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Analytical mechanics approaches in the dynamic modelling of Delta mechanism

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SUMMARY
The increasing importance of computational models for the design of complex mechanical systems raises a discussion on defining some criteria for the selection of adequate modelling methods. This paper aims to contribute to such discussion from an educational point of view. By choosing the Delta parallel mechanism as a typical representative of multi-body mechanical systems, four approaches – one based on the Principle of Virtual Work, two based on Lagrange's formalism, and one based on Kane's formalism – are analysed from the perspective of modelling procedures. Finally, inverse dynamic simulations are carried out along with qualitative comparisons of the considered approaches.

KEYWORDS: Parallel mechanisms; Manipulators; Kinematics; Dynamics; Analytical mechanics; Virtual work; Lagrange; Kane; Numerical simulation.

Nomenclature

a_k	Arbitrary length
$\boldsymbol{a}_k$	Acceleration vector
C	Constraint matrix
$\mathrm{c}_{k,j}$	$\cos(q_{k,j})$
c_{ϕ_k}	$\cos(\phi_k)$
$\mathrm{d}(\cdot)$	Differential operator
$\boldsymbol{F}_k$	Force vector
f_i	Holonomic constraint equation
g	Acceleration of gravity
h	Number of holonomic constraint equations
$\boldsymbol{I}_{k,i}$	Moment of inertia of the k-th rigid body around its i-th principal axis
J	Constraint equations Jacobian
n	Number of generalized coordinates defined
M	Inertia matrix
m_k	Arbitrary mass
Q_i	Generalized active force associated with the coordinate q_i
Q_i^*	Generalized inertia force associated with the coordinate q_i
$\widehat{Q}_i^*$	Generalized giroscopic inertia forces associated with the coordinate q_i
q_i	Generalized coordinate
$\boldsymbol{R}_\ell$	Resultant of active forces

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$\boldsymbol{R}_\ell^*$ Resultant of inertia forces
$\boldsymbol{r}_k$ Position vector
$\mathrm{s}_{k,j}$ $\sin(q_{k,j})$
s_{ϕ_k} $\sin(\phi_k)$
T Kinetic energy
$\boldsymbol{T}_\ell$ Resultant of active torques
$\boldsymbol{T}_\ell^*$ Resultant of inertia torques
u_i Generalized speed
V Potential energy
$\boldsymbol{v}_k$ Velocity vector
$\widetilde{\boldsymbol{v}}_{kj}$ j-th partial velocity of the k-th point
$\boldsymbol{\beta}_{kj}$ See equation (14)
γ Number of constraint equations
$\delta(\cdot)$ Variation operator
λ_r Lagrangian multiplier
μ_i Partial derivative of the constraint equation f_i with respect to time
ν Number of degrees of freedom
τ_k Control torque provided by the k-th actuator
$\boldsymbol{\psi}_{kj}$ See equation (25)
$\boldsymbol{\omega}_\ell$ Angular velocity
$\widetilde{\boldsymbol{\omega}}_{\ell j}$ j-th partial angular velocity of the ℓ-th point
$\overline{\delta W}$ Virtual work
$\overline{\delta \xi}$ Arbitrary column vector diffeomorphic to a vector of ν independent variations of generalized coordinates

1. Introduction

Recent advances in the development of new computational tools are increasingly simplifying the modelling of complex mechanical systems, taking advantage of systematic algorithms that can be built using approaches from several methods of analytical mechanics. Commercial software packages such as ADAMS®, SIMPACK®, SD/FAST® and others are some examples of computational tools that use analytical mechanics-based algorithms. However, modelling some simpler mechanical systems does not require the use of specific commercial software packages. In some cases, complete derivation of dynamic models and corresponding numerical simulations can be performed using generic computational tools such as Wolfram Mathematica®, Maple®, wxMaxima, MATLAB®, GNU Octave, Scilab and others. Although such an approach requires some programming skills, it allows a wider comprehension of the modelling process (avoiding erroneous analysis of the systems due to modelling mistakes) and more versatility (concerning modifications in parameters and input/output variables). The ability to select an adequate method to derive the equations of motion of a mechanical system is of paramount importance to explore such advantages. Obviously, the suitability of an approach depends on particular characteristics of a mechanical system so that no method will be the best for every system modelled.

The present work proposes a discussion on the use of analytical mechanics-based approaches to model multi-body mechanical systems, comparing qualitatively their applications with Delta mechanism.[2,3] The choice of such mechanism is due to two reasons: it is a typical representative of parallel (closed-loop) kinematic structures and its dynamics was and is still extensively studied in the literature.[1,19] Despite many advantages presented by parallel mechanisms in terms of practical applications when compared with their serial (open-loop) counterparts, the dynamic modelling of these systems is still a difficult task.[15,16]

Essentially, some of the usual approaches applied in the modelling of parallel mechanisms are the adapted versions of successful and traditional methods for serial mechanisms.[5,21] Also, most of these approaches restrict the choice of modelling coordinates and the definitions of generalized speeds, requiring sometimes the elimination of redundant variables to achieve final equations. Other aspect recurrently observed in some of these approaches is the great number of mathematical operations needed to derive dynamic equations, which is sometimes due to an inadequate choice of modelling

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variables. Moreover, it is common to find authors adopting simplifying hypotheses, which are at first unnecessary, in order to avoid some intricate steps of some modelling methods and to eliminate some complex terms from the resulting dynamic equations.

Generally, the methods employ only two sets of coordinates when dealing with parallel mechanisms: one to describe the motion of the end-effector, and the other to describe the motions imposed by the actuators of the system.[6] In a particular case of a Delta parallel mechanism, such an approach results in some difficulties to describe the motion of parallelograms of chains. In order to avoid the influence of such difficulties in the derivation of the dynamic model, some authors adopt simplifying hypotheses with respect to the inertia properties of these components.[4,14] Even when these two sets of coordinates are used along with their respective generalized momenta (which are the generalized speeds of Hamilton's approach), these difficulties on the description of the dynamics of the parallelograms remain, leading to the adoption of the same type of hypotheses.[17]

In such cases, a simple strategy to develop either a comprehensive or a simplified mathematical model is to define redundant generalized coordinates to simplify as much as possible the expressions of velocities of significant points, angular velocities of relevant bodies, and that of all the generalized forces. Staicu[20] uses (passive) joint coordinates, together with end-effector and actuator coordinates, to model the Delta parallel mechanism. Dealing with these coordinates involves a cumbersome number of matrix transformations. However, using methods derived from the Principle of Virtual Work, a comprehensive inverse dynamic analysis is performed, determining the time histories of actuator torques associated with a given motion of the system.

Other usual dynamic approaches that have been successfully adapted from applications to serial mechanisms are the recursive methods. Khan *et al.*[10] present a brief discussion on the use of recursive and non-recursive analytical mechanics methods to be applied to constrained mechanical systems such as parallel mechanisms. The authors also presented modular recursive methods for modelling parallel architecture manipulators, based on the use of the Newton–Euler equations of motion, and of the decoupled natural orthogonal complement matrices associated with constraints. Although preserving all the advantages of recursive methods (both in terms of modelling and numerical simulations), a large number of matrix operations involved can make it difficult to both interpret and find eventual modelling mistakes in the system of equations so obtained.

Concerning this discussion on the suitability of different approaches for modelling complex mechanical systems, this paper aims to have an educational purpose. First of all, it briefly presents three traditional analytical mechanics methods (Principle of Virtual Work, Lagrange's and Kane's equations). Moreover, it proposes an integration of these methods with a general kinematic approach, which is adequate for studying not only parallel mechanisms but also many other holonomic (or simple nonholonomic) constrained mechanical systems. The kinematic techniques presented are both simple and general, not restricting definitions of variables and not favouring the choice of any kind of coordinates as "principal" or "fundamental," relatively to other possibilities. It is also briefly discussed as how the use of redundant generalized coordinates can lead to simplifications in the modelling of these systems. Considering that simulations are performed using computational algorithms, all the presented approaches lead to mathematical models in matrix forms. Also, the concept of "constraint matrix" is explored in a slightly different form when compared with similar approaches found in the literature,[9,10] keeping all the advantages of these techniques, but being more simply interpretable. It is also shown that the non-uniqueness of these constraint matrices opens the possibility of implementing several algorithms to find such matrices, some of these not even requiring the selection of an "independent set" of coordinates to be performed. Finally, a qualitative comparison among the applications of these methods to the Delta mechanism can lead to a rudimentary guide for selecting analytical dynamic approaches to be applied in the modelling of similar mechanical systems.

In Section 2, some aspects about kinematics and constraint equations are discussed. In Section 3, the formalisms to develop dynamic equations using the Principle of Virtual Work, Lagrange's and Kane's approaches are presented and compared. In Section 4, the Delta mechanism is presented, and some considerations about its symmetries and how these can be used to simplify the dynamic model are introduced. In Section 5, the dynamic equations of the Delta mechanism are derived using the approaches presented previously, and some comparisons are carried out regarding simulation methods. In addition, some numerical simulation results are shown along with a summary of

qualitative comparisons among models obtained by different approaches. Finally, Section 6 presents the Conclusions.

2. Constraints and Kinematics

Mechanisms, generally, can be modelled as mechanical systems consisting of some connected solid bodies. Each pair of linked solid bodies constitutes a joint. The joints are classified according to the provided relative degrees-of-freedom (DOF). Generally, a mechanism has two most relevant parts: the base and the end-effector, whose motion relative to the base is the main object of study in a modelling process. In order to identify these relative motions, it is useful to analyse the motion constraints imposed by each chain of bodies linking the base to the end-effector (in a parallel mechanism there are more than one chain linking these main parts, while in a serial mechanism there is only one chain). Such an approach is a mobility analysis based on the method of the Lie group of rigid body displacements.[7] It is worth noting that in mechanisms the solid bodies are generally stiffer than the parts used to assemble them. Thus, it is common to consider such bodies as rigid, which significantly simplifies the modelling process, once all the equations generated are either algebraic or ordinary differential equations (that is, none of the equations of motion is a partial differential equation).

Once all the relative motions in the joints have been correctly identified, and the number of degrees of freedom of the system are properly determined, it is necessary to define a set of generalized coordinates. Some coordinates that naturally emerge from the mobility analysis can be classified in the following three sets:

- The first set consisting of coordinates that describe all the possible motions of the end-effector.
- The second set consisting of coordinates that describe the motions imposed by the actuators.
- The third set consisting of auxiliary coordinates defined to describe motions of links of the mechanism's chains which cannot be described trivially by the coordinates of the two first sets (these variables have to be carefully chosen, otherwise the increase in the number of variables might not lead to a significant benefit).

Sometimes, even the definition of the fourth set of coordinates may be necessary to simplify the expressions of the generalized active forces.

If n generalized coordinates are defined to model a ν-DOF mechanical system, then $\gamma = n - \nu$ independent constraint equations must be satisfied by such coordinates. The constraints of a mechanical system can be classified as holonomic or nonholonomic.[11] The former category includes all the constraints that can be expressed as functions of the coordinates of the system and the time only. These equations cannot involve the differentials of any of these variables. It is worth noting that sometimes integrable linear differential forms will appear as constraint equations. These constraints are holonomic because, being integrable, the differential form can be replaced by a function of the coordinates and time only. Suppose that a mechanical system has h holonomic constraints. These constraints must be expressed as:

$$f_i(q_1, \ldots, q_n, t) = 0 \quad \text{for} \quad i = 1, \ldots, h, \tag{1}$$

or in differential form as:

$$\sum_{j=1}^{n} \frac{\partial f_i}{\partial q_j} \mathrm{d}q_j + \frac{\partial f_i}{\partial t} \mathrm{d}t = 0 \quad \text{for} \quad i = 1, \ldots, h \tag{2}$$

All constraints that do not satisfy these conditions are called nonholonomic (including those that cannot even be expressed as equations). A remarkable case of nonholonomic constraints are those that can be expressed as a linear differential form, where all the coefficients of the differentials of the coordinates are functions of the system coordinates and the time only. These constraints are sometimes referred as simple nonholonomic.[8] Dealing with nonholonomic constraints that are not simple is out of the scope of this work. Considering that the system has $\bar{h} = \gamma - h$ simple nonholonomic constraints,

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it is possible to express these as the following non-integrable differential forms:

$$\sum_{j=1}^{n} J_{ij}\,\mathrm{d}q_j + \mu_i\,\mathrm{d}t = 0 \quad \text{for} \quad i = h+1, \ldots, \gamma. \tag{3}$$

Generally, if in a mechanical system all the constraints can be classified as holonomic and simple nonholonomic (which includes a wide range of systems commonly modelled), it is possible to define $J_{ij} = \partial f_i/\partial q_j$ and $\mu_i = \partial f_i/\partial t$ for $i = 1, \ldots, h$ and $j = 1, \ldots, n$, so that all the constraint equations can be expressed as a simple matrix equation:

$$J\,\mathrm{d}q + \mu\,\mathrm{d}t = 0. \tag{4}$$

The matrix J is the Jacobian of the constraint equations.

However, analytical mechanics-based methods do not deal with the differentials of the system coordinates, but with its variations (which are considered independently in the course of time). Thus, the constraint equations involving such variations can be expressed in the following matrix form:

$$J\,\delta q = 0. \tag{5}$$

Noting that J is a $\gamma \times n$ matrix that can be a function of the system coordinates and the time, there will be configurations of the system for which the rank of the matrix J is γ (which are called non-singular configurations). In such cases it is possible to choose γ columns of J that will constitute an invertible $\gamma \times \gamma$ matrix, which will be called J°. The matrix constituted by the remaining columns is an $\gamma \times \nu$ matrix, which will be called J^{*}, such that the constraint equations can be expressed as:

$$\begin{aligned} &J^{\circ}\delta q^{\circ} + J^{*}\delta q^{*} = 0, \\ &\delta q^{\circ} = -\left(J^{\circ}\right)^{-1} J^{*}\delta q^{*}. \end{aligned} \tag{6}$$

Consider any column vector $\overline{\delta\xi}$ with ν components (not necessarily constituted by exact variation differentials) such that there is a diffeomorphism between $\overline{\delta\xi}$ and δq^{*}. It can be stated that, in any non-singular configuration of a ν-DOF mechanical system, the variations of the n system coordinates (given by the vector δq) can be expressed as a function of ν independent variations only (which are the components of the vector $\overline{\delta\xi}$). If a linear diffeomorphism between $\overline{\delta\xi}$ and δq^{*} is defined, there will be a $n \times \nu$ matrix C, which, in this text, will be named *constraint matrix*, such that:

$$\delta q = C\,\overline{\delta\xi}. \tag{7}$$

It is important to note that a constraint matrix is not unique, once it depends on the choice of the vector $\overline{\delta\xi}$. Also, a constraint matrix is generally a function of the system configuration, that is, it can be a function of the time and the generalized coordinates. Furthermore, it is also remarkable that, once the components of $\overline{\delta\xi}$ are arbitrary, the substitution of Eq. (7) in (5) leads to:

$$J\,C\,\overline{\delta\xi} = 0 \qquad \Rightarrow \qquad J\,C = 0. \tag{8}$$

Conversely, it can be stated that any matrix C satisfying Eq. (8) can be considered a constraint matrix to the system.

In order to illustrate this method, consider a spherical pendulum mounted on a horizontal base that moves vertically (Z-direction) according to a given law $\zeta(t)$, see Fig. 1. The vector basis $\{\boldsymbol{n}_x, \boldsymbol{n}_y, \boldsymbol{n}_z\}$ and the coordinate system $OXYZ$ are attached to an inertial reference frame and $\{\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z\}$ is attached to the pendulum.

As this pendulum has 2 DOF, the angles θ and ϕ are enough to completely describe its motion: θ is the angle between the direction of the bar axis and the Z-axis, and ϕ is the angle between the projection of the bar axis in the OXY plane and the Y-axis. However, suppose that it is also desired to include in the model the rectangular coordinates x, y, z of the point P (measured in the system

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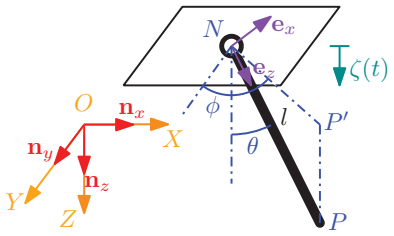

Fig. 1. Spherical pendulum constituted by a rigid thin bar of length l connected to a base which has an imposed vertical motion $\zeta(t)$.

$OXYZ$, with O and N coinciding when $\zeta(t) = 0$). Three constraint equations are then needed and these can be expressed by the following geometrical relations:

$$\begin{cases} x - l\sin\theta\sin\phi = 0 \\ y - l\sin\theta\cos\phi = 0 \\ z - l\cos\theta - \zeta(t) = 0 \end{cases}. \tag{9}$$

These holonomic constraint equations have the following differential form:

$$\begin{cases} \mathrm{d}x - l\cos\theta\sin\phi\,\mathrm{d}\theta - l\sin\theta\cos\phi\,\mathrm{d}\phi = 0 \\ \mathrm{d}y - l\cos\theta\cos\phi\,\mathrm{d}\theta + l\sin\theta\sin\phi\,\mathrm{d}\phi = 0 \\ \mathrm{d}z + l\sin\theta\,\mathrm{d}\theta - \zeta'(t)\,\mathrm{d}t = 0 \end{cases}. \tag{10}$$

In order to illustrate the use of non-exact differential forms to express the constraint equations in the matricial form (7), consider that the angular velocity vector of the pendulum can be expressed as $\boldsymbol{\omega} = -\dot{\theta}\,\boldsymbol{e}_x + \dot{\phi}\sin\theta\,\boldsymbol{e}_y - \dot{\phi}\cos\theta\,\boldsymbol{e}_z$. This expression motivates the definition of the non-exact differential $\overline{\mathrm{d}\psi} = \cos\theta\,\mathrm{d}\phi$. Thus, the variations of the system coordinates can be easily expressed as functions of $\overline{\delta\psi}$ and $\delta\theta$ as follows:

$$\begin{bmatrix} \delta x \\ \delta y \\ \delta z \\ \delta\theta \\ \delta\phi \end{bmatrix} = \begin{bmatrix} l\tan\theta\cos\phi & l\cos\theta\sin\phi \\ -l\tan\theta\sin\phi & l\cos\theta\cos\phi \\ 0 & -l\sin\theta \\ 0 & 1 \\ \sec\theta & 0 \end{bmatrix} \begin{bmatrix} \overline{\delta\psi} \\ \delta\theta \end{bmatrix}. \tag{11}$$

In this case, a linear solution for the constraint equations has been obtained, being possible to define a constraint matrix C such that $\delta q = C\,\overline{\delta\xi}$.

3. Analytical Mechanics-Based Methods

3.1. Methods based on the Principle of Virtual Work

The Principle of Virtual Work[11,13] states that the total virtual work done by the effective forces (which are the sum of active forces and inertia forces of each of the system particles, not including the constraint ones), denoted by $\overline{\delta W}^{(e)}$, is zero for all reversible variations that satisfy the kinematic constraints. Consider, for instance, a system constituted by N mass particles, with $\boldsymbol{F}_k$ denoting the applied forces on the particle k (except the constraint forces), m_k denoting its mass, $\boldsymbol{a}_k$ its acceleration measured in an inertial reference frame and $\delta\boldsymbol{r}_k$ denoting a variation on its position. The Principle of Virtual Work for this system can be stated as:

$$\overline{\delta W}^{(e)} = \sum_{k=1}^{N} (\boldsymbol{F}_k - m_k\boldsymbol{a}_k)\cdot\delta\boldsymbol{r}_k = 0. \tag{12}$$

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It is worth mentioning that the notation $\overline{\delta W}$ is used to highlight that the virtual work is not necessarily an exact variation differential. Moreover, when generalized coordinates are used, all the displacement variations ($\delta \boldsymbol{r}_k, k = 1, \ldots, N$) can be expressed as linear functions of the variations of the system coordinates ($\delta q_i, i = 1, \ldots, n$). Furthermore, if the system coordinates do not constitute an independent set of variables, but all the system constraints are holonomic or simple nonholonomic, as discussed in Section 2, it is possible to define ν variation differentials $\overline{\delta \xi_j}$ and to find a constraint matrix C (function of the system coordinates and time) such that $\delta q = C\,\overline{\delta \xi}$. Thus, it can be stated that:

$$\overline{\delta W}^{(e)} = \sum_{j=1}^{\nu} \left[\sum_{k=1}^{N} (\boldsymbol{F}_k - m_k \boldsymbol{a}_k) \cdot \boldsymbol{\beta}_{kj} \right] \overline{\delta \xi_j} = 0 \tag{13}$$

with the $\boldsymbol{\beta}_{kj}$ ($k = 1, \ldots, N$, $j = 1, \ldots, \nu$) being defined as:

$$\boldsymbol{\beta}_{kj} = \sum_{i=1}^{n} \frac{\partial \boldsymbol{r}_k}{\partial q_i} C_{ij}. \tag{14}$$

Considering that all the $\overline{\delta \xi_j}$ ($j = 1, \ldots, \nu$) are independent variations that satisfy the system constraints, the only way to ensure the nullity of the virtual work of the system for any possible variation is imposing that

$$\sum_{k=1}^{N} (\boldsymbol{F}_k - m_k \boldsymbol{a}_k) \cdot \boldsymbol{\beta}_{kj} = 0 \quad \text{for} \quad j = 1, \ldots, \nu. \tag{15}$$

This method for dealing with the system constraints leads to a set of dynamic equations with as much equations as the number of degrees of freedom of the modelled system.

Particularly, consider that this N mass particles system moves as a single rigid body. Because of the system constraints in this case, there is a variation vector $\overline{\delta \boldsymbol{\theta}}$ (with $\overline{\mathrm{d}\boldsymbol{\theta}} = \boldsymbol{\omega}\,\mathrm{d}t$, where $\boldsymbol{\omega}$ is the angular velocity of this rigid body) such that:

$$\delta \boldsymbol{r}_k = \delta \boldsymbol{r}^* + \overline{\delta \boldsymbol{\theta}} \times \boldsymbol{r}_k^*. \tag{16}$$

In this equation, $\boldsymbol{r}^*$ was taken to denote the position vector of the centre of mass of the system and $\boldsymbol{r}_k^* = \boldsymbol{r}_k - \boldsymbol{r}^*$, so that $\sum_{k=1}^{N} m_k \boldsymbol{r}_k^* = \boldsymbol{0}$. Also, for this system:

$$\boldsymbol{a}_k = \boldsymbol{a}^* + \dot{\boldsymbol{\omega}} \times \boldsymbol{r}_k^* + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \boldsymbol{r}_k^* \right) \tag{17}$$

where $\boldsymbol{a}^*$ is the acceleration of the centre of mass of the system.

Replacing Eqs. (16) and (17) in Eq. (12) leads, after some algebraic simplifications involving the use of some properties of scalar and cross products, to the following expression:

$$\overline{\delta W}^{(e)} = (\boldsymbol{R} + \boldsymbol{R}^*) \cdot \delta \boldsymbol{r}^* + (\boldsymbol{T} + \boldsymbol{T}^*) \cdot \overline{\delta \boldsymbol{\theta}} = 0, \tag{18}$$

where $\boldsymbol{R}$, $\boldsymbol{R}^*$, $\boldsymbol{T}$ and $\boldsymbol{T}^*$ are defined as:

$$\boldsymbol{R} = \sum_{k=1}^{N} \boldsymbol{F}_k, \tag{19}$$

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$$\boldsymbol{R}^* = -\sum_{k=1}^{N} m_k \boldsymbol{a}^* = -m\,\boldsymbol{a}^*, \tag{20}$$

$$\boldsymbol{T} = \sum_{k=1}^{N} \boldsymbol{r}_k^* \times \boldsymbol{F}_k, \tag{21}$$

$$\boldsymbol{T}^* = -\sum_{k=1}^{N} m_k \boldsymbol{r}_k^* \times \left[\dot{\boldsymbol{\omega}} \times \boldsymbol{r}_k^* + \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \boldsymbol{r}_k^*\right)\right] = -\boldsymbol{I} \cdot \dot{\boldsymbol{\omega}} - \boldsymbol{\omega} \times \boldsymbol{I} \cdot \boldsymbol{\omega}. \tag{22}$$

In Eq. (20), m represents the total mass of the system, and in Eq. (22), $\boldsymbol{I}$ represents the inertia dyadic of the system relative to its centre of mass. A complete derivation of Eq. (22) is easily found in the literature.[8]

In analogy to the derivation of Eq. (15), it can be stated that for a system with ν-DOF, comprising N_B rigid bodies, the dynamic equations can be given by:

$$\sum_{\ell=1}^{N_B} \left[\left(\boldsymbol{R}_\ell + \boldsymbol{R}_\ell^*\right) \cdot \boldsymbol{\beta}_{\ell j}^* + \left(\boldsymbol{T}_\ell + \boldsymbol{T}_\ell^*\right) \cdot \boldsymbol{\psi}_{\ell j}\right] = 0 \quad \text{for} \quad j = 1, \ldots, \nu. \tag{23}$$

In this case, $\boldsymbol{\beta}_{\ell j}^*$ and $\boldsymbol{\psi}_{\ell j}$ are given by the following expressions:

$$\boldsymbol{\beta}_{\ell j}^* = \sum_{i=1}^{n} \frac{\partial \boldsymbol{r}_\ell^*}{\partial q_i} C_{ij} = \sum_{i=1}^{n} \frac{\partial \boldsymbol{v}_\ell^*}{\partial \dot{q}_i} C_{ij}, \tag{24}$$

$$\boldsymbol{\psi}_{\ell j} = \sum_{i=1}^{n} \frac{\partial \boldsymbol{\omega}_\ell}{\partial \dot{q}_i} C_{ij}. \tag{25}$$

Equation (23) can be used for the dynamic modelling of every system constituted exclusively by rigid bodies. This approach has two main advantages: It is applicable to systems with redundant generalized coordinates (that is, when $n > \nu$), and there is no need of including any constraint force in the dynamic equations.

3.2. Lagrange's methods based on the extended Hamilton's principle

Hamilton's principle[11,13] states that, for any conservative mechanical system, it is null the variation of the time integral of the Lagrangean function between two defined configurations. For non-conservative mechanical systems, a slightly different version of this principle, called the Extended Hamilton's Principle, has to be used. The derivation of this principle can be easily done by means of the Principle of Virtual Work considering that:

$$m_k \ddot{\boldsymbol{r}}_k \cdot \delta \boldsymbol{r}_k = m_k \frac{\mathrm{d}}{\mathrm{d}t} \left(\dot{\boldsymbol{r}}_k \cdot \delta \boldsymbol{r}_k\right) - \delta T_k, \tag{26}$$

where $T_k = \frac{1}{2} m_k \dot{\boldsymbol{r}}_k \cdot \dot{\boldsymbol{r}}_k$. Defining $T = \sum_{k=1}^{N} T_k$ as the kinetic energy of the system, and replacing this identity in Eq. (12), the following expression is obtained:

$$\overline{\delta W}^{(a)} + \delta T = \sum_{k=1}^{N} m_k \frac{\mathrm{d}}{\mathrm{d}t} \left(\dot{\boldsymbol{r}}_k \cdot \delta \boldsymbol{r}_k\right), \tag{27}$$

with $\overline{\delta W}^{(a)}$ defined as:

$$\overline{\delta W}^{(a)} = \sum_{k=1}^{N} \boldsymbol{F}_k \cdot \delta \boldsymbol{r}_k. \tag{28}$$

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Consider now the time integral of Eq. (27) between two instants of time t_1 and t_2 such that $\delta \boldsymbol{r}_k(t_1) = \delta \boldsymbol{r}_k(t_2) = 0$ for $k = 1, \ldots, N$,

$$\int_{t_1}^{t_2} \left(\overline{\delta W}^{(a)} + \delta T \right) \mathrm{d}t = \sum_{k=1}^{N} \left[m_k \dot{\boldsymbol{r}}_k \cdot \delta \boldsymbol{r}_k \right]_{t_1}^{t_2} = 0. \tag{29}$$

The virtual work of the active forces of the system can be expressed as:

$$\overline{\delta W}^{(a)} = \sum_{i=1}^{n} Q_i \, \delta q_i. \tag{30}$$

The ith generalized (active) force Q_i is defined as:

$$Q_i = \sum_{k=1}^{N} \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{r}_k}{\partial q_i} = \sum_{k=1}^{N} \boldsymbol{F}_k \cdot \frac{\partial \boldsymbol{v}_k}{\partial \dot{q}_i}. \tag{31}$$

Moreover, using properties from variational calculus,[11] it can be stated that:

$$\delta \int_{t_1}^{t_2} T \, \mathrm{d}t = \int_{t_1}^{t_2} \left[\sum_{i=1}^{n} Q_i^* \delta q_i \right] \mathrm{d}t. \tag{32}$$

The ith generalized inertia force Q_i^* is defined as:

$$Q_i^* = \frac{\partial T}{\partial q_i} - \frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial T}{\partial \dot{q}_i} \tag{33}$$

Thus, Eq. (29) can be written in the following form:

$$\int_{t_1}^{t_2} \left[\sum_{i=1}^{n} (Q_i^* + Q_i) \delta q_i \right] \mathrm{d}t = 0. \tag{34}$$

In a system where redundant generalized coordinates are used, there are two methods for transforming Eq. (34). These methods will lead to different, but equivalent, systems of dynamic equations. The most common approach used in such cases is the Lagrangian multiplier method.[11] It uses the constraint equations (in implicit form) and some undetermined coefficients, named multipliers, to solve variational calculus problems in which some of the variations involved are constrained. Considering that the constraint equations of a system whose constraints are either holonomic or simple nonholonomic can be written simply as $\sum_{i=1}^{n} J_{ri} \, \delta q_i = 0$ for $r = 1, \ldots, \gamma$, the variational Eq. (34) can be solved by adding these constraint equations multiplied by undetermined coefficients λ_r $(r = 1, \ldots, \gamma)$ to the original one as shown:

$$\int_{t_1}^{t_2} \left[\sum_{i=1}^{n} \left(Q_i^* + Q_i + \sum_{r=1}^{\gamma} J_{ri} \lambda_r \right) \delta q_i \right] \mathrm{d}t = 0. \tag{35}$$

In this equation, there are γ undetermined parameters that can be chosen for making null the coefficients of δq_i for $i = \nu + 1, \ldots, n$, for instance. Then there will appear only ν variations δq_i

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in Eq. (35). As the system has ν-DOF, these δq_i $(i = 1, \ldots, \nu)$ must be independent so that their coefficients must be null in order to ensure the nullity of the integral for any possible variation in the system configuration. This procedure will lead to n dynamic equations in the following form:

$$Q_i^* + Q_i + \sum_{r=1}^{\gamma} J_{ri}\lambda_r = 0 \quad \text{for} \quad i = 1, \ldots, n. \tag{36}$$

Introducing the notation Q for the column vector whose components are Q_i, Q^* for the column vector whose components are Q_i^* $(i = 1, \ldots, n)$ and λ for the column vector whose components are λ_r $(r = 1, \ldots, \gamma)$, the dynamic Eqs. (36) can be put in the following matrix form:

$$Q^* + Q + J^{\mathrm{T}}\lambda = 0. \tag{37}$$

These n equations have $n + \gamma$ unknown variables (n coordinates and γ multipliers) so that they have to be solved along with the γ constraint Eqs. (4). In a system with a lot of constraint equations, this method requires the solution of system with much more equations than the number of coordinates, as the undetermined multipliers (generally of no interest) have to be determined along with the relevant variables in the analysis.

Another possibility to deal with Eq. (34) is to use a solution of the constraint equations (which is an explicit matrix form of these equations) such as the one presented in Eq. (7):

$$\int_{t_1}^{t_2} \delta q^{\mathrm{T}}(Q^* + Q)\mathrm{d}t = \int_{t_1}^{t_2} \overline{\delta\xi}^{\mathrm{T}} C^{\mathrm{T}}(Q^* + Q)\mathrm{d}t = 0. \tag{38}$$

As the variations $\overline{\delta\xi}_j$ $(j = 1, \ldots, \nu)$ are all independent, the only way to nullify the value of the integral for all possible independent variations in the configuration of the system is making:

$$C^{\mathrm{T}}(Q^* + Q) = 0. \tag{39}$$

This approach will lead to ν dynamic equations, which have to be solved along with the γ constraint Eqs. (4) in order to determine the solution for the n coordinates of the system. In this case, the number of equations is equal to the number of coordinates defined. This is an evident advantage of the approach using dynamic equations of the form (39) instead of the form (37). In the multi-body dynamics literature, formulations analogous to the presented in the derivation of Eq. (39) are often associated with Maggi's formalism.[12]

3.3. Kane's method

Kane's method can make use of generalized kinematic variables to replace the time derivatives of the generalized coordinates in the dynamic equations, simplifying them mathematically. Sometimes these new variables also favour the interpretation of some terms in the equations, being of more interest than the generalized coordinates themselves. This procedure is also used in Newton–Euler's, Hamilton's[11] and Poincaré's[22] methods. In the Newton–Euler's approach, the variables used to describe the motion of a rigid body are coordinates of position of the centre of mass, Euler angles, components of the velocity of the centre of mass and components of the angular velocity of the body (which, in general, are not equal to the time derivatives of Euler angles). Hamilton's approach replaces time derivatives of coordinates by generalized momenta. Poincaré's approach is a generalization of Lagrange's approach which enables the replacement of time derivatives of generalized coordinates by freely chosen generalized kinematic variables (although in most of the cases they will be the components of velocities, angular velocities or generalized momenta). In this aspect, Kane's approach is similar to Poincaré's approach. In the most general case, these kinematic variables (also called generalized speeds) are the n components of a column vector u, and there is an invertible affine transformation between u and $\dot{q}$ that can be described by means of explicit and known functions $X_{ij}(q, t)$ and $y_i(q, t)$

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as:

$$u_i = \sum_{j=1}^{n} X_{ij}(q,t)\,\dot{q}_j + y_i(q,t) \quad \text{for} \quad i = 1,\ldots,n. \tag{40}$$

This transformation can be written in matrix form as:

$$u = X(q,t)\,\dot{q} + y(q,t), \tag{41}$$

$$\dot{q} = V(q,t)\,u + w(q,t), \tag{42}$$

where $V = X^{-1}$ and $w = -X^{-1}y$. The literature presents several ways to define these new kinematic variables u_i $(i = 1,\ldots,n)$ as affine transformations of the time derivatives of the coordinates in Kane's approach.[8,18]

In a ν-DOF system where n coordinates are defined $(n > \nu)$, not all the variations of the coordinates are independent as are not the n variables u_i. As seen in Section 2, it is possible to find a constraint matrix $C(q,t)$ such that all the variations of the system coordinates can be written as a function of an independent set of variations only, which is expressed in Eq. (7), $\delta q = C\,\overline{\delta\xi}$. Defining $\widetilde{u} = \overline{\mathrm{d}\xi}/\mathrm{d}t$, it can be stated that $\dot{q} = C\,\widetilde{u} + \eta$, where η is a vector that satisfies $J\eta = -\mu$, as can be seen in Eq. (4). Replacing it in Eq. (41) leads to:

$$u = \widetilde{C}(q,t)\,\widetilde{u} + \widetilde{\eta}(q,t), \tag{43}$$

where $\widetilde{C} = XC$ and $\widetilde{\eta} = y + X\eta$. Equations (43) can be interpreted as the constraint equations written in terms of the generalized kinematic variables.

In order to obtain Kane's dynamic equations for a mechanical system, two important definitions are required: Given the velocity $\boldsymbol{v}_k$ of a point and the angular velocity $\boldsymbol{\omega}_\ell$ of a rigid body, the corresponding jth partial velocities and jth partial angular velocities associated are given by:

$$\widetilde{\boldsymbol{v}}_{kj} = \frac{\partial \boldsymbol{v}_k}{\partial\,\widetilde{u}_j} = \sum_{i=1}^{n} \frac{\partial \boldsymbol{v}_k}{\partial u_i}\widetilde{C}_{ij}, \tag{44}$$

$$\widetilde{\boldsymbol{\omega}}_{\ell j} = \frac{\partial \boldsymbol{\omega}_\ell}{\partial\,\widetilde{u}_j} = \sum_{i=1}^{n} \frac{\partial \boldsymbol{\omega}_\ell}{\partial u_i}\widetilde{C}_{ij}. \tag{45}$$

Consider, for instance, a system comprising N mass particles, with $\boldsymbol{F}_k$ denoting the applied force on particle k (except the constraint forces), m_k denoting its mass, $\boldsymbol{a}_k$ its acceleration and $\widetilde{\boldsymbol{v}}_{kj}$ denoting its partial velocity, both measured in an inertial reference frame. Kane's dynamic equations for this system can be written as follows:

$$\widetilde{F}_j^* + \widetilde{F}_j = 0 \quad \text{for} \quad j = 1,\ldots,\nu, \tag{46}$$

where:

$$\widetilde{F}_j = \sum_{k=1}^{N} \boldsymbol{F}_k \cdot \widetilde{\boldsymbol{v}}_{kj}, \tag{47}$$

$$\widetilde{F}_j^* = \sum_{k=1}^{N} (-m_k \boldsymbol{a}_k) \cdot \widetilde{\boldsymbol{v}}_{kj}. \tag{48}$$

Proceeding analogously to the development made in Section 3.1, for a system with ν-DOF,

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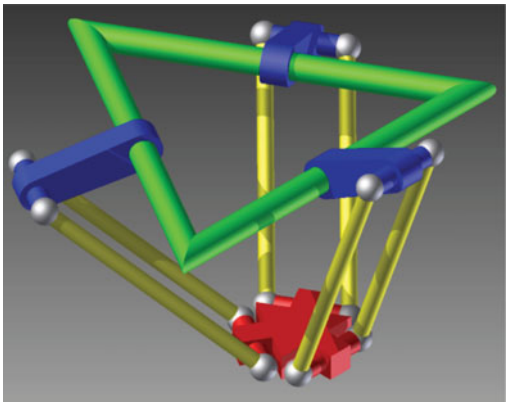

Fig. 2. Representation of a Delta parallel mechanism.

consisting of N_B rigid bodies, Kane's dynamic equations are given by the following expression:

$$\sum_{\ell=1}^{N_B}\left[\left(\boldsymbol{R}_\ell + \boldsymbol{R}_\ell^*\right)\cdot\widetilde{\boldsymbol{v}}_{\ell j}^* + \left(\boldsymbol{T}_\ell + \boldsymbol{T}_\ell^*\right)\cdot\widetilde{\boldsymbol{\omega}}_{\ell j}\right] = 0 \quad \text{for} \qquad j = 1, \ldots, \nu, \tag{49}$$

where the expressions of $\boldsymbol{R}$, $\boldsymbol{R}^*$, $\boldsymbol{T}$ and $\boldsymbol{T}^*$ are given in Eqs. (19)–(22) and $\widetilde{\boldsymbol{v}}_{\ell j}^*$ is the jth partial velocity of the centre of mass of the ℓth rigid body of the system. It is worth noting that if the trivial choice of u_i is made, that is, $u_i = \dot{q}_i$ for $i = 1, \ldots, n$, Kane's dynamic Eqs. (46) will be exactly the same as Eqs. (15), and Eqs. (49) will be exactly the same as Eqs. (18). It means that Kane's method can be interpreted as a generalization of methods based on the Principle of Virtual Work.

4. Delta Parallel Mechanism

Delta is a 3-DOF parallel symmetric mechanism, as represented in Fig. 2. Its three chains are commonly disposed in a triangular symmetry, connecting its base to its end-effector. Each chain is constituted by an arm that forms an active rotational joint with the fixed base, and by a spherical joints parallelogram connecting the other end of this arm to the end-effector.

Applying the Lie group of rigid body displacements approach,[7] it is possible to note that each of the chains individually allows for the end-effector three-dimensional translational motions and a rotational motion (around the same axis as the active rotational joint of the chain) relative to the mechanism base. As these active rotational joint axes are not parallel, no rotational motion of the end-effector relative to the base is allowed in this mechanism. That is, the 3 DOF of this mechanism are related to the three-dimensional translational motions of its end-effector relative to its base.

Thus, it can be stated that the Cartesian coordinates of any point of the end-effector, for instance its centre, are enough to fully describe the motion of the mechanism. Other possibility is to make the description by means of the coordinates representing the angles between the actuated arms and the mechanism's base plane. However, it is easy to note that both possibilities will lead to complex kinematic descriptions of bodies which are not the ones where these coordinates are defined. Another clear disadvantage of modelling this system with a minimum number of coordinates is the impossibility of exploring the existing symmetries among the mechanism's chains, once such coordinates by themselves do not evidence these symmetries.

In order to make an adequate analysis of this mechanism, some definitions are needed. First of all, the mechanism base will be called body 0, the end-effector will be called body 1 and the object manipulated by this mechanism (which is considered as rigidly attached to the end-effector) will be called body 2. The mechanism's chains will be referred as C_1, C_2 and C_3. The actuated arms will be called bodies 3, 5 and 7 respectively, and the pairs of lateral bars of each parallelogram (both describing the same rigid body motion in each parallelogram) will be called bodies 4, 6 and 8 respectively. Then it is necessary to define some coordinate systems in order to choose an adequate

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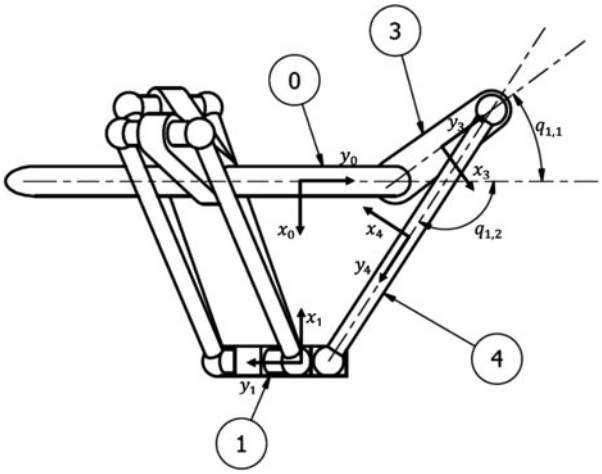

Fig. 3. Defining coordinate systems and generalized coordinates to model the Delta mechanism.

set of generalized coordinates to the model. This is done as shown in Fig. 3: one coordinate system will be defined for each body of the mechanism having its origins on the geometric centres of each of these parts. The coordinate systems of the bodies 5 and 7 are defined analogously to the coordinate system of body 3 as are the ones of bodies 6 and 8 defined analogously to the coordinate system of body 4.

Assuming that the chains are identical and are arranged symmetrically, some parameters can be defined: ϕ_k is the angle between the planes x_0y_0 and $x_{(2k+1)}y_{(2k+1)}$ ($k = 1, 2, 3$), a_1 is the distance between the centre of the base and the axis of any of the active rotational joints, a_2 is half of the length of the arms 3, 5 and 7, a_3 is half of the length of the lateral bars of the parallelograms (bodies 4, 6 and 8) and a_4 is the distance between the centre of the end-effector and the midpoint of the lower side of any parallelogram. The coordinates to be used in the model are defined in the following way: q_1, q_2 and q_3 are the Cartesian coordinates (x_0, y_0 and z_0) of the centre of the end-effector in the coordinate system of the mechanism's base, $q_{(3k+1)} = q_{k,1}$ are the angles between the planes $y_{(2k+1)}z_{(2k+1)}$ and y_0z_0 (measured in the positive direction of the z_0-axis), $q_{(3k+2)} = q_{k,2}$ are the angles between the planes $y_{(2k+2)}z_{(2k+2)}$ and y_0z_0 (measured in the positive direction of the z_0-axis) and $q_{(3k+3)} = q_{k,3}$ are the angles between the planes $x_{(2k+1)}y_{(2k+1)}$ and $x_{(2k+2)}y_{(2k+2)}$ (measured in the positive direction of $x_{(2k+2)}$ axis), for $k = 1, 2, 3$, as exemplified in Fig. 3 for the chain C_1.

5. Dynamic Modelling of Delta Mechanism

Before modelling the Delta mechanism, some simplifying hypotheses are adopted. All the bodies constituting this mechanism are considered perfectly rigid, and all the joints are considered ideal (neither friction effects nor clearances occur). Thus, only gravitational forces and torque inputs provided by the system actuators are considered as active efforts. All torque inputs are also considered as ideal, that is, they have only components τ_k applied in the direction of the respective actuated rotational joint axis of the kth chain, and the actuators never saturate, always providing the required torque for a particular motion of the system.

With these hypotheses, the constraint equations and dynamic equations will be derived using the coordinates defined in the previous section. The approaches to derive these equations will be compared, and some numerical simulations will be presented to verify the consistency of the results and to determine which is the best analytical mechanics approach to model the Delta mechanism.

5.1. Derivation of constraint equations

A strategy to obtain constraint equations of a parallel mechanism consists of the description of the motion of the end-effector using the coordinates defined for each chain. Proceeding in this way, the

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obtained constraint equations will explicitly relate the motion of each chain to the motion of the end-effector. The most important advantage of this approach is that if all the chains of a mechanism are identical, so are the forms of the constraint equations, which saves a large amount of time in its derivations, once it can be restricted to the derivation of these equations for a single chain. Using geometrical relations, the coordinates q_1, q_2 and q_3 of the geometric centre of the end-effector can be written as the functions of $q_{k,1}$, $q_{k,2}$ and $q_{k,3}$ leading to the following constraint equations (for $k = 1, 2, 3$):

$$f_{k,1}(q_i) = q_1 + 2a_2 \mathrm{s}_{k,1} + 2a_3 \mathrm{c}_{k,3} \mathrm{s}_{k,2} = 0, \tag{50}$$

$$f_{k,2}(q_i) = q_2 + 2\mathrm{s}_{\phi_k} a_3 \mathrm{s}_{k,3} - \mathrm{c}_{\phi_k}(a_1 - a_4 + 2a_2 \mathrm{c}_{k,1} + 2a_3 \mathrm{c}_{k,2} \mathrm{c}_{k,3}) = 0, \tag{51}$$

$$f_{k,3}(q_i) = q_3 - 2\mathrm{c}_{\phi_k} a_3 \mathrm{s}_{k,3} - \mathrm{s}_{\phi_k}(a_1 - a_4 + 2a_2 \mathrm{c}_{k,1} + 2a_3 \mathrm{c}_{k,2} \mathrm{c}_{k,3}) = 0. \tag{52}$$

In these equations, $\mathrm{c}_{k,j} = \cos(q_{k,j})$, $\mathrm{s}_{k,j} = \sin(q_{k,j})$, $\mathrm{c}_{\phi_k} = \cos(\phi_k)$ and $\mathrm{s}_{\phi_k} = \sin(\phi_k)$.

As it was shown in the previous sections, it is important to express these constraint equations in a differential form. As none of these constraints is explicitly dependent on time, the differential form of such equations can be represented by $J\,\mathrm{d}q = 0$ (or expressing in terms of the variations of the coordinates, $J\,\delta q = 0$). For this model, the matrix J is given by:

$$J = \begin{bmatrix} I_{3\times3} & J^{\langle1\rangle} & 0 & 0 \\ I_{3\times3} & 0 & J^{\langle2\rangle} & 0 \\ I_{3\times3} & 0 & 0 & J^{\langle3\rangle} \end{bmatrix}, \tag{53}$$

where $I_{3\times3}$ is a 3×3 identity matrix, and $J^{\langle k\rangle}$ ($k = 1, 2, 3$) is a 3×3 matrix whose general expression is:

$$J^{\langle k\rangle} = \begin{bmatrix} J_{1,1}^{\langle k\rangle} & J_{1,2}^{\langle k\rangle} & J_{1,3}^{\langle k\rangle} \\ J_{2,1}^{\langle k\rangle} & J_{2,2}^{\langle k\rangle} & J_{2,3}^{\langle k\rangle} \\ J_{3,1}^{\langle k\rangle} & J_{3,2}^{\langle k\rangle} & J_{3,3}^{\langle k\rangle} \end{bmatrix}, \tag{54}$$

$$\begin{aligned}
J_{1,1}^{\langle k\rangle} &= 2a_2 \mathrm{c}_{k,1}, \\
J_{1,2}^{\langle k\rangle} &= 2a_3 \mathrm{c}_{k,2} \mathrm{c}_{k,3}, \\
J_{1,3}^{\langle k\rangle} &= -2a_3 \mathrm{s}_{k,2} \mathrm{s}_{k,3}, \\
J_{2,1}^{\langle k\rangle} &= 2\mathrm{c}_{\phi_k} a_2 \mathrm{s}_{k,1}, \\
J_{2,2}^{\langle k\rangle} &= 2\mathrm{c}_{\phi_k} a_3 \mathrm{c}_{k,3} \mathrm{s}_{k,2}, \\
J_{2,3}^{\langle k\rangle} &= 2\mathrm{s}_{\phi_k} a_3 \mathrm{c}_{k,3} + 2\mathrm{c}_{\phi_k} a_3 \mathrm{c}_{k,2} \mathrm{s}_{k,3}, \\
J_{3,1}^{\langle k\rangle} &= 2\mathrm{s}_{\phi_k} a_2 \mathrm{s}_{k,1}, \\
J_{3,2}^{\langle k\rangle} &= 2\mathrm{s}_{\phi_k} a_3 \mathrm{c}_{k,3} \mathrm{s}_{k,2}, \\
J_{3,3}^{\langle k\rangle} &= -2\mathrm{c}_{\phi_k} a_3 \mathrm{c}_{k,3} + 2\mathrm{s}_{\phi_k} a_3 \mathrm{c}_{k,2} \mathrm{s}_{k,3}.
\end{aligned}$$

For instance, defining $\xi = [q_1\ q_2\ q_3]^{\mathrm{T}}$, in any non-singular configuration of this system it can be stated that:

$$\delta q = C\,\delta\xi \quad \text{with} \quad C = \begin{bmatrix} I_{3\mathrm{x}3} \\ -(J^{\langle1\rangle})^{-1} \\ -(J^{\langle2\rangle})^{-1} \\ -(J^{\langle3\rangle})^{-1} \end{bmatrix}. \tag{55}$$

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Clearly, as discussed in Section 2, the constraint matrix presented in Eq. (55) can be replaced by any other matrix satisfying the condition $JC = 0$, as established in Eq. (8).

Upon such considerations, the dynamic equations of this system can be readily derived by the approaches presented in Section 3.

5.2. Derivation of the equations of motion

Considering that the coordinates q_1, q_2 and q_3 completely describe any motion performed by the mechanism, it seems natural to choose $\delta\xi = [\delta q_1 \;\; \delta q_2 \;\; \delta q_3]^{\mathrm{T}}$ as a vector of independent variations of the configurations of the system. Such a choice leads to $\widetilde{u}_1 = \dot{q}_1$, $\widetilde{u}_2 = \dot{q}_2$, $\widetilde{u}_3 = \dot{q}_3$. Also, choosing $u = \dot{q}$, the dynamic equations obtained by Kane's methodology will be the same as those obtained by the Principle of Virtual Work, as discussed in Section 3. These equations will be referred in the remaining of the text as model K. Lagrange's equations can be presented in two different forms: one of them using undetermined multipliers as presented in Eq. (37), which will be called model $\mathrm{L_I}$, and the other using an explicit form for the constraint equations, as presented in Eq. (39), which will be called model $\mathrm{L_E}$.

In order to obtain such models, it is relevant to describe the velocities of the centres of mass (considered coincident with the geometric centres of each body) and the angular velocities of each rigid body of the mechanism, which can be given by the following expressions:

$$\boldsymbol{v}_1^* = \boldsymbol{v}_2^* = \dot{q}_1\,\boldsymbol{i}_0 + \dot{q}_2\,\boldsymbol{j}_0 + \dot{q}_3\,\boldsymbol{k}_0, \tag{56}$$

$$\boldsymbol{v}_{2k+1}^* = -a_2 \mathrm{c}_{k,1}\dot{q}_{k,1}\,\boldsymbol{i}_0 - \mathrm{c}_{\phi_k} a_2 \mathrm{s}_{k,1}\dot{q}_{k,1}\,\boldsymbol{j}_0 - \mathrm{s}_{\phi_k} a_2 \mathrm{s}_{k,1}\dot{q}_{k,1}\,\boldsymbol{k}_0, \tag{57}$$

$$\begin{aligned}\boldsymbol{v}_{2k+2}^* = {} & [-2a_2\mathrm{c}_{k,1}\dot{q}_{k,1} - a_3\mathrm{c}_{k,2}\mathrm{c}_{k,3}\dot{q}_{k,2} + a_3\mathrm{s}_{k,2}\mathrm{s}_{k,3}\dot{q}_{k,3}]\,\boldsymbol{i}_0 - [\mathrm{s}_{\phi_k} a_3\mathrm{c}_{k,3}\dot{q}_{k,3} + \mathrm{c}_{\phi_k}\upsilon]\,\boldsymbol{j}_0 \\ & + [\mathrm{c}_{\phi_k} a_3\mathrm{c}_{k,3}\dot{q}_{k,3} - \mathrm{s}_{\phi_k}\upsilon]\,\boldsymbol{k}_0,\end{aligned} \tag{58}$$

$$\upsilon = 2a_2\mathrm{s}_{k,1}\dot{q}_{k,1} + a_3\mathrm{c}_{k,3}\mathrm{s}_{k,2}\dot{q}_{k,2} + a_3\mathrm{c}_{k,2}\mathrm{s}_{k,3}\dot{q}_{k,3},\ \boldsymbol{\omega}_1 = \boldsymbol{\omega}_2 = \boldsymbol{0}, \tag{59}$$

$$\boldsymbol{\omega}_{2k+1} = \dot{q}_{k,1}\,\boldsymbol{k}_{2k+1}, \tag{60}$$

$$\boldsymbol{\omega}_{2k+2} = \dot{q}_{k,3}\,\boldsymbol{i}_{2k+2} + \mathrm{s}_{k,3}\dot{q}_{k,2}\,\boldsymbol{j}_{2k+2} + \mathrm{c}_{k,3}\dot{q}_{k,2}\,\boldsymbol{k}_{2k+2}. \tag{61}$$

Thus, the parcels of kinetic energy associated with each body of this mechanism are given by the following expressions (it is supposed that the local axes of each coordinate system are the respective principal inertia axes of each body so that $I_{i,j}$ represents the moment of inertia of the ith body around the jth principal axis):

$$T_1 + T_2 = \frac{1}{2}(m_1 + m_2)\left(\dot{q}_1^2 + \dot{q}_2^2 + \dot{q}_3^2\right), \tag{62}$$

$$T_{2k+1} = \frac{1}{2}\left(a_2^2 m_3 + I_{3,3}\right)\dot{q}_{k,1}^2, \tag{63}$$

$$\begin{aligned}T_{2k+2} = \frac{1}{2}\Big[& 4a_2^2 m_4\dot{q}_{k,1}^2 + \left(a_3^2 m_4 + I_{4,1}\right)\dot{q}_{k,3}^2 + \left(\mathrm{s}_{k,3}^2 I_{4,2} + \mathrm{c}_{k,3}^2\left(a_3^2 m_4 + I_{4,3}\right)\right)\dot{q}_{k,2}^2 \\ & + 4a_2 a_3 m_4\dot{q}_{k,1}[\mathrm{c}_{k,3}\cos(q_{k,1} - q_{k,2})\dot{q}_{k,2} + \mathrm{s}_{k,3}\sin(q_{k,1} - q_{k,2})\dot{q}_{k,3}]\Big].\end{aligned} \tag{64}$$

It can be stated that there is an n by n matrix $M(q, t)$ and an n column vector $\widehat{Q}^*(q, \dot{q}, t)$ such that:

$$Q^* = -M\ddot{q} + \widehat{Q}^*. \tag{65}$$

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For this particular system:

$$M = \begin{bmatrix} M^{\langle 0\rangle} & 0 & 0 & 0 \\ 0 & M^{\langle 1\rangle} & 0 & 0 \\ 0 & 0 & M^{\langle 2\rangle} & 0 \\ 0 & 0 & 0 & M^{\langle 3\rangle} \end{bmatrix}, \tag{66}$$

$$\widehat{Q}^* = \begin{bmatrix} (\widehat{Q}^*)^{\langle 0\rangle} \\ (\widehat{Q}^*)^{\langle 1\rangle} \\ (\widehat{Q}^*)^{\langle 2\rangle} \\ (\widehat{Q}^*)^{\langle 3\rangle} \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} Q^{\langle 0\rangle} \\ Q^{\langle 1\rangle} \\ Q^{\langle 2\rangle} \\ Q^{\langle 3\rangle} \end{bmatrix}, \tag{67}$$

where $M^{\langle 0\rangle} = (m_1 + m_2)I_{3x3}$, $(\widehat{Q}^*)^{\langle 0\rangle} = 0$ and the components of $M^{\langle k\rangle}$ and $(\widehat{Q}^*)^{\langle k\rangle}$ $(k = 1, 2, 3)$ are given by:

$$M_{1,1}^{\langle k\rangle} = a_2^2\,(m_3 + 4m_4) + I_{3,3}, \tag{68}$$

$$M_{2,2}^{\langle k\rangle} = \mathrm{s}_{k,3}^2 I_{4,2} + \mathrm{c}_{k,3}^2\left(a_3^2 m_4 + I_{4,3}\right), \tag{69}$$

$$M_{3,3}^{\langle k\rangle} = a_3^2 m_4 + I_{4,1}, \tag{70}$$

$$M_{1,2}^{\langle k\rangle} = M_{2,1}^{\langle k\rangle} = 2a_2 a_3 m_4 \mathrm{c}_{k,3}\cos(q_{k,1} - q_{k,2}), \tag{71}$$

$$M_{1,3}^{\langle k\rangle} = M_{3,1}^{\langle k\rangle} = 2a_2 a_3 m_4 \mathrm{s}_{k,3}\sin(q_{k,1} - q_{k,2}), \tag{72}$$

$$M_{2,3}^{\langle k\rangle} = M_{3,2}^{\langle k\rangle} = 0, \tag{73}$$

$$(\widehat{Q}^*)_1^{\langle k\rangle} = 4a_2 a_3 m_4\left(\cos(q_{k,1} - q_{k,2})\mathrm{s}_{k,3}\dot{q}_{k,2}\dot{q}_{k,3} - \frac{1}{2}\mathrm{c}_{k,3}\sin(q_{k,1} - q_{k,2})(\dot{q}_{k,2}^2 + \dot{q}_{k,3}^2)\right), \tag{74}$$

$$(\widehat{Q}^*)_2^{\langle k\rangle} = 2\mathrm{c}_{k,3}(a_2 a_3 m_4 \sin(q_{k,1} - q_{k,2})\dot{q}_{k,1}^2 + \mathrm{s}_{k,3}(a_3^2 m_4 - I_{4,2} + I_{4,3})\dot{q}_{k,2}\dot{q}_{k,3}), \tag{75}$$

$$(\widehat{Q}^*)_3^{\langle k\rangle} = -\mathrm{s}_{k,3}\left(2a_2 a_3 m_4 \cos(q_{k,1} - q_{k,2})\dot{q}_{k,1}^2 + \mathrm{c}_{k,3}\left(a_3^2 m_4 - I_{4,2} + I_{4,3}\right)\dot{q}_{k,2}^2\right). \tag{76}$$

The parcels of gravitational potential energy due to each body of this mechanism are given by:

$$V_1 + V_2 = -g\,(m_1 + m_2)\,q_1, \tag{77}$$

$$V_{2k+1} = g a_2 m_3 \mathrm{s}_{k,1}, \tag{78}$$

$$V_{2k+2} = g m_4\left(2a_2 \mathrm{s}_{k,1} + a_3 \mathrm{c}_{k,3}\mathrm{s}_{k,2}\right). \tag{79}$$

Thus, the generalized forces vector is given by $(k = 1, 2, 3)$:

$$Q^{\langle 0\rangle} = [(m_1 + m_2)g \quad 0 \quad 0]^{\mathrm{T}}, \tag{80}$$

$$Q_1^{\langle k\rangle} = -g a_2\,(m_3 + 2m_4)\,\mathrm{c}_{k,1} + \tau_k, \tag{81}$$

$$Q_2^{\langle k\rangle} = -g a_3 m_4 \mathrm{c}_{k,2}\mathrm{c}_{k,3}, \tag{82}$$

$$Q_3^{\langle k\rangle} = g a_3 m_4 \mathrm{s}_{k,2}\mathrm{s}_{k,3}. \tag{83}$$

Analogously, it is possible to express Kane's dynamic Eqs. (49) in the following matrix form:

$$\widetilde{M}\ddot{q} - \widetilde{Q}^* - \widetilde{Q} = 0. \tag{84}$$

Having the expressions of velocities of centres of mass and angular velocities of each body of the Delta mechanism, the corresponding accelerations, angular accelerations, partial velocities and partial

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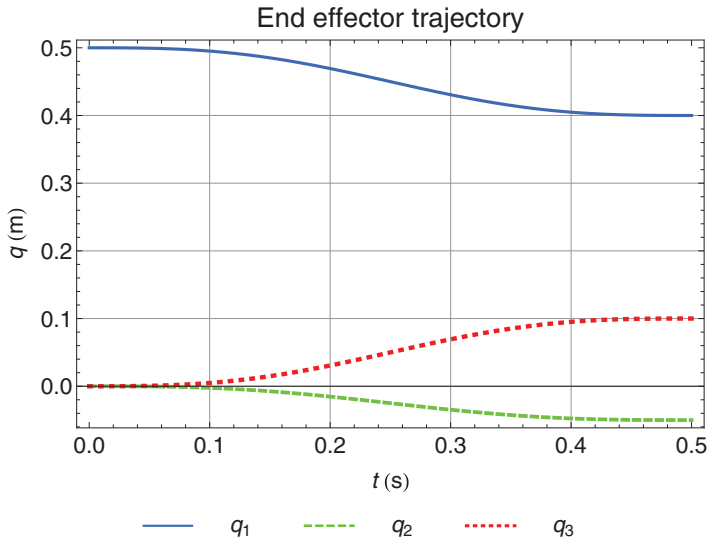

Fig. 4. Trajectory imposed to the end-effector in order to perform inverse dynamic numerical simulations with the models of the Delta mechanism.

angular velocities can be determined. Then, according to the approaches presented in Sections 3.1 and 3.3, the expressions of the matrix $\widetilde{M}$ and the column vectors $\widetilde{Q}^*$ and $\widetilde{Q}$ can also be determined.

Deriving Kane's dynamic equations for the Delta mechanism and simplifying these mathematically, after some factorizations, it can be noted that, as expected, the following identities hold:

$$\widetilde{M} = C^{\mathrm{T}} M, \tag{85}$$

$$\widetilde{Q}^* = C^{\mathrm{T}} \widehat{Q}^*, \tag{86}$$

$$\widetilde{Q} = C^{\mathrm{T}} Q. \tag{87}$$

That is, using the same variables, models K and $\mathrm{L_E}$ lead to exactly the same dynamic equations, although their derivations are slightly different. The former deals with vector functions (forces, torques, velocities, partial velocities, accelerations, angular velocities, partial angular velocities and angular accelerations), and the later with scalar functions, such as kinetic energy, potential energy and generalized forces.

In terms of the derivation of the dynamic equations, considering that the same software was employed (Wolfram Mathematica 8.0[23]), it can be stated that models $\mathrm{L_I}$ and $\mathrm{L_E}$ (Lagrange's equations) required almost the same computational effort, as both need the expressions of matrix M and the column vectors Q^* and Q. Model $\mathrm{L_I}$ is even simpler than model $\mathrm{L_E}$ because only the Jacobian J is required and not a constraint matrix C as in model $\mathrm{L_E}$. This is not a relevant aspect, in this particular system, once a constraint matrix C can be easily expressed in terms of the Jacobian J as shown in Eq. (55), but it may be an important issue for systems in which finding an explicit form of the constraint equations is not so straightforward.

Nevertheless, when comparing models $\mathrm{L_I}$ and $\mathrm{L_E}$ with model K (that in this case represents both the Principle of Virtual Work and Kane's approaches), it is noted that the computational effort to derive the dynamic equations is higher as it requires vector expressions (as accelerations and angular accelerations) that are significantly more complex than the expressions of kinetic energy. Also, it is more difficult to put the resulting equations in a matrix form adequate to numerical simulations.

5.3. Numerical simulations of the dynamic equations

In order to make a numerical performance assessment of the dynamic models, algorithms to perform inverse dynamic simulations were developed for each of these. As previously discussed, models K and $\mathrm{L_E}$ lead to the same dynamic equations, although their derivations are different. Thus, only two different numerical simulations need to be performed. The goal of these simulations is the calculation

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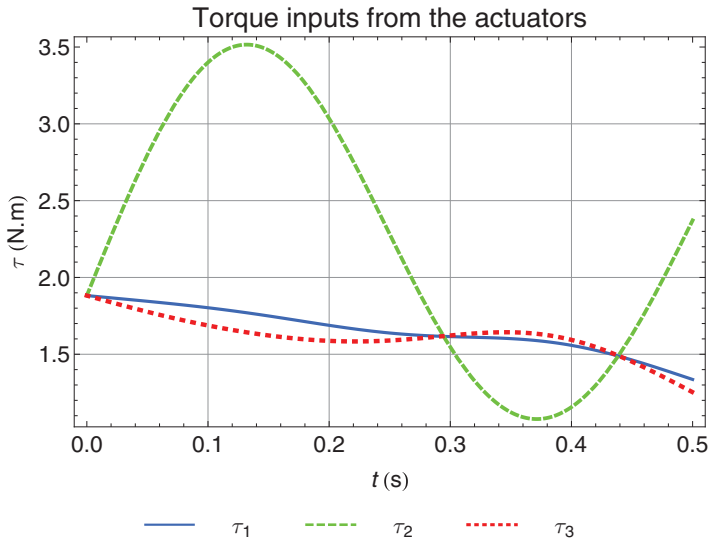

Fig. 5. Time history of torque inputs needed to perform trajectory in Fig. 4.

Table I. Qualitative assessment of models K, L_E and L_I .

Model	Derivation	Interpretability	Numerical computations
K	★★	★★★★★	★★★★★
L_E	★★★★	★★★	★★★★★
L_I	★★★★★	★★★	★★★

of torques in each of the system actuators corresponding to a prescribed motion of the end-effector. The trajectory imposed to the end-effector is shown on Fig. 4 and is defined as a straight line with null velocities and accelerations in its both extremities.

For the simulations, the following values of parameters were used: $a_1 = 0.20$ m, $a_2 = 0.10$ m, $a_3 = 0.25$ m, $a_4 = 0.10$ m, $\phi_1 = 0$, $\phi_2 = 2\pi/3$, $\phi_3 = -2\pi/3$, $m_1 = 0.30$ kg, $m_2 = 1.00$ kg, $m_3 = m_5 = m_7 = 0.40$ kg, $m_4 = m_6 = m_8 = 0.25$ kg. The inertia moments were estimated considering that all the bodies comprised ideal bars, using the corresponding mathematical formulas.

The results obtained by the numerical simulations of both models L_E (which has the same equations as model K) and L_I are, as expected, identical. These results are presented in Fig. 5. It can be stated that the torques required for the motion of this system do not have a much different magnitude compared with the torques required to maintain the equilibrium of the system in both the extremities of this trajectory, that is, large accelerations can be achieved without consuming a large amount of energy, as expected for a parallel mechanism.

5.4. Qualitative assessment of models K, L_E *and* L_I

In order to conclude the discussion on the suitability of each analytical mechanics approach to model the Delta parallel mechanism, a qualitative assessment is performed based on the results presented in the previous sections. This assessment consists of comparing the following three characteristics associated with these approaches:

1. *Derivation*: The effort to obtain dynamic equations of the model in a form that is suitable to numerical simulations.
2. *Interpretability*: The ability to understand the meaning of each term of dynamic equations.
3. *Computational effort in numerical simulations*: The kind of operations needed to perform a numerical simulation of the model.

For each of these characteristics, models K, L_E and L_I receive a grade from one to five stars, being awarded five stars to the method that stands out in a given characteristic in relation to others, as shown in Table I. It must be made clear that this assessment is purely qualitative and entirely based on the

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applications of the approaches to model the Delta mechanism. The intention of presenting this kind of evaluation is to guide the selection of an appropriate method to model other types of mechanical systems that have some similarities with the Delta mechanism (which includes, for example, a wide range of parallel mechanisms).

In terms of derivation, as discussed in Section 5.2, obtaining the expressions of inertia forces by Lagrange's approach is much simpler than by the Principle of Virtual Work and Kane's approaches, once it only involves partial derivatives of the kinetic energy, which is a scalar function, instead of requiring the vector expressions of accelerations and angular accelerations. Moreover, model K also requires an additional factorization to acquire the same matrix form as the model L_E. It is also worth commenting that deriving model L_I is even simpler as it does not require obtaining a constraint matrix.

On the one hand, dealing with the vector expressions of forces, torques, accelerations, angular accelerations, partial velocities and partial angular velocities, represents a disadvantage in terms of the derivation of the Principle of Virtual Work and Kane's formalisms, but on the other hand, it becomes a great advantage in terms of interpretability, once for each term in the dynamic equations the active or inertia force associated can be easily identified. This characteristic makes it easier not only to identify some modelling mistakes during the derivations but also to make eventual simplifications in the model.

Finally, concerning the computational effort in numerical simulations, it is worth commenting that the system of equations corresponding to models K and L_E had a better performance than the system of equations of model L_I for the simulations presented in Section 5.3. These simulations were performed in Wolfram Mathematica 8.0, on the same computer. When an algorithm based on Eq. (8) was used to obtain a constraint matrix, the simulation corresponding to models K and L_E took approximately 83% of the time of the simulation corresponding to model L_I (five simulations were performed for each system of equations, resulting in an average time of 1.38 s for the simulations corresponding to models K and L_E, and an average time of 1.65 s for the simulations corresponding to model L_I, considering only the time necessary to run the inverse dynamics functions). It is worth noting that this additional time observed in the execution of the simulation of model L_I is due to the fact that it has to deal with the inversion of higher order matrices so that in longer simulations, this difference in the simulation time becomes more relevant. Also, these higher order matrices are characteristically sparse (with several zero elements) so that, near singular configurations, it is expected that the inversion of such matrices will be more subjected to numerical errors than the inversion of not so sparse matrices, as those occurring in the system of equations of models K and L_E (this being an evident advantage for these models).

6. Conclusions

Analytical mechanics methods clearly demonstrate several advantages in comparison with those derived from vectorial mechanics, mainly when dealing with complex mechanical systems whose description of constraints is not straightforward. The modelling of the Delta parallel mechanism has shown that, for closed-loop kinematic chains, a lot of advantages can be obtained from the use of redundant generalized coordinates. The most relevant of these is the exploration of symmetries in the system, which significantly simplifies the derivation of both constraint and dynamic equations.

Four modelling methods were analyzed: One based on the Principle of Virtual Work, two based on Lagrange's equations and one based on Kane's formalism. Regarding the two methods based on Lagrange's approaches, one has the form of Lagrange's equations valid for constrained system and therefore uses the Lagrangian multipliers, while the other employs an explicit matricial formulation that eliminates these multipliers. In addition, a comparison was conducted to evaluate the capability of the four methods to deal with constraints and redundant coordinates, the simplicity to implement the software code, the interpretability of the generated equations and the computational effort in numerical simulations.

The methods based on Lagrange's approach have shown a lot of advantages in terms of software-aided derivation, being easier to manipulate the expressions of kinetic energy and generalized forces than to work with complex vector expressions of accelerations and angular accelerations as required in both the Principle of Virtual Work and Kane's approaches. Regarding the last two methods, it is important to note that these are operationally similar, although Kane's formalism adds some algebraic

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technics that enable more options on the choice of variables during the modelling process. In spite of the fact that these options were not explored here when modelling the Delta parallel mechanism, these can be very useful to model even more complex systems. In such cases, use of Poincaré's approach,[22] instead of Lagrange's approach, would be recommended to obtain dynamic equations in the same variables as in Kane's formalism in order to conduct fair comparisons.

In the case of the Delta parallel mechanism, once all models were derived by using the same variables, as expected, Kane's approach and the Principle of Virtual Work (model K) led to the same equations as those obtained by Lagrange's method by using constraint equations in an explicit form (model L_E). The generated model L_E was already in an adequate matrix form to develop simulation algorithms, while model K required additional algebraic work to acquire the same form. On the other hand, it seems easier to interpret the meaning of each term of the dynamic equations in the derivation of model K, when compared with model L_E.

Comparing Lagrange's method that uses the constraint equations in explicit form with the one that uses undetermined multipliers, it is possible to note that the former is slightly more complex in terms of derivation, depending on the type of algorithm chosen to find a constraint matrix for each time instant (once some of these algorithms may require more derivation steps). However, for a symmetrical system such as the Delta parallel mechanism, these further mathematical transformations can be as simple as the one presented in Eq. (55). Also, using the constraint equations in explicit form, the number of equations to be simulated numerically is equal to the total number of coordinates defined, while the use of undetermined multipliers adds more (generally irrelevant) variables and equations, leading to sparse matrix forms of higher orders (which demand a greater computational effort for numerical operations). Although no relevant difference was observed in the results of the performed numerical simulations, this last fact makes Lagrange's method using constraint equations in explicit form (without multipliers) the most interesting choice to be used in the modelling of a closed-loop kinematic chain, such as the Delta parallel mechanism, in terms of numerical simulation computations.

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