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On the conservative pasting lemma

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Abstract. Several perturbation tools are established in the volume-preserving setting allowing for the pasting, extension, localized smoothing and local linearization of vector fields. The pasting and the local linearization hold in all classes of regularity ranging from C^1 to C^∞ (Hölder included). For diffeomorphisms, a conservative linearized version of Franks' lemma is proved in the $C^{r,\alpha}$ ($r \in \mathbb{Z}^+$, $0 < \alpha < 1$) and C^∞ settings, the resulting diffeomorphism having the same regularity as the original one.

Key words: divergence-free vector field, localized smoothing, local linearization, volume-preserving diffeomorphisms, Franks' lemma
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1. *Introduction*

1.1. *Continuous-time dynamics.* One of the basic problems in conservative continuous-time dynamics is the following.

How may a local C^r-perturbation of a divergence-free vector field be extended to a global one?

More precisely (and always in the conservative setting), given a C^r vector field X on a closed connected manifold M and a C^r-perturbation Y of the restriction of X to an open set U, is it possible to find a C^r-perturbation Z of X that still coincides with Y in a slightly smaller set, say, in any chosen compact set $K \subset U$? In the non-conservative context, the solution is trivial; Y can be glued with X using a suitable partition of unity, i.e. we let $\widehat{Z} = \xi Y + (1 - \xi)X$ in U and $\widehat{Z} = X$ in U^c, where the smooth function ξ equals one in a neighbourhood of K and zero in neighbourhood of U^c. Clearly, $\widehat{Z}$ is C^r-close to X if Y is C^r-close to X in U, and the problem is solved.

In the conservative setting, the situation is more delicate, as $\widehat{Z}$ constructed as above fails, in general, to be divergence-free in the transition 'annulus' Ω, i.e., in the set where $0 < \xi < 1$. One obvious way to tackle this difficulty is by trying to find a C^r vector field v supported in $\overline{\Omega}$ whose divergence equals that of $\widehat{Z}$ and then set $Z = \widehat{Z} - v$, which thus cancels the divergence. Provided v can be found C^r-small if $Y - X|_U$ is C^r-small, the question is solved.

The problem is that, in the conservative setting, an obstruction of topological nature may hinder the above procedure: the interplay between the divergence theorem and connected cobordism. To simplify the explanation, all manifolds referred to below are assumed to be compact, connected, orientable and smooth (besides second countable and Hausdorff). Let M, U and K be as above. We start by observing that K may contain a closed $(n-1)$-submanifold γ which is the boundary of *no* n-submanifold contained in U. In this case, the perturbation Y of the restriction of X to U may change the original flux across γ (see Example 1 below). But, simultaneously, there might exist another closed $(n-1)$-submanifold γ', now contained in U^c, that together with γ constitutes the boundary of an n-submanifold W. Note that the divergence-cancelling procedure described above ensures that $Z = X$ in U^c, and thus the original flux of X across γ' is kept unchanged in Z. As a consequence, the flux of Z across the cobordant manifolds γ and γ' will be distinct, which implies (by the divergence theorem) that the divergence of Z cannot identically vanish inside the manifold W bounded by γ and γ'. Therefore, there is no possibility of extending $Y|_K$ in a divergence-free way to the whole M so that the resulting vector field still coincides with X in U^c. At first glance, one may think that the above obstruction might be overcome if one can find an alternative method for the construction of the extension Z of $Y|_K$ that renounces to obtain $Z = X$ in U^c.

Even so, the answer may still be negative. Indeed, the desired divergence-free extension of $Y|_K$ might simply not exist at all (see Example 2 below). Note that while, by hypothesis, γ is the boundary of *no* n-submanifold contained in U, it may still be the boundary of an n-submanifold W *not* contained in U (i.e., γ may be null-cobordant in M). Now, by the divergence theorem, the flux of the original vector field X across γ is zero, but the divergence-free C^r-perturbation Y of the restriction of X to U may change this flux to a non-zero value. But then, no C^1 extension of $Y|_K$ to the whole M can have a divergence that identically vanishes inside W.

These obstructions can be removed at once if we make a simple and natural topological assumption, namely, that $U \backslash K$ is connected. This implies the existence of a compact n-submanifold P with smooth connected boundary such that $K \subset \operatorname{int} P$ and $P \subset U$ (Lemma A.1), which is the key to the construction of the pasting of Y and X by the procedure described above. This pasting result (Theorem 1), which can also be formulated in the Hölder setting (Theorem 3), is then briefly the following.

THEOREM (Conservative C^r pasting lemma) *Let M be a closed connected manifold, let $U \subsetneq M$ be an open neighbourhood of a compact set K such that $U \backslash K$ is connected and let $r \in \mathbb{Z}^+$. In the conservative setting, given any C^r vector field X on M and any C^r-perturbation Y of the restriction of X to U, there exists a C^r-perturbation Z of X that coincides with Y in a neighbourhood of K and with X in U^c.*

Theorem 1 also shows that vector field Z can be obtained so that the C^r norm of $Z - X$ is linearly bounded by that of $Y - X|_U$, for some fixed constant $C > 1$ depending only on r, K and U (and, of course, on the manifold's atlas, which is assumed to be fixed).

The proof is constructive, elementary and self contained. It essentially relies on a simple but ingenious global-to-local reduction procedure originally due to Moser [**MO**]. Besides its simplicity, the main advantage of Moser's direct approach is the guarantee that the auxiliary divergence-cancelling vector field v satisfying $\operatorname{div} v = \operatorname{div} \widehat{Z}$ will be (compactly) supported inside the open 'transition annulus' $\Omega \subset U \backslash K$ (the set where the transition from vector field Y to X is set to take place; in practice, it will correspond to a small neighbourhood of the closure of $\{x \in M : 0 < \xi(x) < 1\}$), and thus extends by zero to the whole M (in the C^r class). This is needed to guarantee that the divergence cancelling operation $\widehat{Z} - v$ producing Z does not change $\widehat{Z}$ outside Ω, so that Z still coincides with Y and X in K and U^c, respectively. Due to the linearity of the divergence operator, the use of optimal regularity tools of Dacorogna–Moser type [**DM**, Theorem 2] (which are crucial in the discrete-time case; see §§1.2 and 4) can be entirely avoided, as there is no regularity loss in the divergence of the initial (non-conservative) pasting perturbation: if X and Y are divergence-free C^r vector fields and $\widehat{Z}$ is a vector field defined as above, then $\operatorname{div} \widehat{Z}$ is still of class C^r and C^r small if Y is C^r close to X.

This conservative pasting lemma permits us to establish several perturbation tools of which three illustrative examples are singled out.

(1) Localized smoothing (Theorem 5): at least for certain useful open sets $\Omega \subset M$ (see Footnote † on page 10), *one may conservatively C^r perturb a divergence-free vector field X in order to make it smooth inside Ω, while keeping X unchanged on the complement of Ω.*

(2) Perturbative extension with increased regularity (Corollary 1): *if a C^r-perturbation Y of the restriction of X to U is of class C^s, $s > r$, C^r being the regularity of X, then $Y|_K$ can be (conservatively) extended to a C^r-perturbation of X that is of class C^s on the whole M.*

(3) Local linearization of 'Franks' lemma type' (Theorem 6): *one may conservatively C^1-perturb a vector field v near a point x (keeping $v(x)$ unaltered), in order to change its derivative at x and make v affine linear near this point, the allowed variation δ of the derivative depending linearly on the required C^1-closeness ϵ of the resulting vector field to v* (this result requires the use of an additional homothety trick).

Other examples could be given. However, the primary intention of this work is to present a few solid basic techniques that might serve as a starting point for the development of more sophisticated conservative tools. Special care has been taken to ensure the following.

(a) The results obtained are the best possible both in terms of the regularity of the resulting vector field or diffeomorphism in terms of the regularity of the closeness of the resulting system to the original one. In the case of volume-preserving diffeomorphisms (see §§1.2 and 4), this endeavour is restricted by the limits of the present knowledge concerning the existence of optimal regularity solutions to the prescribed Jacobian partial differential equation (PDE) (which is an open problem in the C^r case, $r \in \mathbb{Z}^+$ [**CDK**, p. 192]).

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(b) The linear dependence of δ on ϵ is established in all perturbation results (with the exception of Theorems 4 and 5 where this is meaningless). Obtaining this dependence is often crucial in applications.

(c) The proofs presented are constructive whenever possible and complete or at least easily completable following the indications in the text.

The pasting technique for divergence-free vector fields was introduced by Arbieto and Matheus in [**AM**]. It is known, however, that the statements and proofs of the main tools [**AM**, §3.1] are not quite correct (concerning the statements, see Warnings 1 and 2 below). Some of the problems have been identified in [**AM2**], but we are unaware of any reference correctly stating this kind of results and providing sound proofs. The writing of the present work was partially stimulated by the author's encounter with that paper.

1.2. *Discrete-time dynamics.* We now turn to the case of volume-preserving diffeomorphisms. To establish in this setting a C^r-perturbation pasting lemma analogous to Theorem 1 seems beyond the techniques presently available (see (a) above and §4), the main difficulty being that the volume-correcting C^r diffeomorphism (playing the analogous role to the divergence-cancelling vector field v in §1.1) must now be reconstructed from a determinant which is only of class C^{r-1} and C^{r-1}-close to one. Nevertheless, using optimal regularity tools with control of support, such a result can actually be established in the Hölder setting, but special care must be taken due to the pathological continuity behaviour of the composition and inversion operators in these functional spaces. This result will be presented in a separate note [**TE2**]. Here, we shall restrict our work to establishing a quite general conservative linearized version of Franks' lemma, an important feature being that the resulting diffeomorphism will have the same $C^{r,\alpha}$ regularity as the original one ($r \in \mathbb{Z}^+$, $0 < \alpha < 1$). As is well known, to achieve the local affine linearization (and not merely the perturbation of the derivative) it is often essential to guarantee the control of the dynamics near the perturbed fixed point or periodic orbit, especially when the perturbed derivative is non-hyperbolic, as was already evident in the original paper [**FR**]. Another important aspect as far as applications are concerned is to establish the linear dependence of the permitted variation δ of the derivative in terms of the required C^1-closeness ϵ to the original diffeomorphism. As in [**FR**], this linear dependence is also established in Theorem 8. It is interesting to compare the later result both with (a) the original Franks' lemma and with (b) the corresponding result for vector fields (Theorem 6). In all three results, the resulting diffeomorphism or vector field has the same regularity as the original one and the linear dependence of δ on ϵ is established, but while (a) and (b) are quite elementary, the Hölder case of Theorem 8 requires the use of optimal regularity tools with control of support and has much deeper roots, ultimately relying on the elliptical regularity solutions to the Poisson problem with Neumann boundary condition and the corresponding Schauder estimates (see [**DM**]). The solution in the C^∞ case is simpler, relying on Moser's elegant yet powerful flow method. In both cases, the starting point is a homothety trick that proved crucial in establishing Avila's regularization [**AV**]. Note, however, that the results in [**DM**] cannot be directly applied in the present context, due to their lack of control of support (see (ii) below). We use instead their counterparts in [**TE**], where this control is achieved (the proofs of the later results follow closely the original ones in [**DM**]). In the dynamical systems literature, Dacorogna–Moser's powerful theorems have been often misinterpreted and naively applied in several ways. As these flaws are somewhat recurrent, it is perhaps not out of place to draw attention to them here.

(i) In [**DM**], it is necessary to assume that the domain Ω is *connected* (besides bounded). This was omitted by lapse in the statements of the propositions, but it is explicitly assumed at the beginning of page 2.

(ii) In [**DM**, Theorem 1'], the solution diffeomorphism φ, in general, does *not* extend by the identity to the whole $\mathbb{R}^n$ in the $C^{k+1,\alpha}$ class, not even when the determinant f equals one in a neighbourhood of $\partial\Omega$. For instance, in order to guarantee that a volume-correcting diffeomorphism acts only inside the region Ω where the volume distortion takes place (i.e., that $\operatorname{supp}(\varphi - \mathrm{Id}) \subset \overline{\Omega}$), one needs instead the corresponding results with control of support as in [**TE**]. An analogous observation holds for the linearized problem $\operatorname{div} u = h$.

(iii) The optimal regularity statements in [**DM**] and [**TE**] with $C^{k,\alpha}$ replaced by C^k, $k \in \mathbb{Z}^+$, have *not* been established in dimension higher than one (being false for $k = 0$ [**CDK**, pp. 192 and 180]).

(iv) Concerning the regularity of the solution diffeomorphism φ in [**DM**, Theorems 1'] when the determinant f is C^∞, see part (B) in the proof of Lemma 2 below.

2. *Conservative pasting, extension, localized smoothing and local linearization of vector fields*

Convention. Throughout this paper, M is a (second countable, Hausdorff) connected orientable closed C^∞ manifold of dimension $n \geq 2$, equipped with a finite atlas $(V_i,\ \phi_i)_{i\leq m}$ and a C^∞ volume form ω. By [**MO**] †, we can assume that the atlas is *conservative*, i.e., on each local chart, ω pushes forward to the canonical volume form on $\mathbb{R}^n$ and $\phi_i(V_i) = \lambda\mathbb{B}^n$, for some constant $\lambda > 0$; μ is the Lebesgue measure induced by ω on M. We may further assume that the atlas is *regular* in the sense that there is a 'larger' conservative atlas $(W_i,\ \Phi_i)_{i\leq m}$ such that $\overline{V_i} \subset W_i$ and $\Phi_i|_{V_i} = \phi_i$. As usual, $\mathbb{B}^n$ is the (open unit) n-ball in Euclidean space and $\mathbb{D}^n = \overline{\mathbb{B}^n}$ is the n-disk.

Given an open set $U \subset M$, denote by $\mathfrak{X}^s(U)$, $s \in \mathbb{Z}^+ \cup \{\infty\}$, the space of vector fields of class C^s defined on U and by $\mathfrak{X}^s_\mu(U)$ the subspace of those that are divergence-free in relation to ω or, equivalently, whose flows preserve μ. As mentioned in the Introduction, in Theorem 1 we consider vector fields Y defined on open sets $U \subset M$, which are C^r-perturbations of $X|_U$, where X is a vector field in $\mathfrak{X}^s_\mu(M)$. To guarantee that the C^r norms of these Y remain finite, we introduce the following definition.

Definition 1. (C^r-bounded) Let $r,\ s \in \mathbb{Z}^+ \cup \{\infty\}$, $r \leq s$. $Y \in \mathfrak{X}^s(U)$ is C^r*-bounded* if Y and all its derivatives up to order r are bounded on U. $\|\cdot\|_{C^r;U}$ is Whitney C^r norm ($\mathbb{N}_0 \ni r \leq s$) on $\mathfrak{X}^s(U)$ (§A.1). When $U = M$, we simply write $\|\cdot\|_{C^r}$.

† As remarked in [**DM**, pp. 4 and 23], the proof given in [**MO**, Lemma 2] is actually for that proposition with both the hypothesis $\operatorname{supp}(g - h) \subset Q$ and the conclusion $\operatorname{supp}(u - \mathrm{Id}) \subset Q$ removed (if $g,\ h$ are the restrictions to the open n-cube Q of two smooth volume forms defined on $\overline{Q}$ and having the same total volume, the proof produces a smooth diffeomorphism u realizing a pullback between them).

We recall the informal description of Theorem 1. *In the volume-preserving setting*, let X be a vector field of class C^r on a closed manifold M and let $U \subsetneq M$ be an open neighbourhood of a compact set K. Given a C^r perturbation Y of the restriction of X to U, it is possible (provided $U \backslash K$ is connected), to C^r-perturb X inside U only, so that the resulting vector field on M still coincides with Y in some neighbourhood of K. One interesting point is that the perturbation can be made C^∞ in the open set where the control over the dynamics is necessarily lost, i.e., on the 'transition annulus' where the conservative 'harmonization' of the two original vector fields takes place (this being the unavoidable cost of bringing together in the same vector field two more or less 'conflicting' dynamics).

THEOREM 1. (C^s conservative pasting with C^r-closeness) *Let M be a manifold as above. Suppose that K is a compact subset with an open neighbourhood $U \subsetneq M$ such that $U \backslash K$ is connected. Then, given $s \in \mathbb{Z}^+ \cup \{\infty\}$ and an integer $1 \leq r \leq s$, there is an open set $K \subset V \subset U$ and a constant $C = C(r, K, U) > 1$ such that, given $X \in \mathfrak{X}^s_\mu(M)$ and a C^r-bounded $Y \in \mathfrak{X}^s_\mu(U)$, there exists $Z \in \mathfrak{X}^s_\mu(M)$ satisfying:*

(1) $Z = Y$ *in* V*;*

(2) $Z = X$ *in a neighbourhood of* U^c*; and*

(3) $\|Z - X\|_{C^r} \leq C\|Y - X\|_{C^r;U}$.

Moreover, V depends only on K and U and not on r, s and one may further require Z to be C^∞ at every point where it neither coincides with X nor with Y.

Actually the proof establishes a considerably more precise result (as usual, $Y \not\equiv X|_U$ means that $Y(x) \neq X(x)$ for some point $x \in U$).

THEOREM 2. *Let M, K, U, r and s be as above. Then, there is a constant $C = C(r, K, U) > 1$ and there are two disjoint compact n-submanifolds Q and S with smoothly diffeomorphic connected boundaries for which $K \subset \operatorname{int} Q$ and $U^c \subset \operatorname{int} S$ and such that, given $X \in \mathfrak{X}^s_\mu(M)$ and a C^r-bounded $Y \in \mathfrak{X}^s_\mu(U)$ such that $Y \not\equiv X|_U$, there exists $Z \in \mathfrak{X}^s_\mu(M)$ satisfying:*

(1) $Z = Y$ *in* Q*;*

(2) $Z = X$ *in* S*;*

(3) Z *is* C^∞ *in* $\Omega = (Q \cup S)^c$, Ω *being* C^∞ *diffeomorphic to* $\partial Q \times \,]0, 1[$*; and*

(4) $\|Z - X\|_{C^r} \leq C\|Y - X\|_{C^r;U}$.

Moreover, Q and S depend only on K and U and not on r, s.

(Note that if $Y \equiv X|_U$, then inequality (4) implies that $Z \equiv X$ on the whole M, and thus one cannot, in general, guarantee the conclusion (3) in this case.)

Remark 1. (Hölder setting) Theorem 1 is still valid in the Hölder setting (i.e., for divergence-free vector fields of class $C^{s,\beta}$ endowed with a possibly lower $C^{r,\alpha}$ norm), the unique exception being that one may require Z to be C^∞ in the set of points where Z neither coincides with X nor with Y essentially only when $r + \alpha < s + \beta$ (smooth maps being, in general, only $C^{r,\rho}$-dense in the class of $C^{r,\alpha}$ maps, $0 < \rho < \alpha \leq 1$; see Theorem 3 below for the notation). We observe that while the previous density remark implies that the analogue of conclusion (3) in Theorem 2 is impossible to obtain when

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$r = s$ and $0 < \alpha = \beta \leq 1$†, the remaining relevant Hölder case $r + \alpha < s + \beta$ is actually free from these constraints. In particular, using the above mentioned (Euclidean space) Hölder density result in place of the C^r-density of C^∞ in C^r, the proof of Theorem 4 immediately yields that $\mathfrak{X}^\infty_\mu(M)$ is $C^{r,\alpha}$-dense in $\mathfrak{X}^{s,\beta}_\mu(M)$ when $r + \alpha < s + \beta$. Note, however, that, *a priori*, this is not enough to obtain the corresponding Hölder version of Theorem 5 (which, in turn, is used to obtain conclusion (3) in Theorem 2 above), as the resulting vector field Z would still be obtained as the limit of a sequence of $C^{s,\beta}$ vector fields, and this sequence is Cauchy only in relation to the lower $C^{r,\alpha}$-norm, which is not enough to ensure that Z belongs to the higher class $C^{s,\beta}$, as required. Nevertheless, this problem can be overcome by a simple lower semicontinuity reasoning: using [**GT**, (7.14), p. 148 and Lemma 7.3, p. 150] one sees that modifying the proof of Theorem 4 as explained above, the sequence Z_k of smooth, divergence-free vector fields $C^{r,\alpha}$-converging to X has $C^{s,\beta}$ norm uniformly bounded by that of X times a constant. Now, carrying on the proof of Theorem 5 using these smooth approximations to X, it is immediate to check that an analogous uniform boundedness of the $C^{s,\beta}$ norms also holds for all the auxiliary functions and vector fields involved in the construction of the Cauchy sequence Z_k (the universality of the operator Φ in Lemma 1 being essential here). This finally yields that the $C^{s,\beta}$ norms of the vector fields in this sequence are still uniformly bounded by the $C^{s,\beta}$ norm of X times a constant (which is independent of X). This guarantees that the limit vector field Z actually belongs to the $C^{s,\beta}$ class by lower semicontinuity (see, e.g., [**CDK**, p. 358]). We finally observe that the existence of manifolds Q and S satisfying (1)–(3) as in Theorem 2 also holds for Theorem 3, except that (as explained above) one cannot guarantee Z to be C^∞ in Ω when $r = s$ and $0 < \alpha = \beta \leq 1$.

In §3.2 we briefly outline the few changes needed in the proof of Theorem 1 to obtain Theorem 3. There, it is also explained why constant C actually does not depend on the Hölder exponent α, but only on r, K and U.

Given an open set $U \subset M$, $s \in \mathbb{Z}^+$ and $0 < \beta \leq 1$, $\mathfrak{X}^{s,\beta}(U)$ is the subspace of $\mathfrak{X}^s(U)$ consisting of vector fields Y such that, on local charts, each partial derivative of Y of order s is β-Hölder continuous (these derivatives being functions from $\phi_j(V_j \cap U)$ into $\mathbb{R}^n$). One sets $C^{s,0} := C^s$ and $C^{\infty,\beta} := C^\infty$.

THEOREM 3. ($C^{s,\beta}$ conservative pasting with $C^{r,\alpha}$-closeness) *Let M be a manifold as above. Suppose that K is a compact subset with an open neighbourhood $U \subsetneq M$ such that $U \backslash K$ is connected. Then, given $s \in \mathbb{Z}^+ \cup \{\infty\}$, $0 \leq \alpha, \beta \leq 1$, and an integer $1 \leq r \leq s$ such that $r + \alpha \leq s + \beta$, there is an open set $K \subset V \subset U$ and a constant $C = C(r, K, U) > 1$ such that, given $X \in \mathfrak{X}^{s,\beta}_\mu(M)$ and a C^r-bounded $Y \in \mathfrak{X}^{s,\beta}_\mu(U)$, there exists $Z \in \mathfrak{X}^{s,\beta}_\mu(M)$ satisfying:*

(1) *$Z = Y$ in V;*

(2) *$Z = X$ in a neighbourhood of U^c; and*

(3) $\|Z - X\|_{C^{r,\alpha}} \leq C\|Y - X\|_{C^{r,\alpha};U}$.

† The case $r + \alpha = s + \beta$ splits into three subcases: (a) the one just mentioned; (b) $r = s$, $\alpha = \beta = 0$, which is the C^r case (Theorem 1); and (c) $s = r + 1$, $\alpha = 1$, $\beta = 0$, which again reduces to Theorem 1, the norms $C^{r,1}$ and C^{r+1} being equivalent [**CDK**, p. 342]. Thus, only (a) is 'Hölder relevant'.

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Moreover, V depends only on K and U and, with the exception of the case $r = s$ and $0 < \alpha = \beta \leq 1$, one may further require Z to be C^∞ at every point where it neither coincides with X nor with Y.

Warning 1. It should be stressed that if $U \backslash K$ is *not* connected, then cobordism constraints might occur making (in general) impossible the conservative pasting of vector fields X and Y as stated in Theorems 1, 2 and 3 (by *any* method and under *any* regularity assumptions; see Example 1). If $U \backslash K$ fails to be connected, a conservative C^r perturbation Y of the restriction of X to U may actually fail to have a divergence-free extension to the whole M, even if the C^r closeness condition is dropped (Example 2).

Example 1. Represent the flat 2-torus as $M = \mathbb{S}^1 \times (\mathbb{R}/\mathbb{Z})$ with coordinates (s, z) and endow it with the standard volume form. Let X be the vertical vector field $\partial/\partial z$ and consider its ϵ-C^∞ perturbation $Y = (1 + \epsilon)(\partial/\partial z)$, $\epsilon > 0$. Then, there is no $Z \in \mathfrak{X}^1_\mu(M)$ such that (a) $Z = Y$ in $K = \mathbb{S}^1 \times 1/2$ and (b) $Z = X$ in $\gamma = \mathbb{S}^1 \times 1$. The vector fields X and Y have a different flux across the cobordant circles $\mathbb{S}^1 \times z$, and thus the divergence of Z cannot identically vanish inside either of the two annuli bounded by K and γ.

Example 2. Extend the annulus $U = \mathbb{S}^1 \times]-1, 1[\subset \mathbb{R}^3$ to a smoothly embedded 2-sphere S that is invariant under rotation about the z-axis and endowed with the canonical volume form inherited from $\mathbb{R}^3$. Endow S with the rotation vector field $X : (x, y, z) \mapsto (-y, x, 0)$ and consider the ϵ-C^∞ perturbation of the restriction of X to U given by $Y = (-y, x, \epsilon)$, $\epsilon > 0$. Both X and Y are divergence-free but there is no C^1 divergence-free extension of Y to the whole S, as the flux of Y across the boundary circle $\gamma = \mathbb{S}^1 \times 0$ is not zero, the divergence being necessarily positive around some point of the southern hemisphere.

Warning 2. The dependence of constant C on r, K and U is obviously unavoidable, whatever the method employed to achieve the pasting of the vector fields. For instance, given a point $p \in M$, in some local chart set $K = \{p\}$ and $U = B_d(p)$, a small open ball whose closure is contained in the chart. Since $d = \text{dist}(K, U^c)$, the mean value theorem then implies that $C = C(r, K, U) > d^{-r}$. In general, and by the same reason, for K and U as in Theorem 1, a 'thin' $U \backslash K$ implies a quite large C. More precisely, assume for the moment that M is endowed with a Riemannian structure inducing an intrinsic metric (this structure is actually unnecessary for the results obtained here). Suppose that, for each $\epsilon > 0$, U_ϵ is an open neighbourhood of K contained in $B_\epsilon(K)$ with $U_\epsilon \backslash K$ connected. Then $C(r, K, U_\epsilon) \to \infty$ as $\epsilon \to 0$. To get an idea of how the 'geometry' of $U \backslash K$ tends to have an impact on the size of C, and in the specific context of the method employed here to solve equation div $v = h$, observe that, roughly speaking, the 'thinner' and possibly more 'convoluted' the image of $U_\epsilon \backslash K$ on the atlas as ϵ tends to zero, the larger the number $N + 1$ of small cubes U_j needed to achieve the covering

$$\overline{\Omega_1} \subset \bigcup_{j=0}^{N} U_j \subset \Omega \subset U_\epsilon \backslash K$$

(see the proof of Theorem 1), and a large number of small cubes contributes to C with a very large multiplicative factor (the smaller the cubes the larger this factor becomes; see,

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in particular, §3(c) and Footnote † on page 20). Together, this and the previous warning impose double caution on the use of the pasting lemma to attempt general perturbations of divergence-free vector fields with *a priori* unspecified support (however, see Theorem 6). As stated, with δ independent of K and U, Theorem 3.1 in [**AM**] contradicts the mean value theorem, assuming, as implicit, that W^c is non-empty (in the paper's notation).

2.1. *Conservative localized smoothing and extension.* The proof of the next result corrects and generalizes that of [**AM**, Theorem 2.2]. It provides a short alternative way to establish Zuppa's regularization theorem [**ZU**] without the need to introduce a Riemannian structure on the manifold.

THEOREM 4. *Let M be a manifold as above and let $r \in \mathbb{Z}^+$. Then*

$$\mathfrak{X}^\infty_\mu(M) \text{ is } C^r\text{-dense in } \mathfrak{X}^r_\mu(M).$$

Proof. Let $(V_i, \phi_i)_{i \le m}$ be the atlas of M. There is no difficulty in finding a partition of unity $\xi_{i\le m}$ subordinate to $V_{i\le m}$ with $\xi_1 = 1$ in $\phi_1^{-1}((2\lambda/3)\mathbb{D}^n)$ (see the convention above). Let $X_i = (X_i^1, \ldots, X_i^n)$, $i \le m$, be the expressions of $X \in \mathfrak{X}^r_\mu(M)$ in the local charts. Since the atlas is regular (see the convention in §2), using convolutions one can find, for each i, a sequence X_{ik} of smooth vector fields on $\phi_i(V_i) = \lambda\mathbb{B}^n$ C^r-converging to X_i. Observe that, as X_i, each X_{ik} is divergence-free (in relation to the standard volume form on $\mathbb{R}^n$), since the convolution operator $*$ is bilinear and satisfies $\partial_j(\rho * X_i^j) = \rho * \partial_j X_i^j$. To simplify the notation, one still denotes by X_{ik} the pullback $\phi_i^*(X_{ik})$. Define the smooth vector field on M,

$$Y_k = \sum_{i\le m} \xi_i X_{ik},$$

setting $\xi_i X_{ik} := 0$ in V_i^c. Since $\sum_{i\le m} \xi_i = 1$, the estimate for the $|\cdot|_r$ norm of the product (end of §A.1) gives

$$|Y_k - X|_r = \left|\sum_{i\le m} \xi_i (X_{ik} - X)\right|_r \le m2^r \max_{i\le m} |\xi_i|_r \max_{i\le m} |X_{ik} - X|_{V_i}|_r. \tag{2.1}$$

Since X and the X_{ik} are divergence-free in M and V_i, respectively, and ξ_i is compactly supported inside V_i,

$$\operatorname{div} Y_k = \operatorname{div} Y_k - \operatorname{div} X = \operatorname{div}(Y_k - X) = \sum_{i\le m;\, j\le n} (\partial_j \xi_i)(X_{ik}^j - X^j)$$

and

$$|\operatorname{div} Y_k|_r \le mn2^r \max_{i\le m} |\xi_i|_{r+1} \max_{i\le m} |X_{ik} - X|_{V_i}|_r. \tag{2.2}$$

Since the norms $|\cdot|_r$ and $\|\cdot\|_{C^r}$ are equivalent, (§A.1) we work with the former. From (2.1) and (2.2), it follows that

$$|Y_k - X|_r, |\operatorname{div} Y_k|_r \xrightarrow[k\to\infty]{} 0 \text{ since } X_{ik} \xrightarrow[k\to\infty]{C^r} X|_{V_i}. \tag{2.3}$$

Now, $\operatorname{div} Y_k = 0$ in $\mathcal{D} = \phi_1^{-1}((2\lambda/3)\mathbb{D}^n)$, since, $\xi_1|_{\mathcal{D}} = 1$ and thus $Y_k = X_{1k}$ in this set. Let $\Omega = M \backslash \phi_1^{-1}((\lambda/3)\mathbb{D}^n)$ and $\Omega_1 = M \backslash \phi_1^{-1}((\lambda/2)\mathbb{D}^n)$. Let $h_k = \operatorname{div} Y_k$. Clearly,

$\overline{\Omega_1} \subset \Omega$ and supp $h_k \subset \Omega_1$. Observe that Ω is connected since M and $\partial\Omega$ (diffeomorphic to $\mathbb{S}^{n-1}$) are both connected, and the same holds for Ω_1. Moreover, $\int_\Omega h_k\omega = 0$ (ω being the volume form on M), since, by the divergence theorem,

$$\int_\Omega (\operatorname{div} Y_k)\omega = \int_{\partial\Omega} Y_k \lrcorner\, \omega = -\int_{\partial\mathcal{B}} X_{1k} \lrcorner\, \omega = -\int_{\mathcal{B}} (\operatorname{div} X_{1k})\omega = 0,$$

where $\mathcal{B} = \phi_1^{-1}((\lambda_1/3)\mathbb{B}^n)$. Now, by Lemma 1 (below), there is a constant $C = C(r, \Omega_1, \Omega) > 0$ and $v_k \in \mathfrak{X}^\infty(M)$ such that

$$\begin{cases} \operatorname{div} v_k = h_k, \\ \operatorname{supp} v_k \subset \Omega, \\ |v_k|_r \leq C|h_k|_r. \end{cases} \tag{2.4}$$

Let $Z_k = Y_k - v_k$. Then, $Z_k \in \mathfrak{X}^\infty_\mu(M)$ and, finally, by (2.3) and (2.4),

$$|Z_k - X|_r \leq |Y_k - X|_r + |v_k|_r \xrightarrow[k\to\infty]{} 0. \qquad \square$$

At least for certain open sets $\Omega \subset M$†, which turn out to be useful in many important situations, one may conservatively C^r-perturb a divergence-free vector field X in order to make it smooth inside Ω, while keeping X unchanged on the complement of Ω. This result has the advantage of avoiding the occurrence of a 'transition annulus', where typically Z is neither smooth nor coincides with X. If, for instance, one needs to perform a preliminary conservative C^r perturbation of a vector field X in order to increase its regularity, it may be actually be possible to smooth it just where this is really needed for the construction of the subsequent perturbations (e.g., on small open neighbourhoods of certain periodic orbits in dimension $n \geq 3$), while keeping X unchanged on the complement of that set. The advantages in terms of dynamical control are evident.

Given a compact n-submanifold $N \subset M$ ($n = \dim M$) with $C^{r\geq 2}$ boundary, one may construct a C^{r-1} vector field transverse to ∂N and pointing inward, which, in turn, defines a C^{r-1} collar embedding $\zeta : \partial N \times [0, \infty[\hookrightarrow N$, $\zeta(x, 0) = x$. For each $\epsilon > 0$, $\zeta(\partial N \times [0, \epsilon])$ is a (compact C^{r-1}) *collar of* ∂N.

THEOREM 5. (Conservative localized smoothing—special case) *Let M be a manifold as above and let $N \subset M$ be a compact n-submanifold with connected C^3 boundary. Let Ω be either the interior of N or the interior of a (compact C^2) collar of ∂N. Given $X \in \mathfrak{X}^r_\mu(M)$, $r \in \mathbb{Z}^+$, there exists $Z \in \mathfrak{X}^r_\mu(M)$ as C^r-close to X, as desired, satisfying:*

(1) *Z is C^∞ in Ω; and*

(2) *$Z = X$ in Ω^c.*

Proof. *Case $\Omega = \operatorname{int} N$.* Since the norms $|\cdot|_r$ and $\|\cdot\|_{C^r}$ are equivalent, we work with the former. Fix a C^2 collar embedding

$$\zeta : \partial N \times [0, \infty[\hookrightarrow N.$$

† In the preprint arXiv:1611.01694v3 to this paper, it was stated without proof (unnumbered theorem on p. 8) that Theorem 5 below still holds for arbitrary open sets $\Omega \subset M$. It turned out that the proof known to the author contained an error. Therefore, and to the best of our knowledge, the general case remains, so far, conjectural.

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Consider the open covering of $\Omega = \text{int}\, N$ by overlapping 'annuli' given by

$$\begin{cases} \Lambda_0 = \Omega \backslash \zeta(\partial N \times]0, \frac{1}{3}]), \\ \Lambda_k = \zeta\left(\partial N \times \left]\dfrac{1}{2k+3}, \dfrac{1}{2k}\right[\right), \quad k \geq 1, \end{cases}$$

and fix a smooth partition of unity $\xi_{k\geq 0}$ of Ω subordinate to it (Λ_0 is actually a C^2-isotopic copy of Ω). Let

$$\Omega_k = \Lambda_k \cap \Lambda_{k+1}.$$

Note that $\xi_k + \xi_{k+1} = 1$ in Ω_k by subordination to the covering (we suggest to the reader the drawing of a figure). Given $X \in \mathfrak{X}^r_\mu(M)$ and $\epsilon > 0$, we shall construct a sequence $Z_{k\geq 0} \in \mathfrak{X}^r_\mu(M)$ such that, for $k \geq 0$:

(1) $Z_{k+1} = Z_k$ in Λ^c_{k+1};
(2) Z_k is C^∞ in $(\Lambda_0 \cup \cdots \cup \Lambda_k) \backslash \Omega_k$;
(3) $Z_k = X$ in $(\Lambda_0 \cup \cdots \cup \Lambda_k)^c$; and
(4) $|Z_0 - X|_r < \epsilon/2$ and $|Z_{k+1} - Z_k|_r < \epsilon/2^{k+2}$.

It follows that Z_k is a Cauchy sequence converging to $Z \in \mathfrak{X}^r_\mu(M)$ in the Banach space $\mathfrak{X}^r_\mu(M)$, satisfying:

- Z is C^∞ in $\Omega = \bigcup^\infty \Lambda_k$;
- $Z = X$ in $\Omega^c = \bigcap^\infty (\Lambda_0 \cup \cdots \cup \Lambda_k)^c$; and
- $|Z - X|_r < \epsilon$.

(A) *Construction of* Z_0. Let $\widehat{Z_0} = \widehat{\xi}_1 X + \xi_0 X_0 \in \mathfrak{X}^r(M)$, where $\widehat{\xi}_1 = \xi_1$ in Λ_0 and $\widehat{\xi}_1 = 1$ elsewhere. Here $X_0 \in \mathfrak{X}^\infty_\mu(M)$ is a vector field whose C^r-closeness to X will be determined below. Note that $\widehat{Z_0}$ is divergence-free in Ω^c_0. Actually, by subordination of the partition to the covering, there is an open set

$$\Omega^*_0 = \zeta(\partial N \times]\tfrac{1}{3} + \delta_0, \tfrac{1}{2} - \delta_0[),$$

where $0 < \delta_0 < 1/12$, such that $\overline{\Omega^*_0} \subset \Omega_0$ and $\text{supp}\, h_0 \subset \Omega^*_0$ for $h_0 := \text{div} \widehat{Z_0}$. Observe that Ω_0 and Ω^*_0 are connected (∂N being connected) with C^2 boundary, and thus, by the divergence theorem [**LA**, p. 203],

$$\begin{aligned} \int_{\Omega_0} h_0 \omega = \int_{\partial\Omega_0} \widehat{Z_0} \lrcorner\, \omega &= -\int_{\partial N_0} X_0 \lrcorner\, \omega + \int_{\partial N^*_0} X \lrcorner\, \omega \\ &= -\int_{\text{int}\, N_0} (\text{div}\, X_0)\omega + \int_{\text{int}\, N^*_0} (\text{div}\, X)\omega \\ &= -0 + 0 = 0 \end{aligned}$$

since $\partial\Omega_0 = \partial N_0 \sqcup \partial N^*_0$, where for $k \geq 0$,

$$N_k = \Omega \backslash \zeta\left(\partial N \times \left]0, \frac{1}{2k+2}\right[\right) \quad \text{and} \quad N^*_k = \Omega \backslash \zeta\left(\partial N \times \left]0, \frac{1}{2k+3}\right[\right)$$

are manifolds that are C^2-isotopic to N. By Lemma 1 (below), there is a constant $C = C(r, \Omega^*_0, \Omega_0) > 0$ and $v_0 = \Phi(h_0) \in \mathfrak{X}^r(M)$ such that

$$\begin{cases} \text{div}\, v_0 = h_0, \\ \text{supp}\, v_0 \subset \Omega_0, \\ |v_0|_r \leq C|h_0|_r. \end{cases}$$

Then

$$Z_0 = \widehat{Z_0} - v_0 = \widehat{Z_0} - \Phi(\operatorname{div} \widehat{Z_0}) \in \mathfrak{X}^r_\mu(M)$$

is C^∞ in $\Lambda_0 \backslash \Omega_0$ and $Z_0 = X$ in Λ_0^c. Moreover, it is easily seen that if $|X_0 - X|_r$ is small, then $|\widehat{Z_0} - X|_r$, $|h_0|_r$ and, consequently, $|v_0|_r$ are all small (see §2.3 below), and hence for X_0 sufficiently C^r-close to X,

$$|Z_0 - X|_r = |\widehat{Z_0} - v_0 - X|_r \leq |\widehat{Z_0} - X|_r + |v_0|_r < \epsilon/2.$$

(B) *Construction of* Z_1. Let

$$\widehat{Z_1} = \widehat{\xi}_2 X + \xi_1 X_1 + \xi_0 X_0 \in \mathfrak{X}^r(M),$$

where $\widehat{\xi}_2 = \xi_2$ in $\Lambda_0 \cup \Lambda_1$ and $\widehat{\xi}_2 = 1$ elsewhere. Again, $X_1 \in \mathfrak{X}^\infty_\mu(M)$ is a vector field whose C^r-closeness to X is to be specified. Now:

(a) $\widehat{Z_1}$ is divergence-free in $(\Omega_0 \cup \Omega_1)^c$;
(b) $\widehat{Z_1} = Z_0$ in Λ_1^c;
(c) $\widehat{Z_1}$ is C^∞ in $(\Lambda_0 \cup \Lambda_1) \backslash \Omega_1$; and
(d) $\widehat{Z_1} = X$ in $(\Lambda_0 \cup \Lambda_1)^c$.

Using Lemma 1, we proceed exactly as in (A) to eliminate the divergence of $\widehat{Z_1}$ inside Ω_0 and Ω_1, while keeping this vector field unchanged in $(\Omega_0 \cup \Omega_1)^c$, and thus obtaining $Z_1 \in \mathfrak{X}^r_\mu(M)$ as C^r-close to Z_0 as desired and still satisfying (b)–(d) above (to establish $\int_{\Omega_1} h_1 \omega = 0$, where $h_1 = \operatorname{div} \widehat{Z_1}|_{\Omega_1}$, we now use $\partial \Omega_1 = \partial N_1 \sqcup \partial N_1^*$ in order to apply the divergence theorem).

As $\widehat{Z_1} = Z_0$ in Λ_1^c and $\widehat{Z_1} = X_1$ and $Z_0 = X$ in $\Lambda_1 \backslash (\Omega_0 \cup \Omega_1)$, and since we can take X_1 as C^r-close to X as desired, we need only to guarantee that Z_1 is as C^r-close to Z_0 as wished in $\Omega_0 \cup \Omega_1$. With Ω_1 there is no concern, the situation being exactly the same as in (A). To see that for X_1 C^r-close to X one has Z_1 C^r-close to Z_0 in Ω_0, we use the linearity of the operator $\Phi : h \mapsto v$ in Lemma 1. First, note that since X_0 and X_1 are both smooth and in Ω_0, we have $\widehat{Z_1} = \xi_1 X_1 + \xi_0 X_0$, and then, in Ω_0,

$$Z_1 = \widehat{Z_1} - \Phi(\operatorname{div} \widehat{Z_1}) = \xi_1 X_1 + \xi_0 X_0 - \Phi(\operatorname{div}(\xi_1 X_1 + \xi_0 X_0))$$

is also smooth. On the other hand, using the linearity of the divergence and that of the operator Φ, writing $X_1 = X + (X_1 - X)$, we have, in Ω_0,

$$Z_1 = A + B,$$

where

$$A = \xi_1 X + \xi_0 X_0 - \Phi(\operatorname{div}(\xi_1 X + \xi_0 X_0))$$

and

$$B = \xi_1 (X_1 - X) - \Phi(\operatorname{div}(\xi_1 (X_1 - X))).$$

Now, since $\widehat{\xi}_1 = \xi_1$ in Ω_0, one has

$$A = \widehat{Z_0} - \Phi(\operatorname{div} \widehat{Z_0}) = Z_0,$$

while (on local charts)

$$B = \xi_1 (X_1 - X) - \Phi\bigg(\sum_{i \leq n} \partial_i \xi_1 (X_1^i - X^i) \bigg)$$

is C^r-small if $|X_1 - X|_r$ is small. Therefore, $(Z_1 - Z_0)|_{\Omega_0}$ is as C^r-small as wished provided $|X_1 - X|_r$ is small enough.

(C). *Construction of* Z_k, $k \geq 2$. Proceeding exactly in the same way as in (B), we let

$$\widehat{Z_k} = \widehat{\xi}_{k+1} X + \xi_k X_k + \cdots + \xi_0 X_0 \in \mathfrak{X}^r(M),$$

where $X_k \in \mathfrak{X}^\infty_\mu(M)$ is as C^r-close to X as needed below and

$$\widehat{\xi}_{k+1} = \begin{cases} \xi_{k+1} & \text{in } \Lambda_0 \cup \cdots \cup \Lambda_k, \\ 1 & \text{elsewhere}, \end{cases}$$

and then cancel the divergence inside Ω_k and Ω_{k-1} using Lemma 1. Reasoning as in (B), we need only to guarantee that Z_k is as C^r-close to Z_{k-1} as wished in Ω_{k-1}. Again, the fact that $\widehat{\xi}_k$ and ξ_k coincide in Ω_{k-1} guarantees that, in this set,

$$Z_k = \widehat{Z_k} - \Phi(\operatorname{div} \widehat{Z_k}) = Z_{k-1} + B,$$

where B is C^r-small if $|X_k - X|_r$ is small, and, consequently, as in (B), $|Z_k - Z_{k-1}|_{r;\Omega_{k-1}}$ is as small as desired and it straightforward to verify that Z_k satisfies (1)–(4) above.

Case $\Omega =$ *interior of a collar*. The proof is the one given above, modulo the following simple change: we fix a C^2 compact collar embedding $\zeta : \partial N \times [0, \epsilon] \hookrightarrow N$ and define, as in the previous case, a sequence Λ_k of overlapping 'annuli' now indexed by $\mathbb{Z}$, forming an open cover of $\Omega = \zeta(\partial N \times]0, \epsilon[)$, with Λ_k approaching ∂N and $\zeta(\partial N \times \epsilon)$ as k tends to ∞ and $-\infty$, respectively. The construction is then essentially the same, noting that the hypersurfaces $\zeta(\partial N \times \delta)$, $\delta \in]0, 1[$, are of class C^2, and thus the divergence theorem applies when needed. □

The next result shows that if in Theorem 3 we want to have Z satisfying (1) and (3) but are not particularly interested in having (2) $Z = X$ in U^c, then the regularity of Z can be increased to that of Y and it can actually be made C^∞ in U^c.

COROLLARY 1. ($C^{s,\beta}$ conservative extension with C^r-closeness) *Let M be a manifold as above. Suppose that K is a compact subset with an open neighbourhood $U \subsetneq M$ such that $U \backslash K$ is connected. Then, given $s \in \mathbb{Z}^+ \cup \{\infty\}$, $0 \leq \beta \leq 1$, and an integer $1 \leq r \leq s$, there is an open set $K \subset V \subset U$ and a constant $C = C(r, K, U) > 1$ (that of Theorem 3) such that, given $X \in \mathfrak{X}^r_\mu(M)$ and a C^r-bounded $Y \in \mathfrak{X}^{s,\beta}_\mu(U)$ such that $Y \not\equiv X|_U$, there exists $Z \in \mathfrak{X}^{s,\beta}_\mu(M)$ satisfying:*

(1) *$Z = Y$ in $\overline{V}$;*

(2) *Z is C^∞ in a neighbourhood of U^c; and*

(3) *$\|Z - X\|_{C^r} \leq C\|Y - X\|_{C^r;U}$.*

Furthermore, if $\beta = 0$, then Z is C^∞ in $\overline{V}^c$.

Proof. Fix $\widehat{X} \in \mathfrak{X}^\infty_\mu(M)$ such that

$$\|\widehat{X} - X\|_{C^r} \leq \frac{1}{2C}\|Y - X\|_{C^r;U}, \tag{2.5}$$

where $C = C(r, K, U) > 1$ is the constant given in Theorem 3. By the observation preceding that result, there is a compact n-submanifold $Q \subset U$ with smooth connected

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boundary such that $K \subset \operatorname{int} Q$ and a vector field $Z_0 \in \mathfrak{X}^{s,\beta}_\mu(M)$ such that $Z_0 = Y$ in Q, $Z_0 = \widehat{X}$ in a neighbourhood of U^c and

$$\|Z_0 - \widehat{X}\|_{C^r} \le C\|Y - \widehat{X}\|_{C^r;U}.$$

By Remark 4 (§3.1), we may replace constant C by $C - 1$ in the inequality above and get

$$\|Z_0 - \widehat{X}\|_{C^r} \le (C-1)\|Y - \widehat{X}\|_{C^r;U} \le (C-1)(\|Y - X\|_{C^r;U} + \|X - \widehat{X}\|_{C^r}).$$

Combining with (2.5),

$$\begin{aligned}\|Z_0 - X\|_{C^r} &\le \|Z_0 - \widehat{X}\|_{C^r} + \|\widehat{X} - X\|_{C^r}\\ &\le (C - 1/2)\|Y - X\|_{C^r;U}.\end{aligned}$$

Let $V = \operatorname{int} Q$. If $\beta > 0$, then $Z = Z_0$ is the desired vector field. If $\beta = 0$, we get Z as wished by applying Theorem 5 to $Z_0 \in \mathfrak{X}^s_\mu(M)$ and $\Omega = \overline{V}^c$, the interior of $N = V^c$, a compact n-submanifold with smooth connected boundary $\partial N = \partial Q$. □

2.2. *Conservative local linearization.* Theorem 3 can be also used to prove that a divergence-free vector field can be conservatively C^1-perturbed to become linearized near $x \in M$, the perturbation support being a neighbourhood of x as small as desired. Although the main application occurs when the points of Σ are singularities of v, we formulate it in the general case. Special care has been taken to find a δ that directly estimates the permitted variation of the derivative on all local charts. Observe that, given $\epsilon > 0$, the same δ (depending linearly on ϵ) works simultaneously for all divergence-free vector fields on M in all classes of regularity (cf. Theorem 7 below).

THEOREM 6. ($C^{s,\beta}$ conservative local linearization—'Franks' lemma type') *Let M be a manifold as above. Then there is a constant $\chi > 0$ (depending only on the atlas of M) such that, given:*

- *any $\epsilon > 0$;*
- *any $v \in \mathfrak{X}^{s,\beta}_\mu(M)$, $s \in \mathbb{Z}^+ \cup \{\infty\}$, $0 \le \beta \le 1$;*
- *any finite set $\Sigma \subset M$;*
- *any neighbourhood U of Σ; and*
- *any traceless linear maps $A_x \in L(n, \mathbb{R})$, $x \in \Sigma$, satisfying*

$$\|A_x - Dv(x)\| < \chi\epsilon,$$

where $Dv(x)$ is taken in some (reindexed) local chart (V_x, ϕ_x) around x, there exists $Z \in \mathfrak{X}^{s,\beta}_\mu(M)$ satisfying:

(1) *for each $x \in \Sigma$, on local chart (V_x, ϕ_x), $Z(y) = v(x) + A_x(y - x)$ near x;*

(2) *$Z = v$ in U^c; and*

(3) *$\|Z - v\|_{C^1} < \epsilon$.*

Remark 2. (1) Implies, for each $x \in \Sigma$, that $Z(x) = v(x)$, and on local chart (V_x, ϕ_x), $DZ(x) = A_x$ and Z is affine linear near x.

Preview of proof. The attentive reader will notice that Theorem 6 is *not* a particular case of Theorem 3. The result easily reduces to the case when Σ consists of a single point.

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The problem is obviously a local one, the construction being carried out on some chosen local chart (performing a translation, we may assume that $x = 0$). Instead of trying to prove directly that, for any traceless $A \in L(n, \mathbb{R})$ sufficiently close to $Dv(0)$, pasting adequately $Y(y) = v(0) + A(y)$ to v on a sufficiently small neighbourhood U of x (using Theorem 3) we can get a divergence-free vector field C^1 close to v, with the inherent problem of controlling the growth of constant $C = C(1, K, U)$ as U 'blows down' to x, we proceed differently and rescale to the open unit ball $\mathbb{B}^n$, the restrictions of vector fields Y and v to arbitrarily small balls $\lambda\mathbb{B}^n$ (under the action of homotheties $\Phi_\lambda = \lambda^{-1}\mathrm{Id}$). Observing that the C^1 norm of the vector field

$$Y_\lambda - v_\lambda = \Phi_{\lambda_*}(Y - v) \in \mathfrak{X}_\mu^{s,\beta}(\mathbb{B}^n)$$

tends to $\|A - Dv(0)\|$ as $\lambda \to 0$, we perform the pasting on this constant scale, with fixed K, U and $C = C(1, K, U)$ and then pull back (scale down) the resulting vector field to the original real scale, i.e., to a sufficiently small ball $\lambda\mathbb{B}^n$, finally extending it by v to the whole M, the non-increasing behaviour of the C^1 norm under the action of homothetic contractions guaranteeing the desired conclusion.

Remark 3. In the proof of Theorem 6, we will need to apply Theorem 3 with M being an open ball $\eta\mathbb{B}^n \subset \mathbb{R}^n$. Obviously, Theorem 3 remains valid if the manifold M is instead a connected open subset of $\mathbb{R}^n$ equipped with the trivial one chart atlas (M, Id) and both $X, Y \in \mathfrak{X}_\mu^{s,\beta}(M)$ are C^r-bounded (see Definition 1), where μ is the Lebesgue measure induced by the the canonical volume on $\mathbb{R}^n$.

Proof of Theorem 6. Choose a local chart around each $x \in \Sigma$ and fix on it a small closed ball $\overline{B_x}$ centred at x (we identify x with its image on the chart), so that these balls have disjoint preimages on M and are contained in U. Changing U by the union of the interiors of these $\#\Sigma$ balls, it is immediate that the proof reduces to the case of Σ consisting of a single point x. Let $d = d(1, \max_{i,j\leq m}\|\phi_{ji}\|_{C^2}) \geq 1$ be the constant controlling the potential magnification of the local C^1 norm of a vector field under the chart transitions of the atlas (see §3.1(c)). Get constant $C = C(1, \frac{1}{3}\mathbb{D}^n, \frac{2}{3}\mathbb{B}^n)$ given by Theorem 3 for $M = \mathbb{B}^n$, taking Remark 3 into consideration, and let $\chi = 1/(Cd)$. Take a local chart (W, ϕ) around x. Performing a translation, we may assume that $\phi(x) = 0 \in \mathbb{R}^n$. Take $\eta > 0$ such that $\eta\mathbb{D}^n \subset \phi(W)$ and $\phi^{-1}(\eta\mathbb{D}^n) \subset U$. To simplify the notation we still denote by v the vector field $\phi_* v|_W \in \mathfrak{X}_\mu^{s,\beta}(\phi(W))$ (recall that the atlas is regular (see the convention in §2), and hence this local chart expression of v is C^1-bounded; μ is now the Lebesgue measure on $\mathbb{R}^n$). Fix any traceless $A \in L(n, \mathbb{R})$ such that

$$\|A - Dv(0)\| < \chi\epsilon$$

(recall that $\phi(x) = 0$ and $Dv(0)$ is taken on local chart (W, ϕ)). Define, on $\eta\mathbb{B}^n$,

$$Y(y) = v(0) + A(y) - v(y).$$

HOMOTHETY TRICK (Step 1). *Rescaling to the unit scale.* For each $0 < \lambda < \min(1, \eta)$, rescale $Y|_{\lambda\mathbb{B}^n}$ to the unit ball $\mathbb{B}^n$

$$Y_\lambda = (\lambda^{-1}\mathrm{Id})_* Y|_{\lambda\mathbb{B}^n} \in \mathfrak{X}_\mu^{s,\beta}(\mathbb{B}^n).$$

Claim. $\|Y_\lambda\|_{C^1;\mathbb{B}^n} \xrightarrow[\lambda\to 0]{} \|A - Dv(0)\|$.

Recall that $\|Y_\lambda\|_{C^1;\mathbb{B}^n} = \max(\|Y_\lambda\|_{C^0;\mathbb{B}^n}, \sup_{x\in\mathbb{B}^n}\|DY_\lambda\|)$.

(a) The derivative is unchanged by the action of the homothety,

$$DY_\lambda(y) = DY(\lambda y) = A - Dv(\lambda x) \quad \text{for all } y \in \mathbb{B}^n$$

and therefore, since v is C^1,

$$\sup_{y\in\mathbb{B}^n}\|DY_\lambda\| = \sup_{y\in\lambda\mathbb{B}^n}\|DY\| \xrightarrow[\lambda\to 0]{} \|A - Dv(0)\|.$$

(b) As for the C^0 norm,

$$\|Y_\lambda\|_{C^0;\mathbb{B}^n} = \lambda^{-1}\|Y\|_{C^0;\lambda\mathbb{B}^n} \xrightarrow[\lambda\to 0]{} \|A - Dv(0)\|$$

since

$$\lambda^{-1}\|Y\|_{C^0;\lambda\mathbb{B}^n} = \sup_{y\in\lambda\mathbb{B}^n}\lambda^{-1}|v(0) + A(y) - v(y)|$$

$$= \sup_{y\in\lambda\mathbb{B}^n}\left|\frac{v(0) + Dv(0;\, y) - v(y)}{\lambda} + \frac{A(y) - Dv(0;\, y)}{\lambda}\right| \xrightarrow[\lambda\to 0]{} \|A - Dv(0)\|$$

as is immediate to verify: the fraction on the left converges to $0 \in \mathbb{R}^n$ as $\lambda \to 0$, while

$$\sup_{y\in\lambda\mathbb{B}^n}\frac{|A(y) - Dv(0;\, y)|}{\lambda} = \sup_{y\in\mathbb{B}^n}|A(y) - Dv(0;\, y)| = \|A - Dv(0)\|.$$

Therefore, for $0 < \lambda < \eta$ small enough,

$$\|Y_\lambda\|_{C^1;\mathbb{B}^n} < \chi\epsilon.$$

(Step 2). *Performing the pasting.* Letting $X \equiv 0$ on $\mathbb{B}^n$, by Theorem 3 (and Remark 3), there is $Z_1 \in \mathfrak{X}^{s,\beta}_\mu(\mathbb{B}^n)$ such that

$$\begin{cases} Z_1 = Y_\lambda & \text{in } \frac{1}{3}\mathbb{B}^n, \\ Z_1 = 0 & \text{in } \mathbb{B}^n \backslash \frac{2}{3}\mathbb{B}^n, \\ \|Z_1\|_{C^1;\mathbb{B}^n} \le C\|Y_\lambda\|_{C^1;\mathbb{B}^n} < C\chi\epsilon = \epsilon/d. \end{cases}$$

(Step 3). *Scaling down to the real scale.* Pullback Z_1 to the 'real scale' defining

$$Z_0 = (\lambda^{-1}\mathrm{Id})^* Z_1 \in \mathfrak{X}^{s,\beta}_\mu(\lambda\mathbb{B}^n)$$

compactly supported in $\lambda\mathbb{B}^n$. Extend Z_0 by zero to the whole $\eta\mathbb{B}^n$ and define, on this set, $Z = Z_0 + v$. Then $Z = v(0) + A$ in $(\lambda/3)\mathbb{B}^n$ and $Z = v$ in $\eta\mathbb{B}^n \backslash (2\lambda/3)\mathbb{B}^n$. Since $\lambda < 1$, $Z_1 \mapsto Z_0$ is a homothetic contraction, and thus the C^1 norm does not increase and

$$\|Z - v\|_{C^1;\eta\mathbb{B}^n} = \|Z_0\|_{C^1;\lambda\mathbb{B}^n} \le \|Z_1\|_{C^1;\mathbb{B}^n} < \epsilon/d.$$

We finally get the desired $Z \in \mathfrak{X}^{s,\beta}_\mu(M)$ extending the pullback $\phi^*(Z)$ by v to the whole M. Note that $Z - v \in \mathfrak{X}^{s,\beta}_\mu(M)$ is compactly supported inside $\phi^{-1}(\eta\mathbb{B}^n)$, and thus the global C^1 norm of $Z - v$ satisfies

$$(3)\ \|Z - v\|_{C^1} \le d\|Z_0\|_{C^1;\lambda\mathbb{B}^n} < \epsilon,$$

and it is immediate to verify that (1) and (2) are also satisfied. □

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2.3. *Conservative pasting—proof of Theorem 1.*

Preview. Using Lemma A.1 (§A.3) and the existence of collars for manifolds with boundary, fix W_0 and W_1, two compact n-submanifolds with C^∞ boundary such that

$$K \subset \text{int } W_0, \quad W_0 \subset \text{int } W_1, \quad W_1 \subset U, \quad \Omega := (\text{int } W_1) \backslash W_0 \quad \text{is connected.}$$

The transition from Y to X will take place inside the open set Ω. Fix $\xi \in C^\infty(M; [0, 1])$ such that $\xi = 1$ in a small neighbourhood of W_0 and $\xi = 0$ in a small neighbourhood of $(\text{int } W_1)^c$. Now, given any X, Y as in the statement, let

$$\begin{cases} w = \xi Y + (1 - \xi) X & \text{in } U, \\ w = X & \text{in } U^c. \end{cases}$$

Note that $w \in \mathfrak{X}^s(M)$ since $\xi = 0$ in a neighbourhood of U^c and Y is defined and of class C^s on U. Since both $X \in \mathfrak{X}^s_\mu(M)$ and $Y \in \mathfrak{X}^s_\mu(U)$ are divergence-free,

$$h := \text{div } w \in C^s(M) \quad \text{and} \quad h \text{ is } C^r\text{-small if } Y - X|_U \text{ is } C^r\text{-small}$$

as $h = \text{div } X = 0$ in a neighbourhood of U^c and (on local charts)

$$h = \sum_{i=1}^{n} (\partial_i \xi)(Y^i - X^i) \quad \text{in } U. \tag{2.6}$$

Clearly, h is (compactly) supported inside Ω. In order to get $Z_0 \in \mathfrak{X}^s_\mu(M)$ satisfying (1) and (2), it is enough to find $v \in \mathfrak{X}^s(M)$ supported inside Ω such that

$$\text{div } v = h = \text{div } w$$

and then let $Z_0 = w - v$, which cancels the divergence of w inside the 'transition annulus' Ω, while keeping w unaltered outside that open set (in particular, $Z_0 = w = Y$ in a neighbourhood of W_0 and $Z_0 = w = X$ in a neighbourhood of $(\text{int } W_1)^c \supset U^c$). Since the smooth scalar function ξ is fixed, by (2.6) the C^r norm of h is linearly bounded by that of $Y - X|_U$,

$$|h|_r \le n 2^r |\xi|_{r+1} |Y - X|_{r;U}, \tag{2.7}$$

and it can be shown that (3) holds (see §3). The crucial facts that guarantee the existence of cancelling vector field v are: (a) the connectedness of Ω, (b) supp $h \subset \Omega$ and (c) $\int_\Omega h\omega = 0$, this equality following readily from the divergence theorem. Since X, Y are divergence-free vector fields, w coincides with Y and X in ∂W_0 and ∂W_1 (respectively) and $\partial \overline{\Omega} = \partial W_0 \sqcup \partial W_1$, and thus

$$\begin{aligned} \int_\Omega h\omega = \int_{\partial\Omega} w \lrcorner\, \omega = -\int_{\partial W_0} Y \lrcorner\, \omega + \int_{\partial W_1} X \lrcorner\, \omega \\ = -\int_{W_0} (\text{div } Y)\omega + \int_{W_1} (\text{div } X)\omega = -0 + 0 = 0. \end{aligned}$$

The actual construction of v uses the global-to-local reduction technique originally devised by Moser in [**MO**], essentially aiming to solve, under condition (c), equation $\det Df = 1 + h$ on closed manifolds. We shall follow a complete presentation of the

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transposition of this technique to the solution of $\operatorname{div} u = h$ on $\Omega \subset \mathbb{R}^n$ (under specific support premises) given by Csató, Dacorogna and Kneuss [**CDK**, pp. 184–188]. The smoothing of Z_0 inside the transition annulus Ω is the last step of the construction.

As a byproduct of the proof below together with the estimates in §§3.1 and 3.2, we obtain the following useful result on the solutions to the equation $\operatorname{div} v = h$, with control of support (this is applied in the proofs of Theorems 4 and 5). The linearity of the operator $\Phi : h \mapsto v$ is immediate to check from its construction (cf. [**CDK**, pp. 184–188]). Another important aspect is that the operator is universal, i.e., v has always the same regularity as h (the construction being independent of r and α) and its $C^{r,\alpha}$ norm can be estimated in terms of that of h times a constant, i.e., the restriction of linear operator Φ to the subspace of $\mathcal{A}$ consisting of those functions h that are of class $C^{r,\alpha}$ is bounded for the $\|\cdot\|_{C^{r,\alpha}}$ norm.

LEMMA 1. *Let M be a manifold as above. Suppose that Ω_1, Ω are two connected open subsets with $\overline{\Omega_1} \subset \Omega$. Then there exists a linear operator $\Phi : \mathcal{A} \to \mathcal{B} : h \mapsto v$, satisfying* $\operatorname{div} v = h$, *where*

$$\mathcal{A} = \left\{ h \in C^1(M) : \int_\Omega h\, \omega = 0 \text{ and } \operatorname{supp} h \subset \Omega_1 \right\},$$
$$\mathcal{B} = \{ v \in \mathfrak{X}^1(M) : \operatorname{supp} v \subset \Omega \}.$$

Furthermore, if h is of class $C^{r,\alpha}$, $r \in \mathbb{Z}^+$, $0 \le \alpha \le 1$, then v is $C^{r,\alpha}$ and there is a constant $C = C(r, \Omega_1, \Omega) \ge 1$ such that

$$\|v\|_{C^{r,\alpha}} \le C \|h\|_{C^{r,\alpha}}.$$

Proof of Theorem 1. According to the Preview, it remains to define Ω and ξ precisely and then solve

$$\operatorname{div} v = h, \quad v \in \mathfrak{X}^s(M) \text{ supported inside } \Omega.$$

The existence of constant $C = C(r, K, U)$ satisfying (3) is proved in §3. We start by carefully constructing Ω and an auxiliary domain Ω_1, which is needed in our approach.

(A) *Construction of Ω, Ω_1, V and w.* Using Lemma A.1 (§A.3), fix a compact n-submanifold P with *connected* C^∞ boundary such that $K \subset \operatorname{int} P$ and $P \subset U$. By the existence of collars for ∂P [**HI**, p. 113], there are four smoothly isotopic (nested) manifolds $P_{i \le 4}$ satisfying

$$K \subset \operatorname{int} P_1, \quad P_i \subset \operatorname{int} P_{i+1} \quad (i \le 3), \quad P_4 = P$$

and such that

$$\Omega := (\operatorname{int} P_4) \backslash P_1 \quad \text{and} \quad \Omega_1 := (\operatorname{int} P_3) \backslash P_2$$

are both diffeomorphic to $\partial P \times]0, 1[$, and hence are connected open sets. Exactly as described in the Preview, fix a scalar function ξ for $W_0 = P_2$ and $W_1 = P_3$ (the same for all X and Y) and define w and h accordingly. Clearly, $h \in C^s(M)$ is supported inside Ω_1, $\overline{\Omega_1} \subset \Omega$ and $\int_\Omega h\omega = 0$. We set $V = \operatorname{int} P_1$.

(B) *Finding a divergence-cancelling vector field v.* In order to find $v \in \mathfrak{X}^s(M)$ supported inside Ω and satisfying $\operatorname{div} v = h$, we may now apply the procedure in [**CDK**, pp. 184–188], which reduces this problem to the solution of finitely many local equations

$$\operatorname{div} v_j = h_j, \quad v_j \in \mathfrak{X}^s(Q_j) \tag{2.8}$$

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with v_j compactly supported inside the open cube $Q_j \subset \mathbb{R}^n$. The construction in [**CDK**] carries almost verbatim to our closed manifold M, as the integrals involved in the definition of the auxiliary functions h_j are invariant under chart transition (see below).

Briefly, since $\overline{\Omega_1} \subset \Omega$ is compact, it can be covered by finitely many small open sets $U_j \subset \Omega$, $0 \leq j \leq N$, $N \geq 3$, each of them intersecting Ω_1, such that the image of each U_j on some (reindexed) local chart $(V_j,\ \phi_j)$ is an open cube $Q_j \subset \phi_j(V_j) \subset \mathbb{R}^n$ of volume ≤ 1†. Clearly, N depends only on Ω_1 and Ω and thus ultimately only on K and U. Auxiliary functions $h_j \in C^s(M)$ are now constructed exactly as in [**CDK**, p. 185, Lemma 9.9]. These are well defined since the atlas is volume preserving, which thus implies that all integrals of scalar functions involved [**CDK**, p. 187] are invariant under chart transition (these appear in the constants λ_k, see §3.1(b). The scalar functions h_j satisfy [**CDK**, Lemma 9.9]

$$h = \sum_{j=0}^{N} h_j, \quad \operatorname{supp} h_j \subset U_j \subset \Omega, \quad \int_{U_j} h_j \omega = 0.$$

On local chart $(V_j,\ \phi_j)$,

$$\int_{Q_j} h_j = 0, \quad \operatorname{supp} h_j \subset Q_j \subset \mathbb{R}^n.$$

Each local equation (2.8) is now solved by [**CDK**, p. 185, Lemma 9.8] (which is valid for arbitrary open cubes; see Footnote † on page 20) and the pullback $\phi_j^* v_j$, still denoted by v_j, is extended by zero to the whole M. As $h = \sum_{j=0}^{N} \operatorname{div} v_j = \operatorname{div}(\sum_{j=0}^{N} v_j)$ and $\operatorname{supp} v_j \subset \Omega$, $v = \sum_{j=0}^{N} v_j$ is the desired vector field. Observe that, by construction, h_j, v_j and finally v are C^s if h is C^s (i.e., if X, Y are C^s; see §3). We now have $Z_0 = w - v \in \mathfrak{X}^s_\mu(M)$ satisfying (1) and (2). Observe that the above procedure actually gives a construction of the operator Φ in Lemma 1, i.e., $v = \Phi(\operatorname{div} w)$. Still, by construction, if $Y = X|_U$, then $Z_0 = X$ (see Lemma 1 above) and hence $Z = X$. Otherwise, by Remark 4 (§3.1), the estimate $\|Z_0 - X\|_{C^r} \leq C\|Y - X\|_{C^r;U}$ is still valid with constant C replaced by $C - 1$ and we finally get Z still satisfying (1)–(3) and smooth in

$$\Delta = \{x \in M :\ Z(x) \neq X(x),\ Y(x)\}$$

by applying Theorem 5 to Z_0 and Ω (this set being the interior of a compact collar of ∂P). □

3. *Linear bound on $C^{r,\alpha}$ norms*

3.1. *The C^r case.* Instead of the standard Whitney C^r norm $\|\cdot\|_{C^r}$, we adopt the equivalent but more convenient norm $|\cdot|_r$ defined in §A.1. Then estimate (3) in Theorem 1 is proved letting $C = n^{(r+1)/2}C' + 1$ and finding a constant $C' = C'(r,\ K,\ U)$ for which

$$|Z_0 - X|_r \leq C'|Y - X|_{r;U} \tag{3.1}$$

(clearly, $C = C(r,\ K,\ U)$ since $n = \dim M$ is fixed).

† This fact will be used in §3.1(c).

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Remark 4. Note that estimate (3) in Theorem 1 will still be valid if one replaces C by $C - 1$ (as a consequence of adding $+1$ in the definition of C). This is used at the end of the proof of Theorem 1 (in the smoothing step). The same observation holds for Theorem 3 (used in Corollary 1).

As

$$|Z_0 - X|_r = |w - v - X|_r \leq |w - X|_r + |v|_r,$$
$$|w - X|_r = |\xi(Y - X)|_{r;U} \leq 2^r |\xi|_r |Y - X|_{r;U},$$

it is enough to find a constant $C_0 = C_0(r, K, U) > 0$ such that $|v|_r \leq C_0 |Y - X|_{r;U}$ and let $C' = 2^r |\xi|_r + C_0$ (as $|\xi|_r$ depends only on r and Ω and thus ultimately only on r, K and U).

We will obtain a finite chain of linear bounds with constants C_1, C_2, C_3 depending only on r, K and U, which finally leads to the desired inequality.

(a) $|h|_r \leq C_1 |Y - X|_{r;U}$. From the local chart expression of h (see (2.6) and (2.7) in the Preview, §2.3), it follows that this inequality holds for $C_1 = n2^r |\xi|_{r+1}$. Thus, $|\xi|_{r+1}$ depends only on r, K and U, and so does C_1.

(b) $|h_j|_r \leq C_2 |h|_r$. Following the reasoning in [**CDK**, §9.3, pp. 184–188] transposed to M, fix $\psi_j,\ \eta_k \in C^\infty(M; [0, 1])$ as in Lemma 9.9. Note that $\psi_j,\ \eta_k$ depend ultimately only on K and U. Let

$$d_1 = \max_{0 \leq j \leq N} |\psi_j|_r, \quad d_2 = \max_{1 \leq k \leq N} |\eta_k|_r.$$

By definition, $h_j = h\psi_j + \sum_{k=1}^N \lambda_k A_k^j \eta_k$ (see the proof of Lemma 9.9 in [**CDK**, pp. 185–188]), where each A_k^j (depending on the sequence $U_0, \ldots, U_N$) is either -1, 0 or 1 and the λ_k are the constants solving $\sum_{k=1}^N \lambda_k A_k^j = \int_\Omega h\psi_j$, $0 \leq j \leq N$. In order to find the λ_k, we solve the N simultaneous equations corresponding to $1 \leq j \leq N$, as matrix E obtained from $(N + 1) \times N$ matrix $A = (A_k^j)$ by truncating its first line is actually invertible and the solutions thus obtained automatically satisfy the equation corresponding to $j = 0$. Finding λ_k by Cramer rule, $\lambda_k = |B|/|E|$, and expanding determinant $|B|$ along the kth column (knowing that $A_k^k = 1$, $A_k^j = -1$ or 0 if $j < k$, $A_k^j = 0$ if $j > k$ and each column of E contains, at most, two non-zero entries), we immediately get, on the chart containing the cube U_j (recalling that $N \geq 3$),

$$|\lambda_k| \leq N2^{N-3} \max_{0 \leq j \leq N} \left| \int_\Omega h\psi_j \right| \leq N2^{N-3} \text{meas } \Omega |h|_r,$$
$$|h_j|_r \leq |h\psi_j|_r + N \max_{0 \leq j \leq N} |\lambda_k A_k^j \eta_k|_r \leq C_2 |h|_r,$$

where $C_2 = C_2(r, K, U) = 2^r d_1 + N^2 2^{N-3} d_2 \text{meas } \Omega$.

(c) $|v_j|_r \leq C_3 |h_j|_r$. Recall that v_j is found on local chart $(V_j,\ \phi_j)$ as the solution of (2.8) given by [**CDK**, Lemma 9.8, p. 185] and then extending its pullback by zero to the whole M. Clearly, Lemma 9.8 [**CDK**] holds for each cube $Q_j \subset \mathbb{R}^n$†. Since vol $Q_j \leq 1$,

† The proof of Lemma 9.8 in [**CDK**, p. 185] becomes valid for Q_j by performing the obvious translation of the cube and replacing ξ by $\widehat{\xi}_j \in C_0^\infty(]0, \rho_j[)$, $\rho_j = (\text{vol } Q_j)^{1/n} \leq 1$, satisfying $\int_0^{\rho_j} \widehat{\xi}_j = 1$. Each $\widehat{\xi}_j$ is fixed and depends only on vol Q_j, and hence ultimately only on K and U.

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a simple induction argument over the dimension n (carried out on the modified proof of [**CDK**, Lemma 9.8]; see Footnote † on page 20) shows that, on local chart $(V_j,\ \phi_j)$,

$$|v_j|_r \le (2^r |\widehat{\xi}_j|_r)^n |h_j|_r.$$

Now, in order to get the global C^r norm of v_j, we need to take into account the potential magnification of these local norms under chart transitions $(\phi_{ji})_{i,j\le m}$. Since the transitions between the chart expressions of a vector field are of the form

$$X_j|_{\phi_j(V_i\cap V_j)} = \phi_{ji_*} X_i|_{\phi_i(V_i\cap V_j)},$$

it is easily seen that there is a constant

$$d = d(r,\ \max_{i,j\le m}|\phi_{ji}|_{r+1}) \ge 1$$

such that

$$|X_j|_{\phi_j(V_i\cap V_j)}|_r \le d|X|_{\phi_i(V_i\cap V_j)}|_r$$

for any $i,\ \ j \le m$. The global C^r norm of v_j can then be estimated by

$$|v_j|_r \le C_3|h_j|_r \quad \text{where } C_3 = d(2^r d_0)^n,\ d_0 = \max_{0\le j\le N} |\widehat{\xi}_j|_r.$$

As the atlas is fixed, we actually have $C_3 = C_3(r,\ K,\ U)$.

(d) Finally, $v = \sum_{j=0}^{N} v_j$, and hence $|v|_r \le (N+1)\max_{0\le j\le N}|v_j|_r$, and therefore,

$$|v|_r \le (N+1)C_1C_2C_3|Y - X|_{r;U}.$$

As $N+1$, C_1, C_2 and C_3 depend only on r, K and U, the desired constant is $C_0 = (N+1)C_1C_2C_3$.

3.2. *The $C^{r,\alpha}$ case,* $0 < \alpha \le 1$. First, note that a direct inspection of the construction given in the proof of Theorem 1 of the operator Φ in Lemma 1 reveals that the resulting vector field $Z = w - \Phi(\operatorname{div} w)$ is of class $C^{s,\beta}$ if X and Y are $C^{s,\beta}$, $s \in \mathbb{Z}^+$, $0 \le \beta \le 1$. The proof of Theorem 3 is that of Theorem 1 modulo a few changes needed to get estimate (3) that we now indicate. To simplify the estimates, it is preferable to work exclusively with the following $C^{r,\alpha}$ norm, which is equivalent to the usual Whitney–Hölder $C^{r+\alpha}$ norm $\|\cdot\|_{C^{r,\alpha}}$ (see §A.1 for the notation),

$$|X|_{r,\alpha;U} = \max_{i,j;|\sigma|=r}(|X|_{r;U},\ [\partial^\sigma X_j^i]_{\alpha;\phi_j(V_j\cap U)}),$$

the α-Hölder seminorm $[h]_{\alpha;D}$ of a scalar function h on a domain D (with at least two points) being defined in the usual way. On local charts, this is also equivalent to the $C^{r,\alpha}$ norm adopted in [**CDK**, p. 336], which serves as a reference for the estimates invoked below. We will need reasonable estimates for the Hölder norms of the product and composition of functions defined on open subsets $A \subset M$, and these exist provided that: (i) on every local chart, the domain $\phi_i(V_i \cap A)$ of each function involved is a Lipschitz set (see, e.g., [**CDK**, pp. 338, 366, 369]) and (ii) these functions and their derivatives up to order r extend continuously to the boundaries of these domains (we generically denote the space of $C^{r,\alpha}$ functions on A satisfying (ii) by $C^{r,\alpha}(\overline{A})$). With these two conditions, we also guarantee the respective inclusion of Hölder spaces: if $r + \alpha \le s + \beta$, where

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$0 \le r \le s$ are integers and $0 \le \alpha, \beta \le 1$, then $C^{s,\beta}(\overline{A}) \subset C^{r,\alpha}(\overline{A})$ and there is a constant $C = C(s, A) > 0$ such that $|\cdot|_{r,\alpha;A} \le C|\cdot|_{s,\beta;A}$ [**CDK**, p. 342].

Instead of the estimate at the end of §A.1, we now use for the norm of the product of functions in $C^{r,\alpha}(\overline{A})$ (see, e.g., [**CDK**, p. 366]),

$$|hX|_{r,\alpha;A} \le C(r, A)|h|_{r,\alpha;A}|X|_{r,\alpha;A} \tag{3.2}$$

provided each open set $\phi_j(V_j \cap A)$ is Lipschitz. At first sight, this may seem problematic for the estimates involving the vector field Y, whose domain U may not intersect the local charts in Lipschitz sets (also, while C^r-bounded, Y may fail to satisfy condition (ii)). This difficulty is circumvented by the following simple observation (replacing steps (a)–(c) in §3.1).

(a′) Following the proof of Theorem 1, $w = X$ in a neighbourhood of $(\text{int } P)^c$, and thus

$$|w - X|_{r,\alpha} = |w - X|_{r,\alpha;\text{int } P} = |\xi(Y - X)|_{r,\alpha;\text{int } P}.$$

Now, P is a smooth compact n-submanifold with boundary and since the atlas is regular so are the closures $\overline{V_i}$ of the chart domains (these are embedded $\mathbb{D}^n$). Thus each open set $\phi_i(V_i \cap \text{int } P)$ is Lipschitz and so are the domains $\phi_i(V_i \cap V_j)$ of the transition maps ϕ_{ji}. Therefore (as P and ξ depend only on K and U),

$$\begin{aligned}|w - X|_{r,\alpha} &\le C(r, K, U)|\xi|_{r,\alpha}|Y - X|_{r,\alpha;\text{int } P}\\ &= C(r, \alpha, K, U)|Y - X|_{r,\alpha;\text{int } P}\end{aligned}$$

and

$$\begin{aligned}|h|_{r,\alpha} &= |h|_{r,\alpha;\text{int } P}\\ &\le C(r, K, U)|\xi|_{r+1,\alpha}|Y - X|_{r,\alpha;\text{int } P}\\ &= C(r, \alpha, K, U)|Y - X|_{r,\alpha;\text{int } P}.\end{aligned}$$

From now on, we need not concern ourselves with condition (ii) any more, as it is immediate to verify that all functions involved satisfy it.

(b′) The finitely many auxiliary functions ξ, ψ_j, η_j are defined on the whole M, and thus using (3.2) one gets the local estimate (on the chart containing the cube $\phi_j(V_j)$)

$$|h_j|_{r,\alpha} \le C(r, \alpha, K, U)|h|_{r,\alpha}.$$

(c′) The auxiliary functions involved in the construction of the compactly supported solution to $\operatorname{div} v_j = h_j$ on the cube $Q_j = \phi_j(U_j)$ are all defined on (the closure of) this Lipschitz set, and thus (3.2) applies. The deduction of the local estimate

$$|v_j|_{r,\alpha} \le C(r, \alpha, K, U)|h_j|_{r,\alpha}$$

is a bit more subtle than the corresponding C^r case (but still simple), and involves a judicious application of differentiation under the integral sign. Then, as in the C^r case, there is a constant

$$d = d(r, \alpha) = d(r, \max_{i,j \le m}|\phi_{ji}|_{r+1,\alpha}) \ge 1$$

permitting us to estimate the global $C^{r,\alpha}$ norm of v_j in terms of that on the cube times d. To get this constant, one uses (3.2) together with the estimate for the norm of the composition (still subject to conditions (i) and (ii) above; see, e.g., [**CDK**, p. 369]; here $g : A \to B = \text{dom } f$),

$$|f \circ g|_{r,\alpha;A} \le C(r, A, B)|f|_{r,\alpha;B}(1 + |g|^{r+\alpha}_{r,\alpha;A}) \\ \le C(r, A, B)|f|_{r,\alpha;B}(1 + \max(|g|^{r}_{r,\alpha;A}, |g|^{r+1}_{r,\alpha;A})).$$

Finally, we observe that constant C in Theorem 3 actually does not depend on the Hölder exponent α, as C ultimately depends only on r and on the $C^{r,\alpha}$ and $C^{r+1,\alpha}$ norms of finitely many smooth functions depending only on K and U or even only on the atlas (this is the case for the chart transition maps ϕ_{ji}). On local charts, the domains A of these functions are always Lipschitz (see above), and thus, for each such function, all these norms (with α in the range $]0, 1]$) are uniformly estimated in terms of the respective C^{r+2} norm times a constant $C(r, A)$ [**CDK**, p. 342]. Taking the maximum of these constants for the finitely many functions involved, we get a constant $\widehat{C} = \widehat{C}(r, K, U)$, enabling the simultaneous estimate of all these $C^{r,\alpha}$ and $C^{r+1,\alpha}$ norms $(0 < \alpha \le 1)$ in terms of the respective C^{r+2} norms times $\widehat{C}$. Thus C depends only on r, K and U.

4. *Linearized conservative Franks' lemma*

We now state the linearized volume-preserving version of Franks' lemma. Since perturbations of diffeomorphisms are usually carried out via chart representations, as with Theorem 6, care has been taken to find a δ that directly estimates the permitted variation of the derivative on all chart representations (see §A.2.3 for the terminology). We start by stating a simpler topological version of this result. The full strength is achieved in Theorem 8.

THEOREM 7. (Linearized conservative Franks' lemma) *Let M be a manifold as in §2. Fix $r \in \mathbb{Z}^+$ and $0 < \alpha < 1$ and let $\mathcal{U}$ be a C^1 neighbourhood of $f \in \text{Diff}^{r,\alpha}_{\mu}(M)$ in $\text{Diff}^{r,\alpha}_{\mu}(M)$. Then there is a smaller C^1 neighbourhood $\mathcal{U}_0$ of f in $\text{Diff}^{r,\alpha}_{\mu}(M)$ and $\delta = \delta(r, \alpha, f, \mathcal{U}) > 0$ such that, given:*

- *any $g \in \mathcal{U}_0$;*
- *any finite set $\Sigma \subset M$;*
- *any neighbourhood U of Σ; and*
- *any linear maps $A_x \in \text{SL}(n, \mathbb{R})$, $x \in \Sigma$, satisfying*

$$\|A_x - Dg_x(x)\| < \delta,$$

where g_x is some chart representation of g around x, there exists $\widetilde{g} \in \mathcal{U}$ having, for each $x \in \Sigma$, a chart representation $\widetilde{g}_x$ around x comparable with g_x and such that:

(1) $\widetilde{g}_x(y) = g_x(x) + A_x(y - x)$ *near x; and*

(2) $\text{supp}(\widetilde{g} - g) \subset U$.

Furthermore, if g is C^∞, then so is $\widetilde{g}$.

Remark 5. For each $x \in \Sigma$, (1) implies that $\widetilde{g}(x) = g(x)$, $D\widetilde{g}_x(x) = A_x$ and $\widetilde{g}$ is affine linear near x in chart representation $\widetilde{g}_x$.

The proof actually establishes the stronger result stated below. Given a C^1 diffeomorphism f of M onto itself, let $\sup_M \|Df\|$ denote the supremum of $\|Df(y)\|$ for

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all $y \in M$, over all possible chart representations of f around y (see §A.2). As, in chart representations, the derivatives of a conservative diffeomorphism belong to $\mathrm{SL}(n, \mathbb{R})$, imposing a uniform upper bound $\sup_M \|Df\| \le d$ automatically guarantees uniform local bounded distortion for all conservative diffeomorphisms satisfying this inequality: on chart representations, for any $x \in M$, the image of $\mathbb{S}^{n-1}$ under the derivative $Df(x; \cdot)$ is an ellipsoid with major radius $\le d$ and minor radius $\ge d^{-n+1}$ (this is immediate looking at the polar decomposition).

Also, as is shown below in part (C) of the proof of Lemma 2, δ can be made to depend linearly on the required C^1-closeness ϵ of the resulting diffeomorphism $\widetilde{g}$ to g (provided ϵ is small enough). With both observations in mind, Theorem 7 can be reformulated as follows.

THEOREM 8. (Linearized conservative Franks' lemma) *Let M be a manifold as in §2. Fix $r \in \mathbb{Z}^+$, $0 < \alpha < 1$ and $d \ge 1$. Then there is a constant $\chi = \chi(r, \alpha, d) > 0$ such that, given:*

- *any $g \in \mathrm{Diff}_\mu^{r,\alpha}(M)$ with $\sup_M \|Dg\| \le d$;*
- *any $0 < \epsilon \le 1$;*
- *any finite set $\Sigma \subset M$;*
- *any neighbourhood U of Σ; and*
- *any linear maps $A_x \in \mathrm{SL}(n, \mathbb{R})$, $x \in \Sigma$, satisfying*

$$\|A_x - Dg_x(x)\| < \chi\epsilon,$$

where g_x is some chart representation of g around x, then (adopting any local C^1-metrization of $\mathrm{Diff}_\mu^{r,\alpha}(M)$ near g as in §A.2) there exists $\widetilde{g} \in \mathrm{Diff}_\mu^{r,\alpha}(M)$ ϵ-C^1-close to g having, for each $x \in \Sigma$, a chart representation $\widetilde{g_x}$ around x comparable with g_x and such that:

(1) $\widetilde{g_x}(y) = g_x(x) + A_x(y - x)$ *near x; and*

(2) $\mathrm{supp}(\widetilde{g} - g) \subset U$.

Furthermore, if g is C^∞, then so is $\widetilde{g}$.

Remark 6. Avila's localized smoothing [**AV**, Theorem 7] implies that Theorem 7 can be stated for $\mathrm{Diff}_\mu^1(M)$ in place of $\mathrm{Diff}_\mu^{r,\alpha}(M)$ (with $\chi = \chi(d) > 0$), the reduction of the C^1 local linearization to the C^∞ case being then achieved through Lemma 2 below. However, if g is C^k, $k \ge 2$ an integer, one should not be tempted to apply [**AV**, Theorem 7] in order to smooth g near x (getting $\widehat{g}$), and then apply the elementary perturbation lemma [**BC**, Lemma A.4, p. 93] to correct $\widehat{g}(x)$ back to $g(x)$ and finally apply Lemma 2 below to get a C^1 perturbation $\widetilde{g}$, still of class C^k, which is affine linearized near x (in some chart representation) and coincides with g at x and outside any given small neighbourhood of this point. Indeed, [**AV**] does not guarantee the resulting map to be C^2 at the boundary points of the open set Ω where the smoothing takes place, the above reasoning being valid only for $k = 1$.

Obviously, the C^1-closeness of $\widetilde{g}$ to g is the best possible and cannot be upgraded to any of the higher C^{1+} topologies (even if the localized support is dropped and Σ is reduced to a single point). In terms of regularity, Theorems 7 and 8 are also optimal, in the sense that the resulting diffeomorphism $\widetilde{g}$ is still $C^{r,\alpha}$ (respectively, C^∞) as the original one. If one is particularly interested in the class of C^k diffeomorphisms, $k \geq 2$ an integer, it is natural to ask if $\widetilde{g}$ can be found of class C^k as g and not merely of class $C^{k-1,\alpha}$ for any chosen $0 < \alpha < 1$ (a version of this statement appears without proof in [**HHTU**, p. 217]). For $k \geq 2$, a positive answer seems beyond the techniques presently available (if possible at all). The case $k = 1$ is exceptional due to Avila's theorem mentioned above, but no analogous result is known for $k \geq 2$. These difficulties are related to the fact that, in dimension $n \geq 2$, there are, in general, no known C^{r+1} solutions to the prescribed Jacobian PDE, $\det Df = h$, when h is of class C^r, $r \in \mathbb{Z}^+$ (see, e.g., [**CDK**, p. 192], [**RY**, p. 324]).

In virtue of Lemma 2 below, the answer would be positive if g could be C^{k+}-smoothened near 0, i.e., if one could answer affirmatively the following question.

QUESTION (*Local C^{k+}-smoothing with C^1-closeness*) Given any volume-preserving C^k map $g : \mathbb{B}^n \longrightarrow \mathbb{R}^n$, $k \geq 2$ an integer, is there arbitrarily C^1-close to it another volume-preserving C^k map $\widehat{g} : \mathbb{B}^n \longrightarrow \mathbb{R}^n$ which is $C^{k,\alpha}$ near zero (for some $0 < \alpha < 1$) and satisfies $\operatorname{supp}(\widehat{g} - g) \subset\subset \mathbb{B}^n$?

Proof of Theorem 7. We shall reduce the proof to that of Lemma 2 below. Fix a covering system $\{B_l\}_{l \leq \widetilde{m}}$, i, j for f, here called Υ, as in §A.2, and $0 < \epsilon \leq 1$ such that $\mathscr{U}_{\epsilon,\Upsilon}(f) \subset \mathcal{U}$. Let $\mathcal{U}_0 = \mathscr{U}_{\epsilon/2,\Upsilon}(f)$. Recall that, by definition of $\mathscr{U}_{\epsilon,\Upsilon}(f)$, the same covering system works for any $g \in \mathscr{U}_{\epsilon,\Upsilon}(f)$. Let $g \in \mathcal{U}_0$. As one wishes, for each $x \in \Sigma$, to be able to choose freely any chart representation g_x around x where we can perform the local linearization (getting $\widetilde{g_x}$), we will need to estimate $\sup_M \|Dg\|$ for all such g, the supremum of $\|Dg(y)\|$ for all $y \in M$, over all possible chart representations of g around y (see §A.2). As the transitions between chart representations of g around point x are of the form $g_{\widehat{j}\,\widehat{i},B} = \phi_{\widehat{j}j} \circ g_{ji;B} \circ \phi_{i\widehat{i}}$ (§A.2.2), one gets, as $\epsilon \leq 1$, for all $g \in \mathcal{U}_0$,

$$\sup_M \|Dg\| < c := a^2(\sup_M \|Df\| + 1),$$

where

$$a = \max_{i,j \leq m} \sup_{\phi_i(V_i \cap V_j)} \|D\phi_{ji}\|$$

and $\phi_{ji} = \phi_j \circ \phi_i^{-1}$ are the chart transitions of the atlas $(V_i,\ \phi_i)_{i \leq m}$. Note that we need not be concerned with the C^0 norm of $\widetilde{g} - g$ since it becomes as small as wished if $\operatorname{supp}(\widetilde{g} - g)$ is contained in the disjoint union of sufficiently small open balls (on local charts) centred at the points of Σ. This also guarantees that (2) holds. Hence, only the distance between the derivatives of $\widetilde{g}$ and g is of concern. Performing adequate translations in both domain and target of each chart representation g_x around x, it is now easily seen that the problem reduces to proving Lemma 2 below and finding through it the constant $\chi = \chi(r, \alpha, c, n)$ and then letting $\delta = \chi\epsilon_0$, where $\epsilon_0 = \epsilon/2b$. Here, $b \geq 1$ is a multiplicative constant (to be determined below) controlling the possible magnification of the distance $\|D\widetilde{g_x}(y) - Dg_x(y)\|$, $y \in \operatorname{supp}(g_x - \widetilde{g_x})$, when passing from g_x, $\widetilde{g_x}$ to any other pair $\widehat{g}$, $\widehat{\widetilde{g}}$ of comparable chart representations of g and $\widetilde{g}$ around y. This will guarantee, in particular, that, for $g \in \mathcal{U}_0$, $\|\widetilde{g} - g\|_{C^1} < \epsilon/2$ in the local metric induced on $\mathscr{U}_{\epsilon,\Upsilon}(f)$, and therefore that one gets, as wished,

$$\|\widetilde{g} - f\|_{C^1} \leq \|\widetilde{g} - g\|_{C^1} + \|g - f\|_{C^1} < \epsilon/2 + \epsilon/2 = \epsilon.$$

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We now construct $\widetilde{g}$ and proceed to determine the constant b mentioned above. Since this is more subtle than it might seem at first sight, we do it with some detail. To simplify the exposition, we identify a point x in M with its image $\phi_i(x)$ in a local chart. We first select at will, for each $x \in \Sigma$, a chart representation $g_x = g_{ji,D}$ of g around x and fix a small closed ball B_x centred at this point and contained in the (open) domain $\phi_i(D)$ of g_x, such that the B_x are mutually disjoint (i.e., have mutually disjoint preimages in M). Using Lemma 2 below, we find $\delta = \chi\epsilon_0$, where $\epsilon_0 = \epsilon/2b$ and

$$b = n^2 a\Big(a + (c+1)\max_{i,j\le m}\sup_{\phi_i(V_i\cap V_j)}(\|D^2\phi_{ji}\| + 1)\Big)$$

and then, for any given $A_x \in \mathrm{SL}(n, \mathbb{R})$, as in the statement of Theorem 7, we find a volume-preserving $C^{r,\alpha}$ (respectively, C^∞) diffeomorphism onto its image $\widetilde{g_x} : \phi_i(D) \to \phi_j(V_j)$, which is affine linearized by A_x near x and satisfies $\widetilde{g_x}(x) = g_x(x)$, $\mathrm{supp}(\widetilde{g_x} - g_x) \subset B_x$ and $\|\widetilde{g_x} - g_x\|_{C^1} < \epsilon/2b$. In this way, we have $\widetilde{g}$ globally defined: $\widetilde{g} = \widetilde{g_x}$ in B_x and $\widetilde{g} = g$ in $(\bigcup_{x\in\Sigma} B_x)^c$ (again, we simplify the notation identifying $\widetilde{g_x}$ with the corresponding map in M and B_x with its preimage in M). Now, let $\widehat{g} = \widehat{g}_{\widehat{j}\,\widehat{i},E}$ and $\widehat{\widetilde{g}} = \widehat{\widetilde{g}}_{\widehat{j}\,\widehat{i},E}$ be any other pair of comparable chart representations of g and $\widetilde{g}$ around the preimage $\widehat{y} = \phi_i^{-1}(y)$ in M of $y \in \mathrm{supp}(\widetilde{g_x} - g_x)$. We claim that

$$\|D\widehat{\widetilde{g}}(\widehat{y}) - D\widehat{g}(\widehat{y})\| < \epsilon/2, \tag{4.1}$$

as wished. From the expression giving the derivative under chart representation transition,

$$D\widehat{g}(\widehat{y}) = D\phi_{\widehat{j}j}(g_x(y)) \circ Dg_x(y) \circ D\phi_{i\widehat{i}}(\widehat{y}), \qquad y = \phi_{i\widehat{i}}(\widehat{y}),$$

one gets that

$$\|D\widehat{\widetilde{g}}(\widehat{y}) - D\widehat{g}(\widehat{y})\|$$

is less than or equal to

$$\|D\phi_{\widehat{j}j}(\widetilde{g_x}(y)) \circ D\widetilde{g_x}(y) - D\phi_{\widehat{j}j}(g_x(y)) \circ Dg_x(y)\| \cdot \|D\phi_{i\widehat{i}}(\widehat{y})\|. \tag{4.2}$$

(i) If $\widetilde{g_x}(y) = g_x(y)$, then the norm on the left equals

$$\|D\phi_{\widehat{j}j}(g_x(y))\| \cdot \|D\widetilde{g_x}(y) - Dg_x(y)\|$$

and hence

$$\|D\widehat{\widetilde{g}}(\widehat{y}) - D\widehat{g}(\widehat{y})\| \le a^2\|\widetilde{g_x} - g_x\|_{C^1} < a^2\epsilon/2b < \epsilon/2.$$

(ii) If $\widetilde{g_x}(y) \neq g_x(y)$, then, denoting by $\mathcal{M}(y) = [a_{kl}]$ the $n \times n$ matrix in (4.2) inside the norm on the left, we have, for the constant a defined above,

$$\|D\widehat{\widetilde{g}}(\widehat{y}) - D\widehat{g}(\widehat{y})\| \le a\|\mathcal{M}(y)\|.$$

We estimate the absolute value of the entries a_{kl} and then use $\|\mathcal{M}(y)\| \le n \max|a_{kl}|$. Denoting by ϕ^k the kth component of $\phi_{\widehat{j}j}$ and by $\{e_i\}_{i\le n}$ the canonical base of $\mathbb{R}^n$,

$$|a_{kl}| = \Bigg|\sum_{i=1}^{n} \partial_{e_i}\phi^k(\widetilde{g_x}(y)) \cdot \partial_{e_l}\widetilde{g_x}^{\,i}(y) - \partial_{e_i}\phi^k(g_x(y)) \cdot \partial_{e_l} g_x^i(y)\Bigg|.$$

Now, the key step is to write (using the mean value theorem)

$$\partial_{e_i}\phi^k(\widetilde{g_x}(y)) = \partial_{e_i}\phi^k(g_x(y)) + \partial_u\partial_{e_i}\phi^k(z)\cdot|\widetilde{g_x}(y) - g_x(y)|, \tag{4.3}$$

where z is some point in the interior of segment $[\widetilde{g_x}(y),\ g_x(y)]$ and u is the direction $(\widetilde{g_x}(y) - g_x(y))/|\widetilde{g_x}(y) - g_x(y)|$. Since

$$|\partial_{e_l}\widetilde{g_x}^{\,i}(y)| \le \|\widetilde{g_x}\|_{C^1} < \|g_x\|_{C^1} + \epsilon/2b < c+1,$$

a simple calculation shows that

$$|a_{kl}| \le n\Big(a + (c+1)\max_{i,j\le m}\sup_{\phi_i(V_i\cap V_j)}\|D^2\phi_{ji}\|\Big)\|\widetilde{g_x} - g_x\|_{C^1} \tag{4.4}$$

and, since $\|\widetilde{g_x} - g_x\|_{C^1} < \epsilon/2b$, inequality (4.1) follows. The problem with the above reasoning is that the segment $[\widetilde{g_x}(y),\ g_x(y)]$ might *not* be contained in the domain $\phi_j(V_j \cap V_{\widehat{j}})$ of $\phi_{\widehat{j}j}$ and reducing $\operatorname{supp}(\widetilde{g_x} - g_x)$ to an even smaller neighbourhood of x will not help if $g(x) \in \overline{V_{\widehat{j}}}\backslash V_{\widehat{j}}$. To overcome this difficulty, we use the fact that the atlas of M is contained in a larger atlas (see the convention in §2): there is a small $\varrho > 0$ such that, for every chart domain V_k, $\sup\|D^2\Phi_{kj}\|$ evaluated in the ϱ-neighbourhood Δ_{kj} of $\phi_j(V_k \cap V_j)$ is smaller than $\sup\|D^2\phi_{kj}\| + 1$ in $\phi_j(V_k \cap V_j)$. If necessary, we then reduce the radius of the closed ball B_x even further so that $g_x(B_x) \subset \phi_j(V_j)$ has diameter smaller than ϱ. As $y \in \operatorname{supp}(\widetilde{g_x} - g_x) \subset B_x$, both $g_x(y)$ and $\widetilde{g_x}(y)$ are contained in $g_x(B_x)$, and thus the segment $[\widetilde{g_x}(y),\ g_x(y)]$ is entirely contained in $\Delta_{\widehat{j}j}$. It is thus enough to replace $\phi = \phi_{\widehat{j}j}$ in (4.3) by its extension $\Phi_{\widehat{j}j}$ and replace $\|D^2\phi_{ji}\|$ by $\|D^2\phi_{ji}\| + 1$ in (4.4), as is done in the definition of b. We have therefore reduced the proof of Theorem 7 to that of Lemma 2 below. □

From now on, we assume that $\mathbb{R}^n$ and all its subsets are endowed with the standard volume form $dx_1 \wedge \cdots \wedge dx_n$. We write $A \subset\subset B$ for 'A is compact and contained in B'.

LEMMA 2. (Uniform conservative local linearization) *Given any* $n,\ r \in \mathbb{Z}^+$, $0 < \alpha < 1$ *and* $c \ge 1$ *there exists a constant* $\chi = \chi(r, \alpha, c, n) > 0$ *such that, given any*

(a) $0 < \epsilon_0 \le 1$*;*

(b) *any volume-preserving* $C^{r,\alpha}$ *diffeomorphism onto its image*

$$f : \eta\mathbb{B}^n \longrightarrow \mathbb{R}^n, \qquad \eta > 0$$

such that $f(0) = 0$ *and*

$$\|Df(0)\| \le c;$$

(c) *any* $A \in \mathrm{SL}(n, \mathbb{R})$ *such that*

$$\|A - Df(0)\| < \chi\epsilon_0,$$

there exists a volume-preserving $C^{r,\alpha}$ *diffeomorphism onto its image* $f_A : \eta\mathbb{B}^n \to \mathbb{R}^n$ *satisfying:*

(1) $f_A = A$ *near* 0*;*

(2) $\operatorname{supp}(f_A - f) \subset\subset \eta\mathbb{B}^n$*; and*

(3) $\|f_A - f\|_{C^1} < \epsilon_0$.

Furthermore, if f *is* C^∞*, then so is* f_A.

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Proof. We treat the cases (A) $f \in C^{r,\alpha} \backslash C^{\infty}$ and (B) $f \in C^{\infty}$ separately. In order to make the construction of f_A more transparent, we start by establishing in case (A), through a continuity reasoning, the existence for each $\epsilon_0 > 0$ of a $\delta = \delta(r, \alpha, c, n, \epsilon_0) > 0$ such that (1)–(3) hold if $\|A - Df(0)\| < \delta$ and, analogously, $\delta = \delta(c, n, \epsilon_0) > 0$ is found in case (B). Finally, the linear dependence of δ on ϵ_0 for $0 < \epsilon_0 \leq 1$ is established in each case (see (C) and (D) below), getting $\delta = \chi\epsilon_0$ for some constant $\chi = \chi(r, \alpha, c, n) > 0$ in case (A) and for $\chi = \chi(c, n) > 0$ in case (B). We then take χ as the minimum of these two values.

(A) *Case* $f \in C^{r,\alpha} \backslash C^{\infty}$. The following auxiliary fact follows readily from the compactness of

$$\mathrm{SL}_c := \{D \in \mathrm{SL}(n, \mathbb{R}) : \|D\| \leq c\}, \quad c \geq 1$$

and the continuity of the the composition operator for matrices in relation to the standard norm. Together with Fact 2 below, it will ultimately permit us to find, for given $\epsilon_0 > 0$ and $c \geq 1$, a single δ working simultaneously for all f satisfying (b). Proofs of both facts with linear estimates are given in (C).

Fact 1. For any $n \in \mathbb{Z}^+$, $\epsilon > 0$ and $c \geq 1$, there is $\delta > 0$ such that, given any $A \in \mathrm{SL}(n, \mathbb{R})$ and $D \in \mathrm{SL}_c$,

$$\|A - D\| < \delta \Longrightarrow \|A^{-1} \circ D - \mathrm{Id}\| < \epsilon.$$

The precise $\epsilon_0 - \delta$ chain establishing Lemma 2 can be easily reconstructed from the following reasoning, which makes the structure of the proof more transparent. The continuity of the addition and multiplication operators in relation to the $C^{r,\alpha}$ norm and that of the composition and inversion operators in relation to the C^1 norm will be systematically used without mention.

While Lemma 2 is a C^1-closeness result, we will need to work with the $C^{1,\alpha}$ norm until step (A.2) in order to guarantee that the volume-correcting diffeomorphism φ^{-1} is of class $C^{r,\alpha}$ and C^1-close to Id. Then we return to the standard Whitney C^1 norm using $\|\cdot\|_{C^1} \leq n\, |\cdot|_1 \leq n\, |\cdot|_{1,\alpha}$ (see §A.1).

For $h \in C^{r,\alpha}(\overline{\mathbb{B}^n}, \mathbb{R}^n)$, $r \in \mathbb{Z}^+$, $0 < \alpha \leq 1$, we adopt the $C^{r,\alpha}$ norm corresponding to that of §3.2 (for $h \in C^{r,\alpha}(\overline{\mathbb{B}^n})$, the definition is the same but the component superscript i disappears). This is equivalent to the standard Whitney–Hölder $C^{r+\alpha}$ norm $\|\cdot\|_{C^{r,\alpha}}$.

$$|h|_{r,\alpha;\mathbb{B}^n} = \max_{i;|\sigma|=r} (|h|_{r;\mathbb{B}^n}, [\partial^\sigma h^i]_{\alpha;\mathbb{B}^n}).$$

(A.1) *Reducing to the case of diffeomorphisms with domain* $\mathbb{B}^n$ $C^{1,\alpha}$ *-close to* Id *and* $A = \mathrm{Id}$. Let $0 < \lambda < \min(1, \eta)$. For each f of class $C^{r,\alpha}$ satisfying (b), rescale $f|_{\lambda\mathbb{B}^n}$ to the unit ball getting a volume-preserving $C^{r,\alpha}$ diffeomorphism onto its image

$$\begin{aligned} f_\lambda : \mathbb{B}^n &\longrightarrow \mathbb{R}^n \\ z &\longmapsto \lambda^{-1} f(\lambda z). \end{aligned}$$

One has

$$|f_\lambda - Df(0)|_{1,\alpha;\mathbb{B}^n} \underset{\lambda \to 0}{\longrightarrow} 0. \tag{4.5}$$

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(Up to the C^1 norm, the reasoning is the same as in the proof of Theorem 6. Let $\{e_j\}_{j\le n}$ be the canonical base of $\mathbb{R}^n$. Writing ∂_j for ∂_{e_j}, one has, for the partial derivatives of the components f^i_λ of f_λ,

$$\sup_{x,y\in\mathbb{B}^n;x\ne y}\frac{|\partial_j f^i_\lambda(y)-\partial_j f^i_\lambda(x)|}{|y-x|^\alpha}=\sup_{x,y\in\mathbb{B}^n;x\ne y}\lambda^\alpha\frac{|\partial_j f^i(\lambda y)-\partial_j f^i(\lambda x)|}{|\lambda y-\lambda x|^\alpha}$$
$$\le \lambda^\alpha|f|_{1,\alpha;\lambda\mathbb{B}^n}\xrightarrow[\lambda\to 0]{}0,$$

which thus establishes (4.5).) For each $A\in\mathrm{SL}(n,\mathbb{R})$ let

$$h_{A,\lambda}=A^{-1}\circ f_\lambda.$$

By (4.5) (see, e.g., [**CDK**, p. 384]),

$$|h_{A,\lambda}-A^{-1}\circ Df(0)|_{1,\alpha;\mathbb{B}^n}\xrightarrow[\lambda\to 0]{}0. \tag{4.6}$$

Fix $\xi\in C^\infty(\mathbb{B}^n;[0,1])$ (the same for all f and A) with $\xi=0$ in $\frac{1}{3}\mathbb{D}^n$ and $\xi=1$ in $\mathbb{B}^n\backslash\frac{2}{3}\mathbb{B}^n$ and define

$$g_{A,\lambda}=\mathrm{Id}+\xi(h_{A,\lambda}-\mathrm{Id}).$$

Then, noting that, for $L\in L(n,\mathbb{R})$, $|L|_{1,\alpha;\mathbb{B}^n}\le\|L\|$, by (4.6)

$$|h_{A,\lambda}-\mathrm{Id}|_{1,\alpha;\mathbb{B}^n}\xrightarrow[\lambda\to 0]{}|A^{-1}\circ Df(0)-\mathrm{Id}|_{1,\alpha;\mathbb{B}^n}\le\|A^{-1}\circ Df(0)-\mathrm{Id}\| \tag{4.7}$$

and, by Fact 1 above, as $Df(0)\in\mathrm{SL}_c$, for δ small the norm on the right is uniformly small for all f satisfying (b) and all A satisfying (c), and hence, for λ small enough,

$$|g_{A,\lambda}-\mathrm{Id}|_{1,\alpha;\mathbb{B}^n}\text{ is small} \tag{4.8}$$

and, in particular, $g_{A,\lambda}$ is a diffeomorphism of $\mathbb{B}^n$ onto its image.

(A.2) *Correcting the volume distortion.* Dropping the subscripts, for simplicity, let

$$\theta=\theta_{A,\lambda}=\det Dg_{A,\lambda}.$$

Then by (4.8), it follows that (i)

$$|\theta-1|_{0,\alpha;\mathbb{B}^n}\text{ is small,} \tag{4.9}$$

(ii) $\int_{\mathbb{B}^n}\theta=\operatorname{meas} g_{A,\lambda}(\mathbb{B}^n)=\operatorname{meas}\mathbb{B}^n$ and (iii) $\theta=1$ in $\mathscr{C}=\frac{1}{3}\mathbb{D}^n\cup(\mathbb{B}^n\backslash\frac{2}{3}\mathbb{B}^n)$. Now it is easily seen that we can apply [**TE**, Theorem 4] (with $\gamma=\alpha$) to get $\varphi\in\mathrm{Diff}^{r,\alpha}(\mathbb{B}^n)$ such that $\det D\varphi=\theta$ and $\varphi=\mathrm{Id}$ in $\mathscr{C}$, with

$$|\varphi-\mathrm{Id}|_{1;\mathbb{B}^n}\text{ small.} \tag{4.10}$$

Then

$$\widetilde{g_{A,\lambda}}=g_{A,\lambda}\circ\varphi^{-1}$$

is a volume-preserving $C^{r,\alpha}$ diffeomorphism of $\mathbb{B}^n$ onto its image with

$$\begin{cases}\widetilde{g_{A,\lambda}}=\mathrm{Id} & \text{in } \frac{1}{3}\mathbb{D}^n,\\ \widetilde{g_{A,\lambda}}=h_{A,\lambda} & \text{in } \mathbb{B}^n\backslash\frac{2}{3}\mathbb{B}^n,\end{cases}$$

and

$$\|\widetilde{g_{A,\lambda}}-\mathrm{Id}\|_{C^1;\mathbb{B}^n}\text{ is small.} \tag{4.11}$$

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(A.3) *Back to the general case.* Setting

$$\widetilde{f_{A,\lambda}} = A \circ \widetilde{g_{A,\lambda}},$$

it is immediate to verify that

$$\begin{cases} \widetilde{f_{A,\lambda}} = A \text{ near } 0, \\ \operatorname{supp}(\widetilde{f_{A,\lambda}} - f_\lambda) \subset\subset \mathbb{B}^n. \end{cases}$$

By (4.11) and Fact 2 below, for δ (and λ) small,

$$\|\widetilde{f_{A,\lambda}} - Df(0)\|_{C^1;\mathbb{B}^n} < \epsilon_0/2 \tag{4.12}$$

for all f satisfying (b) and all A satisfying (c). For a proof of Fact 2 with a linear estimate, see (C.8) below.

Fact 2. For any $n \in \mathbb{Z}^+$, $\epsilon > 0$ and $c \geq 1$ there is $\delta > 0$ such that, given any $A \in L(n, \mathbb{R})$, $D \in \mathrm{SL}_c$ and a C^1 map $g : \mathbb{B}^n \to \mathbb{R}^n$,

$$\|A - D\|,\ \|g - \mathrm{Id}\|_{C^1;\mathbb{B}^n} < \delta \Longrightarrow \|A \circ g - D\|_{C^1;\mathbb{B}^n} < \epsilon.$$

(A.4) *Scaling down to the real scale.* It remains to scale down $\widetilde{f_{A,\lambda}}$ back to the real scale. Let

$$\begin{aligned} f_{A,\lambda} : \lambda\mathbb{B}^n &\longrightarrow \mathbb{R}^n \\ z &\longmapsto \lambda \widetilde{f_{A,\lambda}}(\lambda^{-1} z). \end{aligned}$$

Since the C^1 norm does not increase under contracting homothetic conjugation and $\lambda < 1$,

$$\|f_{A,\lambda} - Df(0)\|_{C^1;\lambda\mathbb{B}^n} \leq \|\widetilde{f_{A,\lambda}} - Df(0)\|_{C^1;\mathbb{B}^n} < \epsilon_0/2. \tag{4.13}$$

Taking λ even smaller, if necessary, we can further guarantee that

$$\|f - Df(0)\|_{C^1;\lambda\mathbb{B}^n} < \epsilon_0/2.$$

Therefore, as $\operatorname{supp}(f_{A,\lambda} - f|_{\lambda\mathbb{B}^n}) \subset\subset \lambda\mathbb{B}^n$, extending $f_A := f_{A,\lambda}$ by f to the whole $\eta\mathbb{B}^n$, we finally get, by the triangle inequality, that

$$\|f_A - f\|_{C^1;\eta\mathbb{B}^n} < \epsilon_0$$

and it is immediate to check that f_A is $C^{r,\alpha}$ and satisfies all the conclusions of Lemma 2.

(B) *Case* $f \in C^\infty$. Having fixed n, c and ϵ_0, both the determination of $\delta = \delta(c, n, \epsilon_0)$ and the construction of f_A are similar to those in case (A), except that the volume-correcting diffeomorphism φ in (A.2) must be obtained by a different method, as using [**TE**, Theorem 4], there is no guarantee that the solution to $\det D\varphi = \theta$ is smooth when θ is smooth (in the later case, we get a solution φ_r of class C^r, for each $r \in \mathbb{Z}^+$, but *a priori* nothing guarantees that these φ_r coincide to form a C^∞ diffeomorphism. Reciprocally, [**TE**, Theorem 5] and [**CDK**, Lemma 10.4] employed below cannot be applied in case (A) since they do not provide the necessary gain of regularity, from $C^{r-1,\alpha}$(determinant θ) to $C^{r,\alpha}$ (diffeomorphism φ)). Here all functions involved are smooth and

$$|f_\lambda - Df(0)|_{2;\mathbb{B}^n} \xrightarrow[\lambda \to 0]{} 0$$

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(up to the C^1 norm; see (A.1) above). For each multiindex σ of order two, one has, for the partial derivatives of the components of f_λ,

$$\sup_{x\in\mathbb{B}^n} |\partial^\sigma f^i_\lambda(x)| = \sup_{x\in\mathbb{B}^n} \lambda|\partial^\sigma f^i(\lambda x)| \le \lambda |f|_{2;\lambda\mathbb{B}^n} \xrightarrow[\lambda\to 0]{} 0.$$

Therefore,

$$|h_{A,\lambda} - A^{-1}\circ Df(0)|_{2;\mathbb{B}^n} \xrightarrow[\lambda\to 0]{} 0 \tag{4.14}$$

and, consequently, reasoning as in (A),

$$|g_{A,\lambda} - \mathrm{Id}|_{2;\mathbb{B}^n} \text{ is small}$$

and hence

$$|\theta - 1|_{1;\mathbb{B}^n} \text{ is small.}$$

Then, we apply [**TE**, Theorem 5] and [**CDK**, Lemma 10.4] to get a C^∞ solution diffeomorphism to $\det D\varphi = \theta$ with $\varphi = \mathrm{Id}$ in $\mathscr{C}$ and

$$|\varphi - \mathrm{Id}|_{1;\mathbb{B}^n} \text{ small.}$$

It can be verified that, in [**TE**, Theorem 5], if the volume form θ is smooth, the solution diffeomorphism φ is also smooth. This follows from the fact that the solution to the linearized problem $\operatorname{div} u = \theta - 1$ in [**TE**, Theorem 3] is smooth since it depends only on θ and not on r, α (see [**TE**, Remark 3 and Footnote 3]) and from the way in which φ is found (integrating the time dependent vector field $u_t = u/((1-t)\theta + t)$; cf. [**DM**, Lemma 2], [**CDK**, pp. 209–210]). One then uses the estimate in [**TE**, Theorem 3] and that in [**CDK**, Lemma 10.4] to get the estimate $|\varphi - \mathrm{Id}|_{1;\mathbb{B}^n} \le C|\theta - 1|_{1;\mathbb{B}^n}$, for some constant $C = C(n) > 0$. The construction then follows that of case (A). As shown in (D) below, the more general (and abstract) result [**CDK**, Lemma 10.4] can actually entirely replace the use of [**TE**, Theorem 5] above.

(C) *Linear dependence* $\delta = \chi\epsilon_0$ *for* $0 < \epsilon_0 \le 1$ *in the case* $f \in C^{r,\alpha}\backslash C^\infty$. The case of $f \in C^\infty$ is similar; the changes needed being indicated in (D) below. As in §3.1, we shall establish a finite chain of linear bounds that finally lead to the determination of the constant χ. We emphasize that $|\cdot|_{r,\alpha}$ in (C.2)–(C.5) is the $C^{r,\alpha}$ norm defined in (A) above and $\|\cdot\|_{C^1}$ in (C.5)–(C.8) is the standard Whitney C^1 norm (§A.1) in which Lemma 2 is formulated.

We start by establishing the actual estimate in Fact 1.

(C.1) *Given any* $n \in \mathbb{Z}^+$, $c \ge 1$, $A \in \mathrm{SL}(n,\mathbb{R})$ *and* $Df(0) \in \mathrm{SL}_c$,

$$\|A - Df(0)\| < \delta \le 1 \Longrightarrow \|A^{-1}\circ Df(0) - \mathrm{Id}\| < (c+1)^{n-1}\delta = C_1(c,n)\delta.$$

We have

$$\|A^{-1}\circ Df(0) - \mathrm{Id}\| = \|A^{-1}\circ(Df(0) - A)\| < \|A^{-1}\|\cdot\delta.$$

Since $A \in \mathrm{SL}(n,\mathbb{R})$ and $\|A\| < c+1$, looking at its polar decomposition, one sees that

$$\min_{x\in\mathbb{S}^{n-1}} |A(x)| > (c+1)^{-n+1}$$

and thus $\|A^{-1}\| < (c+1)^{n-1}$ and the assertion follows.

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In what follows, C, C' and C'' denote auxiliary generic constants (varying from step to step), whose existence follows from standard Hölder estimates [**CDK**, pp. 342 and 366] or is evident from the context.

(C.2) $\|A^{-1} \circ Df(0) - \mathrm{Id}\| < \delta \Longrightarrow |g_{A,\lambda} - \mathrm{Id}|_{1,\alpha;\mathbb{B}^n} < C_2(n)\delta$. The partition function ξ is fixed for each dimension n and $|\xi|_{1,\alpha;\mathbb{B}^n} \le C(n)|\xi|_{2;\mathbb{B}^n}$ [**CDK**, p. 342], and therefore one has, by (4.7), for λ small enough,

$$\begin{aligned}|g_{A,\lambda} - \mathrm{Id}|_{1,\alpha;\mathbb{B}^n} &= |\xi(h_{A,\lambda} - \mathrm{Id})|_{1,\alpha;\mathbb{B}^n}\\ &\le C'(n)|\xi|_{1,\alpha;\mathbb{B}^n}|h_{A,\lambda} - \mathrm{Id}|_{1,\alpha;\mathbb{B}^n}\\ &\le C''(n)|h_{A,\lambda} - \mathrm{Id}|_{1,\alpha;\mathbb{B}^n}\\ &< C''(n)\delta = C_2(n)\delta.\end{aligned}$$

(C.3) $|g_{A,\lambda} - \mathrm{Id}|_{1,\alpha;\mathbb{B}^n} < \delta \le 1 \Longrightarrow |\theta - 1|_{1,\alpha;\mathbb{B}^n} < C_3(n)\delta$. One has

$$|\theta - 1|_{0,\alpha;\mathbb{B}^n} = \max(|\theta - 1|_{0;\mathbb{B}^n}, [\theta]_{\alpha;\mathbb{B}^n}).$$

Clearly, $|\theta - 1|_{0;\mathbb{B}^n} < C(n)\delta$ for $\theta - 1$ is the sum of $n!$ terms of the form

$$\pm((\widehat{a}_1 + \delta_1)(\widehat{a}_2 + \delta_2) \cdots (\widehat{a}_n + \delta_n) - \widehat{a}_1\widehat{a}_2 \cdots \widehat{a}_n),$$

where each $\widehat{a}_i = 0$ or 1 is an entry of the Id matrix and $|\delta_i| < \delta \le 1$, and thus $|\theta - 1|_{0;\mathbb{B}^n} < n!(2^n - 1)\delta$.

To simplify the notation, we write g for the generic component $g^i_{A,\lambda}$ of $g_{A,\lambda}$ and $\partial^k g$ for its generic partial derivative of order k.

In abridged notation, the determinant $\theta = \det Dg_{A,\lambda}$ is the sum of $n!$ monomials of the form $\pm(\partial g)^n$. Using the the following estimate for the α-Hölder seminorm of the product of scalar functions [**CDK**, p. 366],

$$[h_1 \cdots h_n]_\alpha \le n \max_j |h_j|_0^{n-1} \cdot \max_j [h_j]_\alpha,$$

and since, by hypothesis,

$$\sup_{\mathbb{B}^n} |\partial g| \le |g_{A,\lambda}|_{1,\alpha;\mathbb{B}^n} < |\mathrm{Id}|_{1,\alpha;\mathbb{B}^n} + 1 = 2$$

and $[\partial g]_{\alpha;\mathbb{B}^n} < \delta$, one has (in abridged form)

$$[\theta]_{\alpha;\mathbb{B}^n} \le \sum^{n!} [(\partial g)^n]_{\alpha;\mathbb{B}^n} < n! n 2^{n-1}\delta = C'(n)\delta.$$

Thus (C.3) holds.

(C.4) Let $\Omega = \mathbb{B}^n \setminus \frac{1}{4}\mathbb{D}^n$ and $U = (\mathbb{D}^n \setminus \frac{2}{3}\mathbb{B}^n) \cup (\frac{1}{3}\mathbb{D}^n \setminus \frac{1}{4}\mathbb{B}^n)$. Let $\widehat{\epsilon} = \widehat{\epsilon}(r, \alpha, n) = \widehat{\epsilon}(r, \alpha, U, \Omega)$ and $C_4 = C_4(r, \alpha, n) = c(r, \alpha, U, \Omega)$ be the corresponding constants in [**TE**, Theorem 4]. One has, for the solution diffeomorphism $\varphi \in \mathrm{Diff}^{r,\alpha}(\mathbb{B}^n)$ obtained via [**TE**, Theorem 4] in (A.2) above,

$$|\theta - 1|_{0,\alpha;\mathbb{B}^n} < \delta \le \epsilon(r, \alpha, n) \Longrightarrow |\varphi - \mathrm{Id}|_{1;\mathbb{B}^n} < C_4(r, \alpha, n)\delta.$$

(C.5) We now return to the Whitney C^1 norm. Since $\|\cdot\|_{C^1} \le n|\cdot|_1$ for maps $\mathbb{B}^n \to \mathbb{R}^n$ (§A.1), one has

$$|\varphi - \mathrm{Id}|_{1;\mathbb{B}^n} < \delta \Longrightarrow \|\varphi - \mathrm{Id}\|_{C^1;\mathbb{B}^n} < n\delta = C_5(n)\delta.$$

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(C.6) Let $C_6 = 3$. Then

$$\|\varphi - \mathrm{Id}\|_{C^1;\mathbb{B}^n} < \delta \leq 1/2 \Longrightarrow \|\varphi^{-1} - \mathrm{Id}\|_{C^1;\mathbb{B}^n} < C_6\delta.$$

Since φ^{-1} is a diffeomorphism of $\mathbb{B}^n$ onto itself, one has

$$\begin{aligned}\|\varphi^{-1} - \mathrm{Id}\|_{C^1;\mathbb{B}^n} = \|(\varphi - \mathrm{Id}) \circ \varphi^{-1}\|_{C^1;\mathbb{B}^n} &\leq \|\varphi - \mathrm{Id}\|_{C^1;\mathbb{B}^n}(1 + \|\varphi^{-1}\|_{C^1;\mathbb{B}^n}) \\ &< \delta(1 + \|\varphi^{-1}\|_{C^1;\mathbb{B}^n}),\end{aligned}$$

$$\begin{aligned}\|\varphi - \mathrm{Id}\|_{C^1;\mathbb{B}^n} < 1/2 &\Longrightarrow \min_{u \in \mathbb{S}^{n-1}} |D\varphi(x; u)| > 1/2 \quad \text{for all } x \in \mathbb{B}^n \\ &\Longrightarrow \sup_{\mathbb{B}^n} \|D\varphi^{-1}\| < 2,\end{aligned}$$

and therefore, as $\|\varphi^{-1}\|_{C^0;\mathbb{B}^n} = 1$ it follows that $\|\varphi^{-1}\|_{C^1;\mathbb{B}^n} < 2$, and thus (C.6) holds.

(C.7) Let $C_7 = 4$. Then

$$\|\varphi^{-1} - \mathrm{Id}\|_{C^1;\mathbb{B}^n}, \|g_{A,\lambda} - \mathrm{Id}\|_{C^1;\mathbb{B}^n} < \delta \leq 1 \Longrightarrow \|g_{A,\lambda} \circ \varphi^{-1} - \mathrm{Id}\|_{C^1;\mathbb{B}^n} < C_7\delta.$$

Let $\widehat{g} := g_{A,\lambda}$. Then

$$\begin{aligned}\|\widehat{g} \circ \varphi^{-1} - \mathrm{Id}\|_{C^1;\mathbb{B}^n} &\leq \|(\widehat{g} - \mathrm{Id}) \circ \varphi^{-1}\|_{C^1;\mathbb{B}^n} + \|\varphi^{-1} - \mathrm{Id}\|_{C^1;\mathbb{B}^n} \\ &< \|\widehat{g} - \mathrm{Id}\|_{C^1;\mathbb{B}^n}(1 + \|\varphi^{-1}\|_{C^1;\mathbb{B}^n}) + \delta \\ &< \delta(1 + \|\mathrm{Id}\|_{C^1;\mathbb{B}^n} + 1) + \delta = 4\delta.\end{aligned}$$

(C.8) Let $C_8 = C_8(c) = c + 2$. Then

$$\|A - Df(0)\|, \|g_{A,\lambda} - \mathrm{Id}\|_{C^1;\mathbb{B}^n} < \delta \leq 1 \Longrightarrow \|A \circ g_{A,\lambda} - Df(0)\|_{C^1;\mathbb{B}^n} < C_8\delta.$$

We use the following basic estimate: given any linear map $L \in L(n, \mathbb{R})$ and any C^1-bounded map $h : \mathbb{B}^n \to \mathbb{R}^n$,

$$\|L \circ h\|_{C^1;\mathbb{B}^n} \leq \|L\| \cdot \|h\|_{C^1;\mathbb{B}^n}.$$

Now, writing $\widehat{g}$ for $g_{A,\lambda}$ and D for $Df(0)$,

$$\begin{aligned}\|A \circ \widehat{g} - D\|_{C^1;\mathbb{B}^n} &= \|(A - D) \circ \widehat{g} + D \circ (\widehat{g} - \mathrm{Id})\|_{C^1;\mathbb{B}^n} \\ &\leq \|A - D\| \cdot \|\widehat{g}\|_{C^1;\mathbb{B}^n} + \|D\| \cdot \|\widehat{g} - \mathrm{Id}\|_{C^1;\mathbb{B}^n} \\ &< \delta(\|\mathrm{Id}\|_{C^1;\mathbb{B}^n} + 1) + c\delta = (c + 2)\delta.\end{aligned}$$

(C.9) Let $\widehat{\epsilon} = \widehat{\epsilon}(r, \alpha, n)$ be the constant obtained in (C.4). Note that we may assume that all constants C_k above are ≥ 2. Then, following the above chain of linear estimates, it is immediate to verify that the constant $\chi = \chi(r, \alpha, c, n)$ below satisfies the conclusions of Lemma 2 when $f \in C^{r,\alpha} \backslash C^\infty$: i.e.,

$$\chi = \frac{1}{C_1C_2C_3} \min\left(\widehat{\epsilon}, \frac{1}{C_4C_5C_6C_7C_8}\right).$$

It remains only to verify that the $C^{r,\alpha}$ map $g_{A,\lambda} = \mathrm{Id} + \xi(h_{A,\lambda} - \mathrm{Id})$ is, in fact, a diffeomorphism. It is easily seen (see below) that

$$\|g_{A,\lambda} - \mathrm{Id}\|_{C^1;\mathbb{B}^n} \leq 1/4 \Longrightarrow g_{A,\lambda} \text{ is a diffeomorphism onto its image.} \tag{4.15}$$

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Now $\delta = \chi\epsilon_0 \le \chi$ since $\epsilon_0 \le 1$, therefore

$$\|g_{A,\lambda} - \mathrm{Id}\|_{C^1;\mathbb{B}^n} \le n\|g_{A,\lambda} - \mathrm{Id}\|_{1,\alpha;\mathbb{B}^n} \le nC_2C_1\chi < 1/4$$

as $C_5 = n$ and all constants C_k are ≥ 2. Thus

$$nC_2C_1\chi \le \frac{1}{C_3C_4C_6C_7C_8} \le 2^{-5}.$$

It remains to prove (4.15). It is immediate from the hypothesis that the derivative is everywhere non-singular, and thus only the injectivity of $\widehat{g} := g_{A,\lambda}$ needs to be established. We show that, for any $x,\ y \in \mathbb{B}^n$, $|\widehat{g}(y) - \widehat{g}(x)| \ge \frac{1}{6}|y - x|$. The hypothesis implies that, for any $v \in \mathbb{R}^n$, $|D\widehat{g}(0; v)| \ge \frac{2}{3}|v|$. Let $h(x) = \widehat{g}(x) - \widehat{g}(0) - D\widehat{g}(0; x)$. Then

$$\sup_{\mathbb{B}^n}\|Dh\| = \sup_{\mathbb{B}^n}\|D\widehat{g} - D\widehat{g}(0)\| \le \sup_{\mathbb{B}^n}\|D\widehat{g} - \mathrm{Id}\| + \sup_{\mathbb{B}^n}\|\mathrm{Id} - D\widehat{g}(0)\| \le \tfrac{1}{4} + \tfrac{1}{4} = \tfrac{1}{2}$$

and thus, for any $x,\ y \in \mathbb{B}^n$, $|h(y) - h(x)| \le \frac{1}{2}|y - x|$. Hence

$$\begin{aligned}|\widehat{g}(y) - \widehat{g}(x)| &= |D\widehat{g}(0; y - x) + h(y) - h(x)| \\ &\ge |D\widehat{g}(0; y - x)| - |h(y) - h(x)| \\ &\ge \tfrac{2}{3}|y - x| - \tfrac{1}{2}|y - x| = \tfrac{1}{6}|y - x|.\end{aligned}$$

Therefore $\widehat{g} = g_{A,\lambda}$ is injective and the proof of (C) is complete.

(D) *Linear dependence* $\delta = \chi\epsilon_0$ *for* $0 < \epsilon_0 \le 1$ *in the case* $f \in C^\infty$.

(D.1) The estimate in (C.1) above carries unchanged to the present C^∞ case.

(D.2) From (4.14), reasoning as in (C.2) and now applying the estimate for the $|\cdot|_r$ norm of the product (the end of §A.1), we immediately get

$$\|A^{-1} \circ Df(0) - \mathrm{Id}\| < \delta \Longrightarrow |g_{A,\lambda} - \mathrm{Id}|_{2;\mathbb{B}^n} < 2^2|\xi|_{2;\mathbb{B}^n}\delta = C_2(n)\delta.$$

(D.3) $|g_{A,\lambda} - \mathrm{Id}|_{2;\mathbb{B}^n} < \delta \le 1 \Longrightarrow |\theta - 1|_{1;\mathbb{B}^n} < C_3(n)\delta$. The estimate $|\theta - 1|_{0;\mathbb{B}^n} < n!(2^n - 1)\delta$ was obtained in (C.3). In the abridged notation adopted there, the components $\partial_i\theta = (\nabla\theta)^i$ of $\nabla\theta$ are of the form

$$\sum^{n!}\sum^{n} \pm(\partial^2 g)(\partial g)^{n-1}.$$

Since, by hypothesis, $\sup_{\mathbb{B}^n}|\partial g| < 2$ and $\sup_{\mathbb{B}^n}|\partial^2 g| < \delta$, it follows that

$$\max_i \sup_{\mathbb{B}^n}|\partial_i\theta| \le n!n2^{n-1}\delta,$$

which, together with the estimate above for $|\theta - 1|_{0;\mathbb{B}^n}$, finally gives

$$|\theta - 1|_{1;\mathbb{B}^n} < n!n2^{n-1}\delta = C_3(n)\delta.$$

(D.4) $|\theta - 1|_{1;\mathbb{B}^n} < \delta \Longrightarrow |\theta - 1|_{0,1/2;\mathbb{B}^n} < \sqrt{2n}\delta = C_4(n)\delta$. Reasoning as in §A.1 (equivalence of norms $|\cdot|_r$ and $\|\cdot\|_{C^r}$), we have

$$\max_i \sup_{\mathbb{B}^n}|\partial_i\theta| < \delta \Longrightarrow \sup_{\mathbb{B}^n}\|\nabla\theta\| < \sqrt{n}\delta$$

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and thus, by the mean value inequality,

$$[\theta - 1]_{1/2;\mathbb{B}^n} = \sup_{x,y\in\mathbb{B}^n;\, x\neq y} \frac{|\theta(y)-\theta(x)|}{\sqrt{|y-x|}} \le \sup_{x,y\in\mathbb{B}^n} \sqrt{|y-x|}\, \sup_{\mathbb{B}^n}\|\nabla\theta\| < \sqrt{2n}\delta.$$

Therefore, since $|\theta - 1|_{0;\mathbb{B}^n} < \delta$, (D.4) follows.

(D.5) $|\theta - 1|_{0,1/2;\mathbb{B}^n} < \delta \Longrightarrow |u|_{1;\mathbb{B}^n} < C_5(n)\delta$. Let $u \in \mathfrak{X}^\infty(\mathbb{B}^n)$ be the solution to

$$\begin{cases} \operatorname{div} u = \theta - 1, & \\ u = 0 & \text{in } \mathscr{C}, \end{cases}$$

obtained via [**TE**, Theorem 3] (see (A.2) for the meaning of $\mathscr{C}$ and (B) for the regularity of u), which satisfies

$$|u|_{1;\mathbb{B}^n} \le C(n)|\theta - 1|_{0,1/2;\mathbb{B}^n} < C(n)\delta.$$

(D.6) $|\theta - 1|_{1;\mathbb{B}^n}$, $|u|_{1;\mathbb{B}^n} < \delta \le 1/2 \Longrightarrow |\varphi - \mathrm{Id}|_{1;\mathbb{B}^n} < C_6(n)\delta$. For $t \in [0, 1]$, let $f_t = (1-t)\theta + t$ and $u_t = u/f_t$. Using [**CDK**, Lemma 10.4] with $\Omega = \mathbb{B}^n$, $r = 1$, $\alpha = 0$ and $T = 1$, and since $u = 0$ in $\mathscr{C}$, we obtain a solution $\varphi := \varphi_1 \in \mathrm{Diff}^\infty(\mathbb{B}^n)$ to

$$\begin{cases} \det D\varphi = \theta, & \\ \varphi = \mathrm{Id} & \text{in } \mathscr{C}, \end{cases}$$

(for the regularity of φ see (B) above). Moreover (see below),

$$|\theta - 1|_{1;\mathbb{B}^n}, \quad |u|_{1;\mathbb{B}^n} < \delta \le 1/2 \Longrightarrow |u_t|_{1;\mathbb{B}^n} < 8\delta \le 4 \quad \text{for all } t \in [0, 1]. \tag{4.16}$$

Therefore (still by [**CDK**, Lemma 10.4]),

$$|\varphi - \mathrm{Id}|_{1;\mathbb{B}^n} \le C(n) \int_0^1 |u_t|_{1;\mathbb{B}^n}\, dt \le C(n)8\delta.$$

It remains to show that (4.16) holds.

(0) $\max_{t\in[0,1]} |u_t|_{0;\mathbb{B}^n} \le 2|u|_{0;\mathbb{B}^n} < 2\delta$ since, by hypothesis,

$$\min_{t\in[0,1]} \inf_{\mathbb{B}^n} f_t > 1/2.$$

(1) The partial derivatives of the components of u_t are of the form

$$\partial_j u_t^i = \frac{(\partial_j u^i)((1-t)\theta + t) - u^i(1-t)\partial_j\theta}{((1-t)\theta + t)^2}$$

and therefore

$$\max_{i,j;\, t\in[0,1]} \sup_{\mathbb{B}^n} |\partial_j u_t^i| \le \left(\frac{3\delta}{2} + \frac{\delta}{2}\right) \Big/ \frac{1}{4} = 8\delta$$

since $|\partial_j u^i|$, $|u^i| < \delta$, $\max_{t\in[0,1]} \sup_{\mathbb{B}^n} |f_t| < 3/2$, $t \in [0, 1]$ and $\sup_{\mathbb{B}^n} |\partial_j\theta| < 1/2$.

(D.7) From this point onward the estimates are the same as in (C.5)–(C.8) and, accordingly, we reindex the constants C_5, C_6, C_7, C_8 there as C_7, C_8, C_9, C_{10}, respectively. Again, we may assume that $C_k \ge 2$ for $1 \le k \le 10$ and following the chain of estimates it is immediate to verify that the constant $\chi = \chi(c, n)$ below satisfies the conclusions of Lemma 2 when $f \in C^{\infty}$: i.e.,

$$\chi = \frac{1}{C_1 C_2 C_3 \cdots C_{10}}.$$

Since $C_3 > n$, reasoning as in (C.9), it is immediate to verify that also, in this case, $g_{A,\lambda}$ is, in fact, a diffeomorphism onto its image. The proof of Lemma 2 is complete. □

A. *Appendix.*

A.1. *C^r norms of vector fields and maps.* Let $|\cdot|$ be the Euclidean norm on $\mathbb{R}^n$. Fix a (finite) regular C^{∞} atlas $(V_j, \phi_j)_{j \le m}$ of M. Let $A \subset M$ be an open set and let $X \in \mathfrak{X}^r(A)$ be a vector field of class C^r, $r \in \mathbb{Z}^+$, defined on A. On each (partial) local chart associated with A, $(V_j \cap A, \phi_j|_{V_j \cap A})$, X has an expression

$$X_j : \phi_j(V_j \cap A) \longrightarrow \mathbb{R}^n.$$

X is C^r*-bounded* on A (see §2) if the Whitney C^r norm of X is finite: i.e.,

$$\|X\|_{C^r;A} := \max_{j;0 \le k \le r} \sup_{\phi_j(V_j \cap A)} \|D^k X_j\| < \infty.$$

As the atlas is regular, C^r vector fields defined on M are always C^r-bounded. Here, $\|D^0 X_j(x)\| := |X_j(x)|$ and $\|D^k X_j(x)\| := \max_{u_i \in \mathbb{S}^{n-1}} |D^k X_j(x; u_1, \ldots, u_k)|$). In §3, we work with the equivalent norm

$$|X|_{r;A} := \max_{i,j;\, 0 \le |\sigma| \le r} \sup_{\phi_j(V_j \cap A)} |\partial^{\sigma} X_j^i|,$$

where $X_j = (X_j^1, \ldots, X_j^n)$ and σ runs over all multiindices $\sigma = (\sigma_1, \ldots, \sigma_n) \in \mathbb{N}_0^n$ for which $|\sigma| = \sum \sigma_i \le r$. It is easily seen that

$$|\cdot|_{r;A} \le \|\cdot\|_{C^r;A} \le n^{(r+1)/2} |\cdot|_{r;A},$$

noting that $\max_{x \in \mathbb{S}^{n-1}} \sum_{i=1}^n |x_i|$ is attained when $|x_1| = \cdots = |x_n| = n^{-1/2}$, which thus implies that $\lambda \le \|D^k X_j(x)\| \le n^{(k+1)/2}\lambda$ for $\lambda = \max_{i;|\sigma|=k} |\partial^{\sigma} X_j^i(x)|$.

With the obvious changes, the same definitions are adopted for the C^r norms of maps $X \in C^r(A; \mathbb{R}^q)$ (the local chart expressions of X being then of the form $X_j = X \circ \phi_i^{-1}$), provided we restrict ourselves to the subspace of those that are C^r bounded. In this context, if $h \in C^r(A)$ and either $X \in \mathfrak{X}^r(A)$ or $X \in C^r(A; \mathbb{R}^q)$, then by the Leibniz product rule,

$$|hX|_{r;A} \le 2^r |h|_{r;A} |X|_{r;A},$$

which is an inequality systematically used in §3.1.

A.2. *Local C^1-metrization of* $\mathrm{Diff}_{\mu}^{r,\alpha}(M)$ *and chart representations.*

Definition A.1. (We recall the convention $C^{r,0} := C^r$ and $C^{\infty,\alpha} := C^{\infty}$). Fix a conservative regular atlas $(V_i, \phi_i)_{i \le m}$ of M as before (see the convention in §2). Given $r \in \mathbb{Z}^+ \cup \{\infty\}$, $0 \le \alpha \le 1$, $\mathrm{Diff}_{\mu}^{r,\alpha}(M)$ is the group (under composition) of the $C^{r,\alpha}$ diffeomorphisms f of M onto itself preserving the volume form, $\omega = f^*(\omega)$, or, equivalently, the Lebesgue measure μ induced by it on M. These are the bijections $f : M \to M$ satisfying, for each pair $i, j \le m$:

(1) the map

$$f_{ji} = \phi_j \circ f \circ \phi_i^{-1} : \phi_i(V_i \cap f^{-1}(V_j)) \longrightarrow \mathbb{R}^n$$

is of class $C^{r,\alpha}$, and the same holds for f^{-1} in place of f; and

(2) $\det Df_{ji} \equiv 1$.

A.2.1. *Covering system for* $f \in \mathrm{Diff}_\mu^{r,\alpha}(M)$ *and local* C^1 *metrization.* Given $f \in \mathrm{Diff}_\mu^{r,\alpha}(M)$, by the compactness of M, one can find a finite open cover $B_{l\leq\widetilde{m}}$ of M and two maps

$$i, j : \{1, \ldots, \widetilde{m}\} \longrightarrow \{1, \ldots, m\}$$

such that

$$\overline{B_l} \subset V_{i(l)} \quad \text{and} \quad f(\overline{B_l}) \subset V_{j(l)}.$$

The triple $B_{l\leq\widetilde{m}}, i, j$ is called a *covering system for* f and will be denoted by Υ. For each $\epsilon > 0$, let $\mathscr{U}_{\epsilon,\Upsilon}(f)$ be the set of those $g \in \mathrm{Diff}_\mu^{r,\alpha}(M)$ such that, for all $l \leq \widetilde{m}$,

$$g(\overline{B_l}) \subset V_{j(l)} \quad \text{and} \quad \|g_l - f_l\|, \|D(g_l - f_l)\| < \epsilon,$$

where

$$g_l = \phi_{j(l)} \circ g \circ \phi_{i(l)}^{-1}|_{B_{l*}} \quad \text{and} \quad B_{l*} := \phi_{i(l)}(B_l),$$

f_l being defined in the same way. These $\mathscr{U}_{\epsilon,\Upsilon}(f)$ induce a C^1-topology on $\mathrm{Diff}_\mu^{r,\alpha}(M)$ (see, e.g., [**PR**, p. 262]), making it locally metrizable by the standard Whitney C^1 norm: for any $h, g \in \mathscr{U}_{\epsilon,\Upsilon}(f)$,

$$d_{C^1}(h, g) := \|h - g\|_{C^1} = \max_{l\leq\widetilde{m}} \|h_l - g_l\|_{C^1}.$$

Clearly, a covering system for f also works for any $g \in \mathscr{U}_{\epsilon,\Upsilon}(f)$, $\epsilon > 0$.

A.2.2. *Chart representations of* $f|_B$. Given $f \in \mathrm{Diff}_\mu^{r,\alpha}(M)$, suppose that $B \subset M$ is an open set such that $\overline{B} \subset V_i$ and $f(\overline{B}) \subset V_j$ for some $i, j \leq m$. Then

$$\widehat{f} = f_{ji,B} = \phi_j \circ f \circ \phi_i^{-1}|_{\phi_i(B)}$$

is a *chart representation of* $f|_B$ with domain $\phi_i(B) \subset \phi_i(V_i)$ and target $\phi_j(V_j)$. If $x \in B$, we call $\widehat{f}$ a *chart representation of* f *around* x. To simplify the notation, we abbreviate by x the point $\phi_i(x)$ representing x in the domain of $\widehat{f}$.

A.2.3. *Comparable chart representations.* Given any other $g \in \mathrm{Diff}_\mu^{r,\alpha}(M)$ such that $g(\overline{B}) \subset V_j$, $\widehat{f} = f_{ji,B}$ and $\widehat{g} = g_{ji,B}$ are called *comparable chart representations of* f *and* g *on* B (alternatively, *comparable chart representations of* $f|_B$ *and* $g|_B$). By the continuity of the composition operator in relation to the C^1 norm, if $\|\widehat{f} - \widehat{g}\|_{C^1}$ is small, then $\|\widetilde{f} - \widetilde{g}\|_{C^1}$ is small for any other pair of comparable chart representations of $f|_B$ and $g|_B$. Thus a C^1 perturbation of a chart representation of $f|_B$ results in C^1 perturbations of all other chart representations of $f|_B$, the transition between two such chart representations being explicitly given by

$$f_{\widehat{j}\,\widehat{\imath},B} = \phi_{\widehat{j}j} \circ f_{ji;B} \circ \phi_{i\widehat{\imath}},$$

where $\phi_{kl} = \phi_k \circ \phi_l^{-1}$ are the chart transition maps.

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A.3. *Statement and proof of Lemma A.1.*

LEMMA A.1. *Let M be a (second countable, Hausdorff) connected, boundaryless C^∞ n-manifold. Given a compact subset K with an open neighbourhood $U \subsetneq M$ such that $U \backslash K$ is connected, there is a compact n-submanifold V with connected C^∞ boundary such that $K \subset \operatorname{int} V$ and $V \subset U$.*

Proof. Take a finite cover $B_1, \ldots, B_j$ of K by open Euclidean balls† such that $V_0 := \bigcup_{i \leq j} \overline{B_i} \subset U$. Slightly perturbing the $\overline{B_i}$ if necessary, we can assume that the smooth $(n-1)$-spheres $\partial \overline{B_i}$ intersect transversely so that V_0 is a compact n-submanifold with piecewise smooth boundary. Smooth out the 'edges' of V_0 so that the resulting n-submanifold V_1 has C^∞ boundary and still satisfies $K \subset \operatorname{int} V_1$ and $V_1 \subset U$ (this is clearly possible since the smoothing can be performed arbitrarily near ∂V_0). Assume that ∂V_1 is disconnected (otherwise the proof is complete). The idea is to use the connectedness of $U \backslash K$ to connect successively and inside $U \backslash K$ all the components of ∂V_1, and thus create a new submanifold satisfying the desired conclusions. Needless to say, care must be taken to avoid the intercrossing of the 'connecting tubes', the nature of the 'connecting surgery' depending, at each step i, on whether the tube connecting two components of ∂V_i is contained in $V_i \backslash K$ or in $U \backslash \operatorname{int} V_i$ (see below).

There is no difficulty in showing that, given any component b_0 of ∂V_1, there is a distinct component b_1 and an injective C^∞ path $\gamma : [0, 1] \to U \backslash K$, $\gamma'(t) \neq 0$, such that

$$\gamma(0) \in b_0, \quad \gamma(]0, 1[) \cap \partial V_1 = \emptyset, \quad \gamma(1) \in b_1$$

and γ is transverse to ∂V_1 at $\gamma(0)$, $\gamma(1)$. Clearly, $\gamma^* := \gamma(]0, 1[)$ is contained either in (I) $(\operatorname{int} V_1) \backslash K$ or in (II) $U \backslash V_1$. Thicken the embedded segment $\gamma([0, 1])$ to a thin C^∞ embedded 'tube' $\mathbb{D}^{n-1} \times [0, 1] \overset{f_1}{\hookrightarrow} U \backslash K$ with its bases $\mathbb{D}^{n-1} \times 0$ and $\mathbb{D}^{n-1} \times 1$ attached (respectively) to b_0 and b_1 so that:

(1) the 'outer cylinder' $\mathbb{S}^{n-2} \times [0, 1]$ is smoothly attached to b_0 and b_1; and
(2) as γ^*, $C = f_1(\mathbb{D}^{n-1} \times]0, 1[)$ is disjoint from ∂V_1.

Now, as γ^*, C is contained either in (I) or in (II). In the first case, let

$$V_2 = V_1 \backslash f_1(\mathbb{B}^{n-1} \times [0, 1]) \quad \text{('worm-hole drilling')}$$

and, in the second,

$$V_2 = V_1 \cup f_1(\mathbb{D}^{n-1} \times [0, 1]) \quad \text{('solid handle attaching')}.$$

Since V_2 is obtained from V_1 modifying inside $U \backslash K$ only, it is immediate that V_2 is also an n-submanifold with C^∞ boundary still satisfying $K \subset \operatorname{int} V_2$ and $V_2 \subset U$, but ∂V_2 has one component less than ∂V_1. If ∂V_2 is still disconnected, then use a finite induction argument: we do with V_2 exactly what was done with V_1, decreasing again the number of boundary components by one. After $k-1$ steps (k = number of components of ∂V_1), we get a manifold $V = V_k$, as desired. □

† $D \subset M$ is an Euclidean open ball if there is some local chart (V_i, ϕ_i) such that $\overline{D} \subset V_i$ and, up to a translation, $\phi_i(D) = \lambda \mathbb{B}^n$ for some $\lambda > 0$.

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