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Determination of minimum magnetic energy states in frozen plasmas

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Abstract. As is known, the minimum magnetic energy state for a frozen plasma, subject to the infinite topological constraints, corresponds to a nonlinear force-free field. The magnetic flux invariance in ideal magnetohydrodynamics is possible in important astrophysical applications. We develop a method for explicitly obtaining a minimum energy state starting from an arbitrary initial state. This method does not require the explicit use of the invariance of the differential magnetic helicities. It is particularly useful when the minimum magnetic energy state for the given topological structure is unique. We show examples of the application of the method for this kind of system.

1. Introduction

An important problem in magnetohydrodynamics (MHD) consists of obtaining the minimum magnetic energy states in a magnetized perfect conducting plasma. The conditions of ideal MHD can often be applied in several astrophysical problems. The solenoidal character of the magnetic field, together with the frozen flux condition, allow an analysis based on topological invariants; the most simple is the magnetic helicity related with the linkage of the magnetic induction lines (Moffatt 1992).

One of the first results was obtained by Woltjer (Woltjer 1958; see also Priest 1982) who showed that in a closed system the magnetic helicity evaluated over the whole volume is a temporal invariant; after that, he showed that the minimum magnetic energy, subject to the constant helicity condition, corresponds to a linear force-free field. However, this variational principle is not complete, because in a perfectly conducting plasma there exist infinite topological invariants (Bhattacharjee and Dewar 1982); a differential magnetic helicity, which remains constant, can be associated to each closed field line (Taylor 1974). Taylor asserted that taking into account all these invariants as constraints, the minimum energy state corresponds to a nonlinear force-free field. In fact, if the only forces acting on the system are the magnetic ones, an equilibrium state requires that the current density and the magnetic induction be parallel. For a rigorous proof of this statement, Taylor suggested an extension of the Lagrange multipliers technique for the case of infinite constraints (Taylor 1986, 1992). A related problem consists of obtaining the magnetic induction field of the force-free state which corresponds to arbitrary initial conditions. Following his idea, Taylor noted that in order to determine the

Lagrange multipliers, one has to take into account the infinite differential helicities of the initial state (Taylor 1986). Nevertheless, a variational theory according to this outline is far from being developed (Laurence and Avellaneda 1991).

In this work, we analyze the possibility of finding a minimum magnetic energy state, starting from an arbitrary initial state, by means of a technique which is different to Taylor's proposition. Our aim consists of determining a feasible way to obtain a nonlinear force-free state, to which a system could evolve with arbitrary initial conditions. We introduce a new point of view based on the local analysis of the frozen condition.

The frozen flux condition can be interpreted as a coupling between the magnetic induction lines and the matter, in such a way so as to preserve the differential magnetic helicity on each closed field line, limiting the fluid motion. Taking into account that the magnetic helicity is a measure of the linkage between the magnetic induction lines, the differential magnetic helicity invariance on all closed field lines occurs if and only if the plasma is frozen (Moffatt 1992). In this sense we say that the frozen condition is equivalent to the infinite topological constraints. Therefore, an alternative way to solve the extremal problem is to take into account the restrictions on the magnetic induction variations due to the frozen condition. Following this outline we develop a method for obtaining a local minimum magnetic energy state which is nonlinear force-free and whose topological structure is identical to that of an arbitrary specified initial state.

When a plasma has turbulence, its evolution towards the minimum magnetic energy state could be described by means of Taylor's relaxation theory (Taylor 1986); in this case, the final state is linear force-free. However, if there is no turbulence, the relaxation mechanism must preserve the topological structure; it is precisely in this kind of system that the method described in this paper has its main application.

2. Variation of the magnetic induction under virtual displacements

We analyze an infinite conducting plasma contained in a volume of perfect conducting and rigid walls. Let us define a continuous field $\delta\mathbf{r}(\mathbf{r})$ with continuous first derivatives, which represents a virtual displacement of the fluid. In order to avoid singular currents, the condition

$$\nabla \cdot [(\delta\mathbf{r} \cdot \breve{u})\breve{u}] > -1 \tag{1}$$

must be satisfied for every direction $\breve{u}$. If a continuous displacement field does not fulfill this condition, it will be sufficient to multiply $\delta\mathbf{r}$ by a constant of scale of small value. Except for this scale constant and the boundary condition, the virtual displacement field can be arbitrarily chosen.

Let us consider a magnetic induction field $\mathbf{B}(\mathbf{r})$ which is coupled with the plasma. We choose an arbitrary loop C_1 into the plasma and an open surface S_1, limited by C_1. When a virtual displacement field $\delta\mathbf{r}(\mathbf{r})$ is applied, the magnetic induction $\mathbf{B}(\mathbf{r})$ must transform into $\mathbf{B}'(\mathbf{r})$ satisfying the frozen condition. Furthermore C_1 is transformed into C_2; we define S_2 as the union of S_1 and the surface limited by C_1 and C_2, tangent to the displacements $\delta\mathbf{r}$. The frozen condition requires that the flux of $\mathbf{B}$ through S_1 must be equal to the flux of $\mathbf{B}'$ through S_2. Then

$$\int_{S_1} \mathbf{B} \cdot d\mathbf{S} = \int_{S_1} \mathbf{B}' \cdot d\mathbf{S} + \oint_{C_1} \mathbf{B}' \cdot (\delta\mathbf{r} \times d\mathbf{l}). \tag{2}$$

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Introducing $\delta\mathbf{B}$ as

$$\delta\mathbf{B}(\mathbf{r}) = \mathbf{B}'(\mathbf{r}) - \mathbf{B}(\mathbf{r}), \tag{3}$$

and neglecting terms of order greater than one in the variations, using Stokes theorem and the arbitrariness of C_1 and S_1 we have

$$\delta\mathbf{B}(\mathbf{r}) = \nabla \times [\delta\mathbf{r}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})]. \tag{4}$$

This result must not be confused with that which one obtains for real displacements $d\boldsymbol{\xi}$. Taking into account that for real displacements $\mathbf{v} = d\boldsymbol{\xi}/dt$, from the induction equation we obtain the well-known result

$$d\mathbf{B}(\mathbf{r}, t) = \nabla \times [d\boldsymbol{\xi}(\mathbf{r}, t) \times \mathbf{B}(\mathbf{r}, t)]. \tag{5}$$

Real displacements were used by Bernstein et al. (1958) in their treatment of stability. The main difference between (4) and (5) is that the former preserves the topological structure while the latter also describes temporal evolutions.

Using the result (4) we reobtain the minimum magnetic energy condition for an infinitely conducting plasma, contained in a volume of perfect conducting and rigid walls:

$$\nabla \times \mathbf{B} = \alpha(\mathbf{r})\mathbf{B}. \tag{6}$$

This result was proposed originally by Taylor (1974, 1986).

3. Determination of a local minimum magnetic energy state

In this section we develop a method for the determination of a local minimum magnetic energy state, starting from arbitrary initial conditions.

The set of real states obtained by the successive application of (5) does not include, in general, the minimum magnetic energy state. The importance of (4) is made evident here; it allows, from the initial state, a sequence of states connected by infinitesimal virtual displacements to be built, whose limit is a state of local minimum magnetic energy. In order to obtain these virtual displacements we start writing the magnetic energy of the state with magnetic induction $\mathbf{B} + \delta\mathbf{B}$

$$\frac{1}{2\mu_0}\int_V |\mathbf{B} + \delta\mathbf{B}|^2\, dV = \frac{1}{2\mu_0}\int_V B^2\, dV + \frac{1}{\mu_0}\int_V \mathbf{B}\cdot\delta\mathbf{B}\, dV + \frac{1}{2\mu_0}\int_V |\delta\mathbf{B}|^2\, dV, \tag{7}$$

this energy will be lower than the one previous to the displacement only if

$$\int_V \mathbf{B}\cdot\delta\mathbf{B}\, dV < 0. \tag{8}$$

From (4) we obtain the result

$$\begin{aligned}\int_V \mathbf{B}\cdot\delta\mathbf{B}\, dV &= \int_V \nabla\cdot[(\delta\mathbf{r}\times\mathbf{B})\times\mathbf{B}]\, dV + \int_V (\delta\mathbf{r}\times\mathbf{B})\cdot(\nabla\times\mathbf{B})\, dV \\ &= \oint_S [(\delta\mathbf{r}\times\mathbf{B})\times\mathbf{B}]\cdot d\mathbf{S} + \int_V [\mathbf{B}\times(\nabla\times\mathbf{B})]\cdot\delta\mathbf{r}\, dV,\end{aligned} \tag{9}$$

taking into account the boundary conditions

$$\delta\mathbf{r}(\mathbf{r}_s)\cdot d\mathbf{S} = 0, \tag{10}$$

$$\mathbf{B}(\mathbf{r}_s)\cdot d\mathbf{S} = 0, \tag{11}$$

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the surface integral vanishes. Then, condition (8) reads

$$\int_V [\mathbf{B} \times (\nabla \times \mathbf{B})] \cdot \delta\mathbf{r}\, dV < 0, \tag{12}$$

resulting in the virtual displacements which carry more efficiently to states of lower energy having equal sense as $(\nabla \times \mathbf{B}) \times \mathbf{B}$.

Therefore, the virtual displacements can be written

$$\delta\mathbf{r} = [(\nabla \times \mathbf{B}) \times \mathbf{B}] f(\mathbf{r}, \epsilon)\delta\epsilon, \tag{13}$$

where $f(\mathbf{r}, \epsilon)$ is a regular, positive scalar function that must be chosen in such a way that $\delta\mathbf{r}$ be null on the boundaries. Apart from this, it can be built in order to simplify the resolution of the problem. Furthermore, ϵ is a parameter independent of the position that labels the magnetic induction field $\mathbf{B}(\mathbf{r}, \epsilon)$ of the different states; the initial state corresponds to $\mathbf{B}(\mathbf{r}, 0)$.

Successive application of (13) together with (4), gives

$$\mathbf{B}(\mathbf{r}, \epsilon) = \mathbf{B}(\mathbf{r}, 0) + \int_0^{\epsilon} \nabla \times \{f(\mathbf{r}, \epsilon')[(\nabla \times \mathbf{B}(\mathbf{r}, \epsilon')) \times \mathbf{B}(\mathbf{r}, \epsilon')] \times \mathbf{B}(\mathbf{r}, \epsilon')\}\, d\epsilon', \tag{14}$$

in the limit for $\epsilon \to \infty$, we arrive at a local minimum magnetic energy state. Then, this method connects the specified initial state with one of the local minimum energy states with the same topological structure. This method is particularly useful when the problem, for a given initial condition, admits only one force-free solution. In the next section we analyze examples of this kind of topological structure.

4. Examples with cylindrical symmetry

We consider a frozen plasma confined between two cylindrical, rigid and perfectly conducting walls, of radii R_i and R_e. Let us suppose that two states with $\mathbf{B}_1(r)$ and $\mathbf{B}_2(r)$ can be connected by virtual displacements that preserve the topology. Taking into account the symmetry, the magnetic flux conservation drives to

$$r_2 A_{2\varphi}(r_2) = r_1 A_{1\varphi}(r_1), \tag{15}$$

$$A_{2z}(r_2) = A_{1z}(r_1), \tag{16}$$

where $\mathbf{A}$ is the vector potential and r_1 and r_2 are the radial positions of the same plasma element, before and after the displacement, respectively. From (15) and (16) the following invariant can be obtained:

$$\frac{B_{2\varphi}(r_2)}{r_2 B_{2z}(r_2)} = \frac{B_{1\varphi}(r_1)}{r_1 B_{1z}(r_1)} = h(r_1). \tag{17}$$

If $\mathbf{B}_1(r_1)$ is known, one can evaluate the function $h(r_1)$. The relation (17) can be written as

$$r_2 B_{2\varphi}(r_2) = r_2^2 B_{2z}(r_2) h(r_1). \tag{18}$$

Let us suppose now that $\mathbf{B}_2(r_2)$ is a force-free field; there exists a scalar function $\alpha(r_2)$ such that

$$\frac{d}{dr_2}[r_2 B_{2\varphi}(r_2)] = r_2 \alpha(r_2) B_{2z}(r_2), \tag{19}$$

$$\frac{d}{dr_2} B_{2z}(r_2) = -\alpha(r_2) B_{2\varphi}(r_2) = -r_2 \alpha(r_2) h(r_1) B_{2z}(r_2). \tag{20}$$

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Replacing the differential relations (19) and (20) in the derivative of (18) we obtain the following expression for $\alpha(r_2)$:

$$\alpha(r_2) = \left[2h(r_1) + r_2 \frac{dh(r_1)}{dr_1}\frac{dr_1}{dr_2}\right]\left[1 + r_2^2 h^2(r_1)\right]^{-1}. \tag{21}$$

We consider now the particular case in which the state 1 is such that $h(r_1)$ is a constant

$$h(r_1) = C. \tag{22}$$

In this case, α is a specific function of r_2,

$$\alpha(r_2) = \frac{2C}{1 + C^2 r_2^2} \tag{23}$$

and (20) is a linear differential equation for B_{2z}. Taking into account the frozen condition, its solution is

$$B_{2z}(r_2) = \left[\ln\left(\frac{1 + C^2 R_e^2}{1 + C^2 R_i^2}\right)\right]^{-1} \frac{2C^2 D}{1 + C^2 r_2^2} = \frac{B_{2\varphi}}{C r_2} \tag{24}$$

where

$$D = \int_{R_i}^{R_e} r_1 B_{1z}(r_1)\, dr_1. \tag{25}$$

Therefore, when the initial state satisfies that $B_{1\varphi}/B_{1z}$ is proportional to r_1, the corresponding force-free solution is unique. The following examples are particular cases of this kind of topological structure.

4.1. Example 1

We show a simple example in which the application of the method described in Sec. 3 can be performed analytically and numerically.

We consider the initial magnetic induction given by

$$\mathbf{B}_1(\mathbf{r}) = B_0 \breve{\varphi}, \quad R_i < r < R_e. \tag{26}$$

In order to preserve the cylindrical symmetry, it is convenient to choose the function $f(\mathbf{r}, \epsilon)$ independent of φ and z; thus, for the successive states

$$\mathbf{B}(\mathbf{r}, \epsilon) = B_\varphi(r, \epsilon)\breve{\varphi}. \tag{27}$$

Then, the differential equation associated with the integral equation (14), takes the simple form

$$\begin{aligned}\frac{\partial B_\varphi}{\partial \epsilon}(r, \epsilon) &= \frac{\partial}{\partial r}\left[\frac{f(r, \epsilon)}{r} B_\varphi^2 \frac{\partial}{\partial r}(r B_\varphi)\right] \\ &= \frac{\partial}{\partial r}\left[g(r, \epsilon)\frac{\partial}{\partial r}(r B_\varphi)\right].\end{aligned} \tag{28}$$

An appropriate choice of $g(r, \epsilon)$ allows us to resolve this equation; we find that if

$$g(r, \epsilon) = (1 + \epsilon)^{-1}\left[(R_e - R_i)\frac{\ln(r/R_i)}{\ln(R_e/R_i)} - (r - R_i)\right], \tag{29}$$

the solution of (28) is

$$B_\varphi(r, \epsilon) = \frac{B_0}{1 + \epsilon}\left[1 + \frac{\epsilon(R_e - R_i)}{\ln(R_e/R_i)}\frac{1}{r}\right]. \tag{30}$$

Table 1. $B_{2\varphi}/B_0$ for example 1 as a function of r/R_e ($R_i = 0.2\ R_e$), as obtained with the analytical solution and the numerical calculations.

r/R_e	Analytic	(A)	(B)	(C)
0.20	0.2444	0.2435	0.2424	0.2485
0.25	0.1958	0.1950	0.1942	0.1988
0.30	0.1633	0.1627	0.1620	0.1657
0.35	0.1400	0.1396	0.1390	0.1420
0.40	0.1226	0.1222	0.1217	0.1243
0.45	0.1091	0.1087	0.1083	0.1105
0.50	0.0982	0.0979	0.0975	0.0994
0.55	0.0894	0.0891	0.0888	0.0903
0.60	0.0820	0.0818	0.0815	0.0820
0.65	0.0758	0.0756	0.0752	0.0764
0.70	0.0704	0.0703	0.0700	0.0710
0.75	0.0658	0.0657	0.0654	0.0662
0.80	0.0618	0.0617	0.0614	0.0621
0.85	0.0583	0.0581	0.0579	0.0584
0.90	0.0551	0.0550	0.0548	0.0552
0.95	0.0523	0.0522	0.0520	0.0523
1.00	0.0497	0.0496	0.0495	0.0497

We want to remark that with this expression for $g(r, \epsilon)$, the corresponding $f(r, \epsilon)$ vanishes at R_i and R_e and is positive in (R_i, R_e).

The minimum magnetic energy state is obtained by making $\epsilon \to \infty$ in (30):

$$\mathbf{B}_2(\mathbf{r}) = \mathbf{B}(\mathbf{r}, \infty) = \frac{B_0(R_e - R_i)}{\ln(R_e/R_i)} \frac{1}{r} \breve{\varphi}, \quad R_i < r < R_e. \tag{31}$$

This expression coincides with (24) for the particular case in which $C \to \infty$, $D \to 0$ and $CD \to B_0(R_e - R_i)$.

The complex form of $f(\mathbf{r}, \epsilon)$ that results from (29) suggests that analytical solutions are not always obtainable; perhaps only trivial topologies can be solved in this way. Nevertheless, convergent numerical solutions are obtained with simple forms of $f(\mathbf{r}, \epsilon)$. We have chosen three different expressions for resolving this example:

(A)

$$f(\mathbf{r}, \epsilon) = \epsilon \left[\sin\left(\pi \frac{r - R_i}{R_e - R_i} \right) \right]^{1/2}, \tag{32}$$

(B)

$$f(\mathbf{r}, \epsilon) = \epsilon \left[\sin\left(\pi \frac{r - R_i}{R_e - R_i} \right) + 0.15 \sin\left(7\pi \frac{r - R_i}{R_e - R_i} \right) \right]^{1/2}, \tag{33}$$

(C)

$$f(\mathbf{r}, \epsilon) = \epsilon \left[\sin\left(\pi \frac{r - R_i}{R_e - R_i} \right) + 0.15 \sin\left(15\pi \frac{r - R_i}{R_e - R_i} \right) \right]^{1/2}. \tag{34}$$

In Table 1, values of the non-null component of the magnetic induction, $B_{2\varphi}(r)$, are shown for different distances from the axis. The values obtained with the analytical solution are given in the second column and in the other columns the numerical calculations performed with the three functions given above are shown.

Table 2. $B_{2\varphi}/B_0$ for example 2 as a function of r/R_e, as obtained with the analytical solution and the numerical calculations.

r/R_e	Analytic	(A)	(B)	(C)
0.20	0.2823	0.2796	0.2795	0.2795
0.25	0.3454	0.3420	0.3419	0.3419
0.30	0.4040	0.4002	0.4001	0.4001
0.35	0.4577	0.4537	0.4536	0.4537
0.40	0.5062	0.5023	0.5022	0.5022
0.45	0.5493	0.5457	0.5456	0.5457
0.50	0.5872	0.5840	0.5839	0.5840
0.55	0.6199	0.6174	0.6173	0.6173
0.60	0.6476	0.6460	0.6459	0.6460
0.65	0.6708	0.6701	0.6701	0.6701
0.70	0.6896	0.6901	0.6901	0.6901
0.75	0.7046	0.7062	0.7063	0.7063
0.80	0.7161	0.7189	0.7190	0.7190
0.85	0.7244	0.7285	0.7287	0.7286
0.90	0.7299	0.7352	0.7354	0.7353
0.95	0.7330	0.7393	0.7394	0.7394
1.00	0.7340	0.7408	0.7408	0.7410

Table 3. B_{2z}/B_0 for example 2 as a function of r/R_e, as obtained with the analytical solution and the numerical calculations.

r/R_e	Analytic	(A)	(B)	(C)
0.20	1.411	1.396	1.395	1.395
0.25	1.381	1.367	1.366	1.366
0.30	1.346	1.333	1.333	1.333
0.35	1.307	1.296	1.296	1.296
0.40	1.265	1.255	1.255	1.255
0.45	1.220	1.212	1.212	1.212
0.50	1.174	1.168	1.168	1.168
0.55	1.127	1.122	1.122	1.122
0.60	1.079	1.077	1.077	1.077
0.65	1.032	1.031	1.031	1.031
0.70	0.985	0.986	0.986	0.986
0.75	0.939	0.942	0.942	0.942
0.80	0.895	0.899	0.899	0.899
0.85	0.852	0.857	0.857	0.857
0.90	0.811	0.817	0.817	0.817
0.95	0.771	0.778	0.778	0.779
0.10	0.734	0.741	0.741	0.741

4.2. Example 2

We consider the same system of example 1, with different initial conditions. We suppose that the initial magnetic induction field has non-null components $B_{1\varphi}(r)$ and $B_{1z}(r)$, given by

$$B_{1\varphi}(r) = B_0 \frac{r}{R_e}, \tag{35}$$

$$B_{1z}(r) = B_0. \tag{36}$$

This problem was solved by computational calculations, using the integration functions (A), (B) and (C) of example 1. In Tables 2 and 3 we show the results obtained

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for these cases for $B_{2\varphi}$ and B_{2z} respectively. These are compared with the analytical solution given by (24).

5. Comments and conclusions

The most important contribution of this paper consists of the formulation of a method to obtain a local minimum magnetic energy state, starting from an arbitrary initial state, for a perfectly conducting plasma. According to Taylor, this state is nonlinear force-free. This method takes into account the topological constraints of a frozen plasma and does not require the explicit use of the invariance of the differential magnetic helicities on each closed field line; however, the consequences of the topological linkage, which can be derived from this invariance, are taken into account through the frozen plasma condition.

We want to remark that our method determines a minimum magnetic energy state, but this state will not necessarily be attained by the system for two reasons. First, there could exist more than one local minimum. In addition, a dissipation mechanism is necessary, but it can not be Joule's effect since perfect conductivity has been assumed. Following Moffatt's idea (Moffatt 1992), the magnetic energy can be converted into kinetic energy, which can be dissipated by means of viscous forces, preserving the frozen condition. We want to highlight the difference between this relaxation mechanism and Taylor's (Taylor 1986). Taylor's relaxation requires turbulence; it produces reconnection of the field lines and local ohmic dissipation, breaking the invariance of the differential magnetic helicities. These relaxation processes occur in different physical conditions.

Our method is particularly useful when it can be proved that the problem admits only one force-free solution. We show that cylindrical problems with particular topological structures have only one minimum magnetic energy state. Two examples of application of the method are given. A good agreement of the numerical and analytical results is obtained.

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