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MULTIVARIATE STOCHASTIC COMPARISONS OF SEQUENTIAL ORDER STATISTICS

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In this article we investigate conditions on the underlying distribution functions on which the sequential order statistics are based, to obtain stochastic comparisons of sequential order statistics in the multivariate likelihood ratio, the multivariate hazard rate, and the usual multivariate stochastic orders. Some applications of the main results are also given.

1. INTRODUCTION

Sequential order statistics have been introduced by Kamps [13,14] as an extension of (ordinary) order statistics in order to model sequential k-out-of-n systems, where the failures of components possibly affect remaining ones. A k-out-of-n system is a system with n components that functions if and only if at least k of the components function. The lifetime of a k-out-of-n system is the same as that of the $(n-k+1)$st ordinary order statistic of a set of n independent random variables $X_1, X_2, \ldots, X_n$, where X_i denotes the lifetime of the ith component. In the k-out-of-n system, the failure of any component does not affect the remaining ones. Thus, as a more flexible model, sequential k-out-of-n systems are more applicable to practical situations. A formal definition of sequential order statistics is given in Section 2.

The concept of generalized order statistics was also introduced by Kamps [13,14] as a unified approach to a variety of models of ordered random variables. Choosing the parameters appropriately, several other models of ordered random variables are seen to be particular cases. One may refer to Kamps [14] for ordinary order statis-

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tics, record values, k-record values, and Pfeifers records, to Balakrishnan, Cramer, and Kamps [2] for progressive type II censored order statistics, and to Belzunce, Mercader, and Ruiz [6] and references therein for order statistics under multivariate imperfect repair.

In the past 10 years, a great number of articles have dealt with stochastic comparisons of order statistics and their spacings. It is natural to obtain stochastic comparisons of generalized order statistics and their spacings by analogy with ordinary order statistics. Some recent articles on the subject are by Franco, Ruiz, and Ruiz [9], Korwar [17], Belzunce, Mercader, and Ruiz [5,6], Hu and Zhuang [10,11], Khaledi and Kochar [16], Khaledi [15], and Hu and Zhuang [12].

Sequential order statistics contain generalized order statistics as their special model. It is interesting to study stochastic properties of sequential order statistics. The purpose of this article is to present some results on multivariate stochastic comparisons of sequential order statistics. In Section 2 we recall the definitions of sequential order statistics and of some multivariate and univariate stochastic orders. The main results are given in Section 3. More precisely, let $(X^*_{1,n}, \ldots, X^*_{n,n})$ and $(X^*_{1,n+1}, \ldots, X^*_{n+1,n+1})$ be sequential order statistics based on $\{F_1, \ldots, F_n\}$ and $\{F_1, \ldots, F_{n+1}\}$, respectively. We investigate conditions on $\{F_1, \ldots, F_{n+1}\}$ under which one of the following relationships holds:

$$(X^*_{1,n+1}, \ldots, X^*_{n,n+1}) \le_* (X^*_{1,n}, \ldots, X^*_{n,n}),$$

$$(X^*_{1,n}, \ldots, X^*_{n-1,n}) \le_* (X^*_{2,n}, \ldots, X^*_{n,n}),$$

$$(X^*_{1,n}, \ldots, X^*_{n,n}) \le_* (X^*_{2,n+1}, \ldots, X^*_{n+1,n+1}),$$

where $\le_*$ is any one of the multivariate likelihood ratio order ($\le_{\rm lr}$), the multivariate hazard rate order ($\le_{\rm hr}$), and the usual multivariate stochastic order ($\le_{\rm st}$). In Section 4 some applications of the main results are given. Some discussions on multivariate comparisons of sequential order statistics from two samples are presented in Section 5.

Throughout this article, the terms "increasing" and "decreasing" mean "nondecreasing" and "nonincreasing," respectively; $a/0$ is understood to be ∞ whenever $a > 0$. For any distribution function F, $\bar{F} = 1 - F$ denotes its survival function. All random variables are restricted to be nonnegative. Also, we denote by $[X|A]$ any random variable whose distribution is the conditional distribution of X given event A.

2. PRELIMINARIES

2.1. Sequential Order Statistics

Sequential order statistics are defined by means of a triangular scheme of random variables where the rth line contains $n - r + 1$ random variables with distribution function F_i, $i = 1, \ldots, n$.

DEFINITION 2.1 (Kamps [13, p. 27]): *Let $F_1, \ldots, F_n$ be continuous distribution functions with $F_1^{-1}(1) \le \cdots \le F_n^{-1}(1)$ and let $\{Y_{r,n}^{(j)}, 1 \le j \le n-r+1\}$ be a sequence of independent and identically distributed random variables each distributed according to F_r, where $r = 1, \ldots, n$.*

Let $X_{1,n}^{(j)} = Y_{1,n}^{(j)}$, $1 \le j \le n$, and denote

$$X_{1,n}^* = \min_{j=1}^{n} X_{1,n}^{(j)}.$$

For $r = 2, \ldots, n$, define $X_{r,n}^{(j)} = F_r^{-1}\{F_r(Y_{r,n}^{(j)})[1 - F_r(X_{r-1,n}^*)] + F_r(X_{r-1,n}^*)\}$ and denote

$$X_{r,n}^* = \min_{j=1}^{n-r+1} X_{r,n}^{(j)}.$$

Then $X_{1,n}^*, \ldots, X_{n,n}^*$ are called sequential order statistics based on $\{F_1, \ldots, F_n\}$.

If $F_1, \ldots, F_n$ are absolutely continuous with densities $f_1, \ldots, f_n$, respectively, then the joint density of the first r sequential order statistics $(X_{1,n}^*, \ldots, X_{r,n}^*)$ is given by

$$f_{X_{1,n}^*, \ldots, X_{r,n}^*}(x_1, \ldots, x_r) = \frac{n!}{(n-r)!}\left(\prod_{i=1}^{r-1}\left(\frac{\bar{F}_i(x_i)}{\bar{F}_{i+1}(x_i)}\right)^{n-i} f_i(x_i)\right)[\bar{F}_r(x_r)]^{n-r} f_r(x_r), \tag{2.1}$$

where $x_1 < x_2 < \cdots < x_r$, $1 \le r \le n$, and $\prod_{i=1}^{0}[\cdots] = 1$. In particular, the joint density of $(X_{1,n}^*, \ldots, X_{n,n}^*)$ is given by

$$f_{X_{1,n}^*, \ldots, X_{n,n}^*}(\mathbf{x}) = n!\left[\prod_{i=1}^{n-1}\left(\frac{\bar{F}_i(x_i)}{\bar{F}_{i+1}(x_i)}\right)^{n-i} f_i(x_i)\right] f_n(x_n), \qquad x_1 < x_2 < \cdots < x_n. \tag{2.2}$$

Moreover, sequential order statistics form a Markov chain with transition probabilities

$$\Pr(X_{r,n}^* > t \mid X_{r-1,n}^* = s) = \left[\frac{\bar{F}_r(t)}{\bar{F}_r(s)}\right]^{n-r+1} \quad \text{for } t \ge s \text{ and } r = 2, \ldots, n. \tag{2.3}$$

(See Kamps [14, p. 29] and Cramer and Kamps [7].)

Cramer and Kamps [8] obtained the following recursion formula for the marginal distribution functions $F_{X^*_{1,n}}, \ldots, F_{X^*_{n,n}}$ of the sequential order statistics:

$$F_{X^*_{1,n}}(t) = 1 - [\bar{F}_1(t)]^n,$$

$$F_{X^*_{r,n}}(t) = \begin{cases} F_{X^*_{r-1,n}}(t) - \displaystyle\int_{-\infty}^{t} \frac{\bar{F}_r^{n-r+1}(t)}{\bar{F}_r^{n-r+1}(z)}\, dF_{X^*_{r-1,n}}(z), & F_r(t) < 1, \\ 1, & F_r(t) = 1, \end{cases} \quad \textbf{(2.4)}$$

where $t \in \Re$ and $r = 2, \ldots, n$.

Nonhomogeneous pure birth (NHPB) processes are another useful models of ordered random variables, which arise naturally in many applications of probability (see Belzunce, Lillo, Ruiz, and Shaked [4] and references therein). A counting process $\{N(t), t \geq 0\}$ is a NHPB process with intensity functions $\{r_k(\cdot), k \geq 0\}$ if the following hold:

(a) $\{N(t), t \geq 0\}$ has the Markov property.
(b) $\Pr\{N(t + \Delta t) = k + 1 | N(t) = k\} = r_k(t)\Delta t + \circ(\Delta t)$ for $k \geq 1$.
(c) $\Pr\{N(t + \Delta t) > k + 1 | N(t) = k\} = \circ(\Delta t)$ for $k \geq 1$;

the r_k''s are nonnegative functions that satisfy

$$\int_t^{\infty} r_k(x)\, dx = \infty \quad \text{for all } t \geq 0. \quad \textbf{(2.5)}$$

Condition (2.5) ensures that, with probability 1, the process has a jump after any time point t. In a distributional theoretical sense, there is one-to-one correspondence between sequential order statistics and the first n epoch times of a NHPB process, which is stated in the following proposition.

Proposition 2.1: *Let $\lambda_1(\cdot), \ldots, \lambda_n(\cdot)$ be the hazard rate functions of distribution functions $F_1, \ldots, F_n$, respectively, and let $(X^*_{1,n}, \ldots, X^*_{n,n})$ be the sequential order statistics based on distributions $\{F_1, \ldots, F_n\}$. Define*

$$r_{i-1}(t) = (n - i + 1)\lambda_i(t) \quad \textit{for } i = 1, \ldots, n$$

($r_k(\cdot)$ can be chosen arbitrarily for $k > n$ such that (2.5) is satisfied) and denote by $\{T_k, k \geq 1\}$ the epoch times of a NHPB process with intensity functions $\{r_k(\cdot), k \geq 0\}$. Then

$$(X^*_{1,n}, \ldots, X^*_{n,n}) \stackrel{\text{st}}{=} (T_{1,n}, \ldots, T_{n,n}),$$

where $\stackrel{\text{st}}{=}$ means equality in distribution.

Generalized order statistics are contained in the model of sequential order statistics. Let $(X(1, n, \widetilde{m}, k), \ldots, X(n, n, \widetilde{m}, k))$ be generalized order statistics based on the distribution function F with parameters $k > 0$, $\widetilde{m} = (m_1, \ldots, m_{n-1})$ and $\gamma_{i,n} > 0$

for $i = 1, \ldots, n$. Then $(X(1, n, \widetilde{m}, k), \ldots, X(n, n, \widetilde{m}, k))$ can be regarded as the sequential order statistics $(X^*_{1,n}, \ldots, X^*_{n,n})$ based on distributions $\{F_1, \ldots, F_n\}$, where

$$F_i(x) = 1 - [\bar{F}(x)]^{\alpha_{i,n}}, \qquad \alpha_{i,n} = \frac{\gamma_{i,n}}{n - i + 1} \quad \text{for } i = 1, \ldots, n; \tag{2.6}$$

that is, $F_1, \ldots, F_n$ satisfy the proportional hazard model. This relationship will be used later. For a formal definition of generalized order statistics and their properties, one refer to Kamps [13,14].

2.2. Some Stochastic Orders

Some multivariate stochastic orders that will be used in this article are recalled in the sequel (see Shaked and Shanthikumar [18]).

First, we recall the definition of the usual multivariate stochastic order. Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors. We say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the *usual multivariate stochastic order*, denote by $\mathbf{X} \le_{\mathrm{st}} \mathbf{Y}$, if

$$\mathrm{E}[\phi(\mathbf{X})] \le \mathrm{E}[\phi(\mathbf{Y})]$$

for all increasing function ϕ such that the expectations exist. If X and Y are univariate random variables with distribution functions F and G, respectively, then $X \le_{\mathrm{st}} Y$ if and only if $\bar{F}(t) \le \bar{G}(t)$ for all t.

Next, for the definition of the multivariate hazard rate order, let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional nonnegative random vectors with multivariate conditional hazard rate functions $\eta_{\cdot|\cdot}(\cdot|\cdot)$ and $\lambda_{\cdot|\cdot}(\cdot|\cdot)$ as defined in Shaked and Shanthikumar [18, Sect. 4.C.1]. For any vector $\mathbf{x} \in \Re^n$ and any subset $I = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}$, let $\mathbf{x}_I = (x_{i_1}, \ldots, x_{i_k})$ and $\bar{I}$ be the complement of I in $\{1, \ldots, n\}$. For a random vector $\mathbf{X}$, the interpretation of $\mathbf{X}_I$ is similar. Finally, $\mathbf{e}$ and $\mathbf{0}$ denote the vector of 1's and 0's; the dimensions of $\mathbf{e}$ and $\mathbf{0}$ can be determined from the context. Then $\mathbf{X}$ is said to be less than $\mathbf{Y}$ in the *multivariate hazard rate order*, denoted by $\mathbf{X} \le_{\mathrm{hr}} \mathbf{Y}$, if

$$\eta_{i|I\cup J}(u|\mathbf{s}_{I\cup J}) \ge \lambda_{i|I}(u|\mathbf{t}_I) \quad \text{for all } i \in \overline{I \cup J} \tag{2.7}$$

whenever $I \cap J = \emptyset$, $\mathbf{0} \le \mathbf{s}_I \le \mathbf{t}_I \le u\mathbf{e}$, and $\mathbf{0} \le \mathbf{s}_J \le u\mathbf{e}$. In the univariate case, $X \le_{\mathrm{hr}} Y$ if and only if $\bar{G}(x)/\bar{F}(x)$ is increasing in t.

Finally, we recall the definition of the multivariate likelihood ratio order. Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors with density functions f and g, respectively. We say that $\mathbf{X}$ is less than $\mathbf{Y}$ in the *multivariate likelihood ratio order*, denoted by $\mathbf{X} \le_{\mathrm{lr}} \mathbf{Y}$, if

$$\begin{aligned} &f(x_1, \ldots, x_n)\, g(y_1, \ldots, y_n) \\ &\quad \le f(x_1 \wedge y_1, \ldots, x_n \wedge y_n)\, g(x_1 \vee y_1, \ldots, x_n \vee y_n) \end{aligned}$$

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for all $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$ in $\Re^n$, where

$$x_i \wedge y_i = \min\{x_i, y_i\}, \qquad x_i \vee y_i = \max\{x_i, y_i\}, \qquad i = 1, \ldots, n.$$

In the univariate case, $X \le_{\text{lr}} Y$ if and only if $g(x)/f(x)$ is increasing in x. In the slightly more general case, when $\mathbf{X}$ and $\mathbf{Y}$ are nonnegative, some of the X_i's might be identically zero and the joint distribution of the rest is absolutely continuous or discrete. Suppose that $X_1, \ldots, X_m$ are those that are identically zero for some $0 < m < n$. Also, let f denote the joint density of $(X_{m+1}, \ldots, X_n)$. In that case, we denote $\mathbf{X} \le_{\text{lr}} \mathbf{Y}$, if

$$\begin{aligned} &f(x_{m+1}, \ldots, x_n)\, g(y_1, \ldots, y_n) \\ &\quad \le f(x_{m+1} \wedge y_{m+1}, \ldots, x_n \wedge y_n) \\ &\qquad \times g(y_1, \ldots, y_m, x_{m+1} \vee y_{m+1}, \ldots, x_n \vee y_n) \end{aligned} \tag{2.8}$$

for all $(x_{m+1}, \ldots, x_n)$ and $(y_1, \ldots, y_n)$.

It is well known that the multivariate likelihood ratio order implies the multivariate hazard rate order, which, in turn, implies the usual multivariate stochastic order; that is,

$$\mathbf{X} \le_{\text{lr}} \mathbf{Y} \Rightarrow \mathbf{X} \le_{\text{hr}} \mathbf{Y} \Rightarrow \mathbf{X} \le_{\text{st}} \mathbf{Y}. \tag{2.9}$$

The multivariate orders $\le_{\text{lr}}$ and $\le_{\text{st}}$ are closed under marginalization. However, the multivariate order $\le_{\text{hr}}$ is not closed under marginalization. Such a closure property is very useful for us to establish the univariate comparison result. Sometimes it is difficult for us to obtain an explicit formula of the density functions of X_i and Y_i, but it is easy to get the joint density functions of $\mathbf{X}$ and $\mathbf{Y}$. To obtain $X_i \le_{\text{lr}} Y_i$ for each i, we can establish $\mathbf{X} \le_{\text{lr}} \mathbf{Y}$ first. This idea is illustrated in Subsection 3.1.

If the distribution functions of $\mathbf{X}$ and $\mathbf{Y}$ are F and G, respectively, then $\mathbf{X} \le_* \mathbf{Y}$ is sometimes denoted by $F \le_* G$, where $\le_*$ is any one of the above orders.

3. MAIN RESULTS

In this section, we investigate conditions on the underlying distribution functions on which the sequential order statistics are based, to obtain stochastic comparisons of sequential order statistics in the multivariate likelihood ratio, the multivariate hazard rate, and the usual multivariate stochastic orders.

3.1. Multivariate Likelihood Ratio Ordering

THEOREM 3.1: *Let $F_1, \ldots, F_{n+1}$ be absolutely continuous distribution functions and let $(X^*_{1,n}, \ldots, X^*_{n,n})$ and $(X^*_{1,n+1}, \ldots, X^*_{n+1,n+1})$ be sequential order statistics*

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based on $\{F_1, \ldots, F_n\}$ *and* $\{F_1, \ldots, F_{n+1}\}$, *respectively. If* $F_1 \leq_{\mathrm{hr}} F_2 \leq_{\mathrm{hr}} \cdots \leq_{\mathrm{hr}} F_n$, *then*

$$(X^*_{1,n+1}, \ldots, X^*_{n,n+1}) \leq_{\mathrm{lr}} (X^*_{1,n}, \ldots, X^*_{n,n}). \tag{3.1}$$

Proof: Let $f_1, \ldots, f_{n+1}$ denote the density functions of $F_1, \ldots, F_{n+1}$, respectively. By the definition of the order $\leq_{\mathrm{lr}}$, (2.1), and (2.2), it suffices to verify that, for $x_1 \leq \cdots \leq x_n$ and $y_1 \leq \cdots \leq y_n$,

$$\begin{aligned}
&\left[\prod_{i=1}^{n-1}\left(\frac{\bar{F}_i(x_i)}{\bar{F}_{i+1}(x_i)}\right)^{n+1-i} f_i(x_i)\right]\bar{F}_n(x_n)f_n(x_n)\left[\prod_{i=1}^{n-1}\left(\frac{\bar{F}_i(y_i)}{\bar{F}_{i+1}(y_i)}\right)^{n-i} f_i(y_i)\right]f_n(y_n)\\
&\quad\leq \left[\prod_{i=1}^{n-1}\left(\frac{\bar{F}_i(x_i\wedge y_i)}{\bar{F}_{i+1}(x_i\wedge y_i)}\right)^{n+1-i} f_i(x_i\wedge y_i)\right]\bar{F}_n(x_n\wedge y_n)f_n(x_n\wedge y_n)\\
&\quad\quad\times\left[\prod_{i=1}^{n-1}\left(\frac{\bar{F}_i(x_i\vee y_i)}{\bar{F}_{i+1}(x_i\vee y_i)}\right)^{n-i} f_i(x_i\vee y_i)\right]f_n(x_n\vee y_n).
\end{aligned} \tag{3.2}$$

Let $E_1 = \{1 \leq i \leq n-1 : x_i \geq y_i\}$. Then, (3.2) reduces to

$$\bar{F}_n(x_n \wedge y_n)\prod_{i\in E_1}\frac{\bar{F}_i(y_i)}{\bar{F}_{i+1}(y_i)} \geq \bar{F}_n(x_n)\prod_{i\in E_1}\frac{\bar{F}_i(x_i)}{\bar{F}_{i+1}(x_i)}.$$

which follows from $F_1 \leq_{\mathrm{hr}} \cdots \leq_{\mathrm{hr}} F_n$. This completes the proof. ■

Theorem 3.2: *Let* $F_1, \ldots, F_n$ *be absolutely continuous distribution functions of nonnegative random variables with hazard rates* $\gamma_1(\cdot), \ldots, \gamma_n(\cdot)$, *respectively. Let* $(X^*_{1,n}, \ldots, X^*_{n,n})$ *be sequential order statistics based on* $\{F_1, \ldots, F_n\}$. *If* $F_1 \leq_{\mathrm{lr}} F_2 \leq_{\mathrm{lr}} \cdots \leq_{\mathrm{lr}} F_n$ *and if*

$$\lambda_{i+2}(t) + \lambda_i(t) \geq 2\lambda_{i+1}(t) \quad \textit{for all } t \geq 0 \textit{ and } i = 1, \ldots, n-2, \tag{3.3}$$

then

$$(X^*_{1,n}, \ldots, X^*_{n-1,n}) \leq_{\mathrm{lr}} (X^*_{2,n}, \ldots, X^*_{n,n}). \tag{3.4}$$

Proof: Let $f_1, \ldots, f_n$ denote the density functions of $F_1, \ldots, F_n$, respectively. Since the multivariate likelihood ratio order is closed under marginalization, it is sufficient to verify that

$$(0, X^*_{1,n}, \ldots, X^*_{n-1,n}) \leq_{\mathrm{lr}} (X^*_{1,n}, X^*_{2,n}, \ldots, X^*_{n,n}).$$

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By (2.1), (2.2), and (2.8), we have to prove that, for all $x_2 \le \cdots \le x_n$ and $y_1 \le \cdots \le y_n$,

$$\left[\prod_{i=1}^{n-2}\left(\frac{\bar{F}_i(x_{i+1})}{\bar{F}_{i+1}(x_{i+1})}\right)^{n-i} f_i(x_{i+1})\right]\bar{F}_{n-1}(x_n)f_{n-1}(x_n)\left[\prod_{i=2}^{n-1}\left(\frac{\bar{F}_i(y_i)}{\bar{F}_{i+1}(y_i)}\right)^{n-i} f_i(y_i)\right]f_n(y_n)$$
$$\le \left[\prod_{i=1}^{n-2}\left(\frac{\bar{F}_i(x_{i+1}\wedge y_{i+1})}{\bar{F}_{i+1}(x_{i+1}\wedge y_{i+1})}\right)^{n-i} f_i(x_{i+1}\wedge y_{i+1})\right]\bar{F}_{n-1}(x_n\wedge y_n)f_{n-1}(x_n\wedge y_n)$$
$$\times\left[\prod_{i=2}^{n-1}\left(\frac{\bar{F}_i(x_i\vee y_i)}{\bar{F}_{i+1}(x_i\vee y_i)}\right)^{n-i} f_i(x_i\vee y_i)\right]f_n(x_n\vee y_n). \quad \mathbf{(3.5)}$$

Note that condition (3.3) can be written as

$$\frac{[\bar{F}_{i+1}(t)]^2}{\bar{F}_i(t)\bar{F}_{i+2}(t)} \quad \text{is increasing in } t\in\Re_+ \text{ for } r=1,\ldots,n-2. \quad \mathbf{(3.6)}$$

Let $E_2=\{i: x_{i+1}\ge y_{i+1}, i=1,\ldots,n-2\}$. Then, (3.5) reduces to

$$\left[\prod_{i\in E_2}\left(\frac{\bar{F}_i(x_{i+1})}{\bar{F}_{i+1}(x_{i+1})}\right)^{n-i} f_i(x_{i+1})\right]\bar{F}_{n-1}(x_n)f_{n-1}(x_n)$$
$$\times\left[\prod_{i\in E_2}\left(\frac{\bar{F}_{i+1}(y_{i+1})}{\bar{F}_{i+2}(y_{i+1})}\right)^{n-i-1} f_{i+1}(y_{i+1})\right]f_n(y_n)$$
$$\le\left[\prod_{i\in E_2}\left(\frac{\bar{F}_i(y_{i+1})}{\bar{F}_{i+1}(y_{i+1})}\right)^{n-i} f_i(y_{i+1})\right]\bar{F}_{n-1}(x_n\wedge y_n)f_{n-1}(x_n\wedge y_n)$$
$$\times\left[\prod_{i\in E_2}\left(\frac{\bar{F}_{i+1}(x_{i+1})}{\bar{F}_{i+2}(x_{i+1})}\right)^{n-i-1} f_{i+1}(x_{i+1})\right]f_n(x_n\vee y_n),$$

which follows from (3.6), (2.9), and $F_1 \le_{\mathrm{lr}} F_2 \le_{\mathrm{lr}} \cdots \le_{\mathrm{lr}} F_n$. This completes the proof. ■

THEOREM 3.3: *Let $F_1,\ldots,F_{n+1}$ be absolutely continuous distribution functions of nonnegative random variables with hazard rates $\gamma_1(\cdot),\ldots,\gamma_{n+1}(\cdot)$, respectively, and let $(X^*_{1,n},\ldots,X^*_{n,n})$ and $(X^*_{1,n+1},\ldots,X^*_{n+1,n+1})$ be sequential order statistics based on $\{F_1,\ldots,F_n\}$ and $\{F_1,\ldots,F_{n+1}\}$, respectively. If $F_1\le_{\mathrm{lr}} F_2 \le_{\mathrm{lr}} \cdots \le_{\mathrm{lr}} F_{n+1}$ and if*

$$\lambda_{i+2}(t)+\lambda_i(t)\ge 2\lambda_{i+1}(t) \quad \textit{for all } t \textit{ and } i=1,\ldots,n-1, \quad \mathbf{(3.7)}$$

then

$$(X^*_{1,n},\ldots,X^*_{n,n}) \le_{\mathrm{lr}} (X^*_{2,n+1},\ldots,X^*_{n+1,n+1}). \quad \mathbf{(3.8)}$$

PROOF: Let $f_1, \ldots, f_{n+1}$ denote the density functions of $F_1, \ldots, F_{n+1}$, respectively. Since the multivariate likelihood ratio order is closed under marginalization, it is sufficient to verify that

$$(0, X^*_{1,n}, \ldots, X^*_{n,n}) \le_{\text{lr}} (X^*_{1,n+1}, X^*_{2,n+1}, \ldots, X^*_{n+1,n+1}).$$

Let $E_3 = \{i : x_{i+1} \ge y_{i+1}, i = 1, \ldots, n-1\}$. We have to prove that, for all $x_2 \le \cdots \le x_{n+1}$ and $y_1 \le \cdots \le y_{n+1}$,

$$\begin{aligned}&\left[\prod_{i\in E_3}\left(\frac{\bar F_i(x_{i+1})}{\bar F_{i+1}(x_{i+1})}\right)^{n-i} f_i(x_{i+1})\right] f_n(x_{n+1})\\
&\qquad\times\left[\prod_{i\in E_3}\left(\frac{\bar F_{i+1}(y_{i+1})}{\bar F_{i+2}(y_{i+1})}\right)^{n-i} f_{i+1}(y_{i+1})\right] f_{n+1}(y_{n+1})\\
&\le\left[\prod_{i\in E_3}\left(\frac{\bar F_i(y_{i+1})}{\bar F_{i+1}(y_{i+1})}\right)^{n-i} f_i(y_{i+1})\right] f_n(x_{n+1}\wedge y_{n+1})\\
&\qquad\times\left[\prod_{i\in E_3}\left(\frac{\bar F_{i+1}(x_{i+1})}{\bar F_{i+2}(x_{i+1})}\right)^{n-i} f_{i+1}(x_{i+1})\right] f_{n+1}(x_{n+1}\vee y_{n+1}),\end{aligned}$$

which follows from $F_1 \le_{\text{lr}} F_2 \le_{\text{lr}} \cdots \le_{\text{lr}} F_{n+1}$ and condition (3.7). This completes the proof. ■

Since the multivariate likelihood ratio order is closed under marginalization, we get the following result as a corollary of Theorems 3.1–3.3.

COROLLARY 3.1: *Let* $(X^*_{1,n}, \ldots, X^*_{n,n})$ *and* $(X^*_{1,n+1}, \ldots, X^*_{n+1,n+1})$ *be the same as in Theorem 3.3.*

(1) If $F_1 \le_{\text{hr}} F_2 \le_{\text{hr}} \cdots \le_{\text{hr}} F_n$, *then* $X^*_{i,n+1} \le_{\text{lr}} X^*_{i,n}$ *for* $i = 1, \ldots, n$.
(2) If $F_1 \le_{\text{lr}} F_2 \le_{\text{lr}} \cdots \le_{\text{lr}} F_n$ *and if (3.3) holds, then* $X^*_{i,n} \le_{\text{lr}} X^*_{i+1,n}$ *for* $i = 1, \ldots, n-1$.
(3) If $F_1 \le_{\text{lr}} F_2 \le_{\text{lr}} \cdots \le_{\text{lr}} F_{n+1}$ *and if (3.7) holds, then* $X^*_{i,n} \le_{\text{lr}} X^*_{i+1,n+1}$ *for* $i = 1, \ldots, n$.

Let $(X(1, n, \widetilde m_n, k), \ldots, X(n, n, \widetilde m_n, k))$ and $(X(1, n+1, \widetilde m_{n+1}, k), \ldots, X(n+1, n+1, \widetilde m_{n+1}, k))$ be generalized order statistics based on the distribution function F, where $\widetilde m_{n+1} = (\widetilde m_n, m_n)$. Because of the relationship between sequential and generalized order statistics (see (2.6)), $(X(1, n, \widetilde m_n, k), \ldots, X(n, n, \widetilde m_n, k))$ and $(X(1, n+1, \widetilde m_{n+1}, k), \ldots, X(n+1, n+1, \widetilde m_{n+1}, k))$ are the sequential order statistics based on $\{F_1, \ldots, F_n\}$ and $\{G_1, \ldots, G_{n+1}\}$, respectively, where

$$F_i(x) = 1 - [\bar F(x)]^{\alpha_{i,n}}, \qquad \alpha_{i,n} = \frac{\gamma_{i,n}}{n-i+1}, \qquad i = 1, \ldots, n, \tag{3.9}$$

and

$$G_i(x) = 1 - [\bar{F}(x)]^{\alpha_{i,n+1}}, \qquad \alpha_{i,n+1} = \frac{\gamma_{i,n+1}}{n-i+2}, \qquad i = 1, \ldots, n+1. \tag{3.10}$$

It is seen that F_i and G_i are not, in general, the same. Thus, parts b and c of Theorem 3.1 in Hu and Zhuang [10] cannot be deduced from parts b and c of Corollary 3.1. Furthermore, part a of Theorem 3.1 in Hu and Zhuang [10] cannot be deduced from part a of Corollary 3.1 either.

3.2. Multivariate Hazard Rate Ordering

We now proceed to stochastic comparisons of sequential order statistics in the multivariate hazard rate order.

THEOREM 3.4: *Let $F_1, \ldots, F_{n+1}$ be absolutely continuous distribution functions of nonnegative random variables and let $(X^*_{1,n}, \ldots, X^*_{n,n})$ and $(X^*_{1,n+1}, \ldots, X^*_{n+1,n+1})$ be sequential order statistics based on $\{F_1, \ldots, F_n\}$ and $\{F_1, \ldots, F_{n+1}\}$, respectively. Then*

$$(X^*_{1,n+1}, \ldots, X^*_{n,n+1}) \le_{\text{hr}} (X^*_{1,n}, \ldots, X^*_{n,n}). \tag{3.11}$$

PROOF: The proof is similar to that of Theorem 3.3 in Belzunce et al. [4]. Let $\lambda_1, \ldots, \lambda_{n+1}$ denote the hazard rate functions of $F_1, \ldots, F_{n+1}$, respectively. Denote by $\eta_{\cdot|\cdot}(\cdot|\cdot)$ and $\lambda_{\cdot|\cdot}(\cdot|\cdot)$ the multivariate condition hazard rate functions associated with $(X^*_{1,n+1}, \ldots, X^*_{n,n+1})$ and $(X^*_{1,n}, \ldots, X^*_{n,n})$, respectively. We have to verify (2.7).

Observe that $X^*_{1,n} \le \cdots \le X^*_{n,n}$ together with (2.3). Then the explicit expression of $\lambda_{k|I}(u|\mathbf{t}_I)$ is given by

$$\lambda_{k|I}(u|\mathbf{t}_I) = \begin{cases} (n-r)\lambda_{r+1}(u), & k = r+1 \\ 0, & k > r+1, \end{cases}$$

where I must be of the form $I = \{1, \ldots, r\}$ for some r. Similarly, in $\eta_{k|I\cup J}(u|\mathbf{s}_{I\cup J})$ of (2.7), we must have $I = \{1, \ldots, r\}$ and $J = \{r+1, \ldots, m\}$ for some $m \ge r$, or $J = \emptyset$ (i.e., $m = r$). Thus,

$$\eta_{k|I\cup J}(u|\mathbf{s}_{I\cup J}) = \begin{cases} (n+1-m)\lambda_{m+1}(u), & k = m+1 \\ 0, & k > m+1, \end{cases}$$

where $I = \{1, \ldots, r\}$ and $J = \{r+1, \ldots, m\}$ for $0 \le r \le n-1$.

If $m > r$, then

$$\eta_{k|I\cup J}(u|\mathbf{s}_{I\cup J}) = (n+1-m)\lambda_{m+1}(u) \ge 0 = \lambda_{k|I}(u|\mathbf{t}_I) \quad \text{for } k = m+1,$$

$$\eta_{k|I\cup J}(u|\mathbf{s}_{I\cup J}) = 0 = \lambda_{k|I}(u|\mathbf{t}_I) \quad \text{for } k > m+1;$$

thus, (2.7) holds. If $m = r$ (i.e., $J = \emptyset$), then

$$\eta_{k|I\cup J}(u|\mathbf{s}_{I\cup J}) = (n+1-m)\lambda_{m+1}(u) \geq (n-m)\lambda_{m+1}(u)$$

$$= \lambda_{k|I}(u|\mathbf{t}_I) \quad \text{for } k = m+1,$$

$$\eta_{k|I\cup J}(u|\mathbf{s}_{I\cup J}) = 0 = \lambda_{k|I}(u|\mathbf{t}_I) \quad \text{for } k > m+1;$$

thus, (2.7) also holds. This completes the proof. ■

THEOREM 3.5: *Let $F_1, \ldots, F_n$ be absolutely continuous distribution functions of nonnegative random variables and let $(X^*_{1,n}, \ldots, X^*_{n,n})$ be sequential order statistics based on $\{F_1, \ldots, F_n\}$. If $F_1 \leq_{\rm hr} F_2 \leq_{\rm hr} \cdots \leq_{\rm hr} F_n$, then*

$$(X^*_{1,n}, \ldots, X^*_{n-1,n}) \leq_{\rm hr} (X^*_{2,n}, \ldots, X^*_{n,n}). \qquad \textbf{(3.12)}$$

PROOF: Let $\lambda_1, \ldots, \lambda_n$ denote the hazard rate functions of $F_1, \ldots, F_n$, respectively, and denote by $\eta_{\cdot|\cdot}(\cdot|\cdot)$ and $\lambda_{\cdot|\cdot}(\cdot|\cdot)$ the multivariate condition hazard rate functions associated with $(X^*_{1,n}, \ldots, X^*_{n-1,n})$ and $(X^*_{2,n}, \ldots, X^*_{n,n})$, respectively. Suppose that $F_1 \leq_{\rm hr} \cdots \leq_{\rm hr} F_n$. It suffices to prove that

$$\eta_{k|I\cup J}(u|\mathbf{s}_{I\cup J}) \geq \lambda_{k|I}(u|\mathbf{t}_I) \quad \text{for } r \geq 1 \text{ and } k \geq m+1, \qquad \textbf{(3.13)}$$

$$\eta_{k|I\cup J}(u|\mathbf{s}_{I\cup J}) \geq \lambda_{k|I}(u|\mathbf{t}_I) \quad \text{for } r = 0 \text{ and } k \geq m+1, \qquad \textbf{(3.14)}$$

where $I = \{1, \ldots, r\}$ and $J = \{r+1, \ldots, m\}$, $0 \leq r \leq n-2$ and $\mathbf{s}_I \leq \mathbf{t}_I \leq u\mathbf{e}$, $\mathbf{s}_J \leq u\mathbf{e}$.

If $m > r \geq 1$, then

$$\eta_{k|I\cup J}(u|\mathbf{s}_{I\cup J}) = (n-m)\lambda_{m+1}(u) \geq 0 = \lambda_{k|I}(u|\mathbf{t}_I) \quad \text{for } k = m+1,$$

$$\eta_{k|I\cup J}(u|\mathbf{s}_{I\cup J}) = 0 = \lambda_{k|I}(u|\mathbf{t}_I) \quad \text{for } k > m+1;$$

thus, (3.13) holds. If $m = r \geq 1$ (i.e., $J = \emptyset$), then

$$\eta_{k|I\cup J}(u|\mathbf{s}_{I\cup J}) = (n-m)\lambda_{m+1}(u) \geq (n-m-1)\lambda_{m+2}(u)$$

$$= \lambda_{k|I}(u|\mathbf{t}_I) \quad \text{for } k = m+1,$$

$$\eta_{k|I\cup J}(u|\mathbf{s}_{I\cup J}) = 0 = \lambda_{k|I}(u|\mathbf{t}_I) \quad \text{for } k > m+1;$$

thus, (3.13) holds in this case too.

If $r = 0$, (3.14) can be rewritten as $\eta_{k|J}(u|\mathbf{s}_J) \geq \lambda_{k|\emptyset}(u|\mathbf{t}_\emptyset)$, which is equivalent to

$$\eta_{1|\emptyset}(u|\mathbf{s}_\emptyset) \geq \lambda_{1|\emptyset}(u|\mathbf{t}_\emptyset) \qquad \textbf{(3.15)}$$

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since $\lambda_{k|\emptyset}(u|\mathbf{t}_\emptyset) = 0$ for $k \geq 2$. Thus, it remains to prove that $X^*_{1,n} \leq_{\text{hr}} X^*_{2,n}$. In fact, it follows from (2.4) that

$$\bar{F}_{X^*_{2,n}}(t) = \bar{F}_{X^*_{1,n}}(t) + n\int_o^t \left(\frac{\bar{F}_2(t)}{\bar{F}_2(u)}\right)^{n-1} [\bar{F}_1(u)]^{n-1}\, dF_1(u).$$

Since $F_1 \leq_{\text{hr}} F_2$, we get that

$$\frac{\bar{F}_{X^*_{2,n}}(t)}{\bar{F}_{X^*_{1,n}}(t)} = 1 + \frac{n}{\bar{F}_1(t)} \int_o^t \left(\frac{\bar{F}_2(t)}{\bar{F}_1(t)}\right)^{n-1} \left(\frac{\bar{F}_1(u)}{\bar{F}_2(u)}\right)^{n-1} dF_1(u)$$

is increasing in t. Thus, $X^*_{1,n} \leq_{\text{hr}} X^*_{2,n}$. This completes the proof. ■

Remark 3.1: The assumption that $F_1 \leq_{\text{hr}} F_2 \leq_{\text{hr}} \cdots \leq_{\text{hr}} F_n$ in Theorem 3.5 can be weakened to be

$$(n-i)\lambda_{i+1}(t) \leq (n-i+1)\lambda_i(t) \quad \text{for all } t \text{ and } i = 1, \ldots, n-1, \tag{3.16}$$

as can be seen from its proof.

THEOREM 3.6: *Let* $(X^*_{1,n}, \ldots, X^*_{n,n})$ *and* $(X^*_{1,n+1}, \ldots, X^*_{n+1,n+1})$ *be the same as in Theorem 3.4. If* $F_1 \leq_{\text{hr}} F_2 \leq_{\text{hr}} \cdots \leq_{\text{hr}} F_{n+1}$, *then*

$$(X^*_{1,n}, \ldots, X^*_{n,n}) \leq_{\text{hr}} (X^*_{2,n+1}, \ldots, X^*_{n+1,n+1}). \tag{3.17}$$

PROOF: Let $\lambda_1, \ldots, \lambda_{n+1}$ be the hazard rate functions of $F_1, \ldots, F_{n+1}$, respectively, and denote by $\eta_{\cdot|\cdot}(\cdot|\cdot)$ and $\lambda_{\cdot|\cdot}(\cdot|\cdot)$ the multivariate condition hazard rate functions associated with $(X^*_{1,n}, \ldots, X^*_{n,n})$ and $(X^*_{2,n+1}, \ldots, X^*_{n+1,n+1})$, respectively. Suppose that $F_1 \leq_{\text{hr}} \cdots \leq_{\text{hr}} F_{n+1}$. We have to prove that (3.13) and (3.14) hold whenever $I = \{1, \ldots, r\}$ and $J = \{r+1, \ldots, m\}$, $0 \leq r \leq n-1$, $\mathbf{s}_I \leq \mathbf{t}_I \leq u\mathbf{e}$, $\mathbf{s}_J \leq u\mathbf{e}$.

If $m > r \geq 1$, then

$$\eta_{k|I\cup J}(u|\mathbf{s}_{I\cup J}) = (n-m)\lambda_{m+1}(u) \geq 0 = \lambda_{k|I}(u|\mathbf{t}_I) \quad \text{for } k = m+1,$$

$$\eta_{k|I\cup J}(u|\mathbf{s}_{I\cup J}) = 0 = \lambda_{k|I}(u|\mathbf{t}_I) \quad \text{for } k > m+1.$$

If $m = r \geq 1$ (i.e., $J = \emptyset$), then

$$\eta_{k|I\cup J}(u|\mathbf{s}_{I\cup J}) = (n-m)\lambda_{m+1}(u) \geq (n-m)\lambda_{m+2}(u)$$

$$= \lambda_{k|I}(u|\mathbf{t}_I) \quad \text{for } k = m+1,$$

$$\eta_{k|I\cup J}(u|\mathbf{s}_{I\cup J}) = 0 = \lambda_{k|I}(u|\mathbf{t}_I) \quad \text{for } k > m+1.$$

Thus, (3.13) holds.

For $r = 0$, to prove (3.14), we need to prove that $X^*_{1,n} \leq_{\rm hr} X^*_{2,n+1}$. In fact, it follows from (2.4) that $\bar{F}_{X^*_{1,n}}(t) = [\bar{F}_1(t)]^n$ and

$$\bar{F}_{X^*_{2,n+1}}(t) = [\bar{F}_1(t)]^{n+1} + (n+1)\int_o^t \left(\frac{\bar{F}_2(t)}{\bar{F}_2(u)}\right)^n [\bar{F}_1(u)]^n \, dF_1(u). \tag{3.18}$$

Then $F_1 \leq_{\rm hr} F_2$ implies that

$$\frac{\bar{F}_{X^*_{2,n+1}}(t)}{\bar{F}_{X^*_{1,n}}(t)} = 1 + \int_o^t \left[\left(\frac{\bar{F}_2(t)}{\bar{F}_1(t)} \cdot \frac{\bar{F}_1(u)}{\bar{F}_2(u)}\right)^n (n+1) - 1\right] dF_1(u)$$

is increasing in t. Thus, $X^*_{1,n} \leq_{\rm hr} X^*_{2,n+1}$. This completes the proof. ■

The multivariate hazard rate order is not closed under marginalization. Thus, we could not establish univariate hazard rate ordering between sequential order statistics from Theorems 3.4–3.6 under the conditions stated there.

3.3. Multivariate Usual Stochastic Ordering

Note that the usual multivariate stochastic order is closed under weak convergence. By (2.9), we get the following result as a corollary of Theorems 3.4 without the assumption that $F_1, \ldots, F_{n+1}$ are absolutely continuous.

Corollary 3.2: *Let $(X^*_{1,n}, \ldots, X^*_{n,n})$ and $(X^*_{1,n+1}, \ldots, X^*_{n+1,n+1})$ be sequential order statistics based on $\{F_1, \ldots, F_n\}$ and $\{F_1, \ldots, F_{n+1}\}$, respectively. Then*

$$(X^*_{1,n+1}, \ldots, X^*_{n,n+1}) \leq_{\rm st} (X^*_{1,n}, \ldots, X^*_{n,n}). \tag{3.19}$$

To prove the next two theorems, we need one useful lemma, which gives a sufficient condition of the usual multivariate stochastic ordering between two random vectors. First, we recall one notion of positive dependence from Barlow and Proschan [3]. A random vector $\mathbf{X} = (X_1, \ldots, X_n)$ is said to be *conditionally increasing in sequence* (CIS) if for $i = 2, \ldots, n$,

$$[X_i | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}] \leq_{\rm st} [X_i | X_1 = x^*_1, \ldots, X_{i-1} = x^*_{i-1}]$$

whenever $x_j \leq x^*_j$ for $j = 1, \ldots, i$.

Lemma 3.1 (Shaked & Shanthikumar [18, Thm. 4.B.4]): *Let $\mathbf{X}$ and $\mathbf{Y}$ be two n-dimensional random vectors. If $X_1 \leq_{\rm st} Y_1$ and, for $i = 2, \ldots, n$,*

$$[X_i | X_1 = x_1, \ldots, X_{i-1} = x_{i-1}] \leq_{\rm st} [Y_i | Y_1 = x_1, \ldots, Y_{i-1} = x_{i-1}], \quad \forall (x_1, \ldots, x_{i-1}),$$

and if either $\mathbf{X}$ or $\mathbf{Y}$ is CIS, then $\mathbf{X} \leq_{\rm st} \mathbf{Y}$.

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THEOREM 3.7: *Let* $(X^*_{1,n}, \dots, X^*_{n,n})$ *be sequential order statistics based on* $\{F_1, \dots, F_n\}$. *If* $F_2 \le_{\rm hr} \cdots \le_{\rm hr} F_n$, *then*

$$(X^*_{1,n}, \dots, X^*_{n-1,n}) \le_{\rm st} (X^*_{2,n}, \dots, X^*_{n,n}). \qquad \textbf{(3.20)}$$

PROOF: Note that sequential order statistics have the Markov property. It follows from (2.3) that, for $i = 1, \dots, n-1$,

$$[X^*_{i+1,n} | X^*_{1,n} = x_1, \dots, X^*_{i,n} = x_i] = [X^*_{i+1,n} | X^*_{i,n} = x_i] \qquad \textbf{(3.21)}$$

is increasing with respect to $(x_1, \dots, x_i)$ in the sense of the usual stochastic order. Then $(X^*_{1,n}, \dots, X^*_{n,n})$ is CIS. Since $X^*_{1,n} \le X^*_{2,n}$, by Lemma 3.1, we have to prove that

$$[X^*_{i,n} | X^*_{i-1,n} = s] \le_{\rm hr} [X^*_{i+1,n} | X^*_{i,n} = s] \quad \text{for all } s \text{ and } i = 2, \dots, n-1;$$

that is,

$$\left(\frac{\bar{F}_i(t)}{\bar{F}_i(s)} \right)^{n-i+1} \le \left(\frac{\bar{F}_{i+1}(t)}{\bar{F}_{i+1}(s)} \right)^{n-i} \quad \text{for } i = 2, \dots, n-1 \text{ and } s < t,$$

which follows trivially from the assumption $F_2 \le_{\rm hr} \cdots \le_{\rm hr} F_n$. This completes the proof. ■

Remark 3.2: The result of Theorem 3.7 is also true if the assumption $F_2 \le_{\rm hr} \cdots \le_{\rm hr} F_n$ is replaced by

$$(n-i)\lambda_{i+1}(t) \le (n-i+1)\lambda_i(t) \quad \text{for all } t \text{ and } i = 2, \dots, n-1.$$

THEOREM 3.8: *Let* $(X^*_{1,n}, \dots, X^*_{n,n})$ *and* $(X^*_{1,n+1}, \dots, X^*_{n+1,n+1})$ *be the same as in Corollary 3.2. If* $F_1 \le_{\rm st} F_2$ *and* $F_2 \le_{\rm hr} \cdots \le_{\rm hr} F_{n+1}$, *then*

$$(X^*_{1,n}, \dots, X^*_{n,n}) \le_{\rm st} (X^*_{2,n+1}, \dots, X^*_{n+1,n+1}). \qquad \textbf{(3.22)}$$

PROOF: By a similar argument to that of Theorem 3.7, we have to prove that $X^*_{1,n} \le_{\rm st} X^*_{2,n+1}$ and

$$[X^*_{i,n} | X^*_{i-1,n} = s] \le_{\rm hr} [X^*_{i+1,n+1} | X^*_{i,n+1} = s] \quad \text{for all } s. \qquad \textbf{(3.23)}$$

Now, (3.23) is equivalent to

$$\left(\frac{\bar{F}_i(t)}{\bar{F}_i(s)} \right)^{n-i+1} \le \left(\frac{\bar{F}_{i+1}(t)}{\bar{F}_{i+1}(s)} \right)^{n-i+1} \quad \text{for } s < t \text{ and } i = 2, \dots, n-1,$$

which follows trivially from the assumption $F_2 \leq_{\rm hr} \cdots \leq_{\rm hr} F_n$. On the other hand, it follows from (3.18) and $F_1 \leq_{\rm st} F_2$ that

$$\begin{aligned}
&\bar F_{X^*_{2,n+1}}(t) - \bar F_{X^*_{1,n}}(t)\\
&\quad = \bar F_1^{n+1}(t) - \left[\frac{\bar F_2^n(t)}{\bar F_2^n(z)} \bar F_1^{n+1}(z) \Big|_0^t - \bar F_2^n(t) \int_0^t \bar F_1^{n+1}(z)\, d\, \frac{1}{\bar F_2^n(z)} \right] - \bar F_1^n(t)\\
&\quad = \bar F_2^n(t) + n \bar F_2^n(t) \int_0^t \frac{\bar F_1^{n+1}(z)}{\bar F_2^{n+1}(z)}\, dF_2(z) - \bar F_1^n(t) \geq 0;
\end{aligned}$$

that is, $X^*_{1,n} \leq_{\rm st} X^*_{2,n+1}$. This completes the proof. ■

Since the usual multivariate stochastic order is closed under marginalization, we can get univariate comparison results of sequential order statistics from (3.19)–(3.22).

4. SOME APPLICATIONS

In this section, some applications of the main results in Section 3 are presented.

4.1. Ordinary Order Statistics

Ordinary order statistics based on distribution F correspond to the generalized order statistics with $\gamma_{i,n} = n - i + 1$ for $i = 1, \ldots, n$. Then the F_i's and G_i's defined in (3.9) and (3.10) are all F. Applying Theorems 3.1–3.3, we get the following result.

COROLLARY 4.1: *Let $(X_{1:n}, \ldots, X_{n:n})$ and $(X_{1:n+1}, \ldots, X_{n+1:n+1})$ be ordinary order statistics based on an absolutely continuous distribution F. Then*

$$(X_{1,n+1}, \ldots, X_{n,n+1}) \leq_{\rm lr} (X_{1,n}, \ldots, X_{n,n}),$$

$$(X_{1,n}, \ldots, X_{n-1,n}) \leq_{\rm lr} (X_{2,n}, \ldots, X_{n,n}),$$

$$(X_{1,n}, \ldots, X_{n,n}) \leq_{\rm lr} (X_{2,n+1}, \ldots, X_{n+1,n+1}).$$

4.2. Sequential Order Statistics [Model (2.6)]

Let $\{F_1, F_2, \ldots, F_{n+1}\}$ satisfy the proportional hazard model (2.6); that is,

$$F_i(x) = 1 - [\bar F(x)]^{\alpha_i} \quad \text{for some } F \text{ and } i = 1, \ldots, n+1.$$

If F is absolutely continuous, then $F_i \leq_{\rm lr} F_j$ if and only if $\alpha_i \geq \alpha_j$. On the other hand, conditions (3.3) and (3.7) hold if and only if

$$2\alpha_i \leq \alpha_{i-1} + \alpha_{i+1}, \quad i = 2, \ldots, n. \tag{4.1}$$

Thus, we have the following result as a corollary of Theorems 3.1–3.3.

Corollary 4.2: *Let* $\{F_1, \dots, F_{n+1}\}$ *be as defined above with F absolutely continuous and let* $(X^*_{1,n}, \dots, X^*_{n,n})$ *and* $(X^*_{1,n+1}, \dots, X^*_{n+1,n+1})$ *be sequential order statistics based on* $\{F_1, \dots, F_n\}$ *and* $\{F_1, \dots, F_{n+1}\}$*, respectively. If* $\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_{n+1}$ *and (4.1) holds, then (3.1), (3.4), and (3.8) hold.*

4.3. Progressive Type II Censored Order Statistics

In a progressive type II censoring scheme, N units are placed on a lifetime test. The failure times are described by independent and identically distributed random variables with a common distribution F. A number n $(n \le N)$ of units are observed to fail. A predetermined number R_i of surviving units at the time of the ith failure are randomly selected and removed from further testing. Thus, $\sum_{i=1}^n R_i$ units are progressively censored; hence, $N = n + \sum_{i=1}^n R_i$. The n observed failure times are called progressive type II censored order statistics based on F, denoted by $X^{\mathbf{R}}_{1:n,N} \le X^{\mathbf{R}}_{2:n,N} \le \cdots \le X^{\mathbf{R}}_{n:n,N}$, where $\mathbf{R} = (R_1, \dots, R_n)$. For details on the model of progressive type II censoring, we refer to Korwar [17] and Balakrishnan and Aggarwala [1].

Progressive type II censored order statistics based on F correspond to the generalized order statistics based on distributions $\{F_1, \dots, F_n\}$ defined in (2.6). The parameter $\alpha_{i,n} \equiv \alpha_{i,\mathbf{R}}$ is given by

$$\alpha_{i,\mathbf{R}} = \frac{n - i + 1 + \sum_{j=i}^{n} R_j}{n - i + 1}, \quad i = 1, \dots, n.$$

For censoring policy $\mathbf{R}$ with $R_i = \tau - ic$ for some $\tau > 0$ and $c \ge 0$, we have the following result.

Corollary 4.3: *Let* $\mathbf{R} = (R_1, \dots, R_n)$ *and* $\mathbf{R}' = (R'_1, \dots, R'_n)$ *be two censoring policies with*

$$R_i = \tau - ic \quad \text{and} \quad R'_i = \tau' - ic' \quad \text{for } i = 1, \dots, n,$$

where τ *and* τ' *are positive and* c *and* c' *are nonnegative. If F is absolutely continuous, then the following hold:*

(a) $(X^{\mathbf{R}}_{1:n,N}, X^{\mathbf{R}}_{2:n,N}, \dots, X^{\mathbf{R}}_{n-1:n,N}) \le_{\mathrm{lr}} (X^{\mathbf{R}}_{2:n,N}, X^{\mathbf{R}}_{3:n,N}, \dots, X^{\mathbf{R}}_{n:n,N})$.
(b) For $\tau < \tau'$,

$$(X^{\mathbf{R}}_{1:n,N}, X^{\mathbf{R}}_{2:n,N}, \dots, X^{\mathbf{R}}_{n:n,N}) \le_{\mathrm{hr}} (X^{\mathbf{R}'}_{1:n,N}, X^{\mathbf{R}'_{2:n,N}}, \dots, X^{\mathbf{R}'}_{n:n,N}).$$

Proof: Let $(X^{\mathbf{R}}_{1:n,N}, X^{\mathbf{R}}_{2:n,N}, \dots, X^{\mathbf{R}}_{n:n,N})$ and $(X^{\mathbf{R}'}_{1:n,N}, X^{\mathbf{R}'}_{2:n,N}, \dots, X^{\mathbf{R}'}_{n:n,N})$ be the sequential order statistics based on $\{F_1, \dots, F_n\}$ and $\{G_1, \dots, G_n\}$, respectively, where

$$\bar{F}_i(x) = [\bar{F}(x)]^{\alpha_{i,\mathbf{R}}}, \qquad \bar{G}_i(x) = [\bar{F}(x)]^{\alpha_{i,\mathbf{R}'}}, \qquad i = 1, \dots, n.$$

(a) First,

$$\alpha_{i,\mathbf{R}} = \frac{1}{n-i+1}\left[n-i+1+\sum_{j=i}^{n}(\tau - jc)\right] = \tau + 1 - \frac{(n+i)c}{2}.$$

It is clear that $2\alpha_{i,\mathbf{R}} = \alpha_{i+1,\mathbf{R}} + \alpha_{i-1,\mathbf{R}}$ for $i = 2, \ldots, n-1$ and that $\alpha_{i,\mathbf{R}}$ is decreasing in i. Then the conditions in Theorem 3.2 are fulfilled. Thus, the desired result of part a now follows.

(b) Similarly,

$$\alpha_{i,\mathbf{R}'} = \tau' + 1 - \frac{(n+i)c'}{2}, \qquad i = 1, \ldots, n.$$

Since $N = n(\tau+1) - \frac{1}{2}n(n+1)c = n(\tau'+1) - \frac{1}{2}n(n+1)c'$, it follows that

$$\tau' - \tau = \frac{1}{2}(n+1)[c' - c],$$

and, hence, $c < c'$ when $\tau < \tau'$. Then

$$\alpha_{i,\mathbf{R}'} - \alpha_{i,\mathbf{R}} = (\tau' - \tau) - \frac{1}{2}(n+i)(c'-c) \le 0, \qquad i = 1, \ldots, n,$$

which implies $F_i \le_{\mathrm{lr}} G_i$. Thus, part b follows from Theorem 5.2. ■

4.4. NHPB Processes

Observing the connection between sequential order statistics and epoch times of NHPB processes, we obtain conditions under which epoch times of a NHPB process can be compared in the sense of the multivariate hazard rate order. This result is a corollary of Theorem 3.5 and Proposition 2.1.

COROLLARY 4.4: *Let $\{T_n, n \ge 1\}$ denote the epoch times of a NHPB process $\{N(t), t \ge 0\}$ with intensity functions $\{r_i(t), i \ge 0\}$. If $r_i(t)$ is decreasing in i for each t, then*

$$(T_1, T_2, \ldots, T_{n-1}) \le_{\mathrm{hr}} (T_2, T_3, \ldots, T_n) \quad \textit{for } n \ge 2.$$

PROOF: Define

$$\lambda_i(t) = \frac{r_{i-1}(t)}{n-i+1} \quad \text{for } i = 1, \ldots, n.$$

Since $\int_t^\infty r_i(u)\,du = +\infty$, $\lambda_i(t)$ can be regarded as the hazard rate function of some distribution function F_i. Let $(X^*_{1,n}, \ldots, X^*_{n,n})$ be the sequential order statistics based on $\{F_1, \ldots, F_n\}$. Since $r_i(t)$ is decreasing in i for each t, condition (3.16) is fulfilled.

By Proposition 2.1, we get

$$(T_1, T_2, \ldots, T_n) \stackrel{\text{st}}{=} (X^*_{1,n}, \ldots, X^*_{n,n}).$$

Thus, the desired result now follows from Theorem 3.5 and Remark 3.1. ∎

5. DISCUSSION

Belzunce et al. [4] described various conditions on the parameters of pairs of NHPB processes under which the corresponding epoch times or interepoch intervals are ordered in various senses. Applying Proposition 2.1, we can identify conditions that enable one to compare sequential order statistics from two samples.

Let $(X^*_{1,n}, \ldots, X^*_{n,n})$ and $(Y^*_{1,n}, \ldots, Y^*_{n,n})$ respectively be sequential order statistics based on $\{F_1, \ldots, F_n\}$ and $\{G_1, \ldots, G_n\}$ with

$$F_1^{-1}(1) \leq \cdots \leq F_n^{-1}(1), \quad G_1^{-1}(1) \leq \cdots \leq G_n^{-1}(1).$$

Denote by $\lambda_1(t), \ldots, \lambda_n(t)$ and $\mu_1(t), \ldots, \mu_n(t)$ the hazard rates of $F_1, \ldots, F_n$ and $G_1, \ldots, G_n$, respectively.

From Theorems 3.11–3.13 and 4.10 of Belzunce et al. [4], we have the following results.

THEOREM 5.1: *If $F_1 \leq_{\text{st}} G_1$ and $F_i \leq_{\text{hr}} G_i$ for $i = 2, \ldots, n$, then*

$$(X^*_{1,n}, \ldots, X^*_{n,n}) \leq_{\text{st}} (Y^*_{1,n}, \ldots, Y^*_{n,n}).$$

In particular, $X^*_{i,n} \leq_{\text{st}} Y^*_{i,n}$ for $i = 1, \ldots, n$.

THEOREM 5.2: *If $F_i \leq_{\text{hr}} G_i$ for $i = 1, \ldots, n$, then*

$$(X^*_{1,n}, \ldots, X^*_{n,n}) \leq_{\text{hr}} (Y^*_{1,n}, \ldots, Y^*_{n,n}).$$

THEOREM 5.3: *If $F_i \leq_{\text{lr}} G_i$ for $i = 1, \ldots, n$ and*

$$\mu_{i+1}(t) - \mu_i(t) \geq \lambda_{i+1}(t) - \lambda_i(t) \quad \text{for all } t \geq 0 \text{ and } i = 1, \ldots, n-1, \tag{5.1}$$

then

$$(X^*_{1,n}, \ldots, X^*_{n,n}) \leq_{\text{lr}} (Y^*_{1,n}, \ldots, Y^*_{n,n}). \tag{5.2}$$

In particular, $X^*_{i,n} \leq_{\text{lr}} Y^*_{i,n}$ for $i = 1, \ldots, n$.

THEOREM 5.4: *If $F_1 \leq_{\text{st}} G_1$ and*

$$\lambda_i(u) \geq \mu_i(u + x) \quad \text{for } (u, x) \in \Re^2_+ \text{ and } i \geq 2,$$

then

$$(X^*_{1,n}, X^*_{2,n} - X^*_{1,n}, \ldots, X^*_{n,n} - X^*_{n-1,n}) \leq_{\text{st}} (Y^*_{1,n}, Y^*_{2,n} - Y^*_{1,n}, \ldots, Y^*_{n,n} - Y^*_{n-1,n}).$$

It is worth mentioning that in Theorem 5.3 we weaken the conditions of Theorem 3.13 in Belzunce et al. [4]. Their proof is also valid with minor modification. Because of limited page space, we do not list the comparison results on spacing vectors of sequential order statistics from two samples.

Finally, we give a example, which satisfies all conditions in Theorems 5.1–5.4.

Example 5.1: Let the hazard rate functions of $F_1, \ldots, F_n$ and $G_1, \ldots, G_n$ respectively are given by

$$\lambda_i(t) = \lambda(t) + (i-1)c, \qquad i = 1, \ldots, n,$$

$$\mu_i(t) = \eta + (i-1)c, \qquad i = 1, \ldots, n,$$

where c and η are positive constants, $\lambda(u) \geq \eta$ for all u, and $\lambda(u)$ is decreasing. Obviously, $\lambda_i(t) \geq \mu_i(t)$, and $\mu_i(t)/\lambda_i(t)$ is increasing in t. By Lemma 3.5 of Belzunce et al. [4], we get that $F_i \leq_{\mathrm{lr}} G_i$ for $i = 1, \ldots, n$. Thus, all conditions in Theorems 5.1–5.4 are satisfied.

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