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THE RELATIVE BRUCE–ROBERTS NUMBER OF A FUNCTION ON A HYPERSURFACE

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Abstract We consider the relative Bruce–Roberts number $\mu^-_{\rm BR}(f, X)$ of a function on an isolated hypersurface singularity $(X, 0)$. We show that $\mu^-_{\rm BR}(f, X)$ is equal to the sum of the Milnor number of the fibre $\mu(f^{-1}(0) \cap X, 0)$ plus the difference $\mu(X, 0) - \tau(X, 0)$ between the Milnor and the Tjurina numbers of $(X, 0)$. As an application, we show that the usual Bruce–Roberts number $\mu_{\rm BR}(f, X)$ is equal to $\mu(f) + \mu^-_{\rm BR}(f, X)$. We also deduce that the relative logarithmic characteristic variety $LC(X)^-$, obtained from the logarithmic characteristic variety $LC(X)$ by eliminating the component corresponding to the complement of X in the ambient space, is Cohen–Macaulay.

Keywords: isolated hypersurface singularity; Bruce–Roberts number; logarithmic characteristic variety

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1. Introduction

Let $(X, 0)$ be a germ of complex analytic set in $\mathbb{C}^n$ and $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ a holomorphic function germ. The Bruce–Roberts number of f with respect to $(X, 0)$ was introduced by Bruce and Roberts in [4] and is defined as

$$\mu_{\rm BR}(f, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{{\rm d}f(\Theta_X)},$$

where $\mathcal{O}_n$ is the local ring of holomorphic functions $(\mathbb{C}^n, 0) \to \mathbb{C}$, ${\rm d}f$ is the differential of f and Θ_X is the $\mathcal{O}_n$-submodule of Θ_n of vector fields on $(\mathbb{C}^n, 0)$ which are tangent to $(X,\,0)$ at its regular points. If I_X is the ideal of $\mathcal{O}_n$ of functions vanishing on $(X,\,0)$, then

$$\Theta_X = \{\xi \in \Theta_n \mid \mathrm{d}h(\xi) \in I_X,\ \forall h \in I_X\}.$$

In particular, when $X = \mathbb{C}^n$, $\mathrm{d}f(\Theta_X)$ is the Jacobian ideal of f and thus, $\mu_{\mathrm{BR}}(f,\,X)$ coincides with the classical Milnor number $\mu(f)$. We remark that Θ_X is also denoted in some papers by $\mathrm{Der}(-\log X)$, following Saito's notation [11]. The main properties of $\mu_{\mathrm{BR}}(f,\,X)$ are the following (see [4]):

(a) $\mu_{\mathrm{BR}}(f,\,X)$ is invariant under the action of the group $\mathcal{R}_X$ of diffeomorphisms $\phi : (\mathbb{C}^n,\,0) \to (\mathbb{C}^n,\,0)$ which preserve $(X,\,0)$;

(b) $\mu_{\mathrm{BR}}(f,\,X) < \infty$ if and only if f is finitely determined with respect to the $\mathcal{R}_X$-equivalence;

(c) $\mu_{\mathrm{BR}}(f,\,X) < \infty$ if and only if f restricted to each logarithmic stratum is a submersion in a punctured neighbourhood of the origin.

In general, $\mu_{\mathrm{BR}}(f,\,X)$ is not so easy to compute as the classical Milnor number. The main difficulty comes from the computation of the module Θ_X and most of the times, it is necessary to use a symbolic computer system like Singular [6]. If $(X,\,0)$ is an isolated complete intersection singularity (ICIS) and $\mu_{\mathrm{BR}}(f,\,X)$ is finite, then $(f^{-1}(0) \cap X,\,0)$ is an ICIS [2, Proposition 2.8], therefore it has well-defined Milnor number. In a previous paper, [9] we considered the case that $(X,\,0)$ is an isolated hypersurface singularity (IHS). We showed that

$$\mu_{\mathrm{BR}}(f, X) = \mu(f) + \mu(f^{-1}(0) \cap X, 0) + \mu(X, 0) - \tau(X, 0), \tag{1}$$

where μ and τ are the Milnor and the Tjurina numbers, respectively. Thus, (1) gives an easy way to compute $\mu_{\mathrm{BR}}(f,\,X)$ in terms of well-known invariants. The formula (1) was also obtained independently in [8] and previously in [10] when $(X,\,0)$ is weighted homogeneous.

An important application of (1) allowed us to conclude in [9] that the logarithmic characteristic variety $LC(X)$ is Cohen–Macaulay. We recall that $LC(X)$ is the subvariety of the cotangent bundle $T^*\mathbb{C}^n$ of pairs $(x,\,\alpha)$ such that $\alpha(\xi_x) = 0$, for all $\xi \in \Theta_X$ and for all x in a neighbourhood of 0. When $(X,\,0)$ is holonomic, $LC(X)$ is Cohen–Macaulay if and only if for any Morsification f_t of f we have

$$\mu_{\mathrm{BR}}(f, X) = \sum_{\alpha} m_\alpha n_\alpha,$$

where n_α is the number of critical points of f_t restricted to each logarithmic stratum X_α and m_α is the multiplicity of $LC(X)$ along the irreducible component Y_α associated with X_α (see [4, Corollary 5.8]). When $(X,\,0)$ is an IHS, it always has a finite number of logarithmic strata (i.e., it is holonomic in Saito's terminology) given by $X_0 = \mathbb{C}^n \setminus X$, $X_i \setminus \{0\}$, with $i = 1,\,\ldots,\,k$ and $X_{k+1} = \{0\}$, where $X_1,\,\ldots,\,X_k$ are the irreducible components of X at 0.

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In this paper, we are interested in another important invariant introduced in [4],

$$\mu^-_{\mathrm{BR}}(f, X) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\mathrm{d}f(\Theta_X) + I_X},$$

which we call here the relative Bruce–Roberts number. This is an invariant of the restricted function $f : (X, 0) \to (\mathbb{C}, 0)$ under the induced $\mathcal{R}_X$-action. In fact, as commented in [4], it is equal to the codimension of the $\mathcal{R}_X$-orbit. Moreover, $\mu^-_{\mathrm{BR}}(f, X)$ is finite if and only if f restricted to each logarithmic stratum (excluding X_0) is a submersion in a punctured neighbourhood of the origin.

A natural question is about the relationship between $\mu_{BR}(f, X)$ and $\mu^-_{BR}(f, X)$. It is shown in [4] that if $(X, 0)$ is a weighted homogeneous ICIS then

$$\mu^-_{BR}(f, X) = \mu(f^{-1}(0) \cap X, 0).$$

This, combined with (1) when $(X, 0)$ is a weighted homogeneous IHS, gives that

$$\mu_{BR}(f, X) = \mu(f) + \mu^-_{BR}(f, X). \tag{2}$$

Our main result in § 2 is that if $(X, 0)$ is any IHS and $\mu^-_{BR}(f, X)$ is finite, then

$$\mu^-_{BR}(f, X) = \mu(f^{-1}(0) \cap X, 0) + \mu(X, 0) - \tau(X, 0). \tag{3}$$

In particular, (2) also holds when $\mu_{BR}(f, X)$ is finite, even when $(X, 0)$ is not weighted homogeneous. We also show in Example 3.1 that (2) is not true for higher codimension ICIS.

The relative logarithmic characteristic variety $LC(X)^-$ is obtained from $LC(X)$ by eliminating the component Y_0 associated with the stratum $X_0 = \mathbb{C}^n \setminus X$. In [4], they showed that $LC(X)$ is never Cohen–Macaulay when $(X, 0)$ has codimension > 1 along the points on X_0, but $LC(X)^-$ is always Cohen–Macaulay when $(X, 0)$ is a weighted homogeneous ICIS (of any codimension). Again, Cohen–Macaulayness of $LC(X)^-$ is interesting since it implies that

$$\mu^-_{BR}(f, X) = \sum_{\alpha \neq 0} m_\alpha n_\alpha,$$

for any Morsification f_t of f. As an application of (3), we show in § 3 that $LC(X)^-$ is also Cohen–Macaulay for any IHS $(X, 0)$ (not necessarily weighted homogeneous).

In § 4, we consider any holonomic variety $(X, 0)$ and study characterizations of Cohen–Macaulayness of $LC(X)$ and $LC(X)^-$ in terms of the relative polar curve associated with a Morsification f_t of f. Finally, in § 5, we give a formula which generalizes the classical Thom–Sebastiani formula for the Milnor number of a function defined as a sum of functions with separated variables.

2. The relative Bruce–Roberts number

The main goal of this section is to prove the equality (3). The next lemma is inspired by [2, Proposition 2.8].

Lemma 2.1. *Let $(X, 0)$ be an IHS determined by $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and $f \in \mathcal{O}_n$. The map $(\phi, f) : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$ defines an ICIS if and only if $\mu_{BR}^-(f, X) < \infty$.*

Proof. If $(\phi, f) : (\mathbb{C}^n, 0) \to (\mathbb{C}^2, 0)$ defines an ICIS then $\mu_{BR}^-(f, X)$ is finite because

$$V(\mathrm{d}f(\Theta_X^-)) \subset V(J(f, \phi) + I_X) \subset \{0\}.$$

For the converse, if $\mu_{BR}^-(f, X) < \infty$ then the restriction of f to each logarithmic stratum, excluding $\mathbb{C}^n \setminus X$ is non-singular. The proof is now the same of Proposition 2.8 in [2]. □

The following technical lemma will be used in the proof of the next theorem. Given a matrix A with entries in a ring R, we denote by $I_k(A)$ the ideal in R generated the $k \times k$ minors of A.

Lemma 2.2. *Let $f, g \in \mathcal{O}_n$ be such that $\dim V(J(f, g)) = 1$ and $V(Jf) = \{0\}$, and consider the following matrices*

$$A = \begin{pmatrix} \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \\ \dfrac{\partial g}{\partial x_1} & \cdots & \dfrac{\partial g}{\partial x_n} \end{pmatrix}, \quad A' = \begin{pmatrix} \mu & \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \\ \lambda & \dfrac{\partial g}{\partial x_1} & \cdots & \dfrac{\partial g}{\partial x_n} \end{pmatrix},$$

where $\lambda, \mu \in \mathcal{O}_n$. Let M, M' be the submodules of $\mathcal{O}_n^2$ generated by the columns of A, A' respectively. If $I_2(A) = I_2(A')$ then $M = M'$.

Proof. We see A and A' as homomorphims of modules over $R := \mathcal{O}_n$:

$$A \colon R^n \longrightarrow R^2, \quad A' \colon R^{n+1} \longrightarrow R^2.$$

We consider the R-module $R^2/M = \operatorname{coker}(A)$, which has support $V(I_2(A)) = V(J(f, g))$. Therefore, $\dim(R^2/M) = 1 = n - (n - 2 + 1)$ and hence it is Cohen–Macaulay (see [5]). In particular, it is unmixed. Now, M'/M is a submodule of R^2/M, so the associated primes $\operatorname{Ass}(M'/M)$ are included in $\operatorname{Ass}(R^2/M)$. If $M'/M \neq 0$ then $\operatorname{Ass}(M'/M) \neq \emptyset$ and it follows that $\dim(M'/M) = 1$.

Let U be a neighbourhood of 0 in $\mathbb{C}^n$ such that 0 is the only critical point of f. For all $x \in U \setminus \{0\}$, there exist $i_0 \in \{1, \ldots, n\}$, such that $\partial f/\partial x_{i_0}(x) \neq 0$. We may suppose $i_0 = 1$. Making elementary column operations in the matrices A and A', we obtain

$$B = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ c_1 & c_2 & \cdots & c_n \end{pmatrix}, \quad B' = \begin{pmatrix} \mu & 1 & 0 & \cdots & 0 \\ \lambda & c_1 & c_2 & \cdots & c_n \end{pmatrix}$$

such that

$$I_2(A) = I_2(B), \quad I_2(A') = I_2(B'), \ \operatorname{Im}(A) = \operatorname{Im}(B) \text{ and } \operatorname{Im}(A') = \operatorname{Im}(B').$$

By hypothesis $I_2(A) = I_2(A')$ and consequently $\langle c_2, \ldots, c_n \rangle = \langle \mu c_1 - \lambda, c_2, \ldots, c_n \rangle$. This implies $\lambda = \mu c_1 + \alpha_2 c_2 + \cdots + \alpha_n c_n$, for some $\alpha_2, \cdots, \alpha_n \in R$. Thus,

$$\begin{pmatrix} \mu \\ \lambda \end{pmatrix} = \mu \begin{pmatrix} 1 \\ c_1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ c_2 \end{pmatrix} + \cdots + \alpha_n \begin{pmatrix} 0 \\ c_n \end{pmatrix}.$$

and hence $(M'/M)_x = 0$. This shows that $\operatorname{Supp}(M'/M) \subset \{0\}$ and hence, $M' = M$. □

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Given an IHS $(X, 0)$ defined by a holomorphic function germ $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, we consider the $\mathcal{O}_n$-submodule of the trivial vectors fields, denoted by Θ_X^T, generated by

$$\phi\frac{\partial}{\partial x_i}, \frac{\partial \phi}{\partial x_j}\frac{\partial}{\partial x_k} - \frac{\partial \phi}{\partial x_k}\frac{\partial}{\partial x_j}, \text{ with } i, j, k = 1, \ldots, n; k \neq j.$$

This module was related to the Tjurina number of $(X, 0)$ in [9, 13]. By using different approaches, it is shown that $\tau(X, 0) = \dim_{\mathbb{C}} \Theta_X/\Theta_X^T$. Moreover, in [9], we also proved that $\tau(X, 0) = \dim_{\mathbb{C}} \mathrm{d}f(\Theta_X)/\mathrm{d}f(\Theta_X^T)$ where f is any $\mathcal{R}_X$-finitely determined function germ. The following result generalizes this equality with a weaker hypothesis on f.

Theorem 2.3. *Let $(X, 0)$ be an IHS determined by $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and $f \in \mathcal{O}_n$ such that $\mu_{BR}^-(f, X) < \infty$, then:*

(i) $\frac{\Theta_X}{\Theta_X^T} \approx \frac{df(\Theta_X)+I_X}{df(\Theta_X^T)+I_X}$;

(ii) $\frac{\Theta_X}{\Theta_X^T} \approx \frac{df(\Theta_X)}{df(\Theta_X^T)}$;

(iii) $df(\Theta_X) \cap I_X = JfI_X$;

(iv) $\frac{\mathcal{O}_n}{Jf} \approx \frac{df(\Theta_X^-)}{df(\Theta_X)}$;

(v) $df(\Theta_X) : I_X = Jf$;

(vi) $df(\Theta_X^T) : I_X = Jf$,

where I_X is the ideal generated by ϕ.

Proof. (i) The homomorphism $\Psi : \Theta_X \to \mathrm{d}f(\Theta_X) + I_X$ defined by $\Psi(\xi) = \mathrm{d}f(\xi)$ induces the isomorphism

$$\overline{\Psi} : \frac{\Theta_X}{\Theta_X^T} \to \frac{\mathrm{d}f(\Theta_X) + I_X}{\mathrm{d}f(\Theta_X^T) + I_X}.$$

In fact, it is enough to show that $\Psi^{-1}(\mathrm{d}f(\Theta_X^T) + I_X) \subset \Theta_X^T$. Let $\xi \in \Psi^{-1}(\mathrm{d}f(\Theta_X^T) + I_X)$ then $\Psi(\xi) \in \mathrm{d}f(\Theta_X^T) + I_X$, that is, there exist $\eta \in \Theta_X^T$ and $\mu, \lambda \in \mathcal{O}_n$, such that

$$\begin{cases} \mathrm{d}f(\xi - \eta) = \mu\phi \\ \mathrm{d}\phi(\xi - \eta) = \lambda\phi \end{cases},$$

then

$$\begin{pmatrix} \mu\phi \\ \lambda\phi \end{pmatrix} \in \left\langle \begin{pmatrix} \frac{\partial f}{\partial x_i} \\ \frac{\partial \phi}{\partial x_i} \end{pmatrix} \quad i = 1, \ldots, n \right\rangle$$

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and

$$I_2\begin{pmatrix} \mu\phi & \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \\ \lambda\phi & \dfrac{\partial \phi}{\partial x_1} & \cdots & \dfrac{\partial \phi}{\partial x_n} \end{pmatrix} = I_2\begin{pmatrix} \dfrac{\partial f}{\partial x_1} & \cdots & \dfrac{\partial f}{\partial x_n} \\ \dfrac{\partial \phi}{\partial x_1} & \cdots & \dfrac{\partial \phi}{\partial x_n} \end{pmatrix} = J(f,\phi).$$

Therefore

$$\begin{vmatrix} \mu & \dfrac{\partial f}{\partial x_i} \\ \lambda & \dfrac{\partial \phi}{\partial x_i} \end{vmatrix} \phi \in J(f,\phi)$$

and since ϕ is regular in $\frac{\mathcal{O}_n}{J(f,\phi)}$ then

$$\begin{vmatrix} \mu & \dfrac{\partial f}{\partial x_i} \\ \lambda & \dfrac{\partial \phi}{\partial x_i} \end{vmatrix} \in J(f,\phi), \quad i=1,\ldots,n.$$

By Lemma 2.2, $\lambda \in J\phi$ and using [9, Lemma 3.1], $\xi \in \Theta_X^T$.

(ii) This equality also was proved in [9] with the additional hypothesis that f is $\mathcal{R}_X$-finitely determined.
The epimorphism $\psi : \Theta_X \to \mathrm{d}f(\Theta_X)$ defined by $\psi(\xi) = \mathrm{d}f(\xi)$ induces the isomorphism

$$\overline{\psi} : \frac{\Theta_X}{\Theta_X^T} \to \frac{\mathrm{d}f(\Theta_X)}{\mathrm{d}f(\Theta_X^T)}.$$

In fact, let $\xi \in \ker(\psi)$, then there exist $\lambda \in \mathcal{O}_n$, such that

$$\begin{cases} \mathrm{d}f(\xi) = 0 \\ \mathrm{d}\phi(\xi) = \lambda\phi \end{cases}$$

The rest is similar to the proof of (i).

(iii) Let $\xi \in \Theta_X$ be such that $\mathrm{d}f(\xi) \in I_X$, then there exist μ, $\lambda \in \mathcal{O}_n$, such that

$$\begin{cases} \mathrm{d}f(\xi) = \mu\phi \\ \mathrm{d}\phi(\xi) = \lambda\phi \end{cases}$$

Using the same techniques of the proof of (i), we have

$$\mathrm{d}f(\Theta_X) \cap I_X \subset JfI_X.$$

The other inclusion is immediate.

(iv) It follows from the isomorphisms

$$\frac{\mathrm{d}f(\Theta_X^-)}{\mathrm{d}f(\Theta_X)} = \frac{\mathrm{d}f(\Theta_X) + I_X}{\mathrm{d}f(\Theta_X)} \approx \frac{I_X}{\mathrm{d}f(\Theta_X) \cap I_X} \overset{(iii)}{=} \frac{I_X}{JfI_X} \approx \frac{\mathcal{O}_n}{Jf}.$$

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(v) It follows from (iii).

(vi) It follows from (v) and $Jf \subset \mathrm{d}f(\Theta_X^T) : I_X$.

□

Remark 2.4. The items (ii) and (iv) of Theorem 2.3 seem a bit peculiar since from (iv) the quotient $\mathrm{d}f(\Theta_X^-)/\mathrm{d}f(\Theta_X)$ does not depend on $(X, 0)$ while from (ii), $\mathrm{d}f(\Theta_X)/\mathrm{d}f(\Theta_X^T)$ does not depend on f. Moreover by [9, 13] if $(X, 0)$ is an IHS determined by $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$, then $\dim_{\mathbb{C}} \dfrac{\Theta_X}{\Theta_X^T} = \tau(X, 0)$, therefore

$$\dim_{\mathbb{C}} \frac{\mathrm{d}f(\Theta_X) + I_X}{\mathrm{d}f(\Theta_X^T) + I_X} = \dim_{\mathbb{C}} \frac{\mathrm{d}f(\Theta_X)}{\mathrm{d}f(\Theta_X^T)} = \tau(X, 0).$$

The next theorem is one of the main results of this work.

Theorem 2.5. *Let $(X, 0)$ is an IHS determined by $\phi : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ and $f \in \mathcal{O}_n$ be a function germ such that $\mu_{BR}^-(f, X) < \infty$. Then (ϕ, f) defines an ICIS and*

$$\mu(f^{-1}(0) \cap X, 0) = \mu_{BR}^-(f, X) + \tau(X, 0) - \mu(X, 0).$$

Proof. We consider the exact sequence

$$0 \longrightarrow \frac{\mathrm{d}f(\Theta_X^-)}{\mathrm{d}f(\Theta_X^T) + I_X} \xrightarrow{i} \frac{\mathcal{O}_n}{\mathrm{d}f(\Theta_X^T) + I_X} \xrightarrow{\pi} \frac{\mathcal{O}_n}{\mathrm{d}f(\Theta_X^-)} \longrightarrow 0.$$

Since $(X, 0)$ is an IHS

$$\mathrm{d}f(\Theta_X^T) = J(f, \phi) + JfI_X,$$

hence

$$\begin{aligned} mu_{BR}^-(f, X) &= \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{J(f, \phi) + I_X} - \dim_{\mathbb{C}} \frac{\mathrm{d}f(\Theta_X) + I_X}{\mathrm{d}f(\Theta_X^T) + I_X} \\ &= \mu(f^{-1}(0) \cap X, 0) + \mu(X, 0) - \tau(X, 0). \end{aligned}$$

The last equality is a consequence of the Lê-Greuel formula [3] and Theorem 2.3 (i). □

3. The relative Bruce–Roberts number of a function with isolated singularity

In this section, $(X, 0)$ is an IHS and $f \in \mathcal{O}_n$ is a function germ $\mathcal{R}_X$-finitely determined, then all the results in the previous section are true in this case. In particular from (iv) of Theorem 2.3

$$\mu(f) = \dim_{\mathbb{C}} \frac{\mathrm{d}f(\Theta_X^-)}{\mathrm{d}f(\Theta_X)}. \tag{4}$$

Therefore, by the exact sequence

$$0 \longrightarrow \frac{\mathrm{d}f(\Theta_X^-)}{\mathrm{d}f(\Theta_X)} \xrightarrow{i} \frac{\mathcal{O}_n}{\mathrm{d}f(\Theta_X} \xrightarrow{\pi} \frac{\mathcal{O}_n}{\mathrm{d}f(\Theta_X^-)} \longrightarrow 0,$$

we conclude that

$$\mu_{BR}(f, X) = \mu(f) + \mu_{BR}^-(f, X).$$

The following example shows that the characterization of the Milnor number (4) is not true anymore when $(X, 0)$ is an ICIS with codimension higher than one.

Example 3.1. Let $(X, 0)$ be an ICIS determined by $\phi(x, y, z) = (x^3 + x^2y^2 + y^7 + z^3, xyz)$, and $f(x, y, z) = xy - z^4$, f is a $\mathcal{R}_X$-finitely determined and

$$3 = \mu(f) \neq \dim_{\mathbb{C}} \frac{\mathrm{d}f(\Theta_X^-)}{\mathrm{d}f(\Theta_X)} = 6.$$

As a consequence of the characterization of the Milnor number (4), we prove that $LC(X)^-$ is Cohen–Macaulay when $(X, 0)$ is an IHS.

The logarithmic characteristic variety, $LC(X)$, is defined as follows. Suppose the vector fields $\delta_1, \ldots, \delta_m$ generate Θ_X on some neighbourhood U of 0 in $\mathbb{C}^n$. Let $T_U^*\mathbb{C}^n$ be the restriction of the cotangent bundle of $\mathbb{C}^n$ to U. We define $LC_U(X)$ to be

$$LC_U(X) = \{(x, \xi) \in T_U^*\mathbb{C}^n : \xi(\delta_i(x)) = 0, i = 1, \ldots, m\}.$$

Then $LC(X)$ is the germ of $LC_U(X)$ in $T^*\mathbb{C}^n$ along $T_0^*\mathbb{C}^n$, the cotangent space to $\mathbb{C}^n$ at 0. As $LC(X)$ is independent of the choice of the vector fields δ_i then it is a well-defined germ of analytic subvariety in $T^*\mathbb{C}^n$ (see [4, 11]).

If $(X, 0)$ is holonomic with logarithmic strata $X_0, \ldots, X_k$ then $LC(X)$ has dimension n, and its irreducible components are $Y_0, \ldots, Y_k$, with $Y_i = \overline{N^*X_i}$ as set-germs, where $\overline{N^*X_i}$ is the closure of the conormal bundle N^*X_i of X_i in $\mathbb{C}^n$ (see [4, Proposition 1.14]).

When $(X, 0)$ has codimension higher than one, Bruce and Roberts proved that $LC(X)$ is not Cohen–Macaulay. Then they consider the subspace of $LC(X)$ obtained by deleting the component Y_0 that corresponds to the stratum $X_0 = \mathbb{C}^n \setminus X$, that is

$$LC(X)^- = \bigcup_{i=1}^{k+1} Y_i$$

and as set-germs,

$$LC(X)^- = \bigcup_{i=1}^{k+1} \overline{N^*X_i}.$$

An interesting fact about $LC(X)^-$ is that it may be Cohen–Macaulay even when $LC(X)$ is not Cohen–Macaulay, for example, if $(X, 0)$ is a weighted homogeneous ICIS, then $LC(X)^-$ is Cohen–Macaulay, [4].

Proposition 3.2. *Let $(X, 0)$ be an IHS, then $LC(X)^-$ is Cohen–Macaulay.*

Proof. We consider $(0, p) \in LC(X)^-$, then $(0, p) \in LC(X)$ and there exists $f \in \mathcal{O}_n$ such that $\mathrm{d}f(0) = p$. In [9], we proved that $LC(X)$ is Cohen–Macaulay. Therefore, by [4, Proposition 5.8],

$$\mu_{BR}(f, X) = \sum_{i=0}^{k+1} m_i n_i = m_0 n_0 + \sum_{i=1}^{k+1} m_i n_i = \mu(f) + \sum_{i=1}^{k+1} m_i n_i.$$

where n_i is the number of critical points of a Morsification of f in X_i and m_i is the multiplicity of irreducible component Y_i. Thus,

$$\mu^-_{BR}(f, X) = \mu_{BR}(f, X) - \dim_{\mathbb{C}} \frac{\mathrm{d}f(\Theta^-_X)}{\mathrm{d}f(\Theta_X)} = \mu_{BR}(f, X) - \mu(f) = \sum_{i=1}^{k+1} m_i n_i.$$

and by [4, Proposition 5.11], we obtain that $LC(X)^-$ is Cohen–Macaulay. □

Remark 3.3. We remark that in the proof of the previous proposition, we just used that if $(X, 0) \subset (\mathbb{C}^n, 0)$ is a hypersurface such that $\dim_{\mathbb{C}} \mathrm{d}f(\Theta^-_X)/\mathrm{d}f(\Theta_X) = \mu(f)$ for all f $\mathcal{R}_X$-finitely determined then $LC(X)^-$ is Cohen–Macaulay if and only if $LC(X)$ is Cohen–Macaulay.

4. Polar curves and logarithmic characteristic varieties

It is important to know whether the logarithmic characteristic variety of an analytic variety is Cohen–Macaulay. In [9], we showed that this is the case for IHS. For non-isolated singularities, it is an open problem. In this section, we give one more step in order to solve it: we study the polar curve and the relative polar curve of a holomorphic function germ over a holonomic analytic variety. We show that these curves are Cohen–Macaulay if and only if the logarithmic characteristic variety and the relative logarithmic characteristic variety (respectively) are Cohen–Macaulay. As a consequence, we have the principle of conservation for the Bruce–Roberts number.

Definition 4.1. Let $f \in \mathcal{O}_n$ be a $\mathcal{R}_X$-finitely determined function germ and $F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$, $F(t, x) = f_t(x)$,
a 1-parameter deformation of f. The *polar curve* of F in $(X, 0)$ is

$$C = \{(x, t) \in \mathbb{C}^n \times \mathbb{C};\ \mathrm{d}f_t(\delta_i(x)) = 0,\ \forall i = 1, \ldots, m\},$$

where $\Theta_X = \langle \delta_1, \ldots, \delta_m \rangle$.

In [1], it was proved that if $LC(X)$ is Cohen–Macaulay then the polar curve C is Cohen–Macaulay.

Proposition 4.2. *Let $(X, 0)$ be a holonomic analytic variety. If any $\mathcal{R}_X$-finitely determined function germ has a Morsification whose polar curve is Cohen–Macaulay then $LC(X)$ is Cohen–Macaulay.*

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Proof. Let $(0, p) \in LC(X)$, then there exists an $\mathcal{R}_X$-finitely determined function germ $f \in \mathcal{O}_n$, such that $\mathrm{d}f(0) = p$. Let $F : (\mathbb{C}^n \times \mathbb{C}) \to (\mathbb{C}, 0)$, $F(x, t) = f_t(x)$,

be a Morsification of f. By hypothesis $\mathcal{O}_{n+1}/\mathrm{d}f_t(\Theta_X)$ is Cohen–Macaulay of dimension 1, then by the principle of conservation of number

$$\mu_{BR}(f, X) = \sum_{i=0}^{k+1} \sum_{x \in \Sigma f_t \cap X_i} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\mathrm{d}f_t(\Theta_{X,x})} = \sum_{i=0}^{k+1} \sum_{x \in \Sigma f_t \cap X_i} m_i = \sum_{i=0}^{k+1} n_i m_i$$

because if $x \in X_i$ is a Morse critical point of f_t, then $\mu_{BR}(f_t, X)_x = m_i$, and by [4, Proposition 5.8], $LC(X)$ is Cohen–Macaulay. □

When $LC(X)$ is Cohen–Macaulay, we have

$$\mu_{BR}(f, X) = \sum_{x \in \mathbb{C}^n} \mu_{BR}(f_t, X)_x,$$

where f_t is any 1-parameter deformation of f.

Our purpose now is to prove similar results for $LC(X)^-$. We define the *relative polar curve* by

$$C^- = \{(x, t) \in C;\ x \in X\},$$

where C is the polar curve of F in $(X, 0)$.

The proof of the next proposition is similar to the one of [1, Theorem 3.7].

Proposition 4.3. *Let $(X, 0)$ be a holonomic analytic variety. If $LC(X)^-$ is Cohen–Macaulay then the relative polar curve of every 1-parameter deformation of any $\mathcal{R}_X$-finitely determined function germ is Cohen–Macaulay.*

For the converse, we need the following lemma, which is the analogous of [4, Proposition 5.12] for the relative Bruce–Roberts number.

Lemma 4.4. *Let $(X, 0)$ be a holonomic analytic variety and $f \in \mathcal{O}_n$. We assume that f restricted to $(X, 0)$ is a Morse function. If $x \in X$ is a critical point of f then $\mu_{BR}(f, X)^-_x = m_\alpha$, where m_α is the multiplicity of the irreducible component Y_α corresponding to the logarithmic stratum X_α which contains x.*

Proof. Let $Z_i = Y_i \setminus \bigcup_{j \neq i} Y_j$ where Y_i are the irreducible components of $LC(X)$. We know from [4, Proposition 5.12] that $LC(X)$ is Cohen–Macaulay at points in Z_i, $i = 1, \ldots, k+1$. We see that $LC(X)^-$ coincides locally with $LC(X)$ and hence, $LC(X)^-$ is also Cohen–Macaulay at points in Z_i, $i = 1, \ldots, k+1$.

In fact, let $(0, p) \in Z_i$ with $i \neq 0$, then $(x, p) \notin Y_0$. Let $V := T^*\mathbb{C}^n \setminus Y_0$, which is an open neighbourhood of (x, p). Obviously, we have the equality of sets

$$LC(X) \cap V = LC(X)^- \cap V.$$

Moreover, let I, I^- and I_j be the ideals which define $LC(X)$, $LC(X)^-$ and Y_j, $j = 0, \ldots, k+1$, respectively. Then,

$$I = I_0 \cap I_1 \cap \cdots \cap I_{k+1}, \quad I^- = I_1 \cap \cdots \cap I_{k+1} \text{ and } I_0 = \langle p_1, \ldots, p_n \rangle.$$

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Since $p \neq 0$, I_0 is the total ring at (x, p), so we have an equality between germs of complex spaces.

Finally, we have

$$\mu_{BR}(f, X)_x^- \overset{(*)}{=} \sum_{i=1}^{k+1} m_i n_i \overset{(**)}{=} m_\alpha.$$

The equalities $(*)$ and $(**)$ are consequences of [4, Propositions 5.11 and 5.2], respectively. □

We are ready now to prove the converse of Proposition 4.3.

Proposition 4.5. *Let $(X, 0)$ be a holonomic analytic variety. If the relative polar curve of every 1-parameter deformation of any $\mathcal{R}_X$-finitely determined function germ is Cohen–Macaulay then $LC(X)^-$ is Cohen–Macaulay.*

Proof. Let $(0, p) \in LC(X)^-$, then there exists an $\mathcal{R}_X$-finitely determined function germ $f \in \mathcal{O}_n$, such that $\mathrm{d}f(0) = p$. Let $F : (\mathbb{C}^n \times \mathbb{C}, 0) \to (\mathbb{C}, 0)$ be a Morsification of f and set $f_t(x) = F(x, t)$.

By hypothesis $\mathcal{O}_{n+1}/\mathrm{d}f_t(\Theta_X^-)$ is Cohen–Macaulay of dimension 1. By the principle of the conservation of the multiplicity,

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\mathrm{d}f(\Theta_X^-)} = \sum_{i=1}^{k+1} \sum_{x \in \Sigma f \cap X_i} \dim_{\mathbb{C}} \frac{\mathcal{O}_{n,x}}{\mathrm{d}f_t(\Theta_{X,x}^-)} = \sum_{i=1}^{k+1} \sum_{x \in \Sigma f \cap X_i} m_i = \sum_{i=1}^{k+1} n_i m_i,$$

because if $x \in X_i$ is a Morse critical point of f_t, then $\mu_{BR}(f_t, X)_x^- = m_i$ by Lemma 4.4. By [4, Proposition 5.11], $LC(X)^-$ is Cohen–Macaulay. □

As a consequence of the previous result,

$$\mu_{BR}^-(f, X) = \sum_{x \in \mathbb{C}^n} \mu_{BR}^-(f_t, X)_x,$$

where f_t is any 1-parameter deformation of f.

5. An example with non-isolated singularities

Given natural numbers $0 < k \leq n$, we can see $\mathcal{O}_k$ as a subring of $\mathcal{O}_n$ and Θ_k as a subset of Θ_n. We fix $(x_1, \ldots, x_n)$ as the system of coordinates in $\mathcal{O}_n$ and we use $(x_1, \ldots, x_k)$ as the coordinate system of $\mathcal{O}_k$ and $(x_{k+1}, \ldots, x_n)$ as the one in $\mathcal{O}_{n-k}$.

Let $(X, 0) \subset (\mathbb{C}^k, 0)$ be an analytic variety. We denote by $(\tilde{X}, 0) \subset (\mathbb{C}^n, 0)$ the inclusion of $(X, 0)$ in $(\mathbb{C}^n, 0)$. Then $\Theta_{\tilde{X}} = \mathcal{O}_n \Theta_X + \langle \frac{\partial}{\partial x_{k+1}}, \ldots, \frac{\partial}{\partial x_n} \rangle$ and $LC(\tilde{X}) = LC(X) \times \mathbb{C}^{n-t}$.

Consequently, if $LC(X)$ is Cohen–Macaulay then $LC(\tilde{X})$ is Cohen–Macaulay.

In particular, if $(X, 0)$ is an IHS then $LC(\tilde{X})$ is Cohen–Macaulay.

Let $F \in \mathcal{O}_n$ a function germ with isolated singularity such that $F = f + g$ with $f \in \mathcal{O}_k$ and $g \in \mathcal{O}_{n-k}$. It is known by Sebastiani and Thom [12] that $\mu(F) = \mu(f)\mu(g)$. We prove a similar result for the Bruce–Roberts number,

$$\mu_{BR}(F, \tilde{X}) = \mu(g)\mu_{BR}(f, X).$$

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Proposition 5.1. *Let I and J be ideals in $\mathcal{O}_k$ and $\mathcal{O}_{n-k}$, respectively. If we denote by $I' = I\mathcal{O}_n$ and $J' = J\mathcal{O}_n$ the respective induced ideals in $\mathcal{O}_n$, then*

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I' + J'} < \infty \textit{if and only if } \dim_{\mathbb{C}} \frac{\mathcal{O}_k}{I} < \infty \text{ and } \dim_{\mathbb{C}} \frac{\mathcal{O}_{n-k}}{J} < \infty.$$

Moreover, if these dimensions are finite then

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I' + J'} = \left(\dim_{\mathbb{C}} \frac{\mathcal{O}_k}{I}\right) \left(\dim_{\mathbb{C}} \frac{\mathcal{O}_{n-k}}{J}\right).$$

Proof. The equivalence follows from

$$V(I') = V(I) \times \mathbb{C}^{n-t}, \quad V(J') = \mathbb{C}^t \times V(J) \text{ and } V(I' + J') = V(I) \times V(J).$$

For the equality, by hypothesis there exist positive integer numbers k', k_i and k_j such that

$$\mathcal{M}_n^{k'} \subset I' + J', \quad \mathcal{M}_k^{k_i} \subset I, \; ;\mathcal{M}_{n-k}^{k_j} \subset J,$$

where $\mathcal{M}_\ell$ is the maximal ideal of $\mathcal{O}_\ell$. Let $r = \max\{k', \; k_i, \; k_j\}$, then

$$\frac{\mathcal{O}_n}{I' + J'} \approx \frac{\frac{\mathcal{O}_n}{\mathcal{M}_n^r}}{\frac{I'+J'}{\mathcal{M}_n^r}} = \frac{\frac{\mathbb{C}[z_1,z_2]}{\mathcal{M}_n^r}}{\frac{I''+J''}{\mathcal{M}_n^r}} = \frac{\mathbb{C}[z_1, z_2]}{I'' + J''},$$

where $z_1 = (x_1, \, \ldots, \, x_k)$, $z_2 = (x_{k+1}, \, \ldots, \, x_n)$ and I'' and J'' are the ideals in $\mathbb{C}[z_1, \, z_2]$ generated by the $r - 1$-jets of the generators of I and J, respectively. Analogously,

$$\frac{\mathcal{O}_k}{I} \approx \frac{\mathbb{C}[z_1]}{I'''} \text{ and } \frac{\mathcal{O}_{n-t}}{J} \approx \frac{\mathbb{C}[z_2]}{J'''},$$

where I''' and J''' are the ideals in $\mathbb{C}[z_1]$ and $\mathbb{C}[z_2]$ generated by the $r - 1$-jets of the generators of I and J, respectively. Finally, the equality follows from

$$\frac{\mathbb{C}[z_1]}{I'''} \otimes_{\mathbb{C}} \frac{\mathbb{C}[z_2]}{J'''} = \frac{\mathbb{C}[z_1, z_2]}{I'' + J''},$$

where $\otimes_{\mathbb{C}}$ denotes the tensor product, see [7, Proposition 2.7.13]. □

We observe that the previous result gives a simpler proof to the equality of [12] about the Milnor numbers. Finally, we relate the Bruce–Roberts numbers $\mu_{BR}(F, \tilde{X})$ and $\mu_{BR}(f, X)$.

Corollary 5.2. *Let $(\tilde{X}, 0)$, and $(X, 0)$ as before, and*

$$\begin{aligned} F : (\mathbb{C}^n, 0) &\to (\mathbb{C}, 0), \\ (z_1, z_2) &\mapsto f(z_1) + g(z_2) \end{aligned}$$

then:

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(a) *F is $\mathcal{R}_{\tilde{X}}$-finitely determined if, and only if, f is $\mathcal{R}_X$-finitely determined and g has isolated singularity.*

(b) *If F is $\mathcal{R}_{\tilde{X}}$-finitely determined, $\mu_{BR}(F, \tilde{X}) = \mu(g)\mu_{BR}(f, X)$.*

Proof. It is a consequence of the characterization of $\Theta_{\tilde{X}}$ and the previous theorem. □

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