{char}_{G_{F'/k},\gamma_k}(x_{F'})$ under the natural projection $K_1(\Lambda(F'/k)_{S^*}) \to K_1(\Lambda(F/k)_{S^*})$. Then Lemma 4.5 implies $\mathrm{char}_{G_{F/k},\gamma_k}(x_F)$ is independent of the precise choice of F' and Proposition 4.7 implies $\partial_{G_{F/k}}(\mathrm{char}_{G_{F/k},\gamma_k}(x_F)) = x_F$.

7. Non-commutative main conjectures

In this section we formulate explicit 'main conjectures of non-commutative Iwasawa theory' for certain Tate motives and critical motives. In particular, in the setting of elliptic curves, the conjecture we formulate is finer than that formulated by Coates *et al*. in [**15**] in that we consider interpolation formulae for the leading terms (rather than values) of p-adic L-functions at Artin representations.

Henceforth we will fix an isomorphism of fields $j : \mathbb{C} \cong \mathbb{C}_p$ and often simply omit it from the notation.

7.1. Tate motives

In this subsection we fix a totally real number field k and formulate a main conjecture for class groups associated to fields in $\mathcal{F}_k^+$. We therefore fix a field K in $\mathcal{F}_k^+$ and a finite set of places Σ of k that contains the archimedean places and all places that ramify in K/k. For each Artin representation ρ of $G_{K/k}$ we write $\rho^{j^{-1}}$ for the complex representation of $G_{K/k}$ induced by j^{-1} and $L_\Sigma(s, \rho^{j^{-1}})$ for the Artin L-function of $\rho^{j^{-1}}$ that is truncated by removing the Euler factors attached to places in Σ.

To formulate a main conjecture we must multiply the leading term $L_\Sigma^*(1, \rho^{j^{-1}})$ in the Taylor expansion of $L_\Sigma(s, \rho^{j^{-1}})$ at $s = 1$ by an appropriate period. To define this period we let E be any finite degree Galois extension of k with $E \subset K$ and $G_{K/E} \subseteq \ker(\rho)$. We set $E_\infty := \mathbb{R} \otimes_\mathbb{Q} E \cong \prod_{\mathrm{Hom}(E,\mathbb{C})} \mathbb{R}$ and write $\log_\infty(\mathcal{O}_E^\times)$ for the inverse image of $\mathcal{O}_E^\times \hookrightarrow E_\infty^\times$ under the (componentwise) exponential map $\exp_\infty : E_\infty \to E_\infty^\times$. Then the Dirichlet Unit Theorem implies that $\log_\infty(\mathcal{O}_E^\times)$ is a full lattice in the $\mathbb{R}$-space generated by $E_0 := \{x \in E : \mathrm{Tr}_{E/\mathbb{Q}}(x) = 0\}$ and so there is a canonical isomorphism of $\mathbb{C}[G_{E/k}]$-modules $\mu_\infty : \mathbb{C} \otimes_\mathbb{Z} \log_\infty(\mathcal{O}_E^\times) \cong \mathbb{C} \otimes_\mathbb{Q} E_0$. In addition, if we write $S_p(E)$ for the set of p-adic places of E, then the composite homomorphism

$$\mathbb{Z}_p \otimes_\mathbb{Z} \log_\infty(\mathcal{O}_E^\times) \xrightarrow{\exp_\infty} \mathbb{Z}_p \otimes_\mathbb{Z} \mathcal{O}_E^\times \to \prod_{w \in S_p(E)} U_{E_w}^1 \xrightarrow{(u_w)_w \mapsto (\log_p(u_w))_w} \prod_{w \in S_p(E)} E_w \cong \mathbb{Q}_p \otimes_\mathbb{Q} E$$

(where the second arrow is the natural diagonal map) factors through the inclusion $\mathbb{Q}_p \otimes_\mathbb{Q} E_0 \subset \mathbb{Q}_p \otimes_\mathbb{Q} E$ and hence induces a homomorphism $\mu_p : \mathbb{C}_p \otimes_\mathbb{Z} \log_\infty(\mathcal{O}_E^\times) \cong \mathbb{C}_p \otimes_\mathbb{Q} E_0$ of $\mathbb{C}_p[G_{E/k}]$-modules. The resulting period

$$\Omega_j(\rho) := \det_{\mathbb{C}_p}(\mu_p \circ (\mathbb{C}_p \otimes_{\mathbb{C},j} \mu_\infty)^{-1})^\rho \in \mathbb{C}_p$$

depends upon j and ρ but is independent of the choice of E.

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We write χ_{cyc} for the cyclotomic character $G_k \to \Gamma_k \to \mathbb{Z}_p^\times$ and for any Artin representation ρ of $G_{K/k}$ we write $\langle \rho, 1\rangle$ for the multiplicity with which the trivial representation of $G_{K/k}$ occurs in ρ.

Conjecture 7.1. *Assume that $G_{K/k}$ has no element of order p. Then the Galois group $X_\Sigma(K)$ of the maximal pro-p abelian extension of K that is unramified outside Σ belongs to $\mathfrak{M}_{S^*}(G_{K/k})$ and there exists an element ξ of $K_1(\Lambda(K/k)_{S^*})$ which satisfies both of the following conditions.*

(a) *At each Artin representation ρ of $G_{K/k}$ the value of ξ at ρ is equal to*

$$\xi(\rho) = (\log_p(\chi_{\mathrm{cyc}}(\gamma_k))^{\langle \rho,1\rangle} c_{\rho,k})^{-1} \Omega_j(\rho) L^*_\Sigma(1, \rho^{j^{-1}})^j$$

with $c_{\rho,k} := 1$ if either ρ is trivial or $H_{K/k} \not\subset \ker(\rho)$ and $c_{\rho,k} := 1 - \rho(\gamma_k^{-1})$ otherwise.

(b) $\partial_{G_{K/k}}(\xi) = [X_\Sigma(K)]$.

Remark 7.2. For each Artin representation ρ of $G_{K/k}$ the '(Σ-truncated) p-adic Artin L-function' of ρ is the unique p-adic meromorphic function $L_{p,\Sigma}(\cdot, \rho) : \mathbb{Z}_p \to \mathbb{C}_p$ with the property that for each strictly negative integer n and each isomorphism $j : \mathbb{C} \cong \mathbb{C}_p$ one has $L_{p,\Sigma}(n, \rho) = L_\Sigma(n, (\rho \otimes \omega^{n-1})^{j^{-1}})^j$ where $\omega : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ is the Teichmüller character. Then the 'p-adic Stark conjecture at $s = 1$', as formulated by Serre in [**29**] and discussed by Tate in [**32**, Chapter VI, § 5], asserts that the term $\Omega_j(\rho) L^*_\Sigma(1, \rho^{j^{-1}})^j$ in Conjecture 7.1 (a) is equal to the leading term of $L_{p,\Sigma}(s, \rho)$ at $s = 1$. See [**13**, Remark 5.3] for more details.

Remark 7.3. There are several different motivations lying behind our precise formulation of Conjecture 7.1. It is firstly a natural analogue for the ideal class groups of totally real fields of the main conjecture for elliptic curves without complex multiplication that is formulated by Coates *et al*. in [**15**]. Following Remark 7.2 it is also, modulo the validity of Serre's p-adic Stark conjecture at $s = 1$, a natural non-commutative generalization of the classical main conjecture of commutative Iwasawa theory. Indeed, in Conjecture 8.3 we will formulate a generalization of Conjecture 7.1 that allows $G_{K/k}$ to have an element of order p and it can be shown that if $G_{K/k}$ has rank one and Serre's conjecture is valid, then Conjecture 8.3 is compatible with the central conjecture formulated by Ritter and Weiss in [**27**, § 4] and in the case that $G_{K/k}$ is also abelian with the equivariant main conjecture studied by Greither and the first named author in [**12**]. In particular, if $G_{K/k}$ is abelian of rank one, Serre's p-adic Stark conjecture at $s = 1$ is valid and certain μ-invariants vanish, then the validity of Conjecture 8.3 can be deduced from Wiles's proof of the main conjecture for totally real fields via the approach of either [**26**] or [**12**]. As a further motivation for Conjecture 7.1 we will later prove (in Theorem 8.1) that it also combines with Theorem 2.2 to imply the relevant case of the equivariant Tamagawa number conjecture with respect to any finite degree Galois extension F/k such that $F \subset K$ and Leopoldt's Conjecture is valid for F at p.

7.2. Critical motives

7.2.1. *Preliminaries*

For further background on this (standard) material we refer the reader to either [**18**, §§ 2.2, 2.4], [**10**, § 3] or [**35**, § 2].

We fix a finite extension F of $\mathbb{Q}$ and a motive M that is defined over $\mathbb{Q}$ and has coefficients F. As usual we write M_{B}, M_{dR}, M_ℓ and M_λ for the Betti, de Rham, ℓ-adic and λ-adic realizations of M, where ℓ ranges over rational primes and λ over non-archimedean places of F. We also let t_M denote the tangent space $M_{\mathrm{dR}}/M^0_{\mathrm{dR}}$ of M. For any ring R and $R[\mathrm{Gal}(\mathbb{C}/\mathbb{R})]$-module X we denote by X^+ and X^- the R-submodule of X upon which complex conjugation acts as multiplication by $+1$ and -1 respectively.

In our later calculations we will use each of the following isomorphisms.

- The comparison isomorphisms between the Betti and λ-adic realizations of M induce canonical isomorphisms of F_λ-modules, respectively F_ℓ-modules, of the form
$$g^+_\lambda : F_\lambda \otimes_F M^+_{\mathrm{B}} \cong M^+_\lambda, \quad \text{respectively } g^+_\ell : F_\ell \otimes_F M^+_{\mathrm{B}} \cong M^+_\ell. \tag{7.1}$$
- We set $F_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Q}} F$. Then the comparison isomorphism between the de Rham and Betti realizations of M induces a canonical $F_{\mathbb{R}}$-equivariant period map
$$\mathbb{R} \otimes_{\mathbb{Q}} M^+_{\mathrm{B}} \xrightarrow{\alpha_M} \mathbb{R} \otimes_{\mathbb{Q}} t_M. \tag{7.2}$$
- For each p-adic place λ of F, the comparison isomorphism between the p-adic and de Rham realizations of M induces a canonical isomorphism of F_λ-modules of the form
$$t_p(M_\lambda) = D_{\mathrm{dR}}(M_\lambda)/D^0_{\mathrm{dR}}(M_\lambda) \xrightarrow{g^t_{\mathrm{dR}}} F_\lambda \otimes_F t_M. \tag{7.3}$$

We further recall that the 'motivic cohomology groups' $H^0_f(M) := H^0(M)$ and $H^1_f(M)$ of M are F-modules that can be defined either in terms of algebraic K-theory or motivic cohomology in the sense of Voevodsky (cf. [**10**]). They are both conjectured to be finite dimensional.

Now let M be a critical motive over $\mathbb{Q}$ that has good ordinary reduction at p. Then its p-adic realization $V = M_p$ has a unique $\mathbb{Q}_p$-subspace $\hat{V}$ that is stable under the action of $G_{\mathbb{Q}_p}$ and such that $D^0_{\mathrm{dR}}(\hat{V}) = t_p(V) := D_{\mathrm{dR}}(V)/D^0_{\mathrm{dR}}(V)$. Now let ρ be an Artin representation defined over a number field B and $[\rho]$ the corresponding Artin motive. We fix a p-adic place λ of B, set $L := B_\lambda$ and write $\mathcal{O}$ for the valuation ring of L. Then the λ-adic realization
$$W := W_\rho := N_\lambda = V \otimes_{\mathbb{Q}_p} [\rho]^*_\lambda \tag{7.4}$$
of the motive $N := M(\rho^*) := M \otimes [\rho]^*$ is an L-adic representation and contains the $G_{\mathbb{Q}_p}$-subrepresentation $\hat{W} = \hat{V} \otimes_{\mathbb{Q}_p} [\rho]^*_\lambda$. The *algebraic rank* of $M(\rho^*)$ is defined as
$$r(M)(\rho) := r(N) := \dim_L(H^1_f(\mathbb{Q}, (W_\rho)^*(1))) - \dim_L(H^0_f(\mathbb{Q}, (W_\rho)^*(1))). \tag{7.5}$$

Let Σ be a finite set of places of $\mathbb{Q}$ containing p, ∞ and all places at which M has bad reduction. We fix a field K in $\mathcal{F}_{\mathbb{Q}}$ that is unramified outside Σ. By Υ we denote the set of those primes $\ell \neq p$ such that the ramification index of ℓ in $K/\mathbb{Q}$ is infinite.

For a B-motive N over $\mathbb{Q}$ we denote by $\Omega_\infty(N)$ and $\Omega_p(N)$ the associated complex and p-adic periods and by $R_p(N)$ and $R_\infty(N)$ the associated complex and p-adic regulators, see again [**18**] or [**13**, Theorem 6.5] for more details.

Let $\gamma' = (\gamma'_i)_i$ and $\delta' = (\delta_i)_i$ denote a choice of 'good bases' (in the sense of [**18**, 4.2.24(3)]) of N_B^+ and t_N respectively. Furthermore, we let $P^\vee = (P_1^\vee, \dots, P_{d(N)}^\vee)$ and $P = (P_1, \dots, P_{d(N)})$ be B-bases of $H^1_f(N)$ and $H^1_f(N^*(1))$ respectively. We fix an embedding of B into $\mathbb{C}$. We let $\Omega_\infty(N)$ denote the determinant of the canonical isomorphism

$$\alpha_N : (N_B^+)_{\mathbb{C}} \to (t_N)_{\mathbb{C}} \tag{7.6}$$

with respect to the bases γ' and δ', and $R_\infty(N)$ the determinant of the inverse of the canonical (height-pairing) isomorphism

$$h_\infty(N) : (H^1_f(N^*(1))^*)_{\mathbb{C}} \to H^1_f(N)_{\mathbb{C}} \tag{7.7}$$

with respect to the dual basis $P^d := (P_1^d, \dots, P_{d(N)}^d)$ of P and the basis $P^\vee$ respectively. Similarly, $R_p(N) = \det_L(\mathrm{ad}(h_p(W)))$ is induced from Nekovář's height pairing

$$\mathrm{ad}(h_p(W)) : H^1_f(\mathbb{Q}, W) \to H^1_f(\mathbb{Q}, W^*(1))^*, \tag{7.8}$$

while the definition of $\Omega_p(N)$ is more complicated, because ϵ-constants are involved, see [**18**].

We recall that $\Omega_\infty(N) \neq 0$ if N is critical and that $R_\infty(N) \neq 0$ if the (complex) height pairing of N is non-degenerate. Furthermore, for a $\mathbb{Q}_p$-linear continuous $G_{\mathbb{Q}_p}$-representation Z we write $\Gamma(Z)$ for its Γ-factor [**18**]. Finally, for any L-linear continuous representation V and prime number ℓ we define an element of the polynomial ring $L[u]$ by setting

$$P_\ell(V,u) := P_{L,\ell}(V,u) := \begin{cases} \det_L(1 - \varphi_\ell u \mid V^{I_\ell}), & \text{if } \ell \neq p, \\ \det_L(1 - \varphi_p u \mid D_{\mathrm{cris}}(V)), & \text{if } \ell = p, \end{cases}$$

where φ_ℓ denotes the geometric Frobenius automorphism of ℓ.

As shown by Fukaya and Kato in [**18**, Theorem 4.2.26], the behaviour of local ϵ-factors implies that p-adic L-functions can exist only after a suitable extension of scalars. To describe this we must assume the following hypothesis:

$$\text{the maximal absolutely abelian subfield } K^{ab,p} \text{ of } K \text{ in which } p \text{ is unramified is finite.} \tag{7.9}$$

Under this hypothesis we let A denote the valuation ring of the completion at any p-adic place of the field $K^{ab,p}$. We set $\Lambda_A(K/\mathbb{Q}) := A \otimes_{\mathbb{Z}_p} \Lambda(K/\mathbb{Q})$ and $\Lambda_A(K/\mathbb{Q})_{S^*} := A \otimes_{\mathbb{Z}_p} \Lambda(K/\mathbb{Q})_{S^*}$ and write $\partial_{A,G_{K/\mathbb{Q}}} : K_1(\Lambda_A(K/\mathbb{Q})_{S^*}) \to K_0(\Lambda_A(K/\mathbb{Q}), \Lambda_A(K/\mathbb{Q})_{S^*})$ for the corresponding connecting homomorphism.

7.2.2. *Elliptic curves*

We first consider the case of the motive $M = h^1(E)(1)$ of an elliptic curve E over $\mathbb{Q}$ which has good ordinary reduction at p. We take K to be the extension $\mathbb{Q}(E(p))$ of $\mathbb{Q}$ which arises by adjoining the p-power division points of E and note that K belongs to $\mathcal{F}_{\mathbb{Q}}$. We assume that $G_{K/\mathbb{Q}}$ has no element of order p since in this case one can formulate a very explicit (refined) main conjecture by using the Pontryagin dual $X(E_{/K})$ of the (p-primary) Selmer group; later we will give another formulation for general critical motives involving Selmer complexes. In the present situation one knows that the condition (7.9) is satisfied (cf. [**15**, just before Conjecture 5.7]) and also that, if $\text{Ш}(E_{/K^{\ker(\rho)}})$ is finite, then one has

$$r(M)(\rho^*) = \dim_{\mathbb{C}_p}(e_{\rho^*}(\mathbb{C}_p \otimes_{\mathbb{Z}} E(K^{\ker(\rho)}))). \tag{7.10}$$

We now formulate the refined main conjecture in this setting.

Conjecture 7.4. *Set $K = \mathbb{Q}(E(p))$ and let Σ denote the finite set of places of $\mathbb{Q}$ which ramify in $K/\mathbb{Q}$. Assume that $G_{K/\mathbb{Q}}$ has no element of order p. Then the module $X(E_{/K})$ belongs to $\mathfrak{M}_{S^*}(G_{K/\mathbb{Q}})$. Further, there exists an element $\mathcal{L} = \mathcal{L}(E)$ of $K_1(\Lambda_A(K/\mathbb{Q})_{S^*})$ which satisfies both of the following conditions.*

(a) *At each Artin representation ρ of $G_{K/\mathbb{Q}}$, the value at $T = 0$ of $T^{-r(M)(\rho^*)}\Phi_\rho(\mathcal{L})$ is equal to*

$$(-1)^{r(M)(\rho^*)}\frac{L^*_{B,\Upsilon}(M(\rho^*))}{\Omega_\infty(M(\rho^*))R_\infty(M(\rho^*))} \cdot \Omega_p(M(\rho^*))R_p(M(\rho^*)) \cdot \frac{P_{L,p}(\hat{W}^*_\rho(1), 1)}{P_{L,p}(\hat{W}_\rho, 1)},$$

where $L^*_{B,\Upsilon}(M(\rho^*))$ is the leading coefficient at $s = 0$ of the complex L-function of $M(\rho^*)$, truncated by removing Euler factors for all primes in Υ.

(b) $\partial_{A,G_{K/\mathbb{Q}}}(\mathcal{L}) = [\Lambda_A(K/\mathbb{Q}) \otimes_{\Lambda(K/\mathbb{Q})} X(E_{/K})]$.

Remark 7.5 (independence of the choice of γ). In formulating Conjecture 7.4 we have implicitly fixed the choice of a topological generator γ of $\Gamma_{\mathbb{Q}}$. However the validity of the conjecture is independent of this choice. Indeed, both sides of the equality in (a) change in the same way if one replaces γ by $\gamma' = \gamma^u$ for any $u \in \mathbb{Z}_p^*$. More precisely, if $R_{p,\gamma} = R_p(N)$ denotes the p-adic regulator of N with respect to γ, then $R_{p,\gamma} = u^r R_{p,\gamma'}$ with $r = r(N)$. This is due to the fact that Nekovář's height pairing $\tilde{h}_{\pi,1,1}$ in [**25**, § 11.1.4] takes values in $L \otimes_{\mathbb{Z}_p} \Gamma_{\mathbb{Q}}$ which has been identified with the L-valued height pairing $h_p(W)$ in [**13**, § 6.3] via the identification $\Gamma_{\mathbb{Q}} \cong \mathbb{Z}_p$ which sends γ^c to c. Thus the claim follows from the definition $R_{p,\gamma} = \det_L(\mathrm{ad}(h_p))$. Obviously, the leading term of $\Phi_\rho(\mathcal{L})$ changes in the same way if $T = \gamma - 1$ is replaced by $T' = \gamma' - 1$. In fact in order to obtain an interpolation formula in (a) (or a definition of the leading coefficient and the p-adic height pairing) which is independent of the choice of γ one has to renormalize the leading coefficient and the p-adic height pairing by multiplying by the factor $(\log_p(\chi_{\mathrm{cyc}}(\gamma)))^r$.

Remark 7.6. The conjectured value at $T = 0$ of $T^{-r(M)(\rho^*)}\Phi_\rho(\mathcal{L})$ in Conjecture 7.4 should be the leading term $\mathcal{L}^*(\rho)$ at $T = 0$ of $\Phi_\rho(\mathcal{L})$ only if $R_p(M(\rho^*)) \neq 0$.

Remark 7.7. The primary motivation for believing Conjecture 7.4 is that its precise formulation is what results if one incorporates into the main conjecture of Coates *et al*. [**15**, Conjecture 5.8] the explicit formulae for the leading terms of zeta isomorphisms that were derived from the conjectures of Fukaya and Kato [**18**] by the present authors in [**13**]. (We discuss the link between Conjecture 7.4 and [**15**, Conjecture 5.8] in Proposition 7.8 below.) However, in § 8.2 we will also show that Theorem 2.2 implies that both Conjecture 7.4 and its natural extension to more general critical motives (Conjecture 7.9) are compatible with all relevant cases of the equivariant Tamagawa number conjecture.

Before stating the next result we recall that the explicit interpolation formula given in the main conjecture of [**15**, Conjecture 5.8] requires minor modification. To be precise, one must interchange all occurrences of ρ and $\hat{\rho}$ on the right-hand side of the equality of [**15**, (107)] except for the term '$e_p(\rho)$' (for further details see the footnote at the end of § 6.0 in [**35**]).

Proposition 7.8. *Assume the hypotheses of Conjecture 7.4 and also that* $\text{Ш}(E_{/F})$ *is finite for every finite Galois extension* F *of* $\mathbb{Q}$ *in* K*. Further assume that for all Artin representations* ρ *of* $G_{K/\mathbb{Q}}$ *the 'order of vanishing part' of the Birch–Swinnerton-Dyer Conjecture for* $E(\rho^*)$ *holds. Then Conjecture 7.4 implies the 'main conjecture of non-commutative Iwasawa theory' of [**15**, Conjecture 5.8] (corrected as described above).*

Proof. We first reinterpret the interpolation formula of Conjecture 7.4 (a) in terms of the classical Hasse–Weil L-functions and their twists $L(E, \rho^*, s)$ in the sense of [**15**, (102)] (which is the same as the L-function attached to the B-motive $h^1(E) \otimes [\rho]^*$). To do this we let u in $\mathbb{Z}_p$ be the unit root of the polynomial $1 - a_pX + pX^2$ where, as usual, $p + 1 - a_p = \#\tilde{E}_p(\mathbb{F}_p)$ with $\tilde{E}_p$ denoting the reduction of E modulo p. Furthermore, we write p^{f_ρ} for the p-part of the conductor of ρ and $\epsilon_p(\rho)$ for the local ϵ-factor of ρ at the prime p. Moreover, let $d_+(\rho)$ and $d_-(\rho)$ denote the dimension of the subspace of $[\rho]_\lambda$ on which complex conjugation acts by $+1$ and -1, respectively. We denote the periods of E by

$$\Omega_+(E) := \int_{\gamma^+} \omega, \qquad \Omega_-(E) := \int_{\gamma^-} \omega,$$

where ω is the Néron differential and γ^+ and γ^- denote a generator for the submodule of $\mathrm{H}_1(E(\mathbb{C}), \mathbb{Z})$ on which complex conjugation acts as $+1$ and -1 respectively. Finally, we write $R_\infty(E, \rho^*)$ and $R_p(E, \rho^*)$ for the complex and p-adic regulators of E twisted by ρ^*. Then the displayed expression in Conjecture 7.4 (a) is equal to

$$(-1)^{\dim_{\mathbb{C}_p}(e_{\rho^*}(\mathbb{C}_p \otimes_{\mathbb{Z}} E(K^{\ker(\rho)})))} \frac{L_R^*(E, \rho^*)}{\Omega_+(E)^{d_+(\rho)} \Omega_-(E)^{d_-(\rho)} R_\infty(E, \rho^*)} \times \epsilon_p(\rho) u^{-f_\rho} R_p(E, \rho^*) \frac{P_{L,p}([\rho]_\lambda, u^{-1})}{P_{L,p}([\rho]^*_\lambda, up^{-1})}. \qquad (7.11)$$

Here $L_R^*(E, \rho^*)$ is the leading coefficient at $s = 1$ of the L-function $L_R(E, \rho^*, s)$ obtained from the Hasse–Weil L-function of E twisted by ρ^* by removing the Euler factors at p

and at all primes ℓ at which the j-invariant j_E of E in non-integral. For the calculation of $\Omega_p(M(\rho^*))$ see [**18**, Remark 4.2.27 and the proof of Theorem 4.2.26(1) with $u = \alpha$].

Next we note that in this case the set Υ that occurs in Conjecture 7.4 (and was defined in general just after (7.5)) comprises the prime p and all prime numbers q with $\mathrm{ord}_q(j_E) < 0$ (see also [**18**, 4.5.3] or [**35**, Remark 6.5]). Hence, given the reinterpretation (7.11) of the interpolation formula of Conjecture 7.4 (a), the only essential difference between Conjecture 7.4 and [**15**, Conjecture 5.8] is that Conjecture 7.4 involves an interpolation formula for $((r(M)(\rho^*)!)^{-1}\times)$ the value at $T = 0$ of the $r(M)(\rho^*)$th derivative of $\Phi_\rho(\mathcal{L})$ rather than merely for the value at $T = 0$ of $\Phi_\rho(\mathcal{L})$ itself as in [**15**, Conjecture 5.8].

Now, since $Ш(E_{/K^{\ker(\rho)}})$ is assumed to be finite, Conjecture 7.4 (a) combines with the equality (7.10) to imply that

$$r(\Phi_\rho(\mathcal{L})) \geqslant r(M)(\rho^*) = \dim_{\mathbb{C}_p}(e_{\rho^*}(\mathbb{C}_p \otimes_{\mathbb{Z}} E(K^{\ker(\rho)}))) \geqslant 0$$

because the given interpolation formula has no pole. In particular, $\mathcal{L}$ does not have ∞ as its value at any representation ρ. We also note that the 'order of vanishing part' of the Birch–Swinnerton-Dyer Conjecture for $E(\rho^*)$ implies that the order of vanishing of $L_R(E, \rho^*, s)$ at $s = 1$ is equal to $\dim_{\mathbb{C}_p}(e_{\rho^*}(\mathbb{C}_p \otimes_{\mathbb{Z}} E(K^{\ker(\rho)})))$.

We now assume that $e_{\rho^*}(\mathbb{C}_p \otimes_{\mathbb{Z}} E(K^{\ker(\rho)}))$ vanishes. Then both $R_p(M(\rho^*)) = 1$ and $R_\infty(M(\rho^*)) = 1$. Also, the leading term $L^*_R(E, \rho^*)$ is in this case equal to the value at $s = 1$ of $L_R(E, \rho^*, s)$. Hence, the interpolation formula (7.11) coincides with that given in [**15**, Conjecture 5.8].

On the other hand, if $e_{\rho^*}(\mathbb{C}_p \otimes_{\mathbb{Z}} E(K^{\ker(\rho)})) \neq 0$, then $r(\Phi_\rho(\mathcal{L})) > 0$ and so the value of $\mathcal{L}$ at ρ is equal to 0. In addition, in this case the function $L_R(E, \rho^*, s)$ vanishes at $s = 1$ and so the interpolation formula of [**15**, Conjecture 5.8] also implies that the value of $\mathcal{L}$ at ρ is equal to 0, as required. □

7.2.3. *The general case*

We return to the more general case discussed in § 7.2.1. We fix a full Galois stable $\mathbb{Z}_p$-sublattice T of V and define a $G_{\mathbb{Q}_p}$-stable $\mathbb{Z}_p$-sublattice of $\hat{V}$ by setting $\hat{T} := T \cap \hat{V}$. As before we let $\mathbb{T}$ denote the Galois representation $\Lambda(K/\mathbb{Q}) \otimes_{\mathbb{Z}_p} T$ and set $\hat{\mathbb{T}} := \Lambda(K/\mathbb{Q}) \otimes_{\mathbb{Z}_p} \hat{T}$ similarly. Then $\hat{\mathbb{T}}$ is a $G_{\mathbb{Q}_p}$-stable $\Lambda(K/\mathbb{Q})$-submodule of $\mathbb{T}$. For the definition of the Selmer complex $\mathrm{SC}_U := \mathrm{SC}_U(\hat{\mathbb{T}}, \mathbb{T})$ (and the related one $\mathrm{SC}(\hat{\mathbb{T}}, \mathbb{T})$), which is originally due to Nekovář [**25**], we refer the reader to either [**18**, 4.1.2] or [**13**, (31)]. There are several reasons for using Selmer complexes in this context rather than the Pontryagin dual of the Selmer group: firstly, the latter module is not in general of finite projective dimension over $\Lambda(K/\mathbb{Q})$ and so the formalism of (non-commutative) determinants cannot be applied; secondly, Selmer complexes have better functorial properties than do Selmer groups; thirdly, Nekovář has shown that the Selmer complex is able to explain subtle phenomena (for example, concerning exceptional zeros). Also, by [**18**, Proposition 4.2.35], the Pontryagin dual X of the Selmer group is closely related to the cohomology of the Selmer complex $\mathrm{SC}(\hat{\mathbb{T}}, \mathbb{T})$ and indeed the difference between the classes of X and $\mathrm{SC}(\hat{\mathbb{T}}, \mathbb{T})$ in $K_0(\mathfrak{M}_{S^*}(G_{K/\mathbb{Q}}))$ can be controlled explicitly and in fact often vanishes [**18**, 4.3.15/18].

Conjecture 7.9 (general formulation for critical motives). *Fix a field K in $\mathcal{F}_{\mathbb{Q}}$ that is unramified outside a finite set of places Σ. Then, under the above conditions, the complex SC_U belongs to $D^{\mathrm{p}}_{S^*}(\Lambda(K/\mathbb{Q}))$. Further, there exists an element $\xi = \xi(U, M)$ of $K_1(\Lambda_A(K/\mathbb{Q})_{S^*})$ which satisfies both of the following conditions.*

(a) *At each Artin representation $\rho : G_{K/\mathbb{Q}} \to \mathrm{GL}_n(\mathcal{O})$ for which neither $P_{L,p}(\hat{W}_\rho, 1)$ or $P_{L,p}(W_\rho, 1)$ is equal to 0 the value at $T = 0$ of $T^{-r(M)(\rho^*)}\Phi_\rho(\xi)$ is equal to*

$$(-1)^{r(M)(\rho^*)} \frac{L^*_{B,\Sigma}(M(\rho^*))}{\Omega_\infty(M(\rho^*))R_\infty(M(\rho^*))} \cdot \Omega_p(M(\rho^*))R_p(M(\rho^*)) \cdot \Gamma(\hat{V})^{-1} \cdot \frac{P_{L,p}(\hat{W}^*_\rho(1), 1)}{P_{L,p}(\hat{W}_\rho, 1)},$$

where $L^*_{B,\Sigma}(M(\rho^*))$ is the leading coefficient of the complex L-function of $M(\rho^*)$, truncated by removing Euler factors for all primes in $\Sigma \setminus \{\infty\}$.

(b) $\partial_{A,G_{K/\mathbb{Q}}}(\xi) = \chi(\Lambda_A(K/\mathbb{Q}) \otimes_{\Lambda(K/\mathbb{Q})} \mathrm{SC}_U)$.

Proposition 7.10. *If $M = h^1(E)(1)$ and $K = \mathbb{Q}(E(p))$ are as in Conjecture 7.4, then Conjecture 7.9 is equivalent to Conjecture 7.4.*

Proof. We note first that [**18**, Proposition 4.3.7] implies that the complex SC_U belongs to $D^{\mathrm{p}}_{S^*}(\Lambda(K/\mathbb{Q}))$ precisely when the module $X(E_{/K})$ belongs to the category $\mathfrak{M}_{H_{K/\mathbb{Q}}}(G_{K/\mathbb{Q}}) = \mathfrak{M}_{S^*}(G_{K/\mathbb{Q}})$. Also, SC_U differs from the complex $\mathrm{SC}(\hat{\mathbb{T}}, \mathbb{T})$ in [**18**] only by local terms which belong to $\mathfrak{M}_{S^*}(G_{K/\mathbb{Q}})$ (by [**18**, Proposition 4.3.6]) and have characteristic elements (denoted $\zeta(\ell, K/\mathbb{Q})$ in [**18**]) that correspond to the Euler factors $P_{L,\ell}(W_\rho, s)$ and whose values $P_{L,\ell}(W_\rho, 1)$ at ρ are neither 0 or ∞ (by [**18**, Lemma 4.2.23]). To deduce the claimed result from here one need only note that $\Gamma_{\mathbb{Q}_p}(\hat{V}) = 1$ in this case and recall (from [**18**, Propositions 4.3.15–4.3.18]) that the class of $\mathrm{SC}(\hat{\mathbb{T}}, \mathbb{T})$ in $K_0(\mathfrak{M}_{S^*}(G_{K/\mathbb{Q}}))$ is equal to $[X(E_{/K})]$. □

8. Equivariant Tamagawa numbers

Let F/k be a finite Galois extension of number fields. Then for any motive M defined over k the equivariant Tamagawa number conjecture of [**10**, Conjecture 4.1(iv)] asserts the vanishing of an element $T\Omega(M_F, \mathbb{Z}[G_{F/k}])$ of $K_0(\mathbb{Z}[G_{F/k}], \mathbb{R}[G_{F/k}])$ that is constructed (in general modulo the validity of certain standard conjectures) from the various realizations and comparison isomorphisms associated to the motive $M_F := F \otimes_k M$. Here M_F is regarded as defined over k and endowed with a natural left action of $\mathbb{Q}[G_{F/k}]$ (via the first factor).

Now the product over all primes p and all field isomorphisms $j : \mathbb{C} \cong \mathbb{C}_p$ of the natural composite homomorphism

$$j_* : K_0(\mathbb{Z}[G_{F/k}], \mathbb{R}[G_{F/k}]) \to K_0(\mathbb{Z}[G_{F/k}], \mathbb{C}[G_{F/k}]) \to K_0(\mathbb{Z}[G_{F/k}], \mathbb{C}_p[G_{F/k}]) \to K_0(\mathbb{Z}_p[G_{F/k}], \mathbb{C}_p[G_{F/k}])$$

is injective, where the second map is induced by j (cf. [**4**, Lemma 2.1]). To prove [**10**, Conjecture 4.1(iv)] it therefore suffices to prove that $j_*(T\Omega(M_F,\mathbb{Z}[G_{F/k}]))=0$ for every such j. This reduction has the further advantage that the element $j_*(T\Omega(M_F,\mathbb{Z}[G_{F/k}]))$ can be directly defined without assuming the 'Coherence Hypothesis' of [**10**, § 3.3] that is necessary to define $T\Omega(M_F,\mathbb{Z}[G_{F/k}])$ (cf. [**10**, Remark 8]). However, even if one assumes the standard compatibility conjectures concerning the definition of Euler factors (cf. [**10**, Conjecture 3]), the definition of $j_*(T\Omega(M_F,\mathbb{Z}[G_{F/k}]))$ is in general still conditional, being dependent upon the conjectural existence of a fundamental exact sequence relating the motivic cohomology spaces of M_F and its Kummer dual [**10**, Conjecture 1] and of canonical p-adic Chern class isomorphisms [**10**, Conjecture 2]. In particular, since we are assuming here that the element $j_*(T\Omega(M_F,\mathbb{Z}[G_{F/k}]))$ is well-defined, the results that we prove in this section will not shed any new light on either of [**10**, Conjecture 1, Conjecture 2].

8.1. Tate motives

In this subsection we fix a finite Galois extension F/k of totally real number fields and write $\mathbb{Q}(1)_F$ for the motive $h^0(\operatorname{Spec} F)(1)$, regarded as defined over k and with coefficients $\mathbb{Q}[G_{F/k}]$. We recall that all of the conjectures necessary for the definition of $T\Omega(\mathbb{Q}(1)_F,\mathbb{Z}[G_{F/k}])$ are known to be valid and hence that $j_*(T\Omega(\mathbb{Q}(1)_F,\mathbb{Z}[G_{F/k}]))$ is defined unconditionally as an element of $K_0(\mathbb{Z}_p[G_{F/k}],\mathbb{C}_p[G_{F/k}])$. For a discussion of various explicit consequences of the vanishing of $T\Omega(\mathbb{Q}(1)_F,\mathbb{Z}[G_{F/k}])$ see [**4**] or [**7**].

Theorem 8.1. *Let K be any field which belongs to $\mathcal{F}_k^+$, contains F and is such that $G_{K/k}$ has no element of order p (such a field K exists by virtue of Lemma 6.1). If K validates Conjecture 7.1 and F validates Leopoldt's Conjecture (at p), then one has $j_*(T\Omega(\mathbb{Q}(1)_F,\mathbb{Z}[G_{F/k}]))=0$.*

Proof. For any quotient $\mathcal{G}$ of $G_{K/k}$ we write $\Lambda(\mathcal{G})^\#(1)$ for the $\Lambda(\mathcal{G})$-module $\Lambda(\mathcal{G})$ endowed with the following action of G_k: each σ in G_k acts on $\Lambda(\mathcal{G})^\#(1)$ as right multiplication by the element $\chi_{\mathrm{cyc}}(\sigma)\bar{\sigma}^{-1}$, where $\bar{\sigma}$ denotes the image of σ in $\mathcal{G}$. If K' is the subfield of K with $G_{K'/k}=\mathcal{G}$ and $\mathcal{O}_{k,\Sigma}$ is the subring of k comprising elements that are integral at all places outside Σ then, following Fukaya and Kato [**18**, § 2.1.1] and Nekovář [**25**], the compact support cohomology complex $C_{K'}:=R\Gamma_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},\Lambda(\mathcal{G})^\#(1))$ is an object of $D^{\mathrm{p}}(\Lambda(\mathcal{G}))$ that lies in a canonical exact triangle in $D(\Lambda(\mathcal{G}))$ of the form

$$C_{K'}\to R\Gamma_{\text{ét}}(\mathcal{O}_{k,\Sigma},\Lambda(\mathcal{G})^\#(1))\to\bigoplus_{v\in\Sigma}R\Gamma_{\text{ét}}(k_v,\Lambda(\mathcal{G})^\#(1))\to C_{K'}[1]. \tag{8.1}$$

Below we use the following facts: there is a natural isomorphism in $D^{\mathrm{p}}(\Lambda(\mathcal{G}))$ of the form

$$\Lambda(\mathcal{G})\otimes^{\mathbb{L}}_{\Lambda(K/k)}C_K\cong C_{K'}; \tag{8.2}$$

in each degree i there is a natural isomorphism $H^i(C_K)\cong\varprojlim_{K'}H^i(C_{K'})$ where K' runs over all finite degree Galois extensions K'/k with $K'\subseteq K$ and the limit is taken with

https://doi.org/10.1017/S147474800900022X Published online by Cambridge University Press

respect to the natural corestriction maps; for each such K' there are natural identifications

$$\begin{aligned} H^i(C_{K'}) &\cong H^i_{c,\text{ét}}(\mathcal{O}_{K',\Sigma}, \mathbb{Z}_p(1)), \\ H^i(R\Gamma_{\text{ét}}(\mathcal{O}_{k,\Sigma}, \Lambda(\mathcal{G})^{\#}(1))) &\cong H^i_{\text{ét}}(\mathcal{O}_{K',\Sigma}, \mathbb{Z}_p(1)), \\ H^i(R\Gamma_{\text{ét}}(k_v, \Lambda(\mathcal{G})^{\#}(1))) &\cong \bigoplus_{w|v} H^i_{\text{ét}}(K'_w, \mathbb{Z}_p(1)). \end{aligned}$$

In particular, by a standard computation (involving Kummer theory, class field theory and arithmetic duality) one obtains canonical identifications

$$H^i(C_{K'}) \cong \begin{cases} \ker(\lambda_{K'}), & i = 1, \\ X_\Sigma(K'), & i = 2, \\ \mathbb{Z}_p, & i = 3, \\ 0, & \text{otherwise}, \end{cases} \tag{8.3}$$

where $\lambda_{K'}$ is the diagonal map from $\mathcal{O}_{K'}[1/p]^\times \otimes \mathbb{Z}_p$ to the direct sum over $w \in S_p(K')$ of the pro-p-completion $(K'_w)^\times \hat{\otimes} \mathbb{Z}_p$ of $(K'_w)^\times$ and $X_\Sigma(K')$ is the Galois group of the maximal abelian pro-p extension of K' that is unramified outside Σ. By passing to the limit over $K' \subset K$ one finds that $H^i(C_K)$ is acyclic outside degrees 2 and 3 and that its cohomology in degrees 2 and 3 is canonically isomorphic to $X_\Sigma(K)$ and $\mathbb{Z}_p$ respectively. The first claim of Conjecture 7.1 is therefore equivalent to asserting that C_K belongs to $D^{\mathrm{p}}_{S^*}(\Lambda(K/k))$. In addition, since $\chi(C_K) = \sum_{i \in \mathbb{Z}} (-1)^i [H^i(C_K)]$, Conjecture 7.1 (b) asserts that $\partial_{G_{K/k}}(\xi) = \chi(C_K) + [\mathbb{Z}_p]$ or equivalently, by Proposition 4.7 (ii)(a), (b), that

$$\partial_{G_{K/k}}(\xi') = \chi(C_K) \tag{8.4}$$

with $\xi' := \mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0]) \cdot \xi$.

In the next result we set $c_k := \log_p(\chi_{\mathrm{cyc}}(\gamma_k))$.

Lemma 8.2. *Assume Leopoldt's Conjecture is valid for F (at p) and fix an Artin representation $\rho : G_{K/k} \to \mathrm{GL}_n(\mathcal{O})$ such that V_ρ is an irreducible representation of $G_{F/k}$.*

(i) *Conjecture 7.1 (a) implies that $(\xi')^*(\rho) = c_k^{-\langle \rho, 1 \rangle} \Omega_j(\rho) L^*_\Sigma(1, \rho^{j^{-1}})^j$.*

(ii) *If ρ is non-trivial, then $\mathbb{Q}_p \hat{\otimes}^{\mathbb{L}}_{\Lambda(\Gamma_k)} C_{K,\rho}$ is acyclic and hence C_K is semisimple at ρ, $r_{G_{K/k}}(C_K)(\rho) = 0$ and $t(C_{K,\rho})$ is the canonical morphism.*

(iii) *If ρ is trivial, then $\mathbb{Q}_p \hat{\otimes}^{\mathbb{L}}_{\Lambda(\Gamma_k)} C_{K,\rho}$ is acyclic outside degrees 2 and 3 and its cohomology in degrees 2 and 3 identifies with $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{cok}(\lambda_k)$ and $\mathbb{Q}_p$ respectively. Further, C_K is semisimple at ρ, $r_{G_{K/k}}(C_K)(\rho) = 1$ and $(-1) \times t(C_{K,\rho})$ is induced by the isomorphism $\beta : \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \mathrm{cok}(\lambda_k) \to \mathbb{Q}_p$ that sends each element $(x_v)_v$ of $\prod_{v \in S_p(k)} k_v^\times$ to $c_k^{-1} \sum_{v \in S_p(k)} \log_p(\mathrm{N}_v(x_v))$ with N_v the field-theoretic norm $k_v^\times \to \mathbb{Q}_p^\times$.*

Proof. Leopoldt's Conjecture implies that the determinant $\Omega_j(\rho)$ and hence also the product $(c_k^{\langle\rho,1\rangle}c_{\rho,k})^{-1}\Omega_j(\rho)L^*_\Sigma(1,\rho^{j^{-1}})^j$ in Conjecture 7.1 (a) is non-zero. Since Conjecture 7.1 (a) asserts that this product is equal to the value $\xi(\rho)$ it therefore also implies that $\xi(\rho)=\xi^*(\rho)$. Next we observe that claim (i) is a consequence of the equality

$$\Phi_\rho(\mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0]))=\begin{cases}1-\rho(\gamma_k^{-1})(1+T)^{-1}, & \text{if } H_{K/k}\subseteq\ker(\rho),\\ 1, & \text{otherwise.}\end{cases}\tag{8.5}$$

Indeed, if (8.5) is true, then $\mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0])^*(\rho)=c_{\rho,k}$ and so claim (i) follows from the obvious equalities $(\xi')^*(\rho)=(\mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0])\cdot\xi)^*(\rho)=\mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0])^*(\rho)\xi^*(\rho)$.

To prove (8.5) we regard the tensor product

$$M_\rho:=\Lambda(\Gamma_k)\otimes_{\mathbb{Z}_p}(\mathcal{O}^n)^t=\Lambda_{\mathcal{O}}(\Gamma_k)\otimes_{\mathcal{O}}(\mathcal{O}^n)^t$$

as a $(\Lambda_{\mathcal{O}}(\Gamma_k),\Lambda(G_{K/k}))$-bimodule, where $\Lambda_{\mathcal{O}}(\Gamma_k)$ acts via multiplication on the left and the (right) action of each element g of $G_{K/k}$ is via $x\otimes y\mapsto x\pi_{\Gamma_k}(g)\otimes y\rho(g)$. Then the definitions of the element $\mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0])$ and homomorphism Φ_ρ combine to imply that

$$\begin{aligned}&\Phi_\rho(\mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0]))\\ &\qquad=\det{}_{Q(\mathcal{O}[\![T]\!])}(\mathrm{id}\otimes\mathrm{id}-\mathrm{id}\otimes\theta\mid Q(\mathcal{O}[\![T]\!])\otimes_{\Lambda_{\mathcal{O}}(\Gamma_k)}(M_\rho\otimes_{\Lambda(H_{K/k})}\mathbb{Z}_p)),\end{aligned}\tag{8.6}$$

where θ is the endomorphism of $M_\rho\otimes_{\Lambda(H_{K/k})}\mathbb{Z}_p\cong\Lambda(\Gamma_k)\otimes_{\mathbb{Z}_p}((\mathcal{O}^n)^t\otimes_{\Lambda(H_{K/k})}\mathbb{Z}_p)$ that sends each element $x\otimes(y\otimes z)$ to $x\gamma_k^{-1}\otimes(y\rho(\tilde\gamma_k^{-1})\otimes z)$ (this recipe is independent of the choice of lift $\tilde\gamma_k$ of γ_k through π_{Γ_k}).

Now V_ρ is irreducible and $H_{K/k}$ is normal in $G_{K/k}$ and so $(\mathcal{O}^n)^t\otimes_{\Lambda(H_{K/k})}\mathbb{Q}_p$ is either canonically isomorphic to $(\mathcal{O}^n)^t\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ or vanishes depending on whether $H_{K/k}\subseteq\ker(\rho)$ or not. Thus, if $H_{K/k}\not\subseteq\ker(\rho)$, then (8.6) implies $\Phi_\rho(\mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0]))$ is the determinant of an endomorphism of the zero space and so equal to 1. On the other hand, if $H_{K/k}\subseteq\ker(\rho)$, then $n=1$ (since Γ_k is abelian and V_ρ is irreducible) and so (8.6) implies $\Phi_\rho(\mathrm{char}_{G_{K/k},\gamma_k}(\mathbb{Z}_p[0]))$ is the determinant of the endomorphism of $Q(\mathcal{O}[\![T]\!])$ given by multiplication by $1-\rho(\gamma_k^{-1})(1+T)^{-1}$. The required equality (8.5) is therefore clear.

To prove claims (ii) and (iii) we note that in each degree i the isomorphism (8.2) induces an identification $H^i(\mathbb{Q}_p\hat\otimes^{\mathbb{L}}_{\Lambda(\Gamma_k)}C_{K,\rho})\cong H^i(C_F)^\rho$. Further, if Leopoldt's Conjecture is valid for F, then $\ker(\lambda_F)$ vanishes and $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}X_\Sigma(F)$ is a trivial $G_{F/k}$-module and so the explicit descriptions of (8.3) with $K'=F$ imply C_F is acyclic outside degrees 2 and 3 and moreover that each space $H^i(C_F)^\rho$ vanishes if ρ is non-trivial. This proves the first assertion of claim (ii) and then all remaining assertions of claim (ii) follow immediately from [**13**, Lemma 3.13]. In addition, the long exact cohomology sequence of (8.1) induces an identification $H^2_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(1))\cong\mathrm{cok}(\lambda_k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and if ρ is trivial and we use (8.2) with $K'=k^{\mathrm{cyc}}$ to identify $C_{K,\rho}$ with $C_{k^{\mathrm{cyc}}}$, then [**13**, Lemma 3.13] implies that all remaining assertions of claim (iii) will follow if we can show that the given isomorphism β is equal to -1 times the Bockstein homomorphism $\beta^2_{\Delta_{k,c}}$ in degree 2 of the canonical exact triangle

$$\Delta_{k,c}:\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\mathrm{R}\Gamma_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},T_k)\xrightarrow{\gamma_k-1}\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\mathrm{R}\Gamma_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},T_k)\to\mathrm{R}\Gamma_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(1))\to,$$

where $T_k := \Lambda(k^{\mathrm{cyc}}/k)^{\#}(1)$. Now the argument of [**13**, § 3.2.1] shows that $\beta^2_{\Delta_{k,c}}$ is equal to the homomorphism $H^2_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(1)) \to H^3_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(1))$ induced by taking the cup product with the element φ_k of $H^1_{\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Z}_p) = \mathrm{Hom}_{\mathrm{cts}}(X_\Sigma(k),\mathbb{Z}_p)$ obtained by composing the projection $X_\Sigma(k) \to \Gamma_k$ with the continuous homomorphism $\Gamma_k \to \mathbb{Z}_p$ that sends γ_k to 1. Since cup products commute with corestriction we therefore obtain a commutative diagram

$$\begin{array}{ccc} H^2_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(1)) & \xrightarrow{\beta^2_{\Delta_{k,c}}} & H^3_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(1)) \\ \downarrow{\scriptstyle\kappa^2} & & \downarrow{\scriptstyle\kappa^3} \\ H^2_{c,\text{ét}}(\mathbb{Z}_{\Sigma'},\mathbb{Q}_p(1)) & \xrightarrow{d_k^{-1}\beta^2_{\Delta_{\mathbb{Q},c}}} & H^3_{c,\text{ét}}(\mathbb{Z}_{\Sigma'},\mathbb{Q}_p(1)) \end{array}$$

in which Σ' is the set of rational places lying below those in Σ, $\Delta_{\mathbb{Q},c}$ denotes the exact triangle obtained from $\Delta_{k,c}$ by replacing k and Σ by $\mathbb{Q}$ and Σ' respectively, the vertical arrows are the natural corestriction maps and d_k occurs in the lower row because the restriction of $\varphi_{\mathbb{Q}} \in H^1_{\text{ét}}(\mathcal{O}_{\mathbb{Q},\Sigma},\mathbb{Z}_p)$ to $H^1_{\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Z}_p)$ is equal to $\varphi_k^{d_k}$ (since $\gamma_k = \gamma_{\mathbb{Q}}^{d_k}$). But, with respect to the canonical identifications $H^2_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(1)) \cong \mathrm{cok}(\lambda_k)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$ and $H^3_{c,\text{ét}}(\mathcal{O}_{k,\Sigma},\mathbb{Q}_p(1)) \cong \mathbb{Q}_p$ (and similarly with k and Σ replaced by $\mathbb{Q}$ and Σ'), the map κ^2 is induced by the norm maps $N_v : k_v^\times \to \mathbb{Q}_p^\times$ and κ^3 is the identity map. Thus, since $d_k \times c_{\mathbb{Q}} = c_k$, it is enough for us to prove that $(-1)\times\beta^2_{\Delta_{\mathbb{Q},c}}$ is induced by the homomorphism $c_{\mathbb{Q}}^{-1}\cdot\log_p : \mathbb{Q}_p^\times \to \mathbb{Q}_p$. To compute $\beta^2_{\Delta_{\mathbb{Q},c}}$ explicitly we use the morphism of natural exact triangles

$$\begin{array}{ccccccc} \mathrm{R}\Gamma_{\text{ét}}(\mathbb{Q}_p,T_{\mathbb{Q}}) & \xrightarrow{\gamma_{\mathbb{Q}}-1} & \mathrm{R}\Gamma_{\text{ét}}(\mathbb{Q}_p,T_{\mathbb{Q}}) & \longrightarrow & \mathrm{R}\Gamma_{\text{ét}}(\mathbb{Q}_p,\mathbb{Z}_p(1)) & \longrightarrow & \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\theta} & & \\ \mathrm{R}\Gamma_{c,\text{ét}}(\mathbb{Z}_{\Sigma'},T_{\mathbb{Q}})[1] & \xrightarrow{\gamma_{\mathbb{Q}}-1} & \mathrm{R}\Gamma_{c,\text{ét}}(\mathbb{Z}_{\Sigma'},T_{\mathbb{Q}})[1] & \longrightarrow & \mathrm{R}\Gamma_{c,\text{ét}}(\mathbb{Z}_{\Sigma'},\mathbb{Z}_p(1))[1] & \longrightarrow & \end{array} \tag{8.7}$$

in which each vertical morphism is induced by the definition of compact support cohomology. Indeed, from the long exact cohomology sequences of the rows in this diagram we obtain a commutative diagram

$$\begin{array}{ccc} H^1_{\text{ét}}(\mathbb{Q}_p,\mathbb{Q}_p(1)) & \xrightarrow{\mathbb{Q}_p\otimes_{\mathbb{Z}_p}H^1(\theta)} & H^2_{c,\text{ét}}(\mathbb{Z}_{\Sigma'},\mathbb{Q}_p(1)) \\ \downarrow{\scriptstyle\beta^1} & & \downarrow{\scriptstyle(-1)\times\beta^2_{\Delta_{\mathbb{Q},c}}} \\ H^2_{\text{ét}}(\mathbb{Q}_p,\mathbb{Q}_p(1)) & \xrightarrow{\mathbb{Q}_p\otimes_{\mathbb{Z}_p}H^2(\theta)} & H^3_{c,\text{ét}}(\mathbb{Z}_{\Sigma'},\mathbb{Q}_p(1)) \end{array}$$

Here $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}H^1(\theta)$ identifies with the natural surjection $H^1_{\text{ét}}(\mathbb{Q}_p,\mathbb{Q}_p(1)) \cong (\mathbb{Q}_p^\times\hat{\otimes}\mathbb{Z}_p)\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \to \mathrm{cok}(\lambda_{\mathbb{Q}})\otimes_{\mathbb{Z}_p}\mathbb{Q}_p \cong H^2_{c,\text{ét}}(\mathbb{Z}_{\Sigma'},\mathbb{Q}_p(1))$, $\mathbb{Q}_p\otimes_{\mathbb{Z}_p}H^2(\theta)$ is induced by the identifications $H^2_{\text{ét}}(\mathbb{Q}_p,\mathbb{Q}_p(1)) \cong \mathbb{Q}_p$ and $H^3_{c,\text{ét}}(\mathbb{Z}_{\Sigma'},\mathbb{Q}_p(1)) \cong \mathbb{Q}_p$ and the identity map on $\mathbb{Q}_p$, β^1 is the Bockstein homomorphism in degree 1 of the image under $\mathbb{Q}_p\otimes_{\mathbb{Z}_p} -$ of the upper row of (8.7) and the factor -1 occurs on the right-hand vertical arrow because of the 1-shift

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in the lower row of (8.7). To complete the proof of claim (iii) it thus suffices to recall that the homomorphism β^1 is induced by $c_{\mathbb{Q}}^{-1}\cdot\log_p$ (for a proof of this fact see, for example, [**11**, p. 352]). $\square$

Returning to the proof of Theorem 8.1 we now apply Theorem 2.2 to the conjectural equality (8.4). By taking into account the canonical isomorphism (8.2) with $K'=F$ and the explicit descriptions given in Lemma 8.2 we therefore deduce that

$$\partial_{G_{F/k}}((c_k^{-\langle\rho,1\rangle}\Omega_j(\rho)L^*_\Sigma(1,\rho^{j^{-1}})^j)_{\rho\in\mathrm{Irr}(G_{F/k})}) = -[\mathbf{d}_{\mathbb{Z}_p[G_{F/k}]}(C_F),\beta_*], \tag{8.8}$$

where β_* is the morphism $\mathbf{d}_{\mathbb{C}_p[G_{F/k}]}(\mathbb{C}_p[G_{F/k}]\hat{\otimes}^{\mathbb{L}}_{\mathbb{Z}_p[G_{F/k}]}C_F)\to\mathbf{1}_{\mathbb{C}_p[G_{F/k}]}$ that is induced by the isomorphism

$$\mathbb{Q}_p\otimes_{\mathbb{Z}_p}H^2(C_F)\cong\mathbb{Q}_p\otimes_{\mathbb{Z}_p}\mathrm{cok}(\lambda_k)\xrightarrow{\beta}\mathbb{Q}_p\cong\mathbb{Q}_p\otimes_{\mathbb{Z}_p}H^3(C_F)$$

coming from Lemma 8.2 (ii), (iii). But the proof of [**13**, Theorem 5.5] shows that (8.8) is equivalent to an equality of the form $[\mathbf{d}_{\mathbb{Z}_p[G_{F/k}]}(C_F),\beta'_*]=0$ where $\beta'_*=(\beta'_\rho)_{\rho\in\mathrm{Irr}(G_{F/k})}$ under the identification (1.4) and each β'_ρ is the explicit morphism described in [**13**, (25)]. The fact that (8.8) implies the vanishing of the element $j_*(T\Omega(\mathbb{Q}(1)_F,\mathbb{Z}[G_{F/k}]))$ then follows directly upon explicitly comparing the definition of $T\Omega(\mathbb{Q}(1)_F,\mathbb{Z}[G_{F/k}])$ with that of each morphism β'_ρ. This therefore completes the proof of Theorem 8.1. $\square$

We end this subsection by noting that the above computations show that the correct generalization of Conjecture 7.1 (to groups with an element of order p) is the following.

Conjecture 8.3. *Fix a totally real number field k and a field K in $\mathcal{F}_k^+$ that is unramified outside a finite set of places Σ. Then C_K belongs to $D^{\mathrm{p}}_{S^*}(\Lambda(K/k))$. Further, there exists an element ξ' of $K_1(\Lambda(K/k)_{S^*})$ which satisfies both of the following conditions.*

(a) *At each Artin representation ρ of $G_{K/k}$ one has*

$$\xi'(\rho)=\log_p(\chi_{\mathrm{cyc}}(\gamma_k))^{-\langle\rho,1\rangle}\Omega_j(\rho)L^*_\Sigma(1,\rho^{j^{-1}})^j.$$

(b) $\partial_{G_{K/k}}(\xi')=\chi(C_K)$.

As already observed in Remark 7.3, it can be shown that this conjecture is compatible with that formulated in the case that $G_{K/k}$ has rank one by Ritter and Weiss in [**27**, § 4].

8.2. Critical motives

In this subsection we assume the notation and hypotheses of Conjecture 7.9 and fix a subfield F of K that is both Galois and of finite degree over $\mathbb{Q}$. We set

$$\hat{\mathbb{T}}_F:=\Lambda(F/\mathbb{Q})\otimes_{\mathbb{Z}_p}\hat{T}\cong\Lambda(F/\mathbb{Q})\otimes_{\Lambda(K/\mathbb{Q})}\hat{\mathbb{T}}.$$

We write $Z=Z_\rho$ and $\tilde{Z}=\tilde{Z}_\rho$ for the Kummer duals $W^*_\rho(1)$ and $\hat{W}^*_\rho(1)$ of W_ρ and $\hat{W}_\rho$ respectively; finally we set $\tilde{W}=\tilde{W}_\rho:=W_\rho/\hat{W}_\rho$. In terms of the notation of [**13**] we consider the following assumption on W_ρ.

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Assumption (W). For each ρ in $\mathrm{Irr}(G_{F/\mathbb{Q}})$ the space $W = W_\rho$ satisfies all of the following conditions:

(A_1) $P_\ell(W,1)P_\ell(Z,1) \neq 0$ for all primes $\ell \neq p$;

(B_1) $P_p(W,1)P_p(Z,1) \neq 0$;

(C_1) $P_p(\tilde{W},1)P_p(\tilde{Z},1) \neq 0$; and

(D_2) $H^0_f(\mathbb{Q},W) = H^0_f(\mathbb{Q},Z) = 0$.

In the following result we write

$$\iota_A : K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}], \mathbb{C}_p[G_{F/\mathbb{Q}}]) \to K_0(A[G_{F/\mathbb{Q}}], \mathbb{C}_p[G_{F/\mathbb{Q}}])$$

for the canonical homomorphism obtained by regarding A as a subring of $\mathbb{C}_p$. We also recall that [**10**, Conjecture 4(iii)] (which is a natural equivariant version of the Deligne–Beilinson Conjecture) for the motive M_F, regarded as defined over $\mathbb{Q}$ and with an action of $\mathbb{Q}[G_{F/\mathbb{Q}}]$, implies that the element $j_*(T\Omega(M_F, \mathbb{Z}[G_{F/\mathbb{Q}}]))$ belongs to the image of the natural map $K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}], \mathbb{Q}_p[G_{F/\mathbb{Q}}]) \to K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}], \mathbb{C}_p[G_{F/\mathbb{Q}}])$.

Theorem 8.4. *Assume that*

- *Assumption (W) is valid;*
- *the complex* SC_U *is semi-simple at all* ρ *in* $\mathrm{Irr}(G_{F/\mathbb{Q}})$*;*
- *an* ϵ*-isomorphism*

$$\epsilon_{p,\mathbb{Z}_p[G_{F/\mathbb{Q}}]}(\hat{\mathbb{T}}_F) : \mathbf{1}_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]} \to \boldsymbol{d}_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]}(\mathrm{R}\varGamma(\mathbb{Q}_p, \hat{\mathbb{T}}_F))\boldsymbol{d}_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]}(\hat{\mathbb{T}}_F)$$

  *in the sense of [*__18__*, Conjecture 3.4.3] exists;*
- *Conjecture 7.9 is valid for the motive* M *and the extension* $K/\mathbb{Q}$*.*

Then $\iota_A(j_*(T\Omega(M_F, \mathbb{Z}[G_{F/\mathbb{Q}}])))$ *vanishes. Further, if* $j_*(T\Omega(M_F, \mathbb{Z}[G_{F/\mathbb{Q}}]))$ *belongs to the image of the natural map* $K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}], \mathbb{Q}_p[G_{F/\mathbb{Q}}]) \to K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}], \mathbb{C}_p[G_{F/\mathbb{Q}}])$*, then* $j_*(T\Omega(M_F, \mathbb{Z}[G_{F/\mathbb{Q}}]))$ *vanishes.*

Proof. We fix an element ξ as in Conjecture 7.9. Since SC_U is semisimple at each ρ in $\mathrm{Irr}(G_{F/\mathbb{Q}})$, the obvious analogue of Theorem 2.2 with A in place of $\mathbb{Z}_p$ combines with Conjecture 7.9 (b) to imply that

$$\partial_{G_{F/\mathbb{Q}}}((\xi^*(\rho))_{\rho\in\mathrm{Irr}(G_{F/\mathbb{Q}})}) = -\iota_A([\mathbf{d}_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]}(\mathrm{SC}_U(\hat{\mathbb{T}}_F, \mathbb{T}_F)), t(\mathrm{SC}_U(\hat{\mathbb{T}}_F, \mathbb{T}_F))_{G_{F/\mathbb{Q}}}]).$$

After unwinding the identification (1.2), this means that there exists a morphism in $V(A[G_{F/\mathbb{Q}}])$

$$\psi : \mathbf{1}_{A[G_{F/\mathbb{Q}}]} \to \mathbf{d}_{A[G_{F/\mathbb{Q}}]}(A[G_{F/\mathbb{Q}}] \otimes_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]} \mathrm{SC}_U(\hat{\mathbb{T}}_F, \mathbb{T}_F))$$

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such that

$$(\xi^*(\rho)^{-1})_{\rho\in\mathrm{Irr}(G_{F/\mathbb{Q}})} = t(\mathrm{SC}_U(\hat{\mathbb{T}}_F,\mathbb{T}_F))_{G_{F/\mathbb{Q}}}\circ\psi_{\mathbb{C}_p[G_{F/\mathbb{Q}}]} \in \mathrm{Aut}_{V(\mathbb{C}_p[G_{F/\mathbb{Q}}])}(\mathbf{1}_{\mathbb{C}_p[G_{F/\mathbb{Q}}]})\cong K_1(\mathbb{C}_p[G_{F/\mathbb{Q}}])$$

under the identification (1.4). After recalling the explicit definition of the morphism $t(\mathrm{SC}_U(\hat{\mathbb{T}}_F,\mathbb{T}_F))_{G_{F/\mathbb{Q}}}$ which occurs in Theorem 2.2 and then taking inverses we obtain a morphism in $V(A[G_{F/\mathbb{Q}}])$

$$\psi^{-1}:\mathbf{1}_{A[G_{F/\mathbb{Q}}]}\to\mathbf{d}_{A[G_{F/\mathbb{Q}}]}(A[G_{F/\mathbb{Q}}]\otimes_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]}\mathrm{SC}_U(\hat{\mathbb{T}}_F,\mathbb{T}_F))^{-1}$$

such that

$$(-1)^{r_G(\mathrm{SC}_U)(\rho)}\xi^*(\rho)=t(\mathrm{SC}_U(\rho^*))^{-1}\circ\psi^{-1}(\rho)\in\mathrm{Aut}_{V(\mathbb{C}_p)}(\mathbf{1}_{\mathbb{C}_p})\cong\mathbb{C}_p^\times$$

for all ρ in $\mathrm{Irr}(G_{F/\mathbb{Q}})$. Here we write $\psi^{-1}(\rho)$ for the ρ-component of the morphism

$$\mathbf{1}_{\mathbb{C}_p[G_{F/\mathbb{Q}}]}\to\mathbf{d}_{\mathbb{C}_p[G_{F/\mathbb{Q}}]}(\mathbb{C}_p[G_{F/\mathbb{Q}}]\otimes_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]}\mathrm{SC}_U(\hat{\mathbb{T}}_F,\mathbb{T}_F))^{-1}$$

that is induced by ψ^{-1}. This is equivalent to asserting the existence of a morphism in $V(A[G_{F/\mathbb{Q}}])$

$$\psi':\mathbf{1}_{A[G_{F/\mathbb{Q}}]}\to\mathbf{d}_{A[G_{F/\mathbb{Q}}]}(A[G_{F/\mathbb{Q}}]\otimes_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]}\mathrm{R}\Gamma_c(U,\mathbb{T}_F))^{-1}$$

such that for all ρ in $\mathrm{Irr}(G_{F/\mathbb{Q}})$ the composite morphism

$$\mathbf{1}_{\mathbb{C}_p}\xrightarrow{\psi'(\rho)_{\mathbb{C}_p}}\mathbf{d}_L(\mathrm{R}\Gamma_c(U,W_\rho))^{-1}_{\mathbb{C}_p}\xrightarrow{\beta(\rho)\epsilon(\hat{\mathbb{T}})^{-1}(\rho)}\mathbf{d}_L(\mathrm{SC}_U(\hat{W}_\rho,W_\rho))^{-1}_{\mathbb{C}_p}\xrightarrow{t(\mathrm{SC}_U(\rho^*))^{-1}_{\mathbb{C}_p}}\mathbf{1}_{\mathbb{C}_p}\qquad(8.9)$$

corresponds to $(-1)^{r_G(\mathrm{SC}_U)(\rho)}\xi^*(\rho)$. In this displayed expression we write $\epsilon(\hat{\mathbb{T}})(\rho)$ for $V_{\rho^*}\otimes_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]}\epsilon_{p,\mathbb{Z}_p[G_{F/\mathbb{Q}}]}(\hat{\mathbb{T}}_F)$ and $\beta(\rho)$ for $V_{\rho^*}\otimes_{\mathbb{Z}_p[G_{F/\mathbb{Q}}]}(\mathbb{Z}_p[G_{F/\mathbb{Q}}]\otimes_{\Lambda(G)}\beta)\cong V_{\rho^*}\otimes_{\Lambda(G)}\beta$ with β the morphism $\mathbf{d}_\Lambda(\mathbb{T}^+)_{\tilde{\Lambda}}\cong\mathbf{d}_\Lambda(\hat{\mathbb{T}})_{\tilde{\Lambda}}$ that is defined in [**13**, (35)], and all underlying identifications are as explained in [**13**, § 6].

Now the hypothesis that SC_U is semisimple at ρ combines with Assumption (W), the duality isomorphism $H^3_f(\mathbb{Q},W)\cong H^0_f(\mathbb{Q},Z)$ and the results of [**13**, Lemmas 6.7 and 3.13(ii)] to imply that the algebraic rank $r(M)(\rho^*)$ defined in (7.5) is equal to $r_G(\mathrm{SC}_U)(\rho)$, that [**13**, Condition (F)] is satisfied and that the value at $T=0$ of $T^{-r(M)(\rho^*)}\Phi_\rho(\xi)$ is equal to the leading term $\xi^*(\rho)$. Conjecture 7.9 (a) therefore gives an explicit formula for $\xi^*(\rho)$. Taking this formula into account, one can compare the composite morphism (8.9) to the first displayed morphism after [**13**, Lemma 6.8]. After unwinding the proof of [**13**, Theorem 6.5] (for which we use Assumption (W)) this comparison shows that

$$\psi'(\rho)=\vartheta_\lambda(M(\rho^*))_{\mathbb{C}_p}\circ\zeta_K(M(\rho^*))_{\mathbb{C}_p}$$

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for all ρ in $\mathrm{Irr}(G_{F/\mathbb{Q}})$, where $\vartheta_\lambda(M(\rho^*))_{\mathbb{C}_p}$ and $\zeta_K(M(\rho^*))_{\mathbb{C}_p}$ are the morphisms that occur in [**13**, Conjecture 4.1]. Finally, we note that the validity of the last displayed equality (for all ρ in $\mathrm{Irr}(G_{F/\mathbb{Q}})$) is equivalent to asserting that the element $\iota_A(j_*(T\Omega(M_F, \mathbb{Z}[G_{F/\mathbb{Q}}])))$ vanishes (by the very definition of the latter element). This proves the first claim of the theorem.

The second claim of Theorem 8.4 will now follow if we can show that the natural composite homomorphism

$$K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}], \mathbb{Q}_p[G_{F/\mathbb{Q}}]) \to K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}], \mathbb{C}_p[G_{F/\mathbb{Q}}]) \xrightarrow{\iota_A} K_0(A[G_{F/\mathbb{Q}}], \mathbb{C}_p[G_{F/\mathbb{Q}}])$$

is injective. To do this we write F for the field of fractions of A (so $F \subset \mathbb{C}_p$). Then, since the natural scalar extension map $K_1(F[G_{F/\mathbb{Q}}]) \to K_1(\mathbb{C}_p[G_{F/\mathbb{Q}}])$ is injective, the exact commutative diagram (1.1) with $R = A[G_{F/\mathbb{Q}}]$, $R' = F[G_{F/\mathbb{Q}}]$ and $R'' = \mathbb{C}_p[G_{F/\mathbb{Q}}]$ implies that the natural homomorphism

$$K_0(A[G_{F/\mathbb{Q}}], F[G_{F/\mathbb{Q}}]) \to K_0(A[G_{F/\mathbb{Q}}], \mathbb{C}_p[G_{F/\mathbb{Q}}])$$

is also injective. It therefore suffices to prove that the natural homomorphism

$$K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}], \mathbb{Q}_p[G_{F/\mathbb{Q}}]) \to K_0(A[G_{F/\mathbb{Q}}], F[G_{F/\mathbb{Q}}])$$

is injective. But, since A is unramified over $\mathbb{Z}_p$, this is an immediate consequence of a result of Taylor [**33**, Chapter 8, Theorem 1.1]. Indeed, to see this one need only note that the groups $K_0(\mathbb{Z}_p[G_{F/\mathbb{Q}}], \mathbb{Q}_p[G_{F/\mathbb{Q}}])$ and $K_0(A[G_{F/\mathbb{Q}}], F[G_{F/\mathbb{Q}}])$ are naturally isomorphic to the Grothendieck groups $K_0T(\mathbb{Z}_p[G_{F/\mathbb{Q}}])$ and $K_0T(A[G_{F/\mathbb{Q}}])$ which occur in [**33**]. □

Remark 8.5. If $M = h^1(A)(1)$ for an abelian variety A that has good ordinary reduction at p and is such that the Tate–Shafarevich group $Ш(A_{/F})$ of A over F is finite, then it is known that the vanishing of $j_*(T\Omega(M_F, \mathbb{Z}[G_{F/\mathbb{Q}}]))$ implies the 'p-part' of a Birch–Swinnerton-Dyer type formula (see, for example, [**35**, § 3.1]). However, we stress that Conjecture 7.9 does *not* itself imply that $Ш(A_{/F})$ is finite.

Remark 8.6. Explicit consequences of Conjecture 7.9 for the value at $s = 1$ of twisted Hasse–Weil L-functions have been described by Coates *et al.* in [**15**], by Kato in [**21**] and by Dokchister and Dokchister in [**17**]. However, all of the consequences described in [**15**, **17**, **21**] become trivial when the L-functions vanish at $s = 1$. One of the key advantages of Theorem 8.4 is that in many of these cases it can be combined with the approach of [**7**] to show that Conjecture 7.9 implies a variety of explicit (and highly non-trivial) congruence relations between values at $s = 1$ of *derivatives* of twisted Hasse–Weil L-functions. Such explicit (conjectural) congruences will be considered elsewhere.

Remark 8.7. Following Theorem 8.4 it is of some interest to study elements in K-theory of the form $T\Omega(h^1(E_{/F})(1), \mathbb{Z}[G_{F/\mathbb{Q}}])$ with E an elliptic curve over $\mathbb{Q}$ and $F/\mathbb{Q}$ a finite non-abelian Galois extension. The study of such elements is however still very much in its infancy. Indeed, the only explicit computation that we are currently aware of is the following. Let E be the elliptic curve $y^2 + y = x^3 - x^2 - 10x - 20$. Then, with F equal

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to the splitting field of the polynomial $x^3 - 4x - 1$, the group $G_{F/\mathbb{Q}}$ is dihedral of order 6 and Navilarekallu [**24**] has proved numerically that if ℓ is any odd prime for which the ℓ-primary component of $Ш(E_{/F})$ is trivial, then the element $j_*(T\Omega(h^1(E_{/F})(1), \mathbb{Z}[G_{F/\mathbb{Q}}]))$ vanishes for any isomorphism $j : \mathbb{C} \cong \mathbb{C}_\ell$.

Appendix A. Determinant functors

In this appendix we recall some details of the formalism of determinant functors introduced by Fukaya and Kato in [**18**] and used in [**13**] (see also [**35**])

We fix an associative unital noetherian ring R. We write $B(R)$ for the category of bounded complexes of (left) R-modules, $C(R)$ for the category of bounded complexes of finitely generated (left) R-modules, $P(R)$ for the category of finitely generated projective (left) R-modules, $C^{\mathrm{p}}(R)$ for the category of bounded (cohomological) complexes of finitely generated projective (left) R-modules. By $D^{\mathrm{p}}(R)$ we denote the category of perfect complexes as full triangulated subcategory of the derived category $D^{\mathrm{b}}(R)$ of $B(R)$. We write $(C^{\mathrm{p}}(R), \mathrm{quasi})$ and $(D^{\mathrm{p}}(R), \mathrm{is})$ for the subcategory of quasi-isomorphisms of $C^{\mathrm{p}}(R)$ and isomorphisms of $D^{\mathrm{p}}(R)$, respectively.

For each complex $C = (C^\bullet, d_C^\bullet)$ and each integer r we define the r-fold shift $C[r]$ of C by setting $C[r]^i = C^{i+r}$ and $d^i_{C[r]} = (-1)^r d_C^{i+r}$ for each integer i.

We first recall that there exists a Picard category $\mathcal{C}_R$ and a determinant functor $\mathbf{d}_R : (C^{\mathrm{p}}(R), \mathrm{quasi}) \to \mathcal{C}_R$ with the following properties (for objects C, C' and C'' of $\mathrm{C}^{\mathrm{p}}(R)$).*

A(d) If $C' \to C \to C''$ is a short exact sequence of complexes, then there is a canonical isomorphism $\mathbf{d}_R(C) \cong \mathbf{d}_R(C')\mathbf{d}_R(C'')$ in $\mathcal{C}_R$ (which we usually take to be an identification). These isomorphisms have the following functorial property: for any commutative diagram of short exact sequences of complexes in $C^{\mathrm{p}}(R)$ of the form

$$\begin{array}{ccccc} D' & \longrightarrow & D & \longrightarrow & D'' \\ \downarrow & & \downarrow & & \downarrow \\ C' & \longrightarrow & C & \longrightarrow & C'' \\ \downarrow & & \downarrow & & \downarrow \\ E' & \longrightarrow & E & \longrightarrow & E'' \end{array}$$

there is a commutative diagram in $\mathcal{C}_R$

$$\begin{array}{ccc} \mathbf{d}_R(C) & \xrightarrow{\cong} & \mathbf{d}_R(C')\mathbf{d}_R(C'') \\ \downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\ \mathbf{d}_R(D)\mathbf{d}_R(E) & \xrightarrow{\cong} & \mathbf{d}_R(D')\mathbf{d}_R(E')\mathbf{d}_R(D'')\mathbf{d}_R(E'') \end{array}$$

* The listing starts with (d) to be compatible with the notation of [**35**].

in which the upper, lower, left and right arrow is the isomorphism induced by the central row, top and bottom rows, central column and left- and right-hand columns in the above diagram respectively.

A(e) If C is acyclic, then the quasi-isomorphism $0 \to C$ induces a canonical isomorphism $\mathbf{1}_R \to \mathbf{d}_R(C)$.

A(f) For any integer r one has $\mathbf{d}_R(C[r]) = \mathbf{d}_R(C)^{(-1)^r}$.

A(g) The functor $\mathbf{d}_R$ factorizes over the image of $C^{\mathrm{p}}(R)$ in $D^{\mathrm{p}}(R)$ and extends (uniquely up to unique isomorphisms) to $(D^{\mathrm{p}}(R), \mathrm{is})$.

A(h) For each C in $D^{\mathrm{b}}(R)$ we write $\mathrm{H}(C)$ for the complex which has $\mathrm{H}(C)^i = H^i(C)$ in each degree i and in which all differentials are 0. If $\mathrm{H}(C)$ belongs to $D^{\mathrm{p}}(R)$ (in which case one says that C is *cohomologically perfect*), then C belongs to $D^{\mathrm{p}}(R)$ and there are canonical isomorphisms

$$\mathbf{d}_R(C) \cong \mathbf{d}_R(\mathrm{H}(C)) \cong \prod_{i \in \mathbb{Z}} \mathbf{d}_R(H^i(C))^{(-1)^i}.$$

(For an explicit description of the first isomorphism see [**22**, § 3] or [**3**, Remark 3.2].)

A(i) If R' is another (associative unital noetherian) ring and Y an (R', R)-bimodule that is both finitely generated and projective as an R'-module, then the functor $Y \otimes_R - : P(R) \to P(R')$ extends to a commutative diagram

$$\begin{array}{ccc} (D^{\mathrm{p}}(R), \mathrm{is}) & \xrightarrow{\mathbf{d}_R} & \mathcal{C}_R \\ {\scriptstyle Y \otimes^{\mathbb{L}}_R -}\downarrow & & \downarrow{\scriptstyle Y \otimes_R -} \\ (D^{\mathrm{p}}(R'), \mathrm{is}) & \xrightarrow{\mathbf{d}_{R'}} & \mathcal{C}_{R'} \end{array}$$

In particular, if $R \to R'$ is a ring homomorphism and C is in $\mathrm{D}^{\mathrm{p}}(R)$, then we often simply write $\mathbf{d}_R(C)_{R'}$ in place of $R' \otimes_R \mathbf{d}_R(C)$.

In [**18**] a 'localized K_1-group' is defined for any full subcategory Σ of $C^{\mathrm{p}}(R)$ which satisfies the following four conditions:

(i) $0 \in \Sigma$;

(ii) if C, C' are in $C^{\mathrm{p}}(R)$ and C is quasi-isomorphic to C', then $C \in \Sigma \;\Leftrightarrow\; C' \in \Sigma$;

(iii) if $C \in \Sigma$, then also $C[n] \in \Sigma$ for all $n \in \mathbb{Z}$;

(iv′) if C' and C'' belong to Σ, then $C' \oplus C''$ belongs to Σ.

Definition A.1 (Fukaya–Kato). Assume that Σ satisfies the conditions (i), (ii), (iii) and (iv′) above. Then the *localized* K_1*-group* $K_1(R, \Sigma)$ is defined to be the (multiplicatively written) abelian group which has as generators symbols of the form $[C, a]_{\mathrm{FK}}$ for each complex $C \in \Sigma$ and morphism $a : \mathbf{1}_R \to \mathbf{d}_R(C)$ in $\mathcal{C}_R$ and the following relations:

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(0) $[0, \mathrm{id}_{\mathbf{1}_R}]_{\mathrm{FK}} = 1$;

(1) $[C', \mathbf{d}_R(f) \circ a]_{\mathrm{FK}} = [C, a]_{\mathrm{FK}}$ if $f : C \to C'$ is an quasi-isomorphism with C (and thus C') in Σ;

(2) if $0 \to C' \to C \to C'' \to 0$ is an exact sequence in Σ, then

$$[C, a]_{\mathrm{FK}} = [C', a']_{\mathrm{FK}} \cdot [C'', a'']_{\mathrm{FK}},$$

where a is the composite of $a' \cdot a''$ with the isomorphism induced by property A(d);

(3) $[C[1], a^{-1}]_{\mathrm{FK}} = ([C, a]_{\mathrm{FK}})^{-1}$.

We now assume to be given a left denominator set S of R and let $R_S := S^{-1}R$ denote the corresponding localization and Σ_S the full subcategory of $C^{\mathrm{p}}(R)$ comprising all complexes C in $C^{\mathrm{p}}(R)$ for which $R_S \otimes_R C$ is acyclic. For each complex C in Σ_S and morphism $a : \mathbf{1}_R \to \mathbf{d}_R(C)$ in $\mathcal{C}_R$ we write $\theta_{C,a}$ for the element of $K_1(R_S)$ which corresponds under the canonical isomorphism $K_1(R_S) \cong \mathrm{Aut}_{\mathcal{C}_{R_S}}(\mathbf{1}_{R_S})$ to the composite

$$\mathbf{1}_{R_S} \xrightarrow{R_S \otimes_R a} \mathbf{d}_{R_S}(R_S \otimes_R C) \to \mathbf{1}_{R_S}, \tag{A.1}$$

where the second morphism results from property A(e) with C and R replaced by $R_S \otimes_R C$ and R_S. Then it can be shown that the assignment $[C, a]_{\mathrm{FK}} \mapsto \theta_{C,a}$ induces an isomorphism of groups

$$\mathrm{ch}_{R,\Sigma_S} : K_1(R, \Sigma_S) \cong K_1(R_S)$$

(cf. [**18**, Proposition 1.3.7]). Hence, if Σ is any subcategory of Σ_S we also obtain a composite homomorphism

$$\mathrm{ch}_{R,\Sigma} : K_1(R, \Sigma) \to K_1(R, \Sigma_S) \cong K_1(R_S).$$

In particular, we often use this construction in the following case: C belongs to Σ_S and Σ is equal to smallest full subcategory Σ_C of $C^{\mathrm{p}}(R)$ that contains C and also satisfies the conditions (i), (ii), (iii) and (iv$'$) that are described above.

Appendix B. Bockstein homomorphisms

Let A be a regular noetherian (associative unital) ring and assume given an exact triangle in $D^{\mathrm{p}}(A)$ of the form

$$\Delta : C \xrightarrow{\theta} C \to D \to C[1].$$

For each integer i we define the 'Bockstein homomorphism in degree i of Δ' to be the composite homomorphism

$$\beta^i_\Delta : H^i(D) \to \ker(H^{i+1}(\theta)) \to H^{i+1}(C) \to \mathrm{cok}(H^{i+1}(\theta)) \to H^{i+1}(D),$$

where the first and fourth maps occur in the long exact sequence of cohomology of Δ and the second and third are tautological, and we write

$$\mathrm{H}_\beta(\Delta) : \cdots \xrightarrow{\beta^{i-1}_\Delta} H^i(D) \xrightarrow{\beta^i_\Delta} H^{i+1}(D) \xrightarrow{\beta^{i+1}_\Delta} \cdots$$

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for the associated complex (with each term $H^i(D)$ placed in degree i). The morphism θ is said to be 'semisimple' if the tautological map $\ker(H^i(\theta)) \to \mathrm{cok}(H^i(\theta))$ is bijective in each degree i. This condition is equivalent to asserting that the complex $\mathrm{H}_\beta(\Delta)$ is acyclic. Hence, if true, there is a composite morphism in $V(A)$ of the form

$$t(\Delta) : \mathbf{d}_A(D) \to \mathbf{d}_A(\mathrm{H}(D)) \to \mathbf{d}_A(\mathrm{H}_\beta(\Delta)) \to \mathbf{1}_{V(A)}, \tag{B.1}$$

where the first map is as in A(h), the second is the obvious map (induced by the fact that the complexes $\mathrm{H}(D)$ and $\mathrm{H}_\beta(\Delta)$ agree termwise) and the third is induced by property A(e) and the fact that $\mathrm{H}_\beta(\Delta)$ is acyclic.

Appendix C. Sign conventions

Due to a difference of conventions between [**13**] and [**18**], which unfortunately had not previously been noticed by the authors, the following sign conflict has arisen: for a discrete valuation ring $\mathcal{O}$ with field of fractions L, an element $c \in \mathcal{O} \setminus \{0\} \subseteq L^\times$ corresponds in [**18**] to the class of $[\mathcal{O} \xrightarrow{c} \mathcal{O}, \mathrm{id}]_{\mathrm{FK}}$ in $K_1(L)$, where the complex $\mathcal{O} \xrightarrow{c} \mathcal{O}$ is concentrated in degree 0 and 1 (while this complex is implicitly concentrated in degrees -1 and 0 in [**13**, Remark 2.4]). With this convention, Fukaya and Kato must normalize the connecting homomorphism as $[C, a]_{\mathrm{FK}} \mapsto -[\![C]\!]$ in order to ensure that the connecting homomorphism $L^\times = K_1(L) \to K_0(\Sigma_{\mathcal{O}\setminus\{0\}}) \cong \mathbb{Z}$ coincides with the valuation map ord_L (cf. [**18**, Remark 1.3.16]). It follows that in [**13**, Remark 2.4] the correct formula is actually

$$\mathrm{ord}_L(c) = -\mathrm{length}_{\mathcal{O}}(A),$$

if we identify A with the complex $A[0]$. For the same reason, the signs in [**13**, Proposition 3.19] are incorrect, the correct versions being

$$\chi_{\mathrm{add}}(G, C(\rho^*)) = -\mathrm{ord}_L(\mathcal{L}^*(\rho))$$

and

$$\chi_{\mathrm{mult}}(G, C(\rho^*)) = |\mathcal{L}^*(\rho)|_p^{[L:\mathbb{Q}_p]}.$$

We note also that $\mathcal{L} := [C, a]_{\mathrm{FK}} \in K_1(\Lambda, \Sigma_{S^*})$ is a characteristic element of $-[\![C]\!] = [\![C[1]]\!]$ (rather than of $[\![C]\!]$) in $K_0(\Sigma_{S^*})$ due to the normalization of the connecting homomorphism in [**18**]. For a similar reason we have to add a sign in the formulae of [**18**, (38), (39)] to obtain the corrected versions

$$\mathcal{L}_{U,\beta} := \mathcal{L}_{U,\beta}(M) : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(\mathrm{SC}_U(\hat{\mathbb{T}}, \mathbb{T}))^{-1} \tag{C.1}$$

and

$$\mathcal{L}_\beta := \mathcal{L}_\beta(M) : \mathbf{1}_\Lambda \to \mathbf{d}_\Lambda(\mathrm{SC}(\hat{\mathbb{T}}, \mathbb{T}))^{-1}. \tag{C.2}$$

Also in the following convention we need a 1-shift: we write $\mathcal{L}_{U,\beta}$ and $\mathcal{L}_\beta$ for the elements $[\mathrm{SC}_U[1], \mathcal{L}_{U,\beta}]_{\mathrm{FK}}$ and $[\mathrm{SC}[1], \mathcal{L}_\beta]_{\mathrm{FK}}$ of $K_1(\Lambda(G), \Sigma_{\mathrm{SC}_U})$ and $K_1(\Lambda(G), \Sigma_{\mathrm{SC}})$ respectively. Finally, in the first displayed formula after [**13**, Lemma 6.8] one has to replace $t(\mathrm{SC}_U(\rho^*))_{\tilde{L}}$ by $t(\mathrm{SC}_U(\rho^*)[1])_{\tilde{L}} = t(\mathrm{SC}_U(\rho^*))_{\tilde{L}}^{-1}$.

The authors would like to apologize for these oversights.

Acknowledgements. We are very grateful indeed to Kazuya Kato for his generous help and encouragement, to Manuel Breuning for advice regarding determinant functors, to Peter Schneider for valuable comments on an earlier draft and to Jan Nekovář for many stimulating discussions. Much of this article was written when the first named author was a Visiting Professor at the Institute of Mathematics of the University of Paris 6 and he is extremely grateful to the Institute for this opportunity.

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