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A Q-WADGE HIERARCHY IN QUASI-POLISH SPACES

VICTOR SELIVANOV

Abstract. The Wadge hierarchy was originally defined and studied only in the Baire space (and some other zero-dimensional spaces). Here we extend the Wadge hierarchy of Borel sets to arbitrary topological spaces by providing a set-theoretic definition of all its levels. We show that our extension behaves well in second countable spaces and especially in quasi-Polish spaces. In particular, all levels are preserved by continuous open surjections between second countable spaces which implies e.g., several Hausdorff–Kuratowski (HK)-type theorems in quasi-Polish spaces. In fact, many results hold not only for the Wadge hierarchy of sets but also for its extension to Borel functions from a space to a countable better quasiorder Q.

§1. Introduction. The classical Borel, Luzin, and Hausdorff hierarchies in Polish spaces, which are defined using set operations, play an important role in descriptive set theory (DST). Recently, these hierarchies were extended and shown to have similar nice properties also in quasi-Polish spaces [4] which include many non-Hausdorff spaces of interest for several branches of mathematics and theoretical computer science.

The Wadge hierarchy, introduced in [33, 34], is nonclassical in the sense that it is based on a notion of reducibility that was not recognized in the classical DST, and on using ingenious versions of Gale-Stewart games rather than on set operations. For subsets A, B of the Baire space $\mathcal{N} = \omega^\omega$, A is *Wadge reducible* to B ($A \leq_W B$), if $A = f^{-1}(B)$ for some continuous function f on $\mathcal{N}$. The quotient-poset of the preorder $(P(\mathcal{N}); \leq_W)$ under the induced equivalence relation $\equiv_W$ on the power-set of $\mathcal{N}$ is called *the structure of Wadge degrees* in $\mathcal{N}$. W. Wadge [34] characterized the structure of Wadge degrees of Borel sets (i.e., the quotient-poset of $(\mathbf{B}(\mathcal{N}); \leq_W)$) up to isomorphism. In particular, this quotient-poset is semi-well-ordered, hence it is well-founded and has no three pairwise incomparable elements. For more information on Wadge degrees see [10, 36].

This gives rise to the Wadge hierarchy $\{\mathbf{\Sigma}_\alpha(\mathcal{N})\}_{\alpha<\upsilon}$ (for a rather large ordinal υ) in $\mathcal{N}$ which is a great refinement of the Borel hierarchy (for more information see the next section where we also give precise definitions of other notions mentioned in this introduction). The Wadge hierarchy was originally defined only for the Baire space. Using the methods of [34] it is easy to check that the structure $(\mathbf{B}(X); \leq_W)$ of Wadge degrees of Borel sets in any zero-dimensional Polish space X remains

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semi-well-ordered and the Wadge hierarchy in such spaces looks rather similar to that in the Baire space.

The Wadge hierarchy of sets was an important development in classical DST not only as a unifying concept (it subsumes all hierarchies known before) but also as a useful tool to investigate second countable zero-dimensional spaces. We illustrate this with two examples. In chapter 4 of [31] a complete classification (up to homeomorphism) of homogeneous zero-dimensional absolute Borel sets was achieved, completing a series of earlier results in this direction. In theorem 2.4 of [32] it was shown that any Borel subspace of the Baire space with more than one point has a nontrivial auto-homeomorphism.

In this paper we attempt to find the "correct" extension of the Wadge hierarchy from Polish zero-dimensional spaces to arbitrary second countable spaces. There are at least three approaches to this problem.

The first approach is to show that Wadge reducibility in such spaces behaves similarly to its behavior in the Baire space, i.e., it is a semi-well-order. Unfortunately, this is not the case: for many natural quasi-Polish spaces X the structure $(\mathbf{B}(X); \leq_W)$ is not well-founded and has antichains with more than two elements (see e.g., [1, 5, 8, 19]). Thus, this approach does not lead to a reasonable extension of the Wadge hierarchy to quasi-Polish spaces.

The second approach was independently suggested in [16, 27]. The approach is based on the characterization of quasi-Polish spaces as the second countable T_0-spaces X such that there is a total admissible representation ξ from $\mathcal{N}$ onto X [4]. Namely, one can *define* the Wadge hierarchy $\{\mathbf{\Sigma}_\alpha(X)\}_{\alpha<\upsilon}$ in X by $\mathbf{\Sigma}_\alpha(X) = \{A \subseteq X \mid \xi^{-1}(A) \in \mathbf{\Sigma}_\alpha(\mathcal{N})\}$. One easily checks that the definition of $\mathbf{\Sigma}_\alpha(X)$ does not depend on the choice of ξ, $\bigcup_{\alpha<\nu} \mathbf{\Sigma}_\alpha(X) = \mathbf{B}(X)$, $\mathbf{\Sigma}_\alpha(X) \subseteq \mathbf{\Delta}_\beta(X)$ for all $\alpha < \beta < \upsilon$, and any $\mathbf{\Sigma}_\alpha(X)$ is downward closed under the Wadge reducibility in X. This definition is short and elegant but it gives no real understanding of how the levels $\mathbf{\Sigma}_\alpha(X)$ look like, in particular their set-theoretic descriptions are completely unclear.

The third approach consists in set-theoretic description of subsequent refinements of the Borel hierarchy. It was thoroughly studied in [4] for the Borel and Hausdorff hierarchies in quasi-Polish spaces. This study was continued in [26, 27] for an increasing sequence of pointclasses $\{\mathbf{\Sigma}_\alpha(X)\}_{\alpha<\lambda}$ which exhaust the sets of finite Borel rank, where $\lambda = sup\{\omega_1, \omega_1^{\omega_1}, \omega_1^{\omega_1^{\omega_1}}, \ldots\}$. These classes were conjectured to coincide with the corresponding classes from the second approach and the conjecture was verified for $\mathbf{\Sigma}_\alpha(X)$ with $\alpha < \omega_1^{\omega_1}$ (it follows from corollary 5.10 in [27] and theorem 2 in [26]). Thus, we proposed a way to achieve a reasonable set-theoretic definition of the Wadge hierarchy in X defined as in the second approach.

In this paper we propose a set-theoretic definition for the whole Wadge hierarchy of Borel sets from the second approach. The definition is an infinitary version of the so called fine hierarchy introduced and studied in a series of my publications (see e.g., [18, 23] for a survey). In fact, this paper develops a "classical" infinitary version of the effective finitary version of the Wadge hierarchy in effective spaces and computable quasi-Polish spaces recently developed in [28]. Arguably, our infinitary fine hierarchy (IFH), and hence also the Wadge hierarchy, is a kind of "iterated difference hierarchy" over levels of the Borel hierarchy; it only remains to make precise how to "iterate" the difference hierarchies.

Along with describing (hopefully) the right version of the Wadge hierarchy (by identifying it with the IFH) in arbitrary spaces we show that it behaves well in second countable spaces and especially in quasi-Polish spaces. E.g., it provides the description of all levels $\mathbf{\Sigma}_\alpha(X)$ in quasi-Polish spaces (Theorem 6). Also, all levels of the IFH are preserved by continuous open surjections between second countable spaces which gives a broad extension of results by Saint Raymond and de Brecht for the Borel and Hausdorff hierarchies [4, 17] (Theorem 2). In §4.3 we show that several HK-type theorems are inherited by the continuous open images which yields some such theorems in arbitrary quasi-Polish spaces.

Notions and results of this paper apply not only to the Wadge hierarchy of sets discussed so far but also to a more general hierarchy of functions $A : X \to Q$ from a space X to an arbitrary quasiorder Q. We identify such functions with *Q-partitions of X* of the form $\{A^{-1}(q)\}_{q\in Q}$ in order to stress their close relation to k-partitions (obtained when $Q = \bar{k} = \{0, \dots, k-1\}$ is an antichain with k-elements) studied e.g., in [7, 21, 26, 27].

For Q-partitions A, B of X, let $A \leq_W B$ mean that there is a continuous function f on X such that $A(x) \leq_Q B(f(x))$ for each $x \in X$. The case of sets corresponds to the case of two-partitions. Let $\mathbf{B}(Q^X)$ be the set of Borel Q-partitions A (for which $A^{-1}(q) \in \mathbf{B}(X)$ for all $q \in Q$). A celebrated theorem of van Engelen, Miller and Steel (see theorem 3.2 in [32]) shows that if Q is a countable better quasiorder (bqo) then $\mathcal{W}_Q = (\mathbf{B}(Q^{\mathcal{N}}); \leq_W)$ is a bqo. Although this theorem gives an important information about the quotient-poset of $\mathcal{W}_Q$, it is far from a characterization.

Many efforts (see e.g., [7, 21, 26, 27] and references therein) to characterize the quotient-poset of $\mathcal{W}_Q$ were devoted to *k-partitions of $\mathcal{N}$*. Our approach in [21, 26, 27] to this problem was to characterize the initial segments $(\mathbf{\Delta}^0_\alpha(k^{\mathcal{N}}); \leq_W)$ for bigger and bigger ordinals $2 \leq \alpha < \omega_1$. To achieve this, we defined structures of iterated labeled trees and forests with the so called homomorphism quasiorder and discovered useful properties of some natural operations on the iterated labeled forests and on Q-partitions.

An important progress was recently achieved by T. Kihara and A. Montalbán in [11] where a full characterization of the quotient-poset of $\mathcal{W}_Q$ for arbitrary countable bqo Q is obtained, using an extended set of iterated labeled trees $(\mathcal{T}_{\omega_1}(Q); \leq_h)$ with the homomorphism quasiorder $\leq_h$. Namely, $(\mathcal{T}_{\omega_1}(Q); \leq_h)$ is equivalent to the substructure of $\mathcal{W}_Q$ formed by the σ-join-irreducible elements (the equivalence means isomorphism of the corresponding quotient-posets) via an embedding $\mu : \mathcal{T}_{\omega_1}(Q) \to \mathcal{W}_Q$. The Wadge hierarchy of Q-partitions of $\mathcal{N}$ may be thus written as the family $\{\mathcal{W}_Q(T)\}_{T\in\mathcal{T}_{\omega_1}(Q)}$, where $\mathcal{W}_Q(T) = \{A \in Q^{\mathcal{N}} \mid A \leq_W \mu(T)\}$, and it exhausts all principal ideals of $\mathcal{W}_Q$ formed by σ-join-irreducible Q-Wadge degrees. For $Q = \bar{2}$ this yields a new characterization of the Wadge hierarchy of sets.

Main results of our paper may be sketched as follows. We define a Q-IFH $\{\widehat{\mathcal{L}}(X, T)\}_{T\in\mathcal{T}_{\omega_1}(Q)}$ of Q-partitions of arbitrary space X via natural set operations and propose it as the right extension of the Q-Wadge hierarchy. Theorem 3.36 and Corollary 3.33 show that its levels are ordered as one would expect from the Q-Wadge hierarchy. Theorems 4.7, 4.8 and 4.12 show that this hierarchy in quasi-Polish spaces satisfies Hausdorff–Kuratowski-type theorems. Theorem 4.10 shows that this hierarchy in the Baire space coincides with the hierarchy in [11]. For a quasi-Polish space X, a continuous open surjection $\xi : \mathcal{N} \to X$, and $T \in \mathcal{T}_{\omega_1}(Q)$, let $\mathcal{W}(X,T) = \{A \in Q^X \mid A \circ \xi \in \mathcal{W}_Q(T)\}$ be the levels of Wadge hierarchy of Q-partitions defined according to the second approach. Theorem 4.11 shows that $\widehat{\mathcal{L}}(X,T) = \mathcal{W}(X,T)$ for every countable bqo Q and every $T \in \mathcal{T}_{\omega_1}(Q)$. For the case of two-partitions we obtain a set-theoretic characterization of the Wadge hierarchy from the second approach which looks different from a description of the Wadge hierarchy in [11] (see also [15, 34]). The characterizations in [11, 15, 34] cannot be straightforwardly extended to arbitrary spaces since they use specific features of the Baire space.

Having papers [24, 27, 28] at hand would probably simplify reading of the present paper because they contain simpler versions of some notions and results based on similar ideas. Technical notions for the infinitary case are harder than for the finitary case [24, 28] but the ideas are the same. After recalling necessary preliminaries in the next section, we define in §3 the Q-IFH and establish its general properties. In §4 we prove the above-mentioned additional properties of the Q-IFH in second countable spaces and in quasi-Polish spaces.

§2. Preliminaries.

In this section we briefly recall some notation, notions and facts used throughout the paper. Some more special information is recalled in the corresponding sections below.

2.1. Well and better quasiorders. We use standard set-theoretical notation. In particular, Y^X is the set of functions from X to Y, $P(X)$ is the class of subsets of a set X, $\check{\mathcal{C}}$ is the class of complements $X \setminus C$ of sets C in $\mathcal{C} \subseteq P(X)$. We assume the reader to be acquainted with the notion of ordinal (see e.g., [13]). Ordinals are denoted by $\alpha, \beta, \gamma, \ldots$. Every ordinal α is the set of smaller ordinals, in particular $\omega = \{0, 1, 2, \ldots\}$. We use some notions and facts of ordinal arithmetic. In particular, $\alpha + \beta$, $\alpha \cdot \beta$ and α^β denote the ordinal addition, multiplication and exponentiation of α and β, respectively. Every positive ordinal α is uniquely representable in the Cantor normal form $\alpha = \omega^{\alpha_0} + \cdots + \omega^{\alpha_n}$ where $n < \omega$ and $\alpha \geq \alpha_0 \geq \cdots \geq \alpha_n$; we denote $\alpha* = \omega^{\alpha_0}$. The first noncountable ordinal is denoted by ω_1.

We use standard notation and terminology on partially ordered sets (posets) and quasiorders (qo's). To avoid complex notation, we sometimes abuse terminology about posets by applying it also to qo's; in such cases we just mean the corresponding quotient-poset. A qo $(P; \leq)$ is *well-founded* if it has no infinite descending chains $a_0 > a_1 > \cdots$. In this case there are a unique ordinal $rk(P)$ and a unique rank function rk_P from P onto $rk(P)$ satisfying $a < b \to rk(a) < rk(b)$. It is defined by induction $rk_P(x) = sup\{rk_P(y) + 1 \mid y < x\}$. The ordinal $rk(P)$ is called the *rank* (or *height*) of P, and the ordinal $rk_P(x)$ is called the *rank of* $x \in P$ *in* P.

A *well quasiorder* (wqo) is a qo $Q = (Q; \leq_Q)$ that has neither infinite descending chains nor infinite antichains. Although wqo's are closed under many finitary constructions like forming finite labeled words or trees, they are not always closed under important infinitary constructions. C. Nash-Williams was able to find a subclass of wqo's, called better quasiorders (bqo's), which contains most of the "natural" wqo's (in particular, all finite qo's) and is closed under many infinitary constructions. We omit a bit technical notion of bqo which is used only in formulations and we refer the reader to [29].

Recall that an (upper) *semilattice* is a structure $(S; \sqcup)$ with binary operation $\sqcup$ such that $(x \sqcup y) \sqcup z = x \sqcup (y \sqcup z)$, $x \sqcup y = y \sqcup x$ and $x \sqcup x = x$, for all $x, y, z \in S$. By $\leq$ we denote the induced partial order on S: $x \leq y$ iff $x \sqcup y = y$. The operation $\sqcup$ can be recovered from $\leq$ since $x \sqcup y$ is the supremum of x, y w.r.t. $\leq$. By *σ-semilattice* we mean a semilattice where also supremums $\bigsqcup y_j = y_0 \sqcup y_1 \sqcup \cdots$ of countable sequences of elements $y_0, y_1, \ldots$ exist. An element x of a σ-semilattice S is *σ-join-irreducible* if it cannot be represented as the countable supremum of elements strictly below x. As first stressed in [22], the σ-join-irreducible elements play a central role in the study of Wadge degrees of k-partitions. They are also of principal importance in [11].

2.2. Classical hierarchies in topological spaces. We assume the reader to be familiar with basic notions of topology [6]. The underlying set of a topological space X will be usually also denoted by X, in abuse of notation. We usually abbreviate "topological space" to "space." A space is *zero-dimensional* if it has a basis of clopen sets. Recall that a *basis* for the topology on X is a set $\mathcal{B}$ of open subsets of X such that for every $x \in X$ and open U containing x there is $B \in \mathcal{B}$ satisfying $x \in B \subseteq U$. We sometimes shorten "countably based T_0-space" to "cb_0-space."

Let ω be the space of non-negative integers with the discrete topology. Let $\mathcal{N} = \omega^\omega$ be the set of all infinite sequences of natural numbers (i.e., of all functions $x\colon \omega \to \omega$). Let ω^* be the set of finite sequences of elements of ω, including the empty sequence ε. For $\sigma \in \omega^*$ and $x \in \mathcal{N}$, we write $\sigma \sqsubseteq x$ to denote that σ is an initial segment of the sequence x. By $\sigma x = \sigma \cdot x$ we denote the concatenation of σ and x, and by $\sigma \cdot \mathcal{N}$ the set of all extensions of σ in $\mathcal{N}$. For $x \in \mathcal{N}$, we can write $x = x(0)x(1)\ldots$ where $x(i) \in \omega$ for each $i < \omega$. For $x \in \mathcal{N}$ and $n < \omega$, let $x \restriction n = x(0) \ldots x(n-1)$ denote the initial segment of x of length n. By endowing $\mathcal{N}$ with the product of the discrete topologies on ω, we obtain the so-called *Baire space*. The product topology coincides with the topology generated by the collection of sets of the form $\sigma \cdot \mathcal{N}$ for $\sigma \in \omega^*$. It is well known that $\mathcal{N} \times \mathcal{N}$ and $\mathcal{N}^\omega$ are homeomorphic to $\mathcal{N}$ (see e.g., [9]).

A *tree* is a nonempty set $T \subseteq \omega^*$ which is closed downwards under the prefix relation $\sqsubseteq$. The empty string ε is the *root* of any tree. A *leaf* of T is a maximal element of $(T; \sqsubseteq)$. A tree is *pruned* if it has no leaf. A *path through* a tree T is an element $x \in \mathcal{N}$ such that $x \restriction n \in T$ for each $n \in \omega$. For any tree and any $\tau \in T$, let $[T]$ be the set of paths through T and $T(\tau) = \{\sigma \mid \tau\sigma \in T\}$. We call a tree T *normal* if $\tau(i+1) \in T$ implies $\tau i \in T$. A tree is *infinite-branching* if with every nonleaf node τ it contains all its successors τi; every infinite branching tree is normal. A tree is *well founded* if there is no path through it (i.e., $(T; \sqsupseteq)$ is well founded). The rank of the latter poset is called the rank of T; the ranks of well founded trees are precisely the countable ordinals. By a *forest* we mean a set of strings $T \setminus \{\varepsilon\}$, for some tree T; usually we assume forests to be nonempty. Sometimes we use other obvious notation on trees. E.g., with any sequence of trees $\{T_0, T_1, \ldots\}$ we associate the tree $T = \{\varepsilon\} \cup 0 \cdot T_0 \cup 1 \cdot T_1 \cup \cdots$ such that $T(i) = T_i$ for each $i < \omega$.

A *pointclass* in a space X is a class $\mathbf{\Gamma}(X) \subseteq P(X)$ of subsets of X. A *family of pointclasses* [25] is a family $\mathbf{\Gamma} = \{\mathbf{\Gamma}(X)\}_X$ indexed by arbitrary topological spaces X (or by spaces in a reasonable class) such that each $\mathbf{\Gamma}(X)$ is a pointclass in X and $\mathbf{\Gamma}$ is closed under continuous preimages, i.e., $f^{-1}(A) \in \mathbf{\Gamma}(X)$ for every $A \in \mathbf{\Gamma}(Y)$ and every continuous function $f: X \to Y$. A basic example of a family of pointclasses is given by the family $\mathcal{O} = \{\tau_X\}_X$ of topologies in arbitrary spaces X.

We will use the following operations on families of pointclasses: the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_\sigma$, where $\mathbf{\Gamma}(X)_\sigma$ is the set of all countable unions of sets in $\mathbf{\Gamma}(X)$, the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_\delta$, where $\mathbf{\Gamma}(X)_\delta$ is the set of all countable intersections of sets in $\mathbf{\Gamma}(X)$, the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_c$, where $\mathbf{\Gamma}(X)_c = \check{\mathbf{\Gamma}}(X)$, the operation $\mathbf{\Gamma} \mapsto \mathbf{\Gamma}_d$, where $\mathbf{\Gamma}(X)_d$ is the set of all differences of sets in $\mathbf{\Gamma}(X)$.

The operations on families of pointclasses enable to provide short uniform descriptions of the classical hierarchies in arbitrary spaces. E.g., the *Borel hierarchy* is the sequence of families of pointclasses $\{\mathbf{\Sigma}^0_\alpha\}_{\alpha<\omega_1}$ defined by induction on α as follows [4, 20]: $\mathbf{\Sigma}^0_0(X) := \{\emptyset\}$, $\mathbf{\Sigma}^0_1 := \mathcal{O}$ (the family of open sets), $\mathbf{\Sigma}^0_2 := (\mathbf{\Sigma}^0_1)_{d\sigma}$, and $\mathbf{\Sigma}^0_\alpha(X) = (\bigcup_{\beta<\alpha} \mathbf{\Sigma}^0_\beta(X))_{c\sigma}$ for $\alpha > 2$. The sequence $\{\mathbf{\Sigma}^0_\alpha(X)\}_{\alpha<\omega_1}$ is called *the Borel hierarchy* in X. We also set $\mathbf{\Pi}^0_\beta(X) = (\mathbf{\Sigma}^0_\beta(X))_c$ and $\mathbf{\Delta}^0_\alpha(X) = \mathbf{\Sigma}^0_\alpha(X) \cap \mathbf{\Pi}^0_\alpha(X)$. The classes $\mathbf{\Sigma}^0_\alpha(X), \mathbf{\Pi}^0_\alpha(X), \mathbf{\Delta}^0_\alpha(X)$ are called *levels* of the Borel hierarchy in X. The class $\mathbf{B}(X)$ of *Borel sets* in X is defined as the union of all levels of the Borel hierarchy in X; it coincides with the smallest σ-algebra of subsets of X containing the open sets. We have $\mathbf{\Sigma}^0_\alpha(X) \cup \mathbf{\Pi}^0_\alpha(X) \subseteq \mathbf{\Delta}^0_\beta(X)$ for all $\alpha < \beta < \omega_1$. We do not recall the well known definition of the Hausdorff difference hierarchy over $\mathbf{\Sigma}^0_\alpha(X)$, $\alpha \geq 1$, which is denoted by $\{D_\beta(\mathbf{\Sigma}^0_\alpha(X))\}_{\beta<\omega_1}$ or by $\{\mathbf{\Sigma}^{-1,\alpha}_\beta(X)\}_{\beta<\omega_1}$. The definitions may be found e.g., in §22.E in [9] or in [27]. We recall some structural properties of pointclasses (see e.g., §22.C in [9]).

Definition 2.1.

1. A pointclass $\mathbf{\Gamma}(X)$ has the *separation property* if for every two disjoint sets $A, B \in \mathbf{\Gamma}(X)$ there is a set $C \in \mathbf{\Gamma}(X) \cap \check{\mathbf{\Gamma}}(X)$ with $A \subseteq C \subseteq X \setminus B$.
2. A pointclass $\mathbf{\Gamma}(X)$ has the *reduction property* if for all $C_0, C_1 \in \mathbf{\Gamma}(X)$ there are disjoint $C'_0, C'_1 \in \mathbf{\Gamma}(X)$ such that $C'_i \subseteq C_i$ for $i < 2$ and $C_0 \cup C_1 = C'_0 \cup C'_1$. The pair (C'_0, C'_1) is called a reduct for the pair (C_0, C_1).
3. A pointclass $\mathbf{\Gamma}(X)$ has the *σ-reduction property* if for each countable sequence $C_0, C_1, \ldots$ in $\mathbf{\Gamma}(X)$ there is a countable sequence $C'_0, C'_1, \ldots$ in $\mathbf{\Gamma}(X)$ (called a reduct of $C_0, C_1, \ldots$) such that $C'_i \cap C'_j = \emptyset$ for all $i \neq j$ and $\bigcup_{i<\omega} C'_i = \bigcup_{i<\omega} C_i$.

It is well-known that if $\mathbf{\Gamma}(X)$ has the reduction property then the dual class $\check{\mathbf{\Gamma}}(X)$ has the separation property, but not vice versa, and that if $\mathbf{\Gamma}(X)$ has the σ-reduction property then $\mathbf{\Gamma}(X)$ has the reduction property but not vice versa. Let X be a cb$_0$-space. It is known (see e.g., theorem 22.16 in [9] and theorem 3.5 in [25]) that any level $\mathbf{\Sigma}^0_{2+\alpha}(X)$, $\alpha < \omega_1$, has the σ-reduction property, and if X is zero-dimensional then also $\mathbf{\Sigma}^0_1(X)$ has the σ-reduction property.

2.3. Quasi-Polish spaces and admissible representations. A space X is *Polish* if it is countably based and metrizable with a metric d such that (X, d) is a complete metric space. Examples of Polish spaces are ω, $\mathcal{N}$, the Cantor space $\mathcal{C}$, the space of reals $\mathbb{R}$ and its Cartesian powers $\mathbb{R}^n$ $(n < \omega)$, the closed unit interval $[0, 1]$, the Hilbert cube $[0, 1]^\omega$ and the space $\mathbb{R}^\omega$.

Quasi-Polish spaces were identified and thoroughly studied by M. de Brecht [4] (see also [3] for additional information). Informally, this is a natural class of

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spaces which contains all Polish spaces, many important non-Hausdorff spaces (like ω-continuous domains) and has essentially the same DST as Polish spaces (see below). Let $P\omega$ be the space of subsets of ω equipped with the Scott topology, a countable basis of which is formed by the sets $\{A \subseteq \omega \mid F \subseteq A\}$, where F ranges over the finite subsets of ω. By a *quasi-Polish space* we mean a space homeomorphic to a $\mathbf{\Pi}^0_2$-subspace of $P\omega$. There are several interesting characterizations of quasi-Polish spaces. For this paper the following characterization in terms of representations is relevant.

A *representation* of a set X is a surjection from a subspace of $\mathcal{N}$ onto X. Such a representation is *total* if its domain is $\mathcal{N}$. Representation μ is (*continuously*) *reducible* to a representation ν ($\mu \leq_c \nu$) if $\mu = \nu \circ f$ for some continuous partial function f on $\mathcal{N}$. Representations μ, ν are (*continuously*) *equivalent* ($\mu \equiv_c \nu$) if $\mu \leq_c \nu$ and $\nu \leq_c \mu$. A basic notion of Computable Analysis [35] is the notion of admissible representation. A representation μ of a space X is *admissible*, if it is continuous and any continuous function $\nu : Z \to X$ from a subset $Z \subseteq \mathcal{N}$ to X is continuously reducible to μ. Clearly, any two admissible representations of a space are continuously equivalent. By theorem 12 in [2], any continuous open surjection from a subspace of $\mathcal{N}$ onto X is an admissible representation of X. In theorems 41 and 49 of [4] the following characterization of quasi-Polish spaces was obtained:

PROPOSITION 2.2. [4] *A cb_0-space X is quasi-Polish iff it has a total admissible representation iff there is a continuous open surjection from $\mathcal{N}$ onto X.*

The classical Borel, Luzin and Hausdorff hierarchies in quasi-Polish spaces have properties very similar to their properties in Polish spaces [4]. In particular, for any uncountable quasi-Polish space X and any $\alpha < \omega_1$, $\mathbf{\Sigma}^0_\alpha(X) \not\subseteq \mathbf{\Pi}^0_\alpha(X)$. For any quasi-Polish space X, the Suslin theorem $\bigcup_{\alpha<\omega_1} \mathbf{\Sigma}^0_{1+\alpha}(X) = \mathbf{B}(X) = \mathbf{\Delta}^1_1(X)$ and the HK theorem [4, 9] (saying that $\bigcup_{\beta<\omega_1} \mathbf{\Sigma}^{-1,\alpha}_\beta(X) = \mathbf{\Delta}^0_{\alpha+1}(X)$ for all $\alpha \geq 1$) hold.

Quasi-Polish spaces also share properties of Polish spaces related to Baire category (see e.g., §I.8 of [9] or §7 of [3] for a general background). According to an extension of a basic fact for Polish spaces from [17], every quasi-Polish space X is completely Baire, in particular every nonempty closed set $F \subseteq X$ is nonmeager in F (see corollary 52 in [4] or corollary 7.8 in [3]). Using the technique of category quantifiers (§8.J in [9]), one can show the following preservation property [3, 4, 17] of levels of the Borel hierarchy.

PROPOSITION 2.3. [3, 4, 17] *Let $f : X \to Y$ be a continuous open surjection between cb_0-spaces, $\alpha < \omega_1$, and $A \subseteq Y$. Then $A \in \mathbf{\Sigma}^0_{1+\alpha}(Y)$ iff $f^{-1}(A) \in \mathbf{\Sigma}^0_{1+\alpha}(X)$. Also, every fiber $f^{-1}(y)$ is quasi-Polish, hence nonmeager in $f^{-1}(y)$.*

2.4. Wadge hierarchy in $\mathcal{N}$. Here we give some additional information on the Wadge hierarchy in the Baire space. In [34] W. Wadge proved that the structure $(\mathbf{B}(\mathcal{N}); \leq_W)$ of Borel sets in the Baire space is semi-well-ordered (i.e., it is well-founded and for all $A, B \in \mathbf{B}(\mathcal{N})$ we have $A \leq_W B$ or $\overline{B} \leq_W A$). In particular, there is no antichain of size 3 in $(\mathbf{B}(\mathcal{N}); \leq_W)$. He has also computed the rank υ of $(\mathbf{B}(\mathcal{N}); \leq_W)$ which we call the Wadge ordinal. Recall that a set A is *self-dual* if $A \leq_W \overline{A}$. W. Wadge has shown that if a Borel set is self-dual (resp. nonself-dual) then any Borel set of the next Wadge rank is nonself-dual (resp. self-dual), a Borel

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set of Wadge rank of countable cofinality is self-dual, and a Borel set of Wadge rank of uncountable cofinality is nonself-dual. This characterizes the structure of Wadge degrees of Borel sets up to isomorphism.

In theorem 2 of [37], and also in [30], the following separation theorem for the Wadge hierarchy was established: For any nonself-dual Borel set A exactly one of the principal ideals $\{X \mid X \leq_W A\}$, $\{X \mid X \leq_W \overline{A}\}$ has the separation property. The mentioned results give rise to the *Wadge hierarchy* which is, by definition, the sequence $\{\mathbf{\Sigma}_\alpha(\mathcal{N})\}_{\alpha<\upsilon}$ of all nonself-dual principal ideals of $(\mathbf{B}(\mathcal{N}); \leq_W)$ that do not have the separation property and satisfy for all $\alpha < \beta < \upsilon$ the strict inclusion $\mathbf{\Sigma}_\alpha(\mathcal{N}) \subset \mathbf{\Delta}_\beta(\mathcal{N})$ where, as usual, $\mathbf{\Delta}_\alpha(\mathcal{N}) = \mathbf{\Sigma}_\alpha(\mathcal{N}) \cap \mathbf{\Pi}_\alpha(\mathcal{N})$.

The Wadge hierarchy subsumes the classical hierarchies in the Baire space, in particular $\mathbf{\Sigma}_\alpha(\mathcal{N}) = \mathbf{\Sigma}^{-1}_\alpha(\mathcal{N})$ for each $\alpha < \omega_1$, $\mathbf{\Sigma}_1(\mathcal{N}) = \mathbf{\Sigma}^0_1(\mathcal{N})$, $\mathbf{\Sigma}_{\omega_1}(\mathcal{N}) = \mathbf{\Sigma}^0_2(\mathcal{N})$, $\mathbf{\Sigma}_{\omega_1^{\omega_1}}(\mathcal{N}) = \mathbf{\Sigma}^0_3(\mathcal{N})$ and so on. Thus, the sets of finite Borel rank coincide with the sets of Wadge rank less than $\lambda = sup\{\omega_1, \omega_1^{\omega_1}, \omega_1^{(\omega_1^{\omega_1})}, \ldots\}$. Note that λ is the smallest solution of the ordinal equation $\omega_1^\kappa = \kappa$. Hence, the reader should carefully distinguish $\mathbf{\Sigma}_\alpha(\mathcal{N})$ and $\mathbf{\Sigma}^0_\alpha(\mathcal{N})$. To give the reader an impression about the Wadge ordinal we note that the rank of the qo $(\mathbf{\Delta}^0_\omega(\mathcal{N}); \leq_W)$ is the ω_1-st solution of the ordinal equation $\omega_1^\kappa = \kappa$ [34]. We summarize properties of the Wadge hierarchy of sets in the Baire space which will be tested for survival under extensions to cb_0-spaces (or to quasi-Polish spaces) and to Q-partitions:

1. The levels of the Wadge hierarchy are semi-well-ordered by inclusion.
2. The Wadge hierarchy does not collapse, i.e., $\mathbf{\Sigma}_\alpha \not\subseteq \mathbf{\Pi}_\alpha$ for all $\alpha < \upsilon$.
3. The Wadge degrees of Borel sets coincide with the sets $\mathbf{\Sigma}_\alpha \setminus \mathbf{\Pi}_\alpha$, $\mathbf{\Pi}_\alpha \setminus \mathbf{\Sigma}_\alpha$, $\mathbf{\Delta}_{\alpha+1} \setminus (\mathbf{\Sigma}_\alpha \cup \mathbf{\Pi}_\alpha)$ (where $\alpha < \upsilon$), and $\mathbf{\Delta}_\lambda \setminus (\bigcup_{\alpha<\lambda} \mathbf{\Sigma}_\alpha)$ (where $\lambda < \upsilon$ is a limit ordinal of countable cofinality).
4. If $\lambda < \upsilon$ is a limit ordinal of uncountable cofinality then $\mathbf{\Delta}_\lambda = \bigcup_{\alpha<\lambda} \mathbf{\Sigma}_\alpha$.
5. All levels are downward closed under Wadge reducibility.
6. The levels in item (3) are precisely those having Wadge complete sets.

§3. Infinitary fine hierarchies in a set.

In this section we define the infinitary fine hierarchy and prove some of its basic properties. The Q-partition version of this hierarchy will be called the Q-IFH, for abbreviation. This section extends (and in fact simplifies) the corresponding material from §5 in [27]. The first three subsections describe some related technical notions.

3.1. Iterated Q-trees.

Here we describe a notation system for levels of the Q-IFH. For any qo Q, a *Q-tree* is a pair (T, t) consisting of an infinite-branching well founded tree $T \subseteq \omega^*$ and a labeling $t : T \to Q$. Let $\mathcal{T}(Q)$ be the set of Q-trees quasi-ordered by the relation: $(T, t) \leq_h (V, v)$ iff there is a monotone function $\varphi : T \to V$ with $\forall v \in T(t(x) \leq_Q v(\varphi(x)))$; such a function φ is called a *morphism from* (T, t) *to* (V, v). As follows from Laver's results in [14], if Q is bqo then so is also $(\mathcal{T}(Q); \leq_h)$ which is usually shortened to $\mathcal{T}(Q)$. Thus, $\mathcal{T}$ is an operator on the class BQO of all bqo's. The operator $\mathcal{T}$ and its iterates like $\mathcal{T} \circ \mathcal{T} \circ \mathcal{T}$ were introduced in [22, 27] and turned out crucial for characterizing some initial segments of $\mathcal{W}_{\bar{k}}$ [26, 27].

In [11] a more powerful iteration procedure was invented which yields the set $\mathcal{T}_{\omega_1}(Q)$ of labeled trees sufficient for characterizing $\mathcal{W}_Q$, as discussed in Introduction.

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We give a slightly different (but equivalent) definition of $\mathcal{T}_{\omega_1}(Q)$ more convenient for our purposes here. The differences are caused by our desire to first work only with trees (introducing forests at the last stage), and to relate the qo $\trianglelefteq$ (defined below) to the qo $\leq_h$.

Let $\sigma = \sigma(Q, \omega_1) = \{q, s_\alpha, F_q, F_\alpha \mid q \in Q, \alpha < \omega_1\}$ be the signature consisting of constant symbols q, unary function symbols s_α, and ω-ary function symbols F_q, F_α (of course we assume all signature symbols to be distinct, in particular $Q \cap \omega_1 = \emptyset$). Let $\mathbb{T}_\sigma$ be the set of σ-terms without variables obtained by the standard inductive definition: Any constant symbol q is a term; if u is a term then so is also $s_\alpha(u)$; if $u_0, u_1, \ldots$ are terms then so are also $F_q(u_0, \ldots), F_\alpha(u_0, \ldots)$. Informally, $F_q(u_0, \ldots)$ and $F_\alpha(u_0, \ldots)$ are interpreted as $q \to (u_0 \sqcup \cdots)$ and $s_\alpha(u_0) \to (u_1 \sqcup \cdots)$ respectively (cf. [11] where e.g., the first expression denotes the tree $\varepsilon \cup 0 \cdot u_0 \cup \cdots$ with the root ε labeled by q), hence our modification simply avoids forests from the inductive definition.

The σ-terms are represented by (or even identified with) the normal well founded trees with constants on the leafs and other signature symbols on the nonleaf nodes such that the nodes labeled with s_α have the unique successor while the nodes labeled by F_q of F_α have all successors; we refer to such trees as *syntactic trees*, in order to distinguish them from trees of another kind. As usual, the *rank of a term u*, denoted $rk(u)$, is the rank of its syntactic tree; ranks enable definitions and proofs by induction on terms because the subterms of u are precisely the terms with syntactic tree of the form $u(\tau)$, see §2.2. Obviously, the set $\mathbb{T}_\sigma$ is partitioned into three parts: *constant terms* (i.e., the terms q for some $q \in Q$), *s-terms* (i.e., the terms $s_\alpha(u)$ for unique $\alpha < \omega_1$ and $u \in \mathbb{T}_\sigma$) and *F-terms* (i.e., the terms $F_q(u_0, \ldots)$ or $F_\alpha(u_0, \ldots)$ for unique $q \in Q$, $\alpha < \omega_1$, and $u_0, u_1, \ldots \in \mathbb{T}_\sigma$). We define by induction on terms the binary relation $\trianglelefteq$ on $\mathbb{T}_\sigma$ as follows (cf. definition 3.1 and its extensions in [11]). The relation $\trianglelefteq$ on $\mathbb{T}_\sigma$ is in fact equivalent to the relation $\trianglelefteq$ in [11] restricted to the tree-terms.

Definition 3.1.

1. $q \trianglelefteq r$ iff $q \leq_Q r$;
2. $q \trianglelefteq s_\alpha(u)$ iff $q \trianglelefteq u$;
3. $q \trianglelefteq F_r(u_0, \ldots)$ iff $q \trianglelefteq r$ or $q \trianglelefteq u_i$ for some $i \geq 0$;
4. $q \trianglelefteq F_\alpha(u_0, \ldots)$ iff $q \trianglelefteq u_i$ for some $i \geq 0$;
5. $s_\alpha(u) \trianglelefteq r$ iff $u \trianglelefteq r$;
6. $s_\alpha(u) \trianglelefteq s_\beta(v)$ iff ($\alpha < \beta$ and $u \trianglelefteq s_\beta(v)$) or ($\alpha = \beta$ and $u \trianglelefteq v$) or ($\alpha > \beta$ and $s_\alpha(u) \trianglelefteq v$);
7. $s_\alpha(u) \trianglelefteq F_r(v_0, \ldots)$ iff $s_\alpha(u) \trianglelefteq r$ or $s_\alpha(u) \trianglelefteq v_i$ for some $i \geq 0$;
8. $s_\alpha(u) \trianglelefteq F_\beta(v_0, \ldots)$ iff $s_\alpha(u) \trianglelefteq s_\beta(v_0)$ or $s_\alpha(u) \trianglelefteq v_i$ for some $i \geq 1$;
9. $F_q(u_0, \ldots) \trianglelefteq r$ iff $q \trianglelefteq r$ and $u_i \trianglelefteq r$ for all $i \geq 0$;
10. $F_q(u_0, \ldots) \trianglelefteq s_\alpha(v)$ iff $q \trianglelefteq s_\alpha(v)$ and $u_i \trianglelefteq s_\alpha(v)$ for all $i \geq 0$;
11. $F_q(u_0, \ldots) \trianglelefteq F_r(v_0, \ldots)$ iff ($q \trianglelefteq r$ and $u_i \trianglelefteq F_r(v_0, \ldots)$ for all $i \geq 0$) or $F_q(u_0, \ldots) \trianglelefteq v_i$ for some $i \geq 0$;
12. $F_q(u_0, \ldots) \trianglelefteq F_\beta(v_0, \ldots)$ iff ($q \trianglelefteq s_\beta(v_0)$ and $u_i \trianglelefteq F_\beta(v_0, \ldots)$ for all $i \geq 0$) or $F_q(u_0, \ldots) \trianglelefteq v_i$ for some $i \geq 1$;
13. $F_\alpha(u_0, \ldots) \trianglelefteq r$ iff $s_\alpha(u_0) \trianglelefteq r$ and $u_i \trianglelefteq r$ for all $i \geq 1$;
14. $F_\alpha(u_0, \ldots) \trianglelefteq s_\beta(v)$ iff $s_\alpha(u_0) \trianglelefteq s_\beta(v)$ and $u_i \trianglelefteq s_\beta(v)$ for all $i \geq 1$;
15. $F_\alpha(u_0, \ldots) \trianglelefteq F_r(v_0, \ldots)$ iff ($s_\alpha(u_0) \trianglelefteq r$ and $u_i \trianglelefteq F_r(v_0, \ldots)$ for all $i \geq 1$) or $F_\alpha(u_0, \ldots) \trianglelefteq v_i$ for some $i \geq 0$;

16. $F_\alpha(u_0, \ldots) \trianglelefteq F_\beta(v_0, \ldots)$ iff $(s_\alpha(u_0) \trianglelefteq s_\beta(v_0)$ and $u_i \trianglelefteq F_\beta(v_0, \ldots)$ for all $i \geq 1)$ or $F_\alpha(u_0, \ldots) \trianglelefteq v_i$ for some $i \geq 1$.

From results in [11] it follows that $(\mathbb{T}_\sigma; \trianglelefteq)$ is a bqo. Let $\mathbb{T}_{q,s}$ be the set of constant terms and s-terms. Then $(\mathbb{T}_{q,s}; \trianglelefteq)$ is bqo, hence $(\mathcal{T}(\mathbb{T}_{q,s}); \leq_h)$ is also bqo. The next definition makes precise the relation between the introduced qo's $\trianglelefteq$ and $\leq_h$.

DEFINITION 3.2. We associate with any $u \in \mathbb{T}_\sigma$ the labeled tree $T(u)$ by induction as follows: $T(q)$ is the singleton tree labeled by q, $T(s_\alpha(u))$ is the singleton tree labeled by $s_\alpha(u)$, $T(F_q(u_0, \ldots)) = q \to (T(u_0) \sqcup T(u_1) \sqcup \cdots)$, $T(F_\alpha(u_0, \ldots)) = s_\alpha(u_0) \to (T(u_1) \sqcup T(u_2) \sqcup \cdots)$.

Please be careful in distinguishing $T(u)$ (which is an element of $\mathcal{T}(\mathbb{T}_{q,s})$) and the tree $T(\tau)$ above $\tau \in T$. Obviously, $T(u)$ is a singleton tree iff $u \in \mathbb{T}_{q,s}$. The next lemma is checked by cases from Definition 3.1 using induction on terms.

LEMMA 3.3. *The function $u \mapsto T(u)$ is an isomorphism between $(\mathbb{T}_\sigma; \trianglelefteq)$ and $(\mathcal{T}(\mathbb{T}_{q,s}); \leq_h)$.*

EXAMPLES 3.4.

1. If u has no entries of symbols s_α, F_α then $T(u)$ is obtained from the syntactic tree of u by replacing any label F_q (on the nonleaf nodes) by label q. Therefore, the set of such trees $T(u)$ essentially coincides with the set $\mathcal{T}(Q)$ from the beginning of this subsection.
2. For $u = s_\beta(v)$ where $v = F_\alpha(q_0, s_\gamma(F_{q_1}(r_0, r_1, \ldots)), s_\delta(q_2), q_3, q_4, \ldots)$, we have: $T(u) = (\{\varepsilon\}, u)$ (the singleton tree labeled by u itself), and $T(v) = \{\varepsilon, 0, 1, 2, 3, \ldots\}$ labeled resp. by $s_\alpha(q_0), s_\gamma(F_{q_1}(r_0, r_1, \ldots)), s_\delta(q_2), q_3, q_4, \ldots$.

The next lemma is immediate by induction on terms.

LEMMA 3.5. *Any term $u \in \mathbb{T}_\sigma$ satisfies precisely one of the following alternatives*:

1. $u = q$ *for a unique* $q \in Q$;
2. $u = s_{\beta_0} \cdots s_{\beta_m}(q)$ *for unique* $m < \omega$, $\beta_0, \ldots, \beta_m < \omega_1$, $q \in Q$;
3. $u = F_q(u_0, \ldots)$ *for unique* $q \in Q$ *and* $u_0, \ldots \in \mathbb{T}_\sigma$;
4. $u = F_\alpha(u_0, \ldots)$ *for unique* $\alpha < \omega_1$, *and* $u_0, \ldots \in \mathbb{T}_\sigma$;
5. $u = s_{\beta_0} \cdots s_{\beta_m}(F_q(u_0, \ldots))$ *for unique* $m < \omega$, $\beta_0, \ldots, \beta_m < \omega_1$, $q \in Q$, $u_0, \ldots \in \mathbb{T}_\sigma$;
6. $u = s_{\beta_0} \cdots s_{\beta_m}(F_\alpha(u_0, \ldots))$ *for unique* $m < \omega$, $\beta_0, \ldots, \beta_m < \omega_1$, $\alpha < \omega_1$, $u_0, \ldots \in \mathbb{T}_\sigma$.

Terms from items (1) and (2) above will be called *singleton terms*. With any singleton term u a unique element $q \in Q$ is associated denoted by $q(u)$. Below we will also need the following technical notions.

DEFINITION 3.6. We associate with any $u \in \mathbb{T}_\sigma$ the ordinal $sh(u)$ and the term $u' \in \mathbb{T}_\sigma$ as follows: if u is not an s-term then $sh(u) = 0$ and $u' = u$, otherwise $sh(u) = \omega^{\beta_0} + \cdots + \omega^{\beta_m}$ and $u' = q, F_q(u_0, \ldots), F_\alpha(u_0, \ldots)$ if u satisfies resp. the alternative (2), (5), or (6) above.

Note that "sh" comes from "shift." We collect some obvious properties of u'.

LEMMA 3.7.

1. *$u' = u$ iff u is not an s-term.*
2. *u' is a subterm of u, so $u' \trianglelefteq u$ and if u is an s-term then $rk(u') < rk(u)$.*
3. *u' is not an s-term, hence $u'' = u'$.*
4. *$u' \in Q$ iff u is a singleton term.*

DEFINITION 3.8. We associate with any nonsingleton term $u \in \mathbb{T}_\sigma$ the set $\mathcal{F}(u)$ of sequences $S = (\tau_0, ...)$ in ω^* constructed as follows: $\tau_0 \in T(u') = (T_0, t_0)$; if $t_0(\tau_0)$ is a singleton term then $S = (\tau_0)$, otherwise $\tau_1 \in T(t_0(\tau_0)') = (T_1, t_1)$; if $t_1(\tau_1)$ is a singleton term then $S = (\tau_0, \tau_1)$, otherwise $\tau_2 \in T(t_1(\tau_1)') = (T_2, t_2)$; and so on.

LEMMA 3.9.

1. *For any $u \in \mathbb{T}_\sigma$ and $\tau \in T(u)$, $t_u(\tau) \trianglelefteq u$, where t_u is the labeling function on $T(u)$, and $rk(t_u(\tau)) \leq rk(u)$.*
2. *If u is not a singleton term then $rk(t_u(\tau)') < rk(u)$ for every $\tau \in T(u)$.*
3. *For any nonsingleton term $u \in \mathbb{T}_\sigma$, every sequence in $\mathcal{F}(u)$ is finite.*

PROOF. (1) For $u \in \mathbb{T}_{q,s}$ the assertion is obvious because $\tau = \varepsilon$ and $t_u(\tau) = u$. Let $u = F_q(u_0, ...)$, then either $\tau = \varepsilon$ or $\tau \in T(u_i)$ for a unique $i \geq 0$. In the first case $t_u(\tau) = q \trianglelefteq u$ and $rk(t_u(\tau)) = 0 < rk(u)$. In the second case by induction we have $t_u(\tau) = t_{u_i}(\tau) \trianglelefteq u_i \trianglelefteq u$ and $rk(t_u(\tau)) = rk(t_{u_i}(\tau)) \leq rk(u_i) < rk(u)$.

Finally, let $u = F_\alpha(u_0, ...)$. Then either $\tau = \varepsilon$ or $\tau \in T(u_i)$ for a unique $i \geq 1$. In the first case $t_u(\tau) = s_\alpha(u_0) \trianglelefteq u$ and $rk(t_u(\tau)) = rk(s_\alpha(u_0)) = rk(u_0) + 1 \leq rk(u)$. In the second case we have: $t_u(\tau) = t_{u_i}(\tau) \trianglelefteq u_i \trianglelefteq u$ and $rk(t_u(\tau)) = rk(t_{u_i}(\tau)) \leq rk(u_i) < rk(u)$.

(2) Since u is not singleton, u is not a q-term. If u is an s-term then $t_u(\tau) = u$, so by Lemma 3.7(2) we have $rk(t_u(\tau)') = rk(u') < rk(u)$. If $u = F_q(u_0, ...)$ then, by the proof of item (1), $rk(t_u(\tau)') \leq rk(t_u(\tau)) < rk(u)$. Finally, let $u = F_\alpha(u_0, ...)$. For $\tau \neq \varepsilon$ the assertion follows again from the proof of item (1). For $\tau = \varepsilon$ we have $t_u(\varepsilon) = s_\alpha(u_0)$, hence, by the proof of item (1) and Lemma 3.7(2), $rk(t_u(\varepsilon)') = rk(s_\alpha(u_0)') < rk(s_\alpha(u_0)) \leq rk(u)$.

(3) Suppose the contrary: the sequence $\tau_0, \tau_1, ...$ from Definition 3.6 is infinite, hence all terms $t_0(\tau_0), t_1(\tau_1), ...$ are not singleton. By item (2) we then have $rk(u') > rk(t_0(\tau_0)') > rk(t_1(\tau_1)') > \cdots$, contradicting the well-foundedness of syntactic trees. ⊣

With any $(\tau_0, \dots, \tau_m) \in \mathcal{F}(u)$ we associate the constant $q(\tau_0, \dots, \tau_m) = t_m(\tau_m) \in Q$. For any $q \in Q$ we set $\mathcal{F}_q(u) = \{(\tau_0, \dots, \tau_m) \in \mathcal{F}(u) \mid q = t_m(\tau_m)\}$.

EXAMPLES 3.10.

1. If u has no entries of symbols s_α, F_α then $sh(u) = 0$, $u' = u$, and $\mathcal{F}(u) = \{(\tau) \mid \tau \in T(u)\}$.
2. For the term $u = s_\beta(v)$ from Examples 3.4(2), we have: $sh(u) = \omega^\beta$, $u' = v$, $T(u') = \{\varepsilon, 0, 1, 2, 3, ...\} = (T_0, t_0)$, $t_0(\varepsilon) = s_\alpha(q_0)$, $t_0(0) = s_\gamma(F_{q_1}(r_0, r_1, ...))$, $t_0(1) = s_\delta(q_2)$, $t_0(2) = q_3$, $t_0(3) = q_4, ...$, $sh(t_0(\varepsilon)) = \omega^\alpha$, $sh(t_0(0)) = \omega^\gamma$, $sh(t_0(1)) = \omega^\delta$, $sh(t_0(2)) = 0$, $sh(t_0(3)) = 0, ...$, $t_0(\varepsilon)' = q_0$, $t_0(0)' = F_{q_1}(r_0, r_1, ...)$, $t_0(1)' = q_2$, $t_0(2)' = q_3$, $t_0(3)' = q_4, ...$. Thus, $\mathcal{F}(u) = \{(\varepsilon, \varepsilon), (0, \varepsilon), (0, 0), (0, 1), \dots, (1, \varepsilon), (2), (3), ...\}$ and the corresponding labels of elements of $\mathcal{F}(u)$ are $q_0, q_1, r_0, r_1, \dots, q_2, q_3, q_4, ...$.

To be more consistent with notation of previous papers and of Introduction, we sometimes denote $\mathcal{T}(\mathbb{T}_{q,s})$ by $\mathcal{T}_{\omega_1}(Q)$ and use the structures from Lemma 3.3 interchangeably. Let $\mathcal{T}^{\sqcup}_{\omega_1}(Q)$ be the set of nonempty labeled forests obtained from trees in $\mathcal{T}_{\omega_1}(Q)$ by deleting the root (alternatively and equivalently, one can think of $\mathcal{T}^{\sqcup}_{\omega_1}(Q)$ as the set of countable disjoint unions of trees in $\mathcal{T}_{\omega_1}(Q)$). The relation $\leq_h$ is extended to the larger structure of forests in the obvious way.

The characterization of $\mathcal{W}_Q$ (see Introduction) in terms of the iterated labeled trees may be now described as follows (see [11] for more details). The relation $\simeq$ below denotes the equivalence of qo's.

Proposition 3.11. [11] *We have* $(\mathcal{T}^{\sqcup}_{\omega_1}(Q); \leq_h) \simeq (\mathbf{\Delta}^1_1(Q^{\mathcal{N}}); \leq_W)$, *for every countable bqo Q. The isomorphism of quotient-posets is induced by a map* $\mu : \mathcal{T}_{\omega_1}(Q) \to \mathbf{\Delta}^1_1(Q^{\mathcal{N}})$ *sending trees onto the σ-join irreducible elements.*

For more details on the map μ see Section 4.4 below. For any $\gamma < \omega_1$, apply the construction above to the smaller signature $\sigma(Q, \gamma) = \{q, s_\alpha, F_q, F_\alpha \mid q \in Q, \alpha < \gamma\}$ in place of $\sigma(Q, \omega_1)$. The resulting set of labeled trees is denoted by $\mathcal{T}_{\omega^\gamma}(Q)$. We obtain an operator $\mathcal{T}_{\omega^\gamma}$ on BQO. Finally, for any $\alpha < \omega_1$ we define the operator $\mathcal{T}_\alpha$ on BQO as follows: $\mathcal{T}_0$ is the identity operator, and for any positive countable ordinal α we set $\mathcal{T}_\alpha = \mathcal{T}_{\omega^{\alpha_0}} \circ \cdots \circ \mathcal{T}_{\omega^{\alpha_n}}$ where $n < \omega$ and $\alpha_0 \geq \cdots \geq \alpha_n$ are the unique ordinals with $\alpha = \omega^{\alpha_0} + \cdots + \omega^{\alpha_n}$. The set of forests $\mathcal{T}^{\sqcup}_\alpha(Q)$ is obtained from $\mathcal{T}_\alpha(Q)$ by the above construction. In particular, $\mathcal{T}_{\alpha+1} = \mathcal{T}_\alpha \circ \mathcal{T}$ where $\mathcal{T}$ is the operator from the beginning of this subsection.

3.2. Hierarchy bases. We recall (see e.g., [27]) the technical notion of a (hierarchy) base. Such bases serve as a starting point for constructing the Q-IFH. They have nothing in common with topological bases.

Definition 3.12. By a base in a set X we mean a sequence $\mathcal{L}(X) = \{\mathcal{L}_\alpha\}_{\alpha<\omega_1}$, $\mathcal{L}_\alpha = \mathcal{L}_\alpha(X) \subseteq P(X)$, such that every $\mathcal{L}_\alpha$ is closed under countable union and finite intersection (in particular, $\emptyset, X \in \mathcal{L}_\alpha$), and $\mathcal{L}_\alpha \cup \check{\mathcal{L}}_\alpha \subseteq \mathcal{L}_\beta \cap \check{\mathcal{L}}_\beta$ for all $\alpha < \beta < \omega_1$.

A major natural example of a hierarchy base in a topological space X is the *Borel base* $\mathcal{L}(X) = \{\mathbf{\Sigma}^0_{1+\alpha}(X)\}_{\alpha<\omega_1}$. There are "unnatural" bases, e.g., the bases $\{\mathbf{B}(X), \mathbf{B}(X), \ldots\}$ and $\{P(X), P(X), \cdots\}$ over which any IFH of sets collapses to the first level.

With any base $\mathcal{L}(X)$ in X we associate some new bases as follows. For any $\beta < \omega_1$, let $\mathcal{L}^\beta(X) = \{\mathcal{L}_{\beta+\alpha}(X)\}_\alpha$; we call this base in X the *β-shift of* $\mathcal{L}(X)$. For any $U \subseteq X$, let $\mathcal{L}(U) = \{\mathcal{L}_\alpha(U)\}$ where $\mathcal{L}_\alpha(U) = \{U \cap S \mid S \in \mathcal{L}_\alpha(X)\}$; we call this base in U the *U-restriction of* $\mathcal{L}(X)$.

Lemma 3.13.

1. $(\mathcal{L}^\beta)^\gamma(X) = \mathcal{L}^{\beta+\gamma}(X)$.
2. *If* $\beta* < \alpha*$ *(see §2.1) then* $\mathcal{L}^\beta_\alpha(X) = \mathcal{L}_\alpha(X)$. *Therefore, many levels of* $\mathcal{L}(X)$ *remain unchanged under the β-shift.*

Proof. (1) Indeed, $((\mathcal{L}^\beta)^\gamma)_\alpha = \mathcal{L}^\beta_{\gamma+\alpha} = \mathcal{L}_{\beta+(\gamma+\alpha)} = \mathcal{L}_{(\beta+\gamma)+\alpha} = \mathcal{L}^{\beta+\gamma}_\alpha$.

(2) Since $\beta + \alpha = \alpha$ by the definition of $\beta*$ and $\alpha*$, $\mathcal{L}^\beta_\alpha(X) = \mathcal{L}_{\beta+\alpha}(X) = \mathcal{L}_\alpha(X)$. ⊣

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By a *morphism* $g : \mathcal{L}(X) \to \mathcal{L}(Y)$ of bases we mean a function $g : P(X) \to P(Y)$ such that $g(\emptyset) = \emptyset$, $g(X) = Y$, $g(\bigcup_n U_n) = \bigcup_n g(U_n)$ for every countable sequence $\{U_n\}$ in $P(X)$ (so, in particular, $U \subseteq V$ implies $g(U) \subseteq g(V)$), and $U \in \mathcal{L}_\alpha(X)$ implies $g(U) \in \mathcal{L}_\alpha(Y)$ for each $\alpha < \omega_1$. Obviously, the identity function on $P(X)$ is a morphism of any base in X to itself, and if $g : \mathcal{L}(X) \to \mathcal{L}(Y)$ and $h : \mathcal{L}(Y) \to \mathcal{L}(Z)$ are morphisms of bases then $h \circ g : \mathcal{L}(X) \to \mathcal{L}(Z)$ is also a morphism. We illustrate the notion of morphism with the following well known fact. Recall that a function $f : X \to Y$ between spaces is $\mathbf{\Sigma}^0_{1+\alpha}$-*measurable* iff $f^{-1}(U) \in \mathbf{\Sigma}^0_{1+\alpha}(X)$ for any open set U in Y.

LEMMA 3.14. *Let $f : X \to Y$ be $\mathbf{\Sigma}^0_{1+\alpha}$-measurable and let $\mathcal{L}(X), \mathcal{L}(Y)$ be the Borel bases in X, Y resp. Then $f^{-1} : P(Y) \to P(X)$ is a morphism from $\mathcal{L}(Y)$ to $\mathcal{L}^\alpha(X)$. In particular, if f is continuous then $f^{-1} : P(Y) \to P(X)$ is a morphism of $\mathcal{L}(Y)$ to $\mathcal{L}(X)$.*

The following class of bases will be frequently mentioned in the sequel.

DEFINITION 3.15. A base $\mathcal{L}(X)$ is reducible if every $\mathcal{L}_\alpha(X)$ has the σ-reduction property.

The next fact follows from results in [9] and [25] mentioned in the end of Subsection 2.2.

LEMMA 3.16. *The Borel base in every zero-dimensional cb_0-space is reducible. The one-shift of the Borel base in every cb_0-space is reducible.*

We conclude this subsection with introducing some auxiliary notions used in the sequel. For any tree $T \subseteq \omega^*$ and a T-family $\{U_\tau\}$ of subsets of X (τ ranges over T), we define the T-family $\{\tilde{U}_\tau\}$ of subsets of X by $\tilde{U}_\tau = U_\tau \setminus \bigcup\{U_{\tau'} \mid \tau \sqsubset \tau' \in T\}$; the sets $\tilde{U}_\tau$ will be called *components* of the family $\{U_\tau\}$. The T-family $\{U_\tau\}$ is *monotone* if $U_\tau \supseteq U_{\tau'}$ for all $\tau \sqsubseteq \tau' \in T$. We associate with any T-family $\{U_\tau\}$ the monotone T-family $\{U'_\tau\}$ by $U'_\tau = \bigcup_{\tau' \sqsupseteq \tau} U_{\tau'}$. Below we mostly work with monotone T-families though the next lemma shows that they are in a sense equivalent to arbitrary ones.

LEMMA 3.17. *Let T be a well founded tree, $\mathcal{L}(X)$ be a base, and $\{U_\tau\}$ be a T-family of $\mathcal{L}_\alpha$-sets. Then the components are differences of $\mathcal{L}_\alpha$-sets (hence they belong to $\mathcal{L}_{\alpha+1} \cap \check{\mathcal{L}}_{\alpha+1}$), $\bigcup_\tau U_\tau = \bigcup_\tau \tilde{U}_\tau$, $\tilde{U}_\tau = \widetilde{U'}_\tau$, and $\tilde{U}_\tau \cap \tilde{U}_{\tau'} = \emptyset$ for $\tau \sqsubset \tau' \in T$.*

PROOF. We check only the second assertion, the proofs of others being even simpler. Since $\tilde{U}_\tau \subseteq U_\tau$, $\bigcup_\tau U_\tau \supseteq \bigcup_\tau \tilde{U}_\tau$. Conversely, let $x \in \bigcup_\tau U_\tau$. Then the set $\{\tau \in T \mid x \in U_\tau\}$ is nonempty. Since $(T; \sqsupseteq)$ is well founded, $x \in U_\tau$ for some maximal element τ of $(\{\tau \in T \mid x \in U_\tau\}; \sqsubseteq)$; but then $x \in \tilde{U}_\tau$. ⊣

The next lemma is also easy.

LEMMA 3.18. *Let T be a well founded tree, $\mathcal{L}(X)$ be a base, $\{U^i_\tau\}_i$ be a sequence of monotone T-families of $\mathcal{L}_\alpha$-sets, and $U_\tau = \bigcup_i U^i_\tau$ for each $\tau \in T$. Then $\{U_\tau\}$ is a monotone T-family of $\mathcal{L}_\alpha$-sets and $\tilde{U}_\tau \subseteq \bigcup_i \widetilde{U^i_\tau}$ for each $\tau \in T$.*

We call a T-family $\{V_\tau\}$ of $\mathcal{L}_\alpha$-sets *reduced* if it is monotone and satisfies $V_{\tau i} \cap V_{\tau j} = \emptyset$ for all $\tau i, \tau j \in T$, $i \neq j$. Obviously, for any reduced T-family $\{V_\tau\}$ of $\mathcal{L}_\alpha$-sets the components $\tilde{V}_\tau$ are pairwise disjoint. The reduced T-families form a very special but important class of the monotone T-families. The next lemma is checked by a top-down (assuming that trees grow downwards) application of the σ-reduction property.

LEMMA 3.19. *Let T be an infinitely-branching well founded tree, $\mathcal{L}(X)$ be a base, $\{U_\tau\}$ be a monotone T-family of $\mathcal{L}_\alpha$-sets, and let $\mathcal{L}_\alpha$ have the σ-reduction property. Then there is a reduced T-family $\{V_\tau\}$ of $\mathcal{L}_\alpha$-sets such that $V_\tau \subseteq U_\tau$, $\bigcup_\tau V_\tau = \bigcup_\tau U_\tau$, $\bigcup_i\{V_{\tau i} \mid \tau i \in T\} = \bigcup_i\{V_\tau \cap U_{\tau i} \mid \tau i \in T\}$, and $\tilde{V}_\tau \subseteq \tilde{U}_\tau$ for each $\tau \in T$.*

PROOF. If $T = \{\varepsilon\}$ is singleton, there is nothing to prove. Otherwise, let $\{V_i\}$ be a reduct of $\{U_i\}$ and let $U'_{i\tau} = V_i \cap U_{i\tau}$ for all $i\tau \in T$. Apply this procedure to the trees $T(i)$ and further downwards whenever possible. Since T is well founded, we will finally obtain a desired reduced family which we call a reduct of $\{U_\tau\}$. ⊣

LEMMA 3.20. *For every well founded tree T, a base $\mathcal{L}(X)$, $\rho \in T$ and $\alpha < \omega_1$, there is a unique reduced T-family $\{U_\tau\}$ of $\mathcal{L}_\alpha$-sets such that $\tilde{U}_\rho = X$ (and then necessarily $\tilde{U}_\tau = \emptyset$ for all $\tau \in T \setminus \{\rho\}$).*

PROOF. Obviously, it is enough to set $U_\tau = X$ if $\tau \sqsubseteq \rho$ and $U_\tau = \emptyset$ otherwise. ⊣

3.3. Defining Q-partitions by iterated families. Here we define the notion of a u-family ($u \in \mathbb{T}_\sigma$) in a given base $\mathcal{L}(X)$ and explain how such (iterated) families determine Q-partitions of X. The definitions use induction on terms in §3.1, induction scheme of Definition 3.2 and Lemma 3.3. The u-families F are defined simultaneously for all $X, \mathcal{L}(X)$ as follows.

DEFINITION 3.21.

1. F is a q-family in $\mathcal{L}(X)$ iff $F = \{X\}$.
2. The $s_\alpha(u)$-families in $\mathcal{L}(X)$ coincide with the u-families in $\mathcal{L}^{\omega^\alpha}(X)$.
3. F is an $F_q(u_0, \ldots)$-family in $\mathcal{L}(X)$ iff $F = (\{U_\tau\}, \{F_\tau\})$ where $\{U_\tau\}$ is a monotone T-family of $\mathcal{L}_0$-sets with $U_\varepsilon = X$ and, for each $\tau \in T$, F_τ is a $t(\tau)$-family in $\mathcal{L}(\tilde{U}_\tau)$, where $(T, t) = T(F_q(u_0, \ldots))$.
4. F is an $F_\alpha(u_0, \ldots)$-family in $\mathcal{L}(X)$ iff $F = (\{U_\tau\}, \{F_\tau\})$ where $\{U_\tau\}$ is a monotone T-family of $\mathcal{L}_0$-sets with $U_\varepsilon = X$ and, for each $\tau \in T$, F_τ is a $t(\tau)$-family in $\mathcal{L}(\tilde{U}_\tau)$, where $(T, t) = T(F_\alpha(u_0, \ldots))$.

Reduced u-families F are defined by taking $\{U_\tau\}, F_\tau$ in (3) and (4) to be reduced. From Lemma 3.5 we obtain the following information on the structure of u-families in $\mathcal{L}(X)$ where we use notions from Definition 3.6.

LEMMA 3.22. *Let F be a u-family in $\mathcal{L}(X)$. If u is a singleton term then $F = \{X\}$, otherwise $F = (\{U_\tau\}, \{F_\tau\})$ where $\{U_\tau\}$ is a monotone $T(u')$-family of $\mathcal{L}_0^{sh(u)}$-sets with $U_\varepsilon = X$ and, for each $\tau \in T(u')$, F_τ is a $t(\tau)$-family in $\mathcal{L}^{sh(u)}(\tilde{U}_\tau)$.*

Now we define (again simultaneously for all $X, \mathcal{L}(X)$) the notion "a u-family F in $\mathcal{L}(X)$ determines a partition $A \in Q^X$".

DEFINITION 3.23.

1. A q-family F in $\mathcal{L}(X)$ determines A iff $A = \lambda x.q$ is the constant function $A(x) = q$.
2. An $s_\alpha(u)$-family F in $\mathcal{L}(X)$ determines A iff F determines A as a u-family in $\mathcal{L}^{\omega^\alpha}(X)$.

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3. For $u \in \{F_q(u_0, \dots), F_\alpha(u_0, \dots)\}$, a u-family $F = (\{U_\tau\}, \{F_\tau\})$ in $\mathcal{L}(X)$ determines A iff for each $\tau \in T(u)$, F_τ determines the restriction $A|_{\tilde{U}_\tau}$ of A to $\tilde{U}_\tau$.

By definitions above and Lemma 3.22, a u-family F in $\mathcal{L}(X)$ that determines a Q-partition A may be interpreted as a mind-change "algorithm" for computing the value $A(x) \in Q$ for any given $x \in X$ as follows. We use the set $\mathcal{F}(u)$ from Definition 3.8 and Lemma 3.9.

If u is a singleton term, $A(x) = q(u)$ is a constant Q-partition. Otherwise, $F = (\{U_{\tau_0}\}, \{F_{\tau_0}\})$ where $\{U_{\tau_0}\}$ is a monotone u'-family of $\mathcal{L}_0^{sh(u)}$-sets with $U_\varepsilon = X$ and, for each $\tau_0 \in T(u')$, F_{τ_0} is a $t_0(\tau_0)$-family in $\mathcal{L}^{sh(u)}(\tilde{U}_{\tau_0})$ (which coincides with the $t_0(\tau_0)'$-family in $\mathcal{L}^{sh(u)+sh(t_0(\tau_0))}(\tilde{U}_{\tau_0})$). Since the components $\tilde{U}_{\tau_0}$ (called first level components of F) cover X (by the definition of a monotone family), $x \in \tilde{U}_{\tau_0}$ for some $\tau_0 \in T(u')$; τ_0 is searched by the usual mind-change procedure working with differences of $\mathcal{L}_0^{sh(u)}$-sets (see Lemma 3.17).

If the term $t_0(\tau_0)$ is singleton, $A|_{\tilde{U}_{\tau_0}}$ is a constant Q-partition and we have computed $A(x) \in Q$. Otherwise, $F_{\tau_0} = (\{U_{\tau_0\tau_1}\}, \{F_{\tau_0\tau_1}\})$ and we can continue the computation as above and find a second level component $\tilde{U}_{\tau_0\tau_1}$ of F containing x; this is a harder mind-change procedure working with differences of $\mathcal{L}_0^{sh(u)+sh(t_0(\tau_0))}$-sets. We continue this process until we reach a sequence $(\tau_0, \dots, \tau_m) \in \mathcal{F}(u)$ such that $x \in \tilde{U}_{\tau_0\cdots\tau_m}$ and $t_m(\tau_m)$ is a singleton term; such components $\tilde{U}_{\tau_0\cdots\tau_m}$ are called *terminating* and have the associated constants $q(\tau_0, \dots, \tau_m) \in Q$. Note that the terminating components cover X and if the family F is reduced then the terminating components form a partition of X. In any case we have: $A^{-1}(q) = \bigcup\{\tilde{U}_{\tau_0\cdots\tau_m} \mid (\tau_0, \dots, \tau_m) \in \mathcal{F}_q(u)\}$ for each $q \in Q$.

If the family F above is reduced then the computation is "linear" since the components of each level are pairwise disjoint and cover the parent component, otherwise the computation is "parallel" since already at the first level x may belong to several components $\tilde{U}_{\tau_0}$ (and F may determine no Q-partition).

The described procedure enables to write a u-family F, where u is not a singleton term, in an explicit (but not completely precise) form of u'-family $(\{U_{\tau_0}\}, \{U_{\tau_0\tau_1}\}, \dots)$ in $\mathcal{L}^{sh(u)}(X)$ which is sometimes more intuitive than the form $(\{U_\tau\}, \{F_\tau\})$ above.

EXAMPLES 3.24.

1. If u has no entries of symbols s_α, F_α then a u-family F in $\mathcal{L}(X)$ is essentially a monotone family $T(u)$-family $\{U_\tau\}$ of $\mathcal{L}_0(X)$-sets with $U_\varepsilon = X$. If F determines a Q-partition A then $A(x) = t(\tau)$ whenever $x \in \widetilde{U}_\tau$ where $T(u) = (T, t)$. This case leads to the extension of the difference hierarchy over $\mathcal{L}_0(X)$ to Q-partitions (for $Q = \bar{k}$ this is described in [21, 27]).
2. For the term $u = s_\beta(v)$ from Examples 3.4(2) and 3.10(2), any u-family has the form $(\{U_\varepsilon, U_0, U_1, U_2, U_3, \dots\}, \{U_{\varepsilon,\varepsilon}\}, \{U_{0,\varepsilon}, U_{0,0}, U_{0,1}, \dots\}, \{U_{1,\varepsilon}\})$ where $\{U_\varepsilon, U_0, U_1, U_2, U_3, \dots\}$ is a monotone $T(u')$-family of $\mathcal{L}_{\omega^\beta}$-sets with $U_\varepsilon = X$, $U_{\varepsilon,\varepsilon} = \widetilde{U}_\varepsilon$, $\{U_{0,\varepsilon}, U_{0,0}, U_{0,1}, \dots\}$ is a monotone $t_0(0)$-family of $\mathcal{L}_{\omega^\beta+\omega^\gamma}$-sets with $U_{0,\varepsilon} = \widetilde{U}_0$, and $U_{1,\varepsilon} = \widetilde{U}_1$.

We formulate some properties of the introduced notions. The next lemma is immediate.

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LEMMA 3.25. *Let u be a nonsingleton term and let $A \in Q^X$ be determined by a u'-family $(\{U_{\tau_0}\}, \{U_{\tau_0\tau_1}\}, \dots)$ in $\mathcal{L}^{sh(u)}(X)$.*

1. *If $u' = F_q(u_0, \dots)$ then the u_i-family $(\{U_{i\sigma_0}\}, \{U_{i\sigma_0\tau_1}\}, \dots)$ in $\mathcal{L}^{sh(u)}(U_i)$ determines $A|_{U_i}$ for each $i \geq 0$.*
2. *If $u' = F_\alpha(u_0, \dots)$ then the u_{i+1}-family $(\{U_{i\sigma_0}\}, \{U_{i\sigma_0\tau_1}\}, \dots)$ in $\mathcal{L}^{sh(u)}(U_{i+1})$ determines $A|_{U_{i+1}}$ for each $i \geq 0$.*

Let $f : X \to Y$ be a function such that f^{-1} is a morphism from $\mathcal{L}(Y)$ to $\mathcal{L}(X)$. Associate with any u-family F in $\mathcal{L}(Y)$ the u-family $f^{-1}(F)$ in $\mathcal{L}(X)$ as follows: if $u = q$ then $f^{-1}(F) = \{X\}$; if $u = s_\alpha(v)$ then $f^{-1}(F)$ is the v-family $f^{-1}(F)$ in $\mathcal{L}^{\omega^\alpha}(X)$; in the remaining cases we have $F = (\{U_\tau\}, \{F_\tau\})$, and we set $f^{-1}(F) = (\{f^{-1}(U_\tau)\}, \{f^{-1}(F_\tau)\})$. Clearly, $f^{-1}(F)$ is indeed a u-family in $\mathcal{L}(X)$. The next lemma is immediate.

LEMMA 3.26. *If a u-family F in $\mathcal{L}(Y)$ determines A then the u-family $f^{-1}(F)$ in $\mathcal{L}(X)$ determines $A \circ f$.*

Now we associate with any u-family F in $\mathcal{L}(X)$ and any $V \subseteq X$ the u-family $F|_V$ in the V-restriction $\mathcal{L}(V)$ (see §3.2) as follows: if $u = q$ then $F|_V = \{V\}$; if $u = s_\alpha(v)$ then $F|_V$ is the v-family $F|_V$ in $\mathcal{L}^{\omega^\alpha}(V)$; in the remaining cases we have $F = (\{U_\tau\}, \{F_\tau\})$, and we set $F|_V = (\{V \cap U_\tau\}, \{F_\tau|_V\})$. Obviously, $F|_V$ is indeed a u-family in $\mathcal{L}(V)$. The next lemma is immediate by induction.

LEMMA 3.27. *If a u-family F in $\mathcal{L}(X)$ determines A then the u-family $F|_V$ in $\mathcal{L}(V)$ determines $A|_V$.*

Let $\{G_i\}$, $G_i = (\{U^i_{\tau_0}\}, \{U^i_{\tau_0\tau_1}\}, \dots)$, be a sequence of u-families (u is a nonsingleton term) in $\mathcal{L}(Y_i)$, $Y_i \subseteq X$. Then $G = (\{U_{\tau_0}\}, \{U_{\tau_0\tau_1}\}, \dots)$, where $U_{\tau_0} = \bigcup_i U^i_{\tau_0}$, $U_{\tau_0\tau_1} = \bigcup_i U^i_{\tau_0\tau_1} \dots$, is a u-family in $\mathcal{L}(Y)$ where $Y = \bigcup_i Y_i$. The next lemma follows from Lemma 3.18.

LEMMA 3.28. *Let $A \in Q^X$. If the u-family G_i in $\mathcal{L}(Y_i)$ determines $A|_{Y_i}$ for each $i \geq 0$ then the u-family G in $\mathcal{L}(Y)$ determines $A|_Y$.*

The next lemma is also clear.

LEMMA 3.29. *Let $A \in Q^X$, $Y \in \mathcal{L}_0(X) \cap \check{\mathcal{L}}_0(X)$, $A(x) = q$ for $x \in X \setminus Y$, let $A|_Y$ be determined by a u-family F in $\mathcal{L}(Y)$, and let $\tilde{U}_{\tau_0\cdots\tau_m}$ be a terminating component of F with $q = q(\tau_0, \dots, \tau_m)$. Then there is a u-family F' in $\mathcal{L}(X)$ such that its $(\tau_0, \dots, \tau_m)$-terminating component is $\tilde{U}_{\tau_0\cdots\tau_m} \cup (X \setminus Y)$, all other terminating components coincide with those of F, and F' determines A.*

Let $F = (\{U_{\tau_0}\}, \{U_{\tau_0\tau_1}\}, \dots)$ and $G = (\{V_{\tau_0}\}, \{V_{\tau_0\tau_1}\}, \dots)$ be u-families in $\mathcal{L}(X)$. We say that G *is a reduct of* F if G is reduced and $\tilde{V}_{\tau_0\cdots\tau_m} \subseteq \tilde{U}_{\tau_0\cdots\tau_m}$ for each $(\tau_0, \dots, \tau_m) \in \mathcal{F}(u)$.

LEMMA 3.30. *Let $\mathcal{L}(X)$ be a reducible base in X and $u \in \mathbb{T}_\sigma$. Then any u-family F in $\mathcal{L}(X)$ has a reduct G. Moreover, if F determines A then any reduct of F determines A.*

PROOF. We follow the procedure of computing $A(x)$ described above. If u is a singleton term, we set $G = F = \{X\}$; then F, G determine the same constant Q-partition. Otherwise, F has the form as above. Let G as above be obtained from F by

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repeated reductions from Lemma 3.19, so in particular $\tilde{V}_{\tau_0\cdots\tau_m} \subseteq \tilde{U}_{\tau_0\cdots\tau_m}$ for each $(\tau_0, \dots, \tau_m) \in \mathcal{F}(u)$.

For the second assertion, let F determine A and let G be a reduct of F. For any $x \in X$, let $\tilde{V}_{\tau_0\cdots\tau_m}$ be the unique terminating component of G containing x. Then also $x \in \tilde{U}_{\tau_0\cdots\tau_m}$, hence $A(x) = q(\tau_0, \dots, \tau_m)$ and G determines A. ⊣

The next lemma follows from the results above.

LEMMA 3.31. *Every u-family F in $\mathcal{L}(X)$ determines at most one Q-partition of X. Every reduced u-family G in $\mathcal{L}(X)$ determines precisely one Q-partition of X.*

PROOF. The second assertion follows from the remark that the terminating components of G form a partition of X. For the first assertion, let F in $\mathcal{L}(X)$ determine Q-partitions A, B of X. Let $x \in X$. If u is a singleton term, F determines a constant Q-partition, so in particular $A(x) = B(x)$. Otherwise, $F = (\{U_\tau\}, \{F_\tau\})$ as specified above. By the procedure of computing $A(x)$, there is a terminating component $\tilde{U}_{\tau_0\cdots\tau_m} \ni x$ of F. By Definition 3.23, $A(x) = q(\tau_0, \dots, \tau_m) = B(x)$. ⊣

3.4. Infinitary fine hierarchy over a base. Here we define the Q-IFH over a given base and prove some of its properties.

DEFINITION 3.32. For any base $\mathcal{L}(X)$ in X, qo Q, and $u \in \mathbb{T}_\sigma$, let $\widehat{\mathcal{L}}(X,u) = \bigcup\{\mathcal{L}(X,v) \mid v \trianglelefteq u\}$ where $\mathcal{L}(X,v)$ is the set of Q-partitions of X determined by some v-family in $\mathcal{L}(X)$. The family $\{\widehat{\mathcal{L}}(X,u)\}_{u\in\mathbb{T}_\sigma}$ is called the *infinitary Q-fine hierarchy over* $\mathcal{L}(X)$.

COROLLARY 3.33. *If Q is bqo then $(\{\widehat{\mathcal{L}}(X,u) \mid u \in \mathbb{T}_\sigma\}; \subseteq)$ is bqo.*

PROOF. Clearly, $u \mapsto \widehat{\mathcal{L}}(X,u)$ is a monotone surjection from bqo $(\mathbb{T}_\sigma; \trianglelefteq)$ onto $(\{\widehat{\mathcal{L}}(X,u) \mid u \in \mathbb{T}_\sigma\}; \subseteq)$. Hence, the latter structure is also bqo. ⊣

The algorithm of computing $A(x)$ described above explains in which sense the Q-IFH over $\mathcal{L}(X)$ may be considered as an “iterated difference hierarchy.” Classes $\mathcal{L}(X,u)$ play a major technical role in the proofs below while properties of classes $\widehat{\mathcal{L}}(X,u)$ capture more properties of the Q-Wadge hierarchy. Clearly, if Q is an antichain then $\mathcal{L}(X,u) = \widehat{\mathcal{L}}(X,u)$ for every $u \in \mathbb{T}_\sigma$. By Lemma 3.3, we can equivalently denote the Q-IFH over $\mathcal{L}(X)$ as $\{\widehat{\mathcal{L}}(X,T)\}_{T\in\mathcal{T}_{\omega_1}(Q)}$, as we did in Introduction. The next lemma describes the behavior of Q-IFH w.r.t. the operations on bases from §3.2.

LEMMA 3.34.

1. *For any $\alpha < \omega_1$, $\mathcal{L}(X, s_\alpha(u)) = \mathcal{L}^{\omega^\alpha}(X,u)$ and $\mathcal{L}(X,u) = \mathcal{L}^{sh(u)}(X,u')$.*
2. *For any $V \subseteq X$, $A \in \mathcal{L}(X,u)$ implies $A|_V \in \mathcal{L}(V,u)$.*
3. *Let u be nonsingleton and A determined by a u-family $(\{U_{\tau_0}\}, \{U_{\tau_0\tau_1}\}, \dots)$ in $\mathcal{L}(X)$. If $u' = F_q(u_0, \dots)$ (resp. $u' = F_\alpha(u_0, \dots)$) then $A|_{U_i} \in \mathcal{L}(X,u_i)$ for each $i \geq 0$ (resp. $i \geq 1$).*
4. *Let $A \in Q^X$, $u_0, u_1, \dots \in \mathbb{T}_\sigma$, and let $\{U_i\}_{i\geq 0}$ be nonempty open sets not exhausting X such that $A|_V = \lambda v.q$ (where $V = \mathcal{N} \setminus \bigcup_i U_i$) and $A|_{U_i} \in \mathcal{L}(U_i,u_i)$ for all $i \geq 0$. Then $A \in \mathcal{L}(X,u)$ where $u = F_q(u_0, \dots)$.*
5. *Let $A \in Q^X$, $u_0, u_1, \dots \in \mathbb{T}_\sigma$, and let $\{U_i\}_{i\geq 1}$ be nonempty open sets not exhausting X such that $A|_V \in \mathcal{L}(X, s_\alpha(u_0))$ (where $V = \mathcal{N} \setminus \bigcup_{i\geq 1} U_i$) and $A|_{U_i} \in \mathcal{L}(U_i,u_i)$ for all $i \geq 1$. Then $A \in \mathcal{L}(X,u)$ where $u = F_\alpha(u_0, \dots)$.*

PROOF. (1), (2) and (3) follow resp. from Definition 3.23, Lemma 3.27, and Lemma 3.25.

(4) Let $A|_{U_i} \in \mathcal{L}(X, u_i)$ be determined by a u_i-family $G_i = (\{U^i_{\tau_0}\}, \{U^i_{\tau_0\tau_1}\}, \dots)$ in $\mathcal{L}(U_i)$, for each $i \geq 0$. By Definition 3.2, $T(u) = q \to (T(u_0) \sqcup T(u_1) \sqcup \cdots)$. We define the u-family $G = (\{V_{\tau_0}\}, \{V_{\tau_0\tau_1}\}, \dots)$ in $\mathcal{L}(X)$ as follows: $V_\varepsilon = X$, $V_{i\tau_0} = U^i_{\tau_0}$, $V_{i\tau_0\tau_1} = U^i_{\tau_0\tau_1}$, and so on. Then G determines A, hence $A \in \mathcal{L}(X, u)$.

(5) Similar to (4). ⊣

Next we discuss inclusions of levels of the Q-IFH.

LEMMA 3.35.

1. $\mathcal{L}(X, u) \subseteq \mathcal{L}(X, s_\alpha(u))$.
2. $\mathcal{L}(X, q) \subseteq \mathcal{L}(X, F_q(u_0, \dots))$.
3. $\mathcal{L}(X, u_i) \subseteq \mathcal{L}(X, F_q(u_0, \dots))$ *for all* $i \geq 0$.
4. $\mathcal{L}(X, s_\alpha(u_0)) \subseteq \mathcal{L}(X, F_\alpha(u_0, \dots))$.
5. $\mathcal{L}(X, u_{i+1}) \subseteq \mathcal{L}(X, F_\alpha(u_0, \dots))$ *for all* $i \geq 0$.
6. *Let* $u, v \in \mathbb{T}_\sigma$, $\beta, \gamma < \omega_1$, *and* $\mathcal{L}^\beta(X, u) \subseteq \mathcal{L}^\gamma(X, v)$ *for all* X *and* $\mathcal{L}(X)$. *Then* $\mathcal{L}^{\alpha+\beta}(X, u) \subseteq \mathcal{L}^{\alpha+\gamma}(X, v)$ *for all* $\alpha < \omega_1, X, \mathcal{L}(X)$.

PROOF. (1) Let $A \in \mathcal{L}(X, u)$, then A is determined by a u-family F in $\mathcal{L}(X)$. By Definition 3.12, F is also a u-family in $\mathcal{L}^{\omega^\alpha}(X)$, hence $A \in \mathcal{L}^{\omega^\alpha}(X, u)$. By Lemma 3.34(1), $A \in \mathcal{L}(X, s_\alpha(u))$.

(2) and (3) follow resp. from Lemmas 3.20 and 3.34(3). Let $F_i = G$. For any $\tau \in T(u) \setminus \{i\}$, let F_τ be the trivial reduced $t(\tau)$-family in $\mathcal{L}(\emptyset)$ with empty components. By Definition 3.2, the family F determines A.

Items (4) and (5) are checked by manipulations similar to those in (2) and (3).

(6) For the base $\mathcal{L}^\alpha(X)$ the given inclusion reads $(\mathcal{L}^\alpha)^\beta(X, u) \subseteq (\mathcal{L}^\alpha)^\gamma(X, v)$. By Lemma 3.13(1), $\mathcal{L}^{\alpha+\beta}(X, u) \subseteq \mathcal{L}^{\alpha+\gamma}(X, v)$. ⊣

The main result about inclusions of levels of the Q-IFH is the following theorem.

THEOREM 3.36. *If* Q *is antichain and* $u \trianglelefteq v$, *then* $\mathcal{L}(X, u) \subseteq \mathcal{L}(X, v)$ *for all* $X, \mathcal{L}(X)$.

PROOF. We argue by induction of Definition 3.1.

(1) Let $q \trianglelefteq r$, then $q \leq_Q r$, hence $q = r$, hence trivially $\mathcal{L}(X, q) \subseteq \mathcal{L}(X, r)$.

(2) Let $q \trianglelefteq s_\alpha(u)$, then $q \trianglelefteq u$, hence by induction and Lemma 3.35(1) $\mathcal{L}(X, q) \subseteq \mathcal{L}(X, u) \subseteq \mathcal{L}(X, s_\alpha(u))$.

(3) Let $q \trianglelefteq F_r(u_0, \dots)$, then $q \trianglelefteq r$ or $q \trianglelefteq u_i$ for some $i \geq 0$, and the inclusion follows by induction and Lemma 3.35(2,3).

(4) Let $q \trianglelefteq F_\alpha(u_0, \dots)$, then $q \trianglelefteq s_\alpha(u_0)$ or $q \trianglelefteq u_i$ for some $i \geq 1$, and the inclusion follows by induction and Lemma 3.35(4,5).

(5) Let $s_\alpha(u) \trianglelefteq r$, then $u \trianglelefteq r$. By induction, $\mathcal{L}^{\omega^\alpha}(X, u) \subseteq \mathcal{L}^{\omega^\alpha}(X, r) = \{\lambda x.r\}$. By Lemma 3.34(1), $\mathcal{L}(X, s_\alpha(u)) = \{\lambda x.r\} \subseteq \mathcal{L}(X, r)$.

(6) Let $s_\alpha(u) \trianglelefteq s_\beta(v)$. Then $(\alpha < \beta$ and $u \trianglelefteq s_\beta(v))$ or $(\alpha = \beta$ and $u \trianglelefteq v)$ or $(\alpha > \beta$ and $s_\alpha(u) \trianglelefteq v)$. In the first case, by induction we have $\mathcal{L}(X, u) \subseteq \mathcal{L}(X, s_\beta(v)) \subseteq \mathcal{L}^{\omega^\beta}(X, v)$. By Lemmas 3.35(6), 3.13(2) and 3.34(1), $\mathcal{L}(X, s_\alpha(u)) = \mathcal{L}^{\omega^\alpha}(X, u) \subseteq \mathcal{L}^{\omega^\alpha+\omega^\beta}(X, v) = \mathcal{L}^{\omega^\beta}(X, v) = \mathcal{L}(X, s_\beta(v))$. In the second case, by

induction we have $\mathcal{L}(X,u) \subseteq \mathcal{L}(X,v)$, hence $\mathcal{L}^{\omega^\alpha}(X,u) \subseteq \mathcal{L}^{\omega^\beta}(X,v)$, hence $\mathcal{L}(X, s_\alpha(u)) \subseteq \mathcal{L}(X, s_\beta(v))$. The third case is even easier.

(7) Let $s_\alpha(u) \trianglelefteq F_r(v_0,\dots)$, then $s_\alpha(u) \trianglelefteq r$ or $s_\alpha(u) \trianglelefteq v_i$ for some $i \geq 0$. The assertion follows by Lemma 3.35(2) or (3), respectively.

(8) Let $s_\alpha(u) \trianglelefteq F_\beta(v_0,\dots)$, then $s_\alpha(u) \trianglelefteq s_\beta(v_0)$ or $s_\alpha(u) \trianglelefteq v_i$ for some $i \geq 1$. The assertion follows by Lemma 3.35(4) or (5), respectively.

(9) Let $F_q(u_0,\dots) \trianglelefteq r$, then $q \trianglelefteq r$ and $u_i \trianglelefteq r$ for all $i \geq 0$. In this case the argument of item (5) works.

(10) Let $F_q(u_0,\dots) \trianglelefteq s_\alpha(v)$, then $q \trianglelefteq s_\alpha(v)$ and $u_i \trianglelefteq s_\alpha(v)$ for all $i \geq 0$. If v is a singleton term, the argument of item (9) works, so let v be a nonsingleton term. Without loss of generality we way think that v is an F-term (otherwise, $\mathcal{L}^{\omega^\alpha}(X,v) = \mathcal{L}^{\omega^\alpha + sh(v)}(X,v')$, and we can work with the F-term v' instead of v).

Let $A \in \mathcal{L}(X, F_q(u_0,\dots))$, we have to show that $A \in \mathcal{L}(X, s_\alpha(v))$. Let $(\{U_{\tau_0}\}, \{U_{\tau_0\tau_1}\},\dots)$ be a u-family in $\mathcal{L}(X)$ that determines A, then $A(x) = q$ for each $x \in \tilde{U}_\varepsilon$ (note that $\tilde{U}_\varepsilon \in \mathcal{L}_0^{\omega^\alpha}(X) \cap \check{\mathcal{L}}_0^{\omega^\alpha}(X)$) and, by Lemma 3.25, $A|_{U_i}$ is determined by the u_i-family $(\{U_{i\tau_1}\},\dots)$ in $\mathcal{L}(U_i)$ for every $i \geq 0$. By induction, $A|_{U_i} \in \mathcal{L}^{\omega^\alpha}(U_i, v)$ for every $i \geq 0$, so let $G_i = (\{V^i_{\tau_0}\}, \{V^i_{\tau_0\tau_1}\},\dots)$ be a v-family in $\mathcal{L}^{\omega^\alpha}(U_i)$ that determines $A|_{U_i}$. By Lemma 3.28, the v-family $G = \bigcup_i G_i = (\{V_{\tau_0}\}, \{V_{\tau_0\tau_1}\},\dots)$ in $\mathcal{L}^{\omega^\alpha}(\bigcup_i U_i)$ determines $A_{\bigcup_i U_i}$. By Lemma 3.29, the $s_\alpha(v)$-family G' determines A, hence $A \in \mathcal{L}(X, s_\alpha(v))$.

(11) Let $F_q(u_0,\dots) \trianglelefteq F_r(v_0,\dots)$, then ($q \trianglelefteq r$ and $u_i \trianglelefteq F_r(v_0,\dots)$ for all $i \geq 0$) or $F_q(u_0,\dots) \trianglelefteq v_i$ for some $i \geq 0$; the second case follows from Lemma 3.35(3), so consider the first case. Since Q is antichain, $q = r$. Let $A \in \mathcal{L}(X, F_q(u_0,\dots))$, we have to show that $A \in \mathcal{L}(X, F_q(v_0,\dots))$. Let $(\{U_{\tau_0}\}, \{U_{\tau_0\tau_1}\},\dots)$ be a u-family in $\mathcal{L}(X)$, where $u = F_q(u_0,\dots)$, that determines A, then $A(x) = q$ for each $x \in \tilde{U}_\varepsilon$, and, by Lemma 3.25, $A|_{U_i}$ is determined by the family $(\{U_{i\tau_1}\},\dots)$ in $\mathcal{L}(U_i)$ for each $i \geq 0$. By induction, $A|_{U_i} \in \mathcal{L}(U_i, v)$ for each $i \geq 0$, where $v = F_q(v_0,\dots)$, so $A|_{U_i}$ is determined by a v-family $G_i = (\{V^i_{\tau_0}\}, \{V^i_{\tau_0\tau_1}\},\dots)$ in $\mathcal{L}(U_i)$. By Lemma 3.28, the v-family $G = (\{V_{\tau_0}\}, \{V_{\tau_0\tau_1}\},\dots)$ in $\mathcal{L}(\bigcup_i U_i)$ determines $A|_{\bigcup_i U_i}$. Correcting the v-family G by changing V_ε to X, we obtain a v-family G' in $\mathcal{L}(X)$ that determines A. Thus, $A \in \mathcal{L}(X, v)$.

Items (12), (15) and (16) are checked similar to (10) and (11), item (13) similar to (9), item (14) similar to (11). ⊣

We conclude this subsection with a result about the reduction property. Let the classes red-$\mathcal{L}(X,u)$ be defined as the classes $\mathcal{L}(X,u)$ in Definition 3.32 but with the reduced families in place of arbitrary families. Note that in many spaces the inclusion red-$\mathcal{L}(X,u) \subset \mathcal{L}(X,u)$ is proper (e.g., this holds for $X = P\omega$ where two nonempty open sets always have nonempty intersection). Let red-$\widehat{\mathcal{L}}(X,u) = \bigcup\{$red-$\mathcal{L}(X,v) \mid v \trianglelefteq u\}$.

Proposition 3.37. *If $\mathcal{L}(X)$ is a reducible base then $\mathcal{L}(X,u)$ =red-$\mathcal{L}(X,u)$ and $\widehat{\mathcal{L}}(X,u)$ =red-$\widehat{\mathcal{L}}(X,u)$ for each $u \in \mathbb{T}_\sigma$.*

Proof. The inclusion $\supseteq$ is obvious. Conversely, let $A \in \mathcal{L}(X,u)$, then A is determined by a u-family F in $\mathcal{L}(X)$. By Lemma 3.30, A is determined by a u-family

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G in $\mathcal{L}(X)$ which is a reduct of F. Thus, A is in red-$\mathcal{L}(X, u)$. The assertion for $\widehat{\mathcal{L}}$ follows. ⊣

§4. Infinitary fine hierarchies in cb$_0$-spaces. In this section we study the Q-IFH in cb$_0$-spaces. We show that some important properties are preserved by continuous open surjections while others are not, and we give the set-theoretic description of the Q-Wadge hierarchy in the Baire space. From now on all bases we discuss are the Borel bases $\mathcal{L}(X) = \{\mathbf{\Sigma}^0_{1+\alpha}(X)\}_{\alpha<\omega_1}$ in cb$_0$-spaces X.

4.1. General properties. Here we collect some general properties of Q-IFH in cb$_0$-spaces. As we know from Lemma 3.16, most of levels of the Borel hierarchy in X have the σ-reduction property. By Proposition 3.37, this implies the following simpler characterization of many levels of the Q-IFH in X.

PROPOSITION 4.1. *For any cb$_0$-space X and any $u \in \mathbb{T}_\sigma$, red-$\mathcal{L}^1(X, u) = \mathcal{L}^1(X, u)$. If X is zero-dimensional then red-$\mathcal{L}(X, u) = \mathcal{L}(X, u)$ for all $u \in \mathbb{T}_\sigma$. Similarly for $\widehat{\mathcal{L}}(X, u)$.*

PROPOSITION 4.2. *Let $f : X \to Y$ be a continuous function and $u \in \mathbb{T}_\sigma$. Then $A \in \mathcal{L}(Y, u)$ implies $A \circ f \in \mathcal{L}(X, u)$, and similarly for $\widehat{\mathcal{L}}(X, u)$.*

PROOF. Let $A \in Q^Y$ be defined by a u-family F in $\mathcal{L}(Y)$. Since the preimage function $f^{-1} : P(Y) \to P(X)$ is a morphism from $\mathcal{L}(Y)$ to $\mathcal{L}(X)$ by Lemma 3.14, $A \circ f$ is determined by the u-family $f^{-1}(F)$ in $\mathcal{L}(X)$ by Lemma 3.26. Therefore, $A \circ f \in \mathcal{L}(X, u)$. The assertion for $\widehat{\mathcal{L}}(X, u)$ follows in the obvious way. ⊣

Next we briefly discuss the relation of Q-IFH in X to the Wadge reducibility $\leq_W$ of Q-partitions of X (see Introduction).

PROPOSITION 4.3.

1. *If Q is antichain (in particular, $Q = \bar{k}$) then the levels $\widehat{\mathcal{L}}(X, u)$ and $\mathcal{L}(X, u)$ are closed downwards under Wadge reducibility.*
2. *For any zero-dimensional space X, a qo Q and $u \in \mathbb{T}_\sigma$, the level $\widehat{\mathcal{L}}(X, u)$ is closed downwards under Wadge reducibility.*
3. *For any cb$_0$-space X, a qo Q and $u \in \mathbb{T}_\sigma$, the level $\widehat{\mathcal{L}}^1(X, u)$ is closed downwards under Wadge reducibility.*

PROOF. (1) Since $\leq_Q$ is the equality on Q, $A \leq_W B$ iff $A = B \circ f$ for some continuous function f on X. Thus, the assertion is a particular case of Proposition 4.2 when $X = Y$.

(2) Let $A \leq_W B$ via f and $B \in \widehat{\mathcal{L}}(X, u)$, so $B \in \mathcal{L}(X, v)$ for some $v \trianglelefteq u$. Then $C = B \circ f \in \mathcal{L}(X, v)$ by Proposition 4.2 where $X = Y$, and $A(x) \leq_Q C(x)$ for each $x \in X$. By Proposition 4.1, C is determined by a reduced v-family $F = (\{U_{\tau_0}\}, \{U_{\tau_0\tau_1}\}, \ldots)$. Any $x \in X$ belongs to a unique terminating component $\tilde{U}_{\tau_0\cdots\tau_m}$ with $C(x) = q(\tau_0, \ldots, \tau_m)$. In the syntactic tree of v, any $q = C(x)$ is either a leaf label or the subscript of an F_q-label. Replacing any such $C(x)$ by $A(x)$, we obtain a term w such that $A \in \mathcal{L}(X, w)$ (because A is determined by F in which terminating labels are changed accordingly). It is easy to see that $w \trianglelefteq v$, hence $A \in \widehat{\mathcal{L}}(X, u)$.

(3) The argument of item (2) works. ⊣

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4.2. Preservation property. Here we show that all levels of the Q-IFH are preserved by continuous open surjections.

With any function $f : X \to Y$ between cb$_0$-spaces we associate the function $A \mapsto f[A]$ from $P(X)$ to $P(Y)$ defined by

$$f[A] = \{y \in Y \mid A \cap f^{-1}(y) \text{ is nonmeager in } f^{-1}(y)\}.$$

Its importance stems from Baire-category properties of cb$_0$-spaces recalled in §2.3. The function $A \mapsto f[A]$ (known as the existential category quantifier, see e.g., §8.J in [9]) was used e.g., in [4, 17, 27]; we changed its notation trying to make it more convenient in our context.

The next two lemmas generalize some results from [4, 17, 27]. Please distinguish $f[A]$ and the image $f(A)$ of A under f.

Lemma 4.4.

1. *The function $A \mapsto f[A]$ is a morphism from $\mathcal{L}(X)$ to $\mathcal{L}(Y)$, and $f[A] \subseteq f(A)$ for each $A \subseteq X$.*
2. *If T is a well founded tree and $\{U_\tau\}$ is a T-family of $\mathbf{\Sigma}^0_{1+\alpha}(X)$-sets then $\{f[U_\tau]\}$ is a T-family of $\mathbf{\Sigma}^0_{1+\alpha}(Y)$-sets, and $\widetilde{f[U_\tau]} \subseteq f[\tilde{U}_\tau]$ for each $\tau \in T$.*

Proof. (1) Let $y \in f[A]$, then $A \cap f^{-1}(y)$ is nonmeager in $f^{-1}(y)$. Then $A \cap f^{-1}(y)$ is nonempty, hence $y \in f(A)$ and $f[A] \subseteq f(A)$. In particular, $f[\emptyset] = \emptyset$. To show that $f[X] = Y$ we have to check that, for any $y \in Y$, $f^{-1}(y)$ is nonmeager in $f^{-1}(y)$, and this follows from quasi-Polishness of $f^{-1}(y)$. The property that $f[\bigcup_n U_n] = \bigcup_n f[U_n]$ for every countable sequence $\{U_n\}$ in $P(X)$ is well known. The (nontrivial) fact that $U \in \mathbf{\Sigma}^0_{1+\alpha}(X)$ implies $f[U] \in \mathbf{\Sigma}^0_{1+\alpha}(Y)$, follows from Proposition 2.3, see [4, 17].

(2) The first assertion follows from (1), so we check the second one. Let $y \in \widetilde{f[U_\tau]}$, i.e., $y \in f[U_\tau] \setminus \bigcup\{f[U_{\tau'}] \mid \tau \sqsubset \tau' \in T\}$. Then $U_\tau \cap f^{-1}(y)$ is nonmeager in $f^{-1}(y)$ and, for each $\tau \sqsubset \tau' \in T$, $U_{\tau'} \cap f^{-1}(y)$ is meager in $f^{-1}(y)$. Then $(\bigcup\{U_{\tau'} \mid \tau \sqsubset \tau' \in T\}) \cap f^{-1}(y)$ is meager in $f^{-1}(y)$, hence $\tilde{U}_\tau = U_\tau \setminus \bigcup\{U_{\tau'} \mid \tau \sqsubset \tau' \in T\}$ is nonmeager in $f^{-1}(y)$, i.e., $y \in f[\tilde{U}_\tau]$. ⊣

We associate with any u-family F in $\mathcal{L}(X)$ the u-family $f[F]$ in $\mathcal{L}(Y)$ by induction as follows: if u is a singleton term (hence $F = \{X\}$) then we set $f[F] = \{Y\}$; otherwise, u' is an F-term and $F = (\{U_\tau\}, \{F_\tau\})$ is a u'-family in $\mathcal{L}^{sh(u)}(X)$; we set $f[F] = (\{f[U_\tau]\}, \{f[F_\tau]\})$ which is a u'-family in $\mathcal{L}^{sh(u)}(Y)$, hence a u-family in $\mathcal{L}(Y)$.

Lemma 4.5. *Let $u \in \mathbb{T}_\sigma$, $A \in Y \to Q$, and $A \circ f \in \mathcal{L}(X, u)$ be determined by a u-family F in $\mathcal{L}(X)$. Then A is determined by the u-family $f[F]$ in $\mathcal{L}(X)$.*

Proof. If u is a singleton term, the assertion is obvious. Otherwise, u' is an F-term and the family F has the form $(\{U_{\tau_0}\}, \{U_{\tau_0\tau_1}\}, \dots)$, so $f[F]$ has the form $(\{f[U_{\tau_0}]\}, \{f[U_{\tau_0\tau_1}]\}, \dots)$. We have to show that A is determined by $f[F]$, i.e., for each $y \in Y$, $A(y) = q(\tau_0, \dots, \tau_m)$, for every terminating component $\widetilde{f[U_{\tau_0\cdots\tau_m}]}$ of $f(F)$ containing y. Such a component exists by induction on the rank of u (see the procedure of computing $A(y)$ described above).

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For any given $y \in Y$ and any such component $f[\widetilde{U_{\tau_0\cdots\tau_m}}]$ we have $y \in f[\tilde{U}_{\tau_0\cdots\tau_m}]$ by Lemma 4.4(2), so $y = f(x)$ for some $x \in \tilde{U}_{\tau_0\cdots\tau_m}$. Thus, $A(y) = (A \circ f)(x) = q(\tau_0, \dots, \tau_m)$. ⊣

As an immediate corollary of Lemmas 4.5 and 3.26 we obtain the following preservation property for levels of the Q-IFH.

Theorem 4.6. *Let $\mathcal{L}(X), \mathcal{L}(Y)$ be Borel bases in cb$_0$-spaces X, Y respectively, $f : X \to Y$ a continuous open surjection, $A : Y \to Q$, and $u \in \mathbb{T}_\sigma$. Then $A \circ f \in \mathcal{L}(X,u)$ iff $A \in \mathcal{L}(Y,u)$, and similarly for $\widehat{\mathcal{L}}(X,u)$.*

Proof. Let $A \in \mathcal{L}(Y,u)$, then A is determined by a u-family F in $\mathcal{L}(Y)$. By Lemma 3.26, $A \circ f \in \mathcal{L}(X,u)$. Conversely, let $A \circ f \in \mathcal{L}(X,u)$, then $A \circ f$ is determined by a u-family F in $\mathcal{L}(X)$. By Lemma 4.5, A is determined by the u-family $f[F]$ in $\mathcal{L}(Y)$, hence $A \in \mathcal{L}(Y,u)$. The assertion for $\widehat{\mathcal{L}}(X,u)$ follows in the obvious way. ⊣

4.3. Inheritance of HK-type theorems. Here we apply the preservation theorem to show that some versions of the Hausdorff–Kuratowski theorem (which we call HK-type theorems for short) are inherited by the continuous open images.

Recall that the Hausdorff theorem in a space X says that $\bigcup_{\beta<\omega_1} \mathbf{\Sigma}^{-1,1}_\beta(X) = \mathbf{\Delta}^0_2(X)$. The difference hierarchy $\{\mathbf{\Sigma}^{-1,1}_\beta(X)\}$ over the open sets in X is usually defined using a difference operator on the transfinite sequences of open sets (see e.g.,, [9, 25]). Since in this paper we promote using labeled trees instead of ordinals, we note that levels $\mathbf{\Sigma}^{-1,1}_\beta(X)$ are easily characterized using $\bar{2}$-labeled trees in $\mathcal{T}(\bar{2})$ (see the beginning of §3.1). Namely, by Proposition 4.9 in [25], there is a tree $T_\beta \in \mathcal{T}(\bar{2})$ of rank β such that $\mathbf{\Sigma}^{-1,1}_\beta(X) = \mathcal{L}(X, T_\beta)$, and any $T \in \mathcal{T}(\bar{2})$ is $\trianglelefteq$-equivalent to one of $T_\beta, \bar{T}_\beta$, where $u \mapsto \bar{u}$ is the automorphism induced by $i \mapsto 1-i$. Thus, the Hausdorff theorem for X may be written as $\bigcup\{\mathcal{L}(X,T) \mid T \in \mathcal{T}(\bar{2})\} = \mathbf{\Delta}^0_2(X)$ (in this subsection it is more convenient to work with labeled trees rather that with terms, see Lemma 3.3).

The Kuratowski theorem extends the Hausdorff theorem to any successor level of the Borel hierarchy in X (see §2.3 for the formulation of this theorem for quasi-Polish spaces). The Kuratowski theorem has a reformulation in terms of $\bar{2}$-labeled trees in just the same way as for the Hausdorff theorem. Namely, the tree form of the HK theorem in X looks like $\bigcup\{\mathcal{L}(X,T) \mid T \in \mathcal{T}_\alpha(\mathcal{T}(\bar{2}))\} = \mathbf{\Delta}^0_{1+\alpha+1}(X)$ for each $\alpha < \omega_1$, where some notation from the end of Section 3.1 is used; in particular, $\mathcal{T}_\alpha \circ \mathcal{T} = \mathcal{T}_{\alpha+1}$.

The tree form of the HK-theorem readily extends to Q-partitions which yields our first example of inheritance of the HK-type theorems. We say that a cb$_0$-space *X satisfies the HK-theorem for Q-partitions in level* $1+\alpha+1 < \omega_1$, iff $\bigcup\{\mathcal{L}(X,T) \mid T \in \mathcal{T}_{\alpha+1}(Q)\} = \mathbf{\Delta}^0_{1+\alpha+1}(Q^X)$. We define the qo $\leq_{co}$ on cb$_0$-spaces by: $Y \leq_{co} X$ iff there is a continuous open surjection from X onto Y.

Theorem 4.7. *If a cb$_0$-space X satisfies the HK-theorem for Q-partitions in level $1+\alpha+1 < \omega_1$, then so does every space $Y \leq_{co} X$.*

Proof. Since the inclusion $\bigcup\{\mathcal{L}(X,T) \mid T \in \mathcal{T}_{\alpha+1}(Q)\} \subseteq \mathbf{\Delta}^0_{1+\alpha+1}(Q^X)$ is easy, we check only the opposite inclusion. Let $A \in \mathbf{\Delta}^0_{1+\alpha+1}(Q^Y)$ and let $f : X \to Y$ be a

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continuous open surjection. Then $A \circ f \in \mathbf{\Delta}^0_{1+\alpha+1}(Q^X)$, hence $A \circ f \in \mathcal{L}(X, T)$ for some $T \in \mathcal{T}_{\alpha+1}(Q)$. By Theorem 4.6, $A \in \mathcal{L}(Y, T)$. ⊣

Our second example is concerned with a version of HK-theorem for limit levels of the Borel hierarchy. The problem of finding a construction principle for the $\mathbf{\Delta}^0_\lambda$-subsets of the Baire space in the case that λ is a positive limit countable ordinal was posed long ago by Luzin and resolved in [34] as an important step to the complete description of the Wadge hierarchy. We state the inheritance property for an extension of this result from sets to Q-partitions. We say that a cb$_0$-space X *satisfies the Wadge property for Q-partitions in a limit level $\lambda < \omega_1$*, iff $\bigcup\{\mathcal{L}(X, T) \mid T \in \mathcal{T}_\lambda(Q)\} = \mathbf{\Delta}^0_\lambda(Q^X)$.

The next result is proved in just the same way as the previous theorem.

THEOREM 4.8. *If a cb$_0$-space X satisfies the Wadge property for Q-partitions in a limit level $\lambda < \omega_1$, then so does every space $Y \leq_{co} X$.*

4.4. Characterizing Q-Wadge hierarchy in the Baire space. Here we show that the Q-IFH in the Baire space coincides with the Wadge hierarchy of Q-partitions. The structure of Wadge degrees of Borel measurable Q-partitions of $\mathcal{N}$ was characterized in [11] (Proposition 3.11 in §3.1). In particular, a set-theoretic characterization of the nonself-dual levels of the Q-Wadge hierarchy (with levels $\mathcal{W}(\mathcal{N}, T)$ from Introduction) was provided (Lemma 3.16 and its extensions), by defining classes Σ_T of Q-partitions using set-theoretic operations and showing that $\mathcal{W}(\mathcal{N}, T) = \widehat{\Sigma}_T$ for each $T \in \mathcal{T}_{\omega_1}(Q)$ where $\widehat{\Sigma}_T = \{A \in Q^{\mathcal{N}} \mid \exists B \in \Sigma_T(A \leq_W B)\}$.

The definition of Σ_T in [11] uses special features of the Baire space and looks a bit different from our general definition of levels of the Q-IFH. The main result of this subsection shows that these classes for the Baire space coincide. For the reader's convenience, we cite necessary notions and results from [11] (see also [12]).

Any nonempty closed set C in $\mathcal{N}$ and any Q-partition $A : C \to Q$ induce a Q-partition $\hat{A} : \mathcal{N} \to Q$ obtained by composing A with the canonical retraction from $\mathcal{N}$ onto C (abusing notation, A and $\hat{A}$ are often identified). Similarly, any $A : U \to Q$, where U is a nonempty open set in $\mathcal{N}$, may be identified with some $\hat{A} : \mathcal{N} \to Q$ (see Observations 3.5 and 3.6 in [11]). We recall (in a slightly different from [11] notation of §3.1) the definition of classes Σ_T (in fact, we define Σ_u for $u \in \mathbb{T}_\sigma$, where $T = T(u)$, see Lemma 3.3, cf. Definition 3.7 and its extensions in [11]).

DEFINITION 4.9.

1. $\Sigma_q = \{\lambda x.q\}$.
2. $\Sigma_{s_\alpha(u)}$ consists of $A \circ g$ where $A \in \Sigma_u$ and g is a $\mathbf{\Sigma}^0_{1+\omega^\alpha}$-measurable function on $\mathcal{N}$.
3. $\Sigma_{F_q(u_0,\dots)}$ consists of $A \in Q^{\mathcal{N}}$ such that for some pairwise disjoint nonempty open sets $U_0, U_1, \dots$ not exhausting $\mathcal{N}$ we have: $A|_V = \lambda v.q$ (where $V = \mathcal{N} \setminus \bigcup_i U_i$) and $A|_{U_i} \in \Sigma_{u_i}$ for all $i \geq 0$.
4. $\Sigma_{F_\alpha(u_0,\dots)}$ consists of $A \in Q^{\mathcal{N}}$ such that for some pairwise disjoint nonempty open sets $U_1, U_2, \dots$ not exhausting $\mathcal{N}$ we have: $A|_V \in \Sigma_{s_\alpha(u_0)}$ (where $V = \mathcal{N} \setminus \bigcup_{i\geq 1} U_i$) and $A|_{U_i} \in \Sigma_{u_i}$ for all $i \geq 1$.

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In [11] the following basic deep fact was established: For any countable ordinal α, there is a $\mathbf{\Sigma}^0_{1+\alpha}$-measurable conciliatory (an important technical notion from [11]) function $\mathcal{U}_\alpha\colon \mathcal{N} \to \mathcal{N}$ which is *universal*; that is, for every $\mathbf{\Sigma}^0_{1+\alpha}$-measurable function $f\colon \mathcal{N} \to \mathcal{N}$, there is a continuous function $g\colon \mathcal{N} \to \mathcal{N}$ such that f is equivalent to $\mathcal{U}_\alpha \circ g$. It was also shown that every σ-join-irreducible Borel function $A\colon \mathcal{N} \to Q$ is Wadge equivalent to a conciliatory function. In fact, for any $u \in \mathbb{T}_\sigma$ there is a special Σ_u-complete conciliatory function $\mu(u)\colon \mathcal{N} \to Q$ defined as follows: $\mu(q) = \lambda x.q$; $\mu(s_\alpha(u)) = \mu(u) \circ \mathcal{U}_{\omega^\alpha}$; $\mu(F_q(u_0,\ldots)) = \mu(q) \to (\mu(u_0) \sqcup \cdots)$; $\mu(F_\alpha(u_0,\ldots)) = \mu(s_\alpha(u_0)) \to (\mu(u_1) \sqcup \cdots)$.

THEOREM 4.10. *For any countable bqo Q we have $\Sigma_u = \mathcal{L}(\mathcal{N}, u)$ and $\widehat{\Sigma}_u = \widehat{\mathcal{L}}(\mathcal{N}, u)$ for every $u \in \mathbb{T}_\sigma$. Thus, in the Baire space the Q-IFH coincides with the Q-Wadge hierarchy.*

PROOF. It suffices to prove the first equality. Clearly, $\Sigma_q = \mathcal{L}(\mathcal{N}, q)$ for $q \in Q$. To prove $\Sigma_{s_\alpha(u)} = \mathcal{L}(\mathcal{N}, s_\alpha(u))$, note that $\Sigma_u = \mathcal{L}(\mathcal{N}, u)$ by induction and $\mathcal{L}(\mathcal{N}, s_\alpha(u)) = \mathcal{L}^{\omega^\alpha}(\mathcal{N}, u)$ by Lemma 3.34(1). Let $A \circ g \in \Sigma_{s_\alpha(u)}$ where $A \in \Sigma_u = \mathcal{L}(\mathcal{N}, u)$ and g is $\mathcal{L}^{\omega^\alpha}(\mathcal{N})$-measurable. By Lemmas 3.14 and 3.26, $A \circ g \in \mathcal{L}^{\omega^\alpha}(\mathcal{N}, u)$, as desired. Conversely, let $A \in \mathcal{L}^{\omega^\alpha}(\mathcal{N}, u)$. By the remarks before the theorem, $\mu(s_\alpha(u)) = \mu(u) \circ \mathcal{U}_{\omega^\alpha}$ is Wadge complete in $\mathcal{L}^{\omega^\alpha}(\mathcal{N}, u)$, hence $A = (\mu(u) \circ \mathcal{U}_{\omega^\alpha}) \circ f$ for some continuous function f on $\mathcal{N}$. Then $A = \mu(u) \circ (\mathcal{U}_{\omega^\alpha} \circ f)$, $\mu(u) \in \mathcal{L}(\mathcal{N}, u)$, and $\mathcal{U}_{\omega^\alpha} \circ f$ is $\mathcal{L}^{\omega^\alpha}(\mathcal{N})$-measurable. Thus, $A \in \Sigma_{s_\alpha(u)}$.

In proving the equality $\Sigma_{F_q(u_0,\ldots)} = \mathcal{L}(\mathcal{N}, F_q(u_0,\ldots))$, by induction we can assume that $\Sigma_{u_i} = \mathcal{L}(\mathcal{N}, u_i)$ for each $i \geq 0$. Let $A \in \Sigma_{F_q(u_0,\ldots)}$, then for some pairwise disjoint nonempty open sets $U_0, U_1, \ldots$ not exhausting $\mathcal{N}$ we have: $A|_V = \lambda v.q$ (where $V = \mathcal{N} \setminus \bigcup_i U_i$) and $A|_{U_i} \in \Sigma_{u_i}$ for all $i \geq 0$. By induction, $A|_{U_i} \in \mathcal{L}(\mathcal{N}, u_i)$ for all $i \geq 0$. By Lemma 3.34(4), $A \in \mathcal{L}(\mathcal{N}, F_q(u_0,\ldots))$. The converse inclusion follows from Lemma 3.34(3) and Definition 4.9(3). The case of F_α-term is considered similarly. ⊣

4.5. Infinitary fine hierarchies in quasi-Polish spaces. Here we summarise some properties of the Q-IFH in quasi-Polish spaces. For any quasi-Polish space X we fix a continuous open surjection ξ from $\mathcal{N}$ onto X (Proposition 2.2). First we give the characterization of the Wadge hierarchy of Q-partitions announced in Introduction (for $Q = \bar{2}$ this of course yields a characterization of the Wadge hierarchy of sets).

THEOREM 4.11. *Let X be a quasi-Polish space, Q a countable bqo, and $T \in \mathcal{T}_{\omega_1}(Q)$. Then $\mathcal{W}(X, T) = \widehat{\mathcal{L}}(X, T)$.*

PROOF. By Theorem 4.10 and Proposition 3.11, $\mathcal{W}(\mathcal{N}, T) = \widehat{\Sigma}_T = \widehat{\mathcal{L}}(\mathcal{N}, T)$. By Theorem 4.6, for any $A : X \to Q$ we have: $A \in \mathcal{W}(X, T)$ iff $A \circ \xi \in \widehat{\mathcal{L}}(\mathcal{N}, T)$ iff $A \in \widehat{\mathcal{L}}(X, T)$. ⊣

Next we show that the HK-type theorems hold in any quasi-Polish space, which extends some known results. From Proposition 2.2 we know that X is a quasi-Polish space iff $X \leq_{co} \mathcal{N}$. This together with Theorems 4.7 and 4.8 implies the following.

THEOREM 4.12. *Every quasi-Polish space satisfies the HK-theorem for Q-partitions in any successor level $1 + \alpha + 1 < \omega_1$ of the Q-IFH, and also the Wadge property for Q-partitions in any limit level $\lambda < \omega_1$ of the Q-IFH.*

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Let us summarize which properties of the Wadge hierarchy in the Baire space (see end of §2.4) hold in arbitrary quasi-Polish spaces. Property (1) holds for the hierarchies of sets and of Q-partitions for bqo Q (if Q has antichain of size 3, the property holds in the weakened bqo-form). The noncollapse property (2) does not automatically hold and requires additional investigation in any concrete space. Property (3) fails in most of natural spaces. Property (4) holds in arbitrary quasi-Polish space (note that this property is in fact an HK-type theorem); it would be interesting to investigate it for cb_0-spaces which are not quasi-Polish. By Proposition 4.3, property (5) holds for many (but not all) levels of the Q-IFH in cb_0-spaces. Property (6) does not automatically hold and requires additional investigation in any concrete space.

We conclude with an additional open question. In this paper we hopefully found a convincing set-theoretic definition of Q-Wadge hierarchy in quasi-Polish spaces, restricting our attention to Borel Q-partitions. For this the axioms of ZFC suffice. A major open question is to extend the results of this paper to a reasonable class beyond the Borel Q-partitions (perhaps even to all Q-partitions). The Wadge hierarchy for arbitrary subsets of the Baire space is well known [36] and requires suitable set-theoretic axioms alternative to ZFC. The definitions of this paper extend straightforwardly (by taking arbitrarily large ordinal γ in the signature $\sigma(Q, \gamma)$ in §3.1) but beyond the Borel Q-partitions proofs could turn out different from those used in this paper. It is currently not clear which set-theoretic axioms should be used.

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A.P. ERSHOV INSTITUTE OF INFORMATICS SYSTEMS SB RAS
AC. LAVRENTYEV AVE. 6, NOVOSIBIRSK, 630090, RUSSIA
E-mail: vseliv@iis.nsk.su