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Rigidity of pressures of Hölder potentials and the fitting of analytic functions through them

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Abstract. The first part of this work is devoted to the study of higher derivatives of pressure functions of Hölder potentials on shift spaces with finitely many symbols. By describing the derivatives of pressure functions via the central limit theorem for the associated random processes, we discover some rigid relationships between derivatives of various orders. The rigidity imposes obstructions on fitting candidate convex analytic functions by pressure functions of Hölder potentials globally, which answers a question of Kucherenko and Quas. In the second part of the work, we consider fitting candidate analytic germs by pressure functions of locally constant potentials. We prove that all 1-level candidate germs can be realised by pressures of some locally constant potentials, as long as the number of symbols in the symbolic set is large enough. There are also some results on fitting 2-level germs by pressures of locally constant potentials obtained in the work.

Key words: thermodynamic formalism, pressure, Hölder potential, central limit theorem
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1. *Introduction*

This work deals with traditional topics in thermodynamic formalism [**Bow**, **Rue1**], which originates from theoretical physics. We focus on shift spaces with finitely many symbols here, which model dynamics of some smooth systems such as Axiom-A diffeomorphisms through Markov partitions. Given a symbolic set Λ of finitely many symbols and a continuous potential (observable) ϕ on the shift space $\Lambda^{\mathbb{N}}$, a core concept in thermodynamic formalism is the pressure $P(\phi)$. People are particularly interested in the pressure function $P(t\phi)$ with the variable $t > 0$ representing the inverse temperature. A sharp change in the pressure function (or other terms) is usually termed a phase transition as t varies, see for example [**IT1**, **IT2**, **KQW**, **Lop1**, **Lop2**, **Sar**].

For Hölder continuous potentials, Ruelle [**Rue2**] proved that the pressure function $P(t\phi)$ is analytic for $t \in (0, \infty)$ (in fact, he proved that $P(\psi)$ depends analytically on ψ for ψ in the Hölder space $C^h(X)$ with X being a transitive subshift space of finite type and $0 < h \leq 1$ being the exponent [**GT**]). A key ingredient in his proof is the use of the Ruelle (transfer) operator [**BDL**, **GLP**] acting on functions in the Hölder space. Moreover, the equilibrium measure of $t\phi$ for any $t > 0$ and Hölder potential ϕ is always unique, so there are in fact no phase transitions in this case. Let

$$P^{(n)}(t) = P^{(n)}(t\phi) = \frac{d^n P(t\phi)}{dt^n}$$

be the nth derivative of the pressure function $P(t\phi)$ with respect to $t \in (0, \infty)$ for some fixed Hölder potential ϕ. We also write

$$P^{(1)}(t) = P'(t),\ P^{(2)}(t) = P''(t),\ P^{(3)}(t) = P'''(t), \ldots$$

intermittently in the following. We discover that there is some rigid relationship between the derivatives of the pressure function.

THEOREM 1.1. *For a Hölder potential ϕ on a full shift space with finitely many symbols, let $P(t) = P(t\phi)$ be its pressure. Then there exists some positive number M_ϕ depending on ϕ, such that*

$$\sqrt{2\pi^3}(P^{(2)}(t))^{3/2}|P^{(3)}(t)| \leq 9|P^{(3)}(t)| + 2|P^{(4)}(t)| + 3\sqrt{2\pi^3}M_\phi(P^{(2)}(t))^{5/2} \tag{1.1}$$

for any $t > 0$.

The constants are chosen for convenience rather than optimality. It would be difficult to obtain explicit optimal bounds, which is not required for our application here.

A potential ϕ is said to be *generic* (or we say it defines a non-lattice distribution, cf. [**CP**, **Fel**, **PP**]) if for any normalised potential ψ, the spectral radius of the complex Ruelle operator $\mathcal{L}_{\psi+it\phi}$ is less than 1 for any $t \neq 0$. These potentials form an open dense set. In particular, the complement is nowhere dense and closed (in both the uniform and Hölder norms) since any function in this complementary set is necessarily cohomologous to a function in $C(\Lambda^{\mathbb{Z}}, 2\pi\mathbb{Z})$, up to a constant. For pressure functions of generic potentials, the following bounds hold.

THEOREM 1.2. *For a generic Hölder potential ϕ on a full shift space with finitely many symbols, let $P(t) = P(t\phi)$ be its pressure. Then there exists some positive number M_ϕ depending on ϕ, such that*

$$|P^{(3)}(t)(1 - \sqrt{2\pi}(P^{(2)}(t))^{3/2})| \leq 3M_\phi P^{(2)}(t) \tag{1.2}$$

for any $t > 0$.

This means the second derivative of the pressure function of a generic Hölder potential imposes some global subtle restriction on its third derivative. It would be interesting to try to interpret the meaning of $P''(t) = 1/\sqrt[3]{2\pi}$ for the pressure function at individual parameters. Let $\sigma : \Lambda^{\mathbb{N}} \to \Lambda^{\mathbb{N}}$ denote the shift map. Both the proofs of Theorems 1.1 and 1.2 require use of the Ruelle operator and the central limit theorem (CLT) for the process $\{f \circ \sigma^n\}_{n\in\mathbb{N}}$, with the latter one depending on a finer CLT in the generic case. Recall that

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there are some expressions on the higher derivatives of the pressure function by Kotani and Sunada in [**KS1**] for smooth systems, and we refer the readers to [**KS2**] for a CLT for random walks on crystal lattices.

It is well known that $P(t\phi)$ is convex and Lipschitz for continuous ϕ, moreover, the supporting lines of its graph must intersect the vertical axis in a closed bounded interval in $[0, \infty)$. Kucherenko and Quas have shown that any such function can be realised by the pressure function of some continuous potential on some shift space [**KQ1**, Theorem 1], whose result fits into Katok's flexibility programme [**BKR**]. However, the continuous potentials constructed in their work are not Hölder, so they ask the following question (their original problem is set in the multidimensional case).

Problem 1.3. (Kucherenko and Quas) Can a convex, Lipschitz analytic function with its supporting lines intersecting the vertical axis in a closed bounded interval in $[0, \infty)$ be realised by the pressure function of some Hölder potential on some shift space with finite symbols?

In this work, we are dedicated to an answer to their problem. We first point out that any convex, Lipschitz analytic function with its supporting lines intersecting the vertical axis in a closed bounded interval in $[0, \infty)$ can be *approximated* by sequences of pressure functions of locally constant potentials (a potential $\phi : \Lambda^{\mathbb{Z}} \to \mathbb{R}$ is locally constant if there exists some integer $k > 0$ such that for any $x = \cdots x_{-1}x_0x_1 \cdots \in \Lambda^{\mathbb{Z}}$, the value $\phi(x)$ depends only on the terms $x_{-k}, \ldots, x_k$) on some shift space with finitely many symbols.

COROLLARY 1.4. *Let $F(t)$ be a convex Lipschitz function on (α, ∞) for some $\alpha > 0$ with Lipschitz constant $L > 0$, such that its supporting lines intersect the vertical axis in $[\underline{\gamma}, \overline{\gamma}]$ with $0 \le \underline{\gamma} \le \overline{\gamma} < \infty$. Then there exists a sequence of locally constant potentials $\{\phi_n\}_{n=1}^{\infty}$ on some shift space with finite symbols, such that*

$$\lim_{n\to\infty} P(t\phi_n) = F(t) \tag{1.3}$$

for any $t \in (\alpha, \infty)$.

Proof. This is an instant corollary of the result of Kucherenko and Quas in [**KQ1**]. Let

$$\Lambda = \{0, 1, \ldots, \lfloor e^{\overline{\gamma}} \rfloor\} \times \{\lfloor \underline{\gamma} \rfloor, \ldots, \lceil \overline{\gamma} \rceil\} \times \{\lfloor -L \rfloor, \ldots, \lceil L \rceil\},$$

where $\lfloor\ \rfloor$ and $\lceil\ \rceil$ represent the floor and ceiling function, respectively. According to [**KQ1**, Theorem 1], there exists a continuous potential $\phi_F : \Lambda^{\mathbb{Z}} \to \mathbb{R}$, such that

$$P(t\phi_F) = F(t)$$

on (α, ∞). Now let

$$\phi_n(x) \doteq \phi_{n,-}(x) = \inf\{\phi_F(x) : x \in [x_{-n}x_{-n+1} \cdots x_n]\}$$

for any $x = \cdots x_{-(n+1)}x_{-n} \cdots x_n x_{n+1} \cdots \in \Lambda^{\mathbb{Z}}$ and $n \in \mathbb{N}$, where $[x_{-n}x_{-n+1} \cdots x_n]$ means the corresponding cylinder set. Here ϕ_n is a locally constant potential for any fixed n. Now fix $t \in (\alpha, \infty)$ by properties of the pressure function (see for example [**Rue1**, 6.8]),

$$|P(t\phi_n) - P(t\phi_F)| \le |t| \, \| \phi_n - \phi_F \|_{\infty}. \tag{1.4}$$

Since ϕ_F is continuous, this implies equation (1.3). □

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One can see that in the above proof, the increasing sequence of pressures $\{P(t\phi_{n,-})\}_{n\in\mathbb{N}}$ satisfies

$$P(t\phi_{n,-}) \nearrow F(t)$$

as $n \to \infty$ since $\{\phi_{n,-}\}_{n\in\mathbb{N}}$ is an increasing sequence tending to ϕ_F (see [**Wal1**, Theorem 9.7(ii)]). Alternatively, one can take

$$\phi_{n,+}(x) = \sup\{\phi_F(x) : x \in [x_{-n}x_{-n+1}\cdots x_n]\},$$

which results in a decreasing sequence of locally constant potentials approximating $\phi_F(x)$, or

$$\phi_{n,\pm}(x) = \frac{\phi_{n,-}(x) + \phi_{n,+}(x)}{2},$$

which also results in a sequence of locally constant potentials approximating $\phi_F(x)$, while their pressure functions both approximate $F(t)$. See Corollary 5.4 for an interpretation of the result from another point of view.

Remark 1.5. The convergence in Corollary 1.4 is uniform for t in a bounded domain since $\Lambda^{\mathbb{Z}}$ is a compact metric space by equation (1.4).

Remark 1.6. A locally constant potential is of course Hölder, so according to Ruelle's result, the pressure functions $\{P(t\phi_{n,-})\}_{n\in\mathbb{N}}$ are all analytic.

The following result confirms that some convex analytic functions cannot be fitted by the pressure of any Hölder potential on any shift space, which gives a negative answer to Problem 1.3.

THEOREM 1.7. *For any $\alpha > 0$, there exist $0 < \underline{\gamma} < \overline{\gamma}$ and a strictly convex analytic function $F(t)$ on (α, ∞), with its supporting lines intersecting the vertical axis in the interval $[\underline{\gamma}, \overline{\gamma}]$, such that there does not exist any Hölder potential ϕ on any shift space with finite symbols satisfying*

$$P(t\phi) = F(t)$$

on (α, ∞).

We note that the supporting lines taking positive intersections with the vertical axis is due to the associated equilibrium states having positive entropy.

For an explicit example of convex analytic functions in Theorem 1.7, one can simply take

$$F_{2,3,1}(t) = \frac{2t^2 + 3t + te^{-t^2} + e^{-t^2}}{t}$$

on (α, ∞) for any $\alpha > 0$. See Proposition 4.2 for a family of such examples. Thus, one can see that there are in fact elementary functions which cannot be fitted by pressures of Hölder potentials on shift spaces with finite symbols.

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Remark 1.8. After this paper was completed, we became aware of an elegant paper of Kucherenko and Quas [**KQ2**] which showed that there is a precise lower bound on the 'speed' that the pressure function of a (cohomologously non-constant) Hölder potential approaches its asymptote. In particular, they used this analysis to give a negative answer to Problem 1.3. We refer the reader to [**KQ2**] for other interesting rigidity results on the pressure functions of Hölder potentials.

In the following, we consider fitting convex analytic functions locally instead of globally, only by pressures of locally constant potentials on shift spaces with finite symbols. Let

$$\Lambda_n = \{1, 2, \ldots, n\}$$

be the symbolic set of n symbols.

THEOREM 1.9. *Let $t_* > 0$ and $(a_0, a_1) \in \mathbb{R}^2$ satisfying*

$$\frac{a_0}{t_*} > a_1. \tag{1.5}$$

Then for any $n \in \mathbb{N}$ large enough, there exist some $0 \le m_{t_*,a_0,a_1,n} < M_{t_*,a_0,a_1,n} < \infty$ depending on t_*, a_0, a_1, n, such that for any $a_2 \in [m_{t_*,a_0,a_1,n}, M_{t_*,a_0,a_1,n}]$, there exists some sequence of reals $\{c_{i,n}\}_{i=1}^n$, such that the locally constant potential

$$\phi(x) = c_{x_0,n}$$

for $x = \cdots x_{-1}x_0x_1 \cdots \in [x_0]$ on the full shift space $\Lambda_n^{\mathbb{Z}}$ satisfies

$$P(t\phi) = a_0 + a_1(t - t_*) + \frac{a_2}{2!}(t - t_*)^2 + O((t - t_*)^3) \tag{1.6}$$

on $[t_* - \delta_n, t_* + \delta_n]$ for some $\delta_n > 0$.

This means we can fit some germs of level 2 at t_* by pressures of some locally constant potentials when the number of symbols of the shift space is large enough. The values δ_n, $\{c_{i,n}\}_{i=1}^n$ all depend on t_*, a_0, a_1, n and a_2 in fact, while we only indicate the dependence of $m_{t_*,a_0,a_1,n}$ and $M_{t_*,a_0,a_1,n}$ as we are particularly interested in their values in the context of Theorem 1.9. There are some results on the values of

$$\{m_{t_*,a_0,a_1,n}, M_{t_*,a_0,a_1,n}\}_{n\in\mathbb{N}}$$

subject to $t_* > 0$ and $(a_0, a_1) \in \mathbb{R}^2$ satisfying equation (1.5) at the end of §5.

We choose to present all our results in the one-dimensional case, although many of these results can in fact be extended naturally to convex Lipschitz or analytic functions $F(t_1, t_2, \ldots, t_m)$ of m variables. Most of our results also hold on transitive subshift spaces of finite type, with some technical adjustments in their proofs involving the transition matrix. We lay emphasis on two-sided shift spaces with finite symbols in this work; however, some concepts and proofs will be given directly on one-sided shift spaces as we are to employ the Ruelle operator in due course. Since every Hölder potential on two-sided shifts induces a cohomologous Hölder potential on one-sided shifts, these extend naturally onto two-sided ones.

The organisation of the work is as follows. In §2, we introduce some basics in thermodynamic formalism and the CLT for the process generated by a potential and the shift map on the symbolic space with finite symbols. We give an explicit bound on the tail term in the CLT. Section 3 is devoted to the proof of Theorems 1.1 and 1.2. We formulate some expression of the derivatives of the pressure (Corollary 3.11) linking directly to the CLT, which allows us to unveil the relationship between derivatives of the pressure function of various orders. Section 4 is devoted to the proof of Theorem 1.7. In §5, we consider fitting 1- and 2-level candidate analytic germs locally by pressure functions of locally constant potentials (Problem 5.2) on symbolic spaces with finite symbols. We conjecture that any reasonable germ of finite level can be fitted by the pressure function of some locally constant potential locally, as long as the number of the symbols is large enough.

2. *Thermodynamic formalism and the CLT*

In this section, we collect some basic notions and results in thermodynamic formalism for later use. We start from the pressure. Let Λ be some symbolic set with finite symbols, and $\Lambda^{\mathbb{N}}$ be the shift space equipped with the metric

$$d(x, y) = \frac{1}{2^{l(x,y)}}$$

for distinct $x = x_0x_1x_2\ldots, y = y_0y_1y_2\ldots \in \Lambda^{\mathbb{N}}$, where

$$l(x, y) = \min\{i \in \mathbb{N} : x_i \neq y_i\}.$$

For a continuous potential $\phi : \Lambda^{\mathbb{N}} \to \mathbb{R}$ on the compact metric space $\Lambda^{\mathbb{N}}$, let

$$S_{m,\phi}(x) = \sum_{i=0}^{m-1} \phi \circ \sigma^i(x)$$

for $m \in \mathbb{N}$, where σ is the shift map.

Definition 2.1. The pressure $P(\phi)$ of a continuous potential ϕ on $\Lambda^{\mathbb{N}}$ is defined to be

$$P(\phi) = \lim_{m\to\infty} \frac{1}{m} \log \sum_{\sigma^m(x)=x} e^{S_{m,\phi}(x)}.$$

One can refer to [**Wal1**, p. 208] for a definition for continuous potentials on general compact metric spaces. It satisfies the well-known variational formula

$$P(\phi) = \sup\left\{h(\mu) + \int \phi\, d\mu : \mu \text{ is a } \sigma\text{-invariant measure on } \Lambda^{\mathbb{N}}\right\}.$$

Let $C^0(\Lambda^{\mathbb{N}})$ be the collection of all the continuous potentials on $\Lambda^{\mathbb{N}}$. Two potentials $\psi, \phi \in C^0(\Lambda^{\mathbb{N}})$ are said to be *cohomologous* [**Wal2**] in the case where there exists a continuous map $\zeta : \Lambda^{\mathbb{N}} \to \mathbb{R}$ such that

$$\psi(x) - \phi(x) = \zeta(x) - \zeta \circ \sigma(x).$$

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We write $\psi \sim \phi$ to denote the equivalence relationship between two potentials cohomologous to each other. The maps in

$$\{\zeta(x) - \zeta \circ \sigma(x) : \zeta \in C^0(\Lambda^{\mathbb{N}})\}$$

are called *coboundaries*. The importance of the cohomologous relationship is revealed in the following result.

PROPOSITION 2.2. *If $\psi \sim \phi$, then $P(\psi) = P(\phi)$. Moreover, ψ and ϕ share the same equilibrium states.*

Another important tool in thermodynamic formalism is the *Ruelle operator*.

Definition 2.3. For a continuous potential $\psi : \Lambda^{\mathbb{N}} \to \mathbb{R}$, define the Ruelle operator $\mathcal{L}_\psi$ acting on $C^0(\Lambda^{\mathbb{N}})$ as

$$(\mathcal{L}_\psi f)(x) = \sum_{y:\sigma(y)=x} e^{\psi(y)} f(y)$$

for $f \in C^0(\Lambda^{\mathbb{N}})$.

One can see easily that its compositions satisfy

$$(\mathcal{L}_\psi^m f)(x) = \sum_{y:\sigma^m(y)=x} e^{S_{m,\psi}(y)} f(y) \tag{2.1}$$

for any $m \in \mathbb{N}$. For $\psi \in C^h(\Lambda^{\mathbb{N}})$, it admits a simple maximum isolated eigenvalue $\lambda = e^{P(\psi)}$ such that

$$(\mathcal{L}_\psi w_\psi)(x) = e^{P(\psi)} w_\psi(x) \tag{2.2}$$

for some eigenfunction $w_\psi(x) \in C^h(\Lambda^{\mathbb{N}})$, refer to [Rue1]. It then follows that

$$(\mathcal{L}_\psi^m w_\psi)(x) = e^{mP(\psi)} w_\psi(x) \tag{2.3}$$

for $w_\psi(x) \in C^h(\Lambda^{\mathbb{N}})$. A potential ψ is said to be *normalised* if

$$P(\psi) = 0 \quad \text{and} \quad w_\psi = 1_{\Lambda^{\mathbb{N}}},$$

where $1_{\Lambda^{\mathbb{N}}}$ is the identity map on $\Lambda^{\mathbb{N}}$. In the case of ψ being not normalised, we call

$$\bar{\psi} = \psi + \log w_\psi - \log w_\psi \circ \sigma - P(\psi)$$

the *normalisation* of ψ. It is easy to check that $\bar{\psi}$ is a normalised potential. Moreover, $\bar{\psi}$ and ψ share the same equilibrium state.

The unique equilibrium measure for a Hölder potential ψ is denoted by μ_ψ in the following. Now we turn to the CLT for the random process $\{\phi \circ \sigma^j(x)\}_{j=0}^\infty$ with the equilibrium measure μ_ψ defined by some Hölder potential ψ, while ϕ is also assumed to be Hölder. It deals with the asymptotic behaviour of the distribution of $S_{m,\phi}/\sqrt{m}$ with respect to μ_ψ as $m \to \infty$. The Ruelle operator comes in here, see [CP, Lal, Rou]. Let

$$G_m(y) = \mu_\psi\left\{x \in \Lambda^{\mathbb{N}} : \frac{S_{m,\phi}(x)}{\sqrt{m}} < y\right\}$$

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for $y \in \mathbb{R}$. For $a, b \in \mathbb{R}$ and $b > 0$, let $N_{a,b}(y)$ be the normal distribution with expectation a and standard deviation $\sqrt{b}$ on $\mathbb{R}$, that is,

$$\frac{dN_{a,b}(y)}{dy} = \frac{1}{\sqrt{2\pi b}} e^{-(y-a)^2/2b}$$

for $y \in \mathbb{R}$. For Hölder potentials ψ, ϕ on a shift space, since the pressure $P(\psi + t\phi)$ is analytic in a small neighbourhood around 0, denote by

$$\Delta_m = P^{(m)}(\psi + t\phi)|_{t=0}$$

for $m \in \mathbb{N}$ for convenience, while the readers can understand its dependence on ψ, ϕ easily from the contexts in the following. Let

$$P(\psi + t\phi) = \sum_{m=0}^{\infty} \frac{\Delta_m}{m!} t^m = \sum_{m=0}^{3} \frac{\Delta_m}{m!} t^m + t^4 \kappa(t),$$

where $\kappa(t) = \sum_{m=0}^{\infty} (\Delta_{m+4}/(m+4)!) t^m$.

We now come to one of the key ingredients in the proofs of Theorems 1.1 and 1.2.

CENTRAL LIMIT THEOREM. *Let ψ, ϕ be Hölder potentials on a shift space with ϕ being not cohomologous to a constant. If $\int \phi \, d\mu_\psi = 0$, we have*

$$G_m(y) = N_{0,\Delta_2}(y) + O(1/\sqrt{m}),$$

where

$$O(1/\sqrt{m}) \leq \frac{9|\Delta_3| + 2|\Delta_4|}{\sqrt{2\pi^3 m}(\Delta_2)^{3/2}}. \tag{2.4}$$

The convergence is uniform with respect to y. In the case of ϕ being generic, we have

$$G_m(y) = N_{0,\Delta_2}(y) + H_m(y) + o(1/\sqrt{m}), \tag{2.5}$$

where $H_m(y) = (\Delta_3/6\sqrt{m})(1 - (y^2/\Delta_2))e^{-(y^2/2\Delta_2)}$.

The bounds of error terms in equations (2.4) and (2.5) will be used in the proofs of Theorems 1.1 and 1.2, respectively. In the following, we will justify equation (2.4), while equation (2.5) follows from existing results.

This fits into special cases of the *Berry–Esseen theorem* [**Fel**]. A significant point in the version here comparing with [**CP**, Theorems 2, 3] or [**PP**, Theorem 4.13] is the explicit bound on the tail term $O(1/\sqrt{m})$ in equation (2.4). In the following, we justify this explicit bound. To do this, let

$$\chi_m(z) = \int e^{iz(S_{m,\phi}/\sqrt{m})} \, d\mu_\psi$$

be the Fourier transformation of $G_m(y)$. Note that the Fourier transformation of $N_{0,\Delta_2}(y)$ is $e^{-(z^2\Delta_2/2)}$.

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LEMMA 2.4. *Let ψ, ϕ be Hölder potentials on a shift space with ϕ being not cohomologous to a constant. For $\epsilon > 0$ small enough, we have*

$$\frac{1}{2\pi}\int_0^{\epsilon\sqrt{m}} \frac{1}{z}|\chi_m(z) - e^{-(z^2\Delta_2/2)}|\, dz \le \frac{\sqrt{2}|\Delta_3|}{12\sqrt{\pi m}(\Delta_2)^{3/2}} \tag{2.6}$$

for any $m \in \mathbb{N}$ large enough.

Proof. According to [**PP**, equation (4.6)], we have

$$\int_0^{\epsilon\sqrt{m}} \frac{1}{z}\left|\chi_m(z) - e^{-(z^2\Delta_2/2)} + \frac{iz^3\Delta_3}{6\sqrt{m}}e^{-(z^2\Delta_2/2)}\right| dz = O(1/m)$$

for $\epsilon > 0$ small enough. So

$$\frac{1}{2\pi}\int_0^{\epsilon\sqrt{m}} \frac{1}{z}\left|\chi_m(z) - e^{-(z^2\Delta_2/2)}\right| dz \le O(1/m) + \frac{|\Delta_3|}{12\pi\sqrt{m}}\int_0^{\epsilon\sqrt{m}} z^2 e^{-(z^2\Delta_2/2)}\, dz. \tag{2.7}$$

By

$$\int_{-\infty}^{\infty} z^2 e^{-(z^2\Delta_2/2)}\, dz = \frac{\sqrt{2\pi}}{(\Delta_2)^{3/2}},$$

we obtain equation (2.6) from equation (2.7). □

Equipped with Lemma 2.4, we can justify the explicit bound on the tail term in the CLT in equation (2.4).

Proof of the tail term in CLT.

Proof. Without loss of generality, suppose ψ is normalised and $\int \phi\, d\mu_\psi = 0$. It suffices for us to justify equation (2.4) by [**CP**, Theorems 2, 3]. Similar to the proof of [**CP**, Theorem 2], apply [**Fel**, Lemma 2] with the cumulative functions $G_m(y)$ and $N_{0,\Delta_2}(y)$, in our case, one gets (cf. [**CP**, (20)])

$$|G_m(y) - N_{0,\Delta_2}(y)| \le \frac{1}{2\pi}\int_0^{\epsilon\sqrt{m}} \frac{1}{z}\left|\chi_m(z) - e^{-(z^2\Delta_2/2)}\right| dz + \frac{24}{\epsilon\sqrt{2m\pi^3\Delta_2}}. \tag{2.8}$$

Now let us take

$$\frac{1}{\epsilon} = \frac{2}{\Delta_2}\left(\frac{|\Delta_3|}{6} + \frac{|\Delta_4|}{24} + \delta\right)$$

for some small $\delta > 0$, such that it satisfies (cf. [**CP**, (10)])

$$\frac{1}{\epsilon} > \max\left\{\frac{2}{\Delta_2}\left(\frac{|\Delta_3|}{6} + t\kappa(t)\right), \frac{2}{\Delta_2}\kappa(t)\right\}$$

for any $|t| < \epsilon$ in equation (2.8). By equation (2.6), we have

$$|G_m(y) - N_{0,\Delta_2}(y)| \le \frac{\sqrt{2}|\Delta_3|}{12\sqrt{\pi m}(\Delta_2)^{3/2}} + \frac{24}{\sqrt{2m\pi^3\Delta_2}}\frac{2}{\Delta_2}\left(\frac{|\Delta_3|}{6} + \frac{|\Delta_4|}{24} + \delta\right)$$

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$$= \frac{\sqrt{2}|\Delta_3|}{12\sqrt{\pi m}(\Delta_2)^{3/2}} + \frac{8|\Delta_3|}{\sqrt{2\pi^3 m}(\Delta_2)^{3/2}} + \frac{2|\Delta_4|}{\sqrt{2\pi^3 m}(\Delta_2)^{3/2}} + \frac{48\delta}{\sqrt{2\pi^3 m}(\Delta_2)^{3/2}} \tag{2.9}$$

$$\leq \frac{9|\Delta_3|}{\sqrt{2\pi^3 m}(\Delta_2)^{3/2}} + \frac{2|\Delta_4|}{\sqrt{2\pi^3 m}(\Delta_2)^{3/2}} + \frac{48\delta}{\sqrt{2\pi^3 m}(\Delta_2)^{3/2}}.$$

Finally, letting $\delta \to 0$ in equation (2.9), we get equation (2.4). □

We will deal with the pressure function $P(\psi + t\phi)$ for $t \geq 0$ and $\psi, \phi \in C^h(\Lambda^{\mathbb{N}})$ for some $0 < h \leq 1$ in the following sections. By [**Rue2**], $P(\psi + t\phi)$ depends analytically on t in the case that ψ, ϕ are Hölder. We will often assume that

$$\int \phi \, d\mu_\psi = 0$$

in the following when dealing with the higher derivatives of $P(\psi + t\phi)$ because if $\int \phi \, d\mu_\psi = c \neq 0$, we have

$$P(\psi + t(\phi - c)) = P(\psi + t\phi) - ct,$$

then

$$\frac{d^n P(\psi + t(\phi - c))}{dt^n} = \frac{d^n P(\psi + t\phi)}{dt^n} \tag{2.10}$$

for any $n \geq 2$ while $\int (\phi - c) \, d\mu_\psi = 0$. We can also assume that ψ is normalised when dealing with the derivatives of $P(\psi + t\phi)$. If this is not the case, we can simply change ψ to its normalisation $\bar{\psi}$ while

$$\frac{d^n P(\psi + t\phi)}{dt^n} = \frac{d^n P(\bar{\psi} + t\phi)}{dt^n} \tag{2.11}$$

for $n \geq 1$ because

$$P(\bar{\psi} + t\phi) = P(\psi + t\phi) - P(\psi)$$

for any $t \in \mathbb{R}$.

3. *Derivatives of the pressures of Hölder potentials*

In this section, we formulate some explicit expressions for the derivatives of the pressure $P(t\phi) = P(t)$ in terms of the derivatives of the eigenfunction of $\mathcal{L}_{t\phi}$ for $\phi \in C^h(\Lambda^{\mathbb{N}})$ with respect to t. We give basically two expressions of the derivatives, one of which allows the introduction of the random stochastic process $\{\phi \circ \sigma^j(x)\}_{j=0}^m$ for $m \in \mathbb{N}$. The CLT for the random process $\{\phi \circ \sigma^j(x)\}_{j=0}^\infty$ takes core role in our proofs of Theorems 1.1 and 1.2.

First we define some basics to deal with the higher derivatives of compositional functions by the *Faà di Bruno's formula*. For an integer $j \in \mathbb{N}$, we say

$$\tau = \tau_1 \tau_2 \cdots \tau_q$$

with $q \in \mathbb{N}$ is a *partition* of j if the non-increasing sequence of positive integers $j \geq \tau_1 \geq \tau_2 \geq \cdots \geq \tau_q \geq 1$ satisfies $\sum_{i=1}^q \tau_i = j$. Denote the collection of all the possible partitions of j by $\mathfrak{P}(j)$. For example, Table 1 lists all the partitions in $\mathfrak{P}(5)$.

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TABLE 1. Partitions of 5.

5	$q = 1$
4,1	$q = 2$
3,2	$q = 2$
3,1,1	$q = 3$
2,2,1	$q = 3$
2,1,1,1	$q = 4$
1,1,1,1,1	$q = 5$

TABLE 2. The coefficients B_5^τ.

$B_5^5 = 1$
$B_5^{4,1} = 5$
$B_5^{3,2} = 10$
$B_5^{3,1,1} = 10$
$B_5^{2,2,1} = 15$
$B_5^{2,1,1,1} = 10$
$B_5^{1,1,1,1,1} = 1$

We sometimes simply write τ to denote the set $\{\tau_1, \tau_2, \ldots, \tau_q\}$ for convenience in the following, so $\#\tau = q$. Now for τ being a partition of $j \geq 1$, let $\{B_j^\tau\}$ be the number of different choices of dividing a set of j different elements into $\#\tau = q$ sets of sizes $\{\tau_i\}_{i=1}^q$ (with no order on the sets of partitions). Set $B_0^0 = 1$ for convenience. For example, consider the cases $j = 5$ and $\tau = 3, 1, 1$, the number of different choices of dividing a set of 5 different elements into $q = 3$ sets of sizes 3, 1, 1 respectively is

$$C_5^3 = 10 = B_5^{3,1,1}.$$

Table 2 lists all the numbers $\{B_5^\tau\}_{\tau\in\mathfrak{P}(5)}$.

For a smooth map $f : \mathbb{R} \to \mathbb{R}$ on the real line (which suffices for our purposes in this work) and some partition $\tau = \tau_1, \tau_2, \ldots, \tau_q \in \mathfrak{P}(j)$ with $j \geq 1$, let

$$f^{(\tau)}(x) = f^{(\tau_1)}(x) f^{(\tau_2)}(x) \cdots f^{(\tau_q)}(x)$$

be the product of the derivatives. For $j = 0$ and $\tau = 0 \in \mathfrak{P}(0)$, set $f^{(0)}(x) = 1$. Then for two smooth functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$, we have

$$\frac{d^j(g \circ f(x))}{dx^j} = \sum_{\tau\in\mathfrak{P}(j)} B_j^\tau g^{(\#\tau)}(f(x)) f^{(\tau)}(x) \tag{3.1}$$

by virtue of Faà di Bruno's formula.

Now we turn to the higher derivatives of the pressure function. We start from some standard case, then extend the result to the general case.

THEOREM 3.1. *Let $\psi, \phi \in C^h(\Lambda^{\mathbb{N}})$ with ψ being normalised for some finite symbolic set Λ. Assume $\int \phi \, d\mu_\psi = 0$, where μ_ψ is the equilibrium state of ψ. Let $w(t, x)$ be the eigenfunction of the maximum isolated eigenvalue $e^{P(\psi+t\phi)}$ of $\mathcal{L}_{\psi+t\phi}$, which depends analytically on t in a small neighbourhood of* 0. *Considering the derivatives of the pressure function $P(\psi + t\phi)$ at $t = 0$, we have*

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$$
\begin{aligned}
P^{(n)}(\psi + t\phi)|_{t=0} = \sum_{j=1}^{n} C_n^j \int_{\Lambda^{\mathbb{N}}} (\phi(x))^j w^{(n-j)}(0, x)\, d\mu_\psi(x) \\
- \sum_{j=2}^{n-2} C_n^j \sum_{\tau \in \mathfrak{P}(j), 1 \notin \tau} B_j^\tau P^{(\tau)}(\psi + t\phi)|_{t=0} \int_{\Lambda^{\mathbb{N}}} w^{(n-j)}(0, x)\, d\mu_\psi(x) \\
- \sum_{\tau \in \mathfrak{P}(n), \{1,n\} \cap \tau = \emptyset} B_n^\tau P^{(\tau)}(\psi + t\phi)|_{t=0}
\end{aligned}
\tag{3.2}
$$

for any $n \geq 2$.

Proof. According to the above notation, note that

$$
(\mathcal{L}_{\psi+t\phi} w(t, \cdot))(x) = e^{P(\psi+t\phi)} w(t, x). \tag{3.3}
$$

The nth derivative of $(\mathcal{L}_{\psi+t\phi} w(t, \cdot))(x) = \sum_{y:\sigma(y)=x} e^{\psi(y)+t\phi(y)} w(t, y)$ gives

$$
\begin{aligned}
&\frac{d^n \mathcal{L}_{\psi+t\phi} w(t, \cdot)(x)}{dt^n} \\
&\quad = \sum_{y:\sigma(y)=x} \sum_{j=0}^{n} C_n^j \frac{d^j e^{(\psi+t\phi)(y)}}{dt^j} w^{(n-j)}(t, y) \\
&\quad = \sum_{y:\sigma(y)=x} \sum_{j=0}^{n} C_n^j e^{(\psi+t\phi)(y)} (\phi(y))^j w^{(n-j)}(t, y) \\
&\quad = \sum_{j=0}^{n} C_n^j \mathcal{L}_{\psi+t\phi}((\phi(\cdot))^j w^{(n-j)}(t, \cdot)).
\end{aligned}
\tag{3.4}
$$

All derivatives are with respect to t. In the case of $t = 0$, this means

$$
\left. \frac{d^n \mathcal{L}_{\psi+t\phi} w(t, \cdot)(x)}{dt^n} \right|_{t=0} = \sum_{j=0}^{n} C_n^j \mathcal{L}_\psi((\phi(\cdot))^j w^{(n-j)}(0, \cdot)). \tag{3.5}
$$

Note that the dual operator $\mathcal{L}_\psi^*$ fixes μ_ψ, so integration of both sides of equation (3.5) gives

$$
\int \left. \frac{d^n \mathcal{L}_{\psi+t\phi} w(t, \cdot)(x)}{dt^n} \right|_{t=0} d\mu_\psi(x) = \sum_{j=0}^{n} C_n^j \int (\phi(x))^j w^{(n-j)}(0, x) \mu_\psi(x). \tag{3.6}
$$

To get the nth derivative of $P(\psi + t\phi)$, differentiating $e^{P(\psi+t\phi)} w(t, x)$ for n times by equation (3.1), we get

$$
\begin{aligned}
&\frac{d^n (e^{P(\psi+t\phi)} w(t, x))}{dt^n} \\
&\quad = \sum_{j=0}^{n} C_n^j \frac{d^j e^{P(\psi+t\phi)}}{dt^j} w^{(n-j)}(t, x)
\end{aligned}
$$

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$$
\begin{aligned}
&= \sum_{j=0}^{n-1} C_n^j \frac{d^j e^{P(\psi+t\phi)}}{dt^j} w^{(n-j)}(t,x) + \frac{d^n e^{P(\psi+t\phi)}}{dt^n} w(t,x) \\
&= \sum_{j=0}^{n-1} C_n^j \sum_{\tau\in\mathfrak{P}(j)} B_j^\tau P^{(\tau)}(\psi+t\phi) e^{P(\psi+t\phi)} w^{(n-j)}(t,x) \\
&\quad + \sum_{\tau\in\mathfrak{P}(n)} B_n^\tau P^{(\tau)}(\psi+t\phi) e^{P(\psi+t\phi)} w(t,x) \\
&= \sum_{j=0}^{n-1} C_n^j \bigg(\sum_{\tau\in\mathfrak{P}(j),1\notin\tau} B_j^\tau P^{(\tau)}(\psi+t\phi) + \sum_{\tau\in\mathfrak{P}(j),1\in\tau} B_j^\tau P^{(\tau)}(\psi+t\phi) \bigg) \\
&\quad \times e^{P(\psi+t\phi)} w^{(n-j)}(t,x) \\
&\quad + \sum_{\tau\in\mathfrak{P}(n),n\notin\tau} B_n^\tau P^{(\tau)}(\psi+t\phi) e^{P(\psi+t\phi)} w(t,x) + P^{(n)}(\psi+t\phi) e^{P(\psi+t\phi)} w(t,x).
\end{aligned} \tag{3.7}
$$

Remember $P(\psi) = 0$ and $w(0,x) = 1$ as ψ is normalised [**PP**, p. 66]. Taking $t = 0$ in equation (3.7), we get

$$
\begin{aligned}
&\frac{d^n (e^{P(\psi+t\phi)} w(t,x))}{dt^n}\bigg|_{t=0} \\
&\quad = \sum_{j=0}^{n-1} C_n^j \bigg(\sum_{\tau\in\mathfrak{P}(j),1\notin\tau} B_j^\tau P^{(\tau)}(\psi+t\phi)|_{t=0} + \sum_{\tau\in\mathfrak{P}(j),1\in\tau} B_j^\tau P^{(\tau)}(\psi+t\phi)|_{t=0} \bigg) \\
&\quad \times w^{(n-j)}(0,x) \\
&\quad + \sum_{\tau\in\mathfrak{P}(n),n\notin\tau} B_n^\tau P^{(\tau)}(\psi+t\phi)|_{t=0} + P^{(n)}(\psi+t\phi)|_{t=0}.
\end{aligned} \tag{3.8}
$$

Since $\int \phi \, d\mu_\psi = P'(\psi+t\phi)|_{t=0} = 0$ and $\int w'(0,x)\, d\mu_\psi = 0$ [**PP**, p. 66], integrating both sides of equation (3.8) with respect to μ_ψ, we get

$$
\begin{aligned}
&\int \frac{d^n (e^{P(\psi+t\phi)} w(t,x))}{dt^n}\bigg|_{t=0} d\mu_\psi \\
&\quad = \sum_{j=0}^{n-1} C_n^j \sum_{\tau\in\mathfrak{P}(j),1\notin\tau} B_j^\tau P^{(\tau)}(\psi+t\phi)|_{t=0} \int w^{(n-j)}(0,x)\, d\mu_\psi \\
&\qquad + \sum_{\tau\in\mathfrak{P}(n),\{1,n\}\cap\tau=\emptyset} B_n^\tau P^{(\tau)}(\psi+t\phi)|_{t=0} + P^{(n)}(\psi+t\phi)|_{t=0}.
\end{aligned} \tag{3.9}
$$

Finally, combining equations (3.6) and (3.9) together, we get equation (3.2). □

Remark 3.2. The terms

$$
-\sum_{j=2}^{n-2} C_n^j \sum_{\tau\in\mathfrak{P}(j),1\notin\tau} B_j^\tau P^{(\tau)}(\psi+t\phi)|_{t=0} \int_{\Lambda^{\mathbb{N}}} w^{(n-j)}(0,x)\, d\mu_\psi(x)
$$

and

$$-\sum_{\tau\in\mathfrak{P}(n),\{1,n\}\cap\tau=\emptyset} B_n^\tau P^{(\tau)}(\psi+t\phi)|_{t=0}$$

in equation (3.2) are null in the case of $n \leq 3$. This also applies to the corresponding terms later.

Remark 3.3. These expressions are inductive formulae, although one can always get explicit expressions through substituting the lower derivatives $P^{(\tau)}(\psi+t\phi)|_{t=0}$ by their non-inductive versions depending only on $\phi(x)$, $\{w^{(j)}(0,x)\}_{j=1}^n$ and $\mu_\psi(x)$. This also applies to Theorem 3.7.

One can find some description of derivatives of the pressure function by *covariance* of the sequence of functions $\{\phi\circ\sigma^j\}_{j\in\mathbb{N}}$ in [**KS1**, Corollary 1] for smooth ϕ. Without the assumptions of ψ being normalised and $\int \phi\, d\mu_\psi = 0$, Theorem 3.1 evolves into the following form.

COROLLARY 3.4. *Let $\psi,\phi\in C^h(\Lambda^{\mathbb{N}})$ with some finite symbolic set Λ. Here, $\mathcal{L}_{\psi+t\phi}$ admits a maximum isolated eigenvalue $e^{P(\psi+t\phi)}$ close to $e^{P(\psi)}$ with eigenfunction $w(t,x)$ whose projection depends analytically on t in a small neighbourhood of* 0. *Considering the derivatives of the pressure $P(\psi+t\phi)$ at $t=0$, we have*

$$\begin{aligned}P^{(n)}(\psi+t\phi)|_{t=0} &= \sum_{j=1}^n C_n^j \int_{\Lambda^{\mathbb{N}}} \bigg(\phi(x)-\int\phi\, d\mu_\psi\bigg)^j w^{(n-j)}(0,x)\, d\mu_\psi(x)\\ &\quad -\sum_{j=2}^{n-2} C_n^j \sum_{\tau\in\mathfrak{P}(j),1\notin\tau} B_j^\tau P^{(\tau)}(\psi+t\phi)|_{t=0}\int_{\Lambda^{\mathbb{N}}} w^{(n-j)}(0,x)\, d\mu_\psi(x)\\ &\quad -\sum_{\tau\in\mathfrak{P}(n),\{1,n\}\cap\tau=\emptyset} B_n^\tau P^{(\tau)}(\psi+t\phi)|_{t=0} \end{aligned}\tag{3.10}$$

for any $n\geq 2$.

Proof. Let

$$\bar{\psi} = \psi + \log w_\psi(x) - \log w_\psi\circ\sigma - P(\psi),$$

where $w_\psi(x)$ is the eigenfunction of $\mathcal{L}_\psi$ corresponding to the eigenvalue $e^{P(\psi)}$. Taking pressure in the following equation:

$$\bar{\psi}+t\phi = \psi + t\phi + \log w_\psi(x) - \log w_\psi\circ\sigma - P(\psi),$$

then applying Proposition 2.2, we see that

$$P(\bar{\psi}+t\phi) = P(\psi+t\phi) - P(\psi).$$

This implies

$$\frac{d^n P(\bar{\psi}+t\phi)}{dt^n} = \frac{d^n P(\psi+t\phi)}{dt^n}\tag{3.11}$$

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for any $n \geq 1$. Now applying Theorem 3.1 to the normalised potential $\bar{\psi}$ and $\phi - \int \phi \, d\mu_\psi$, (note that $\int(\phi - \int \phi \, d\mu_\psi) \, d\mu_\psi = 0$ and $\mu_\psi = \mu_{\bar{\psi}}$), we justify the corollary by equation (3.11). □

In the following, we present some concrete formulae of some special order n by virtue of Theorem 3.1 for later use.

COROLLARY 3.5. *Let $\psi, \phi \in C^h(\Lambda^{\mathbb{N}})$ with ψ being normalised. Let μ_ψ be the equilibrium state of ψ and $\int \phi \, d\mu_\psi = 0$. Let $e^{P(\psi+t\phi)}$ be the maximum eigenvalue of $\mathcal{L}_{\psi+t\phi}$ with eigenfunction $w(t,x)$ for small t. Then we have*

$$P'''(\psi + t\phi)|_{t=0} = 3\int \phi w''(0,x) \, d\mu_\psi + 3\int \phi^2 w'(0,x) \, d\mu_\psi + \int \phi^3 \, d\mu_\psi. \quad (3.12)$$

Proof. This follows instantly from Theorem 3.1 with $n = 3$, along with some direct computations on the Faà di Bruno's coefficients $\{B_3^\tau\}_{\tau\in\mathfrak{P}(3)}$. □

COROLLARY 3.6. *Let $\psi, \phi \in C^h(\Lambda^{\mathbb{N}})$ with ψ being normalised. Let μ_ψ be the equilibrium state of ψ and $\int \phi \, d\mu_\psi = 0$. Let $e^{P(\psi+t\phi)}$ be the maximum eigenvalue of $\mathcal{L}_{\psi+t\phi}$ with eigenfunction $w(t,x)$ for small t. Then we have*

$$\begin{aligned}
&P''''(\psi + t\phi)|_{t=0} \\
&\quad = 4\int \phi w'''(0,x) \, d\mu_\psi + 6\int \phi^2 w''(0,x) \, d\mu_\psi + 4\int \phi^3 w'(0,x) \, d\mu_\psi + \int \phi^4 \, d\mu_\psi \\
&\qquad - 6P''(\psi + t\phi)|_{t=0} \int w''(0,x) \, d\mu_\psi - 3(P''(\psi + t\phi)|_{t=0})^2 \\
&\quad = 4\int \phi w'''(0,x) \, d\mu_\psi + 6\int \phi^2 w''(0,x) \, d\mu_\psi + 4\int \phi^3 w'(0,x) \, d\mu_\psi + \int \phi^4 \, d\mu_\psi \\
&\qquad - 6\bigg(\int \phi^2 \, d\mu_\psi + 2\int \phi w'(0,x) \, d\mu_\psi\bigg) \int w''(0,x) \, d\mu_\psi \\
&\qquad - 3\bigg(\int \phi^2 \, d\mu_\psi + 2\int \phi w'(0,x) \, d\mu_\psi\bigg)^2. \qquad (3.13)
\end{aligned}$$

Proof. The first equality follows instantly from Theorem 3.1 with $n = 4$ along with some direct computations on the Faà di Bruno's coefficients $\{B_4^\tau\}_{\tau\in\mathfrak{P}(4)}$. The second one is true as

$$P''(\psi + t\phi)|_{t=0} = \int \phi^2 \, d\mu_\psi + 2\int \phi w'(0,x) \, d\mu_\psi.$$

The latter description depends only on $\phi(x)$, $\{w^{(j)}(0,x)\}_{j=1}^3$ and $\mu_\psi(x)$. □

One can also get some precise formulae for some particular n in Corollary 3.4, and some non-inductive ones as we indicate in Remark 3.3. While equations (3.2), (3.10), (3.12), (3.13) all give interesting descriptions of the derivatives of the pressure function $P(\psi + t\phi)$, it seems to us difficult to discover any essential rigid restriction on them, or relationships between them. In the following, we turn to the description of them by the random stochastic process $\{\phi \circ \sigma^j(x)\}_{j=0}^\infty$. This is not a new idea on exploring the regularity of the pressure function $P(\psi + t\phi)$, as one can recall it from many others' work

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in thermodynamic formalism. Again, we first consider some standard case, then extend to the general case.

THEOREM 3.7. *Let $\psi, \phi \in C^h(\Lambda^{\mathbb{N}})$ with ψ being normalised. Let μ_ψ be the equilibrium state of ψ and $\int \phi \, d\mu_\psi = 0$. Let $e^{P(\psi+t\phi)}$ be the maximum isolated eigenvalue of $\mathcal{L}_{\psi+t\phi}$ with eigenfunction $w(t,x)$ whose projection depends analytically on t. Considering the derivatives of the pressure $P(\psi+t\phi)$ at $t=0$, we have*

$$\begin{aligned}P^{(n)}&(\psi+t\phi)|_{t=0}\\ &= \lim_{m\to\infty}\frac{1}{m}\bigg(\sum_{j=2}^{n} C_n^j \int_{\Lambda^{\mathbb{N}}} (S_{m,\phi}(x))^j w^{(n-j)}(0,x)\, d\mu_\psi(x)\\ &\quad -\sum_{j=2}^{n-2} C_n^j \sum_{\tau\in\mathfrak{P}(j),1\notin\tau} m^{\#\tau} B_j^\tau P^{(\tau)}(\psi+t\phi)|_{t=0} \int_{\Lambda^{\mathbb{N}}} w^{(n-j)}(0,x)\, d\mu_\psi(x)\\ &\quad -\sum_{\tau\in\mathfrak{P}(n),\{1,n\}\cap\tau=\emptyset} m^{\#\tau} B_n^\tau P^{(\tau)}(\psi+t\phi)|_{t=0}\bigg) \end{aligned} \tag{3.14}$$

for any $n \geq 2$.

Proof. The proof follows the routine of Proof of Theorem 3.1. Considering equation (2.1), we take n-derivatives on both sides of equation (2.3), take $t=0$, then integrate both sides with respect to $\mu_\psi(x)$, divided by m, and we get

$$\begin{aligned}P^{(n)}&(\psi+t\phi)|_{t=0}\\ &= \frac{1}{m}\bigg(\sum_{j=1}^{n} C_n^j \int_{\Lambda^{\mathbb{N}}} (S_{m,\phi}(x))^j w^{(n-j)}(0,x)\, d\mu_\psi(x)\\ &\quad -\sum_{j=2}^{n-2} C_n^j \sum_{\tau\in\mathfrak{P}(j),1\notin\tau} m^{\#\tau} B_j^\tau P^{(\tau)}(\psi+t\phi)|_{t=0} \int_{\Lambda^{\mathbb{N}}} w^{(n-j)}(0,x)\, d\mu_\psi(x)\\ &\quad -\sum_{\tau\in\mathfrak{P}(n),\{1,n\}\cap\tau=\emptyset} m^{\#\tau} B_n^\tau P^{(\tau)}(\psi+t\phi)|_{t=0}\bigg) \end{aligned} \tag{3.15}$$

as equation (3.2). Now since $w^{(n-1)}(0,x)$ is bounded on X, the ergodic theorem guarantees

$$\lim_{m\to\infty}\frac{1}{m}\int_{\Lambda^{\mathbb{N}}} S_{m,\phi}(x) w^{(n-1)}(0,x)\, d\mu_\psi(x) = 0. \tag{3.16}$$

Then equation (3.14) follows from equation (3.15) as $m\to\infty$ by equation (3.16). □

Theorem 3.7 establishes some link between the derivatives of the pressure function and the process $\{\phi\circ\sigma^j(x)\}_{j=0}^\infty$ through $S_{m,\phi}$ with respect to the equilibrium state μ_ψ. We also formulate a general version of the result.

COROLLARY 3.8. *Let $\psi, \phi \in C^h(\Lambda^{\mathbb{N}})$ with μ_ψ be the equilibrium state of ψ. Here, $\mathcal{L}_{\psi+t\phi}$ admits a maximum isolated eigenvalue $e^{P(\psi+t\phi)}$ close to $e^{P(\psi)}$ with eigenfunction $w(t,x)$ whose projection depends analytically on t in a small neighbourhood of 0.*

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Considering the derivatives of the pressure function $P(\psi + t\phi)$ at $t = 0$, we have

$$
\begin{aligned}
&P^{(n)}(\psi + t\phi)|_{t=0} \\
&\quad = \lim_{m\to\infty} \frac{1}{m}\bigg(\sum_{j=2}^{n} C_n^j \int_{\Lambda^{\mathbb{N}}} \bigg(S_{m,\phi} - m \int \phi\, d\mu_\psi \bigg)^j w^{(n-j)}(0,x)\, d\mu_\psi(x) \\
&\qquad - \sum_{j=2}^{n-2} C_n^j \sum_{\tau\in\mathfrak{P}(j), 1\notin\tau} m^{\#\tau} B_j^\tau P^{(\tau)}(\psi + t\phi)|_{t=0} \int_{\Lambda^{\mathbb{N}}} w^{(n-j)}(0,x)\, d\mu_\psi(x) \\
&\qquad - \sum_{\tau\in\mathfrak{P}(n), \{1,n\}\cap\tau=\emptyset} m^{\#\tau} B_n^\tau P^{(\tau)}(\psi + t\phi)|_{t=0} \bigg)
\end{aligned} \tag{3.17}
$$

for any $n \geq 2$.

Proof. Equipped with Theorem 3.7, the proof follows in line with the Proof of Corollary 3.4. □

The following is a classical result on the second derivative of the pressure [**PP**, Ch. 4].

COROLLARY 3.9. *Let $\psi, \phi \in C^h(\Lambda^{\mathbb{N}})$ with ψ being normalised. Let μ_ψ be the equilibrium state of ψ and $\int \phi\, d\mu_\psi = 0$. Let $e^{P(\psi+t\phi)}$ be the maximum eigenvalue of $\mathcal{L}_{\psi+t\phi}$ with eigenfunction $w(t,x)$ for small t. Then we have*

$$
P''(\psi + t\phi)|_{t=0} = \lim_{m\to\infty} \frac{1}{m} \int S_{m,\phi}^2\, d\mu_\psi. \tag{3.18}
$$

Remark 3.10. Here, $P''(\psi + t\phi)|_{t=0}$ is called variance of the random process $\{\phi \circ \sigma^j(x)\}_{j=0}^\infty$, whose name can be interpreted from the CLT. See [**Rue1**, **PP**].

Now we give some precise descriptions of the third and fourth derivatives of $P(\psi + t\phi)$ by virtue of Theorem 3.7.

COROLLARY 3.11. *Let $\psi, \phi \in C^h(\Lambda^{\mathbb{N}})$ with ψ being normalised. Let μ_ψ be the equilibrium state of ψ and $\int \phi\, d\mu_\psi = 0$. Let $e^{P(\psi+t\phi)}$ be the maximum eigenvalue of $\mathcal{L}_{\psi+t\phi}$ with eigenfunction $w(t,x)$ for small t. Then we have*

$$
P'''(\psi + t\phi)|_{t=0} = \lim_{m\to\infty} \frac{3}{m} \int S_{m,\phi}^2 w'(0,x)\, d\mu_\psi + \lim_{m\to\infty} \frac{1}{m} \int S_{m,\phi}^3\, d\mu_\psi. \tag{3.19}
$$

Proof. This follows instantly from Theorem 3.7 with $n = 3$. □

COROLLARY 3.12. *Let $\psi, \phi \in C^h(\Lambda^{\mathbb{N}})$ with ψ being normalised. Let μ_ψ be the equilibrium state of ψ and $\int \phi\, d\mu_\psi = 0$. Let $e^{P(\psi+t\phi)}$ be the maximum eigenvalue of $\mathcal{L}_{\psi+t\phi}$ with eigenfunction $w(t,x)$ for small t. Then we have*

$$
\begin{aligned}
&P^{(4)}(\psi + t\phi)|_{t=0} \\
&\quad = \lim_{m\to\infty} \bigg(\frac{6}{m} \int S_{m,\phi}^2 w''(0,x)\, d\mu_\psi + \frac{4}{m} \int S_{m,\phi}^3 w'(0,x)\, d\mu_\psi + \frac{1}{m} \int S_{m,\phi}^4\, d\mu_\psi \\
&\qquad - 6P''(\psi + t\phi)|_{t=0} \int w''(0,x)\, d\mu_\psi - 3m(P''(\psi + t\phi)|_{t=0})^2 \bigg)
\end{aligned}
$$

$$= \lim_{m\to\infty} \bigg(\frac{6}{m} \int S_{m,\phi}^2 w''(0,x)\, d\mu_\psi + \frac{4}{m} \int S_{m,\phi}^3 w'(0,x)\, d\mu_\psi + \frac{1}{m} \int S_{m,\phi}^4\, d\mu_\psi - \frac{6}{m} \int S_{m,\phi}^2\, d\mu_\psi \int w''(0,x)\, d\mu_\psi - \frac{3}{m}\bigg(\int S_{m,\phi}^2\, d\mu_\psi \bigg)^2 \bigg). \tag{3.20}$$

Proof. The first equality follows instantly from Theorem 3.7 with $n = 4$, while the second one is true by equation (3.18). The last description depends only on $\phi(x)$, $\{w^{(j)}(0,x)\}_{j=1}^2$ and $\mu_\psi(x)$. □

Through the above formulae, we see the importance of the asymptotic distribution of the random variable $S_{m,\phi}$ with respect to μ_ψ, which is described by the CLT for the process $\{\phi \circ \sigma^j(x)\}_{j=0}^\infty$. Equipped with all the above results, now we are in a position to prove the rigidity results on the third derivatives of $P(\psi + t\phi)$ using Corollary 3.11. We first show Theorem 1.2.

Proof of Theorem 1.2. From now on, we fix $t_* \in (0,\infty)$. Let $\psi = t_*\phi$. Simply by making a change of variable, we can see that

$$P^{(n)}(t_*) = P^{(n)}(t\phi)|_{t=t_*} = P^{(n)}(\psi + t\phi)|_{t=0}$$

for any $n \geq 0$. So equation (1.2) is equivalent to

$$|P'''(\psi + t\phi)|_{t=0}(1 - \sqrt{2\pi}(P''(\psi + t\phi)|_{t=0})^{3/2})| \leq 3M_\phi P''(\psi + t\phi)|_{t=0}. \tag{3.21}$$

We can assume ψ is normalised as otherwise we can change it to its normalisation by equation (2.11). Moreover, it suffices for us to prove it under the assumption $\int \phi\, d\mu_\psi = 0$ by virtue of equation (2.10). If $P''(\psi + t\phi)|_{t=0} = 0$, then ϕ is cohomologous to a constant according to [**PP**, Proposition 4.12]. This forces $P'''(\psi + t\phi)|_{t=0} = 0$, so equation (3.21) is satisfied in this case. In the following, we assume $P''(\psi + t\phi)|_{t=0} > 0$. We resort to Corollary 3.11 to justify equation (3.21) under the above assumptions. We first estimate the term $1/m \int S_{m,\phi}^3\, d\mu_\psi$ in equation (3.19). Since we are assuming the potential is generic, we can apply the CLT with equation (2.5):

$$\begin{aligned}
&\frac{1}{m} \int S_{m,\phi}^3\, d\mu_\psi \\
&\quad= \sqrt{m} \int \bigg(\frac{S_{m,\phi}}{\sqrt{m}} \bigg)^3 d\mu_\psi \\
&\quad= \sqrt{m} \int y^3\, dG_m(y) \\
&\quad= \sqrt{m} \int y^3\, dN_{0,P''(\psi+t\phi)|_{t=0}}(y) + \sqrt{m} \int y^3\, dH_m(y) + \sqrt{m} \cdot o(1/\sqrt{m}) \\
&\quad= \sqrt{m} \cdot 0 + \int y^3\, d\bigg(\frac{P'''(\psi + t\phi)|_{t=0}}{6} \bigg(1 - \frac{y^2}{P''(\psi + t\phi)|_{t=0}} \bigg) e^{-y^2/2P''(\psi+t\phi)|_{t=0}} \bigg) \\
&\qquad + \sqrt{m} \cdot o(1/\sqrt{m}) \\
&\quad= P'''(\psi + t\phi)|_{t=0}\sqrt{2\pi}(P''(\psi + t\phi)|_{t=0})^{3/2} + \sqrt{m} \cdot o(1/\sqrt{m}).
\end{aligned}$$

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By taking $m \to \infty$, we get

$$\lim_{m\to\infty} \frac{1}{m} \int S_{m,\phi}^3 \, d\mu_\psi = P'''(\psi + t\phi)|_{t=0}\sqrt{2\pi}(P''(\psi + t\phi)|_{t=0})^{3/2}. \tag{3.22}$$

By equation (3.19), we have

$$P'''(\psi + t\phi)|_{t=0}(1 - \sqrt{2\pi}(P''(\psi + t\phi)|_{t=0})^{3/2}) = \lim_{m\to\infty} \frac{3}{m} \int S_{m,\phi}^2 w'(0, x) \, d\mu_\psi. \tag{3.23}$$

Since $w'(0, x)$ depends continuously on $x \in X$, there exists some M_ϕ depending on ϕ, such that

$$|w'(0, x)| \le M_\phi. \tag{3.24}$$

Now taking absolute values on both sides of equation (3.23), we justify equation (3.21) by equations (3.24) and (3.18). □

The proof of Theorem 1.1 on the pressure functions of non-generic Hölder potentials follows a similar way.

Proof of Theorem 1.1. Fixing $t_* \in (0, \infty)$, we can simply assume $\psi = t_*\phi$ is normalised and $\int \phi \, d\mu_\psi = 0$. In the case where $P''(\psi + t\phi)|_{t=0} = 0$, so ϕ is cohomologous to a constant, equation (1.1) holds obviously. In the following, we assume ϕ is not cohomologous to a constant, so $P''(\psi + t\phi)|_{t=0} > 0$. We again resort to Corollary 3.11 to justify equation (1.1) under these assumptions. Now for the term $1/m \int S_{m,\phi}^3 \, d\mu_\psi$ in equation (3.19), by virtue of the CLT with equation (2.4),

$$\begin{aligned}
&\frac{1}{m} \int S_{m,\phi}^3 \, d\mu_\psi \\
&\quad = \sqrt{m} \int \left(\frac{S_{m,\phi}}{\sqrt{m}}\right)^3 d\mu_\psi \\
&\quad = \sqrt{m} \int y^3 \, dG_m(y) \\
&\quad \le \sqrt{m} \int y^3 \, dN_{0,P''(\psi+t\phi)|_{t=0}}(y) + \sqrt{m}\frac{9|P'''(\psi + t\phi)|_{t=0}| + 2|P^{(4)}(\psi + t\phi)|_{t=0}|}{\sqrt{2\pi^3 m}(P''(\psi + t\phi)|_{t=0})^{3/2}} \\
&\quad = \sqrt{m} \cdot 0 + \frac{9|P'''(\psi + t\phi)|_{t=0}| + 2|P^{(4)}(\psi + t\phi)|_{t=0}|}{\sqrt{2\pi^3}(P''(\psi + t\phi)|_{t=0})^{3/2}} \\
&\quad = \frac{9|P'''(\psi + t\phi)|_{t=0}| + 2|P^{(4)}(\psi + t\phi)|_{t=0}|}{\sqrt{2\pi^3}(P''(\psi + t\phi)|_{t=0})^{3/2}}
\end{aligned} \tag{3.25}$$

for m large enough. By taking $m \to \infty$ in equation (3.25), we get

$$\lim_{m\to\infty} \frac{1}{m} \int S_{m,\phi}^3 \, d\mu_\psi \le \frac{9|P'''(\psi + t\phi)|_{t=0}| + 2|P^{(4)}(\psi + t\phi)|_{t=0}|}{\sqrt{2\pi^3}(P''(\psi + t\phi)|_{t=0})^{3/2}}. \tag{3.26}$$

Taking modulus on both sides of equation (3.26), we get

$$
\begin{aligned}
&|P'''(\psi + t\phi)|_{t=0}| \\
&\le \left| \lim_{m\to\infty} \frac{1}{m} \int S_{m,\phi}^3 \, d\mu_\psi \right| + \left| \lim_{m\to\infty} \frac{3}{m} \int S_{m,\phi}^2 w'(0,x) \, d\mu_\psi \right| \\
&\le \frac{9|P'''(\psi + t\phi)|_{t=0}| + 2|P^{(4)}(\psi + t\phi)|_{t=0}|}{\sqrt{2\pi^3}(P''(\psi + t\phi)|_{t=0})^{3/2}} + 3M_\phi \left| \lim_{m\to\infty} \frac{3}{m} \int S_{m,\phi}^2 \, d\mu_\psi \right| \\
&= \frac{9|P'''(\psi + t\phi)|_{t=0}| + 2|P^{(4)}(\psi + t\phi)|_{t=0}|}{\sqrt{2\pi^3}(P''(\psi + t\phi)|_{t=0})^{3/2}} + 3M_\phi P''(\psi + t\phi)|_{t=0} \qquad (3.27)
\end{aligned}
$$

for some $|w'(0, x)| \le M_\phi$, which results in equation (1.1). □

One can predict from Corollary 3.12, Theorem 3.7, and the proof of Theorems 1.1, 1.2 that some more rigid relationships between higher derivatives of the pressure function $\{P^{(n)}(t\phi)\}_{n\in\mathbb{N}}$ are possible. These rigidity relationships impose restrictions on fitting convex analytic functions whose supporting lines intersect the vertical axis in some bounded set in $[0, \infty)$ by pressures of Hölder potentials.

4. *Global fitting of convex analytic functions via pressures of Hölder potentials*

This section is dedicated to the proof of Theorem 1.7. We start from the following result on some global behaviour of the pressure functions of generic Hölder potentials.

THEOREM 4.1. *Let $\alpha > 0$. If there exists a strictly convex analytic function $F(t)$ on (α, ∞), with its supporting lines intersecting the vertical axis in $[\underline{\gamma}, \overline{\gamma}] \subset [0, \infty)$, such that*

$$
\sup_{t\in(\alpha,\infty)} \left\{ \left| \frac{F'''(t)}{F''(t)} - \sqrt{2\pi F''(t)} \right| \right\} = \infty, \qquad (4.1)
$$

then there does not exist a shift space with finite symbols with a generic Hölder potential ϕ satisfying

$$
P(t\phi) = F(t)
$$

on (α, ∞).

Proof. This follows directly from Theorem 1.2 in fact. Suppose on the contrary that there exist some shift space X with finite symbols and some generic Hölder potential $\phi \in C^h(X)$ satisfying $P(t\phi) = F(t)$ on (α, ∞), then according to Theorem 1.2, we have

$$
\sup_{t\in(\alpha,\infty)} \left\{ \left| \frac{F'''(t)}{F''(t)} - \sqrt{2\pi F''(t)} \right| \right\} \le 3M_\phi
$$

for some finite $M_\phi > 0$. This contradicts equation (4.1). □

Be careful that we cannot exclude the possibility that one can locally fit some convex analytic function through the pressure of some generic Hölder potential on some shift space with finite symbols by Theorem 1.2. This is because for any strictly convex analytic

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function $F(t)$ on (α, ∞) and $\alpha \le \underline{\alpha} \le \overline{\alpha}$, we always have

$$\sup_{\underline{\alpha} \le t \le \overline{\alpha}} \left\{ \left| \frac{F'''(t)}{F''(t)} - \sqrt{2\pi F''(t)} \right| \right\} < \infty.$$

So one cannot exclude the possibility that there exists some generic Hölder potential ϕ on some shift space satisfying

$$P(t\phi) = F(t)$$

on $[\underline{\alpha}, \overline{\alpha}]$ through Theorem 1.2. See §5 for more results on the problem of local fitting of some convex analytic functions through the pressures of Hölder potentials.

Now for $\alpha > 0$, let

$$\mathcal{F}_\alpha = \{F(t) : F(t) \text{ is a strictly convex analytic function on } (\alpha, \infty) \text{ satisfying equation (4.1)}, \text{its supporting lines intersect the vertical axis in a bounded interval in } [0, \infty)\}.$$

We will show that $\mathcal{F}_\alpha \neq \emptyset$ for any $\alpha > 0$ in the following.

PROPOSITION 4.2. *For any $\alpha > 0$, we have*

$$\tilde{\mathcal{F}}_\alpha = \left\{ F_{a,b,c}(t) = \frac{at^2 + bt + te^{-ct^2} + e^{-ct^2}}{t} \bigg|_{(\alpha,\infty)} \right\}_{a,b>0, c>1/2\sqrt{2}} \subset \mathcal{F}_\alpha.$$

Proof. The restricted functions on (α, ∞) are of course analytic. By considering the second derivative of a function $F_{a,b,c}(t) \in \tilde{\mathcal{F}}_\alpha$, we have

$$F''_{a,b,c}(t) = 4c^2t^2e^{-ct^2} + 4c^2te^{-ct^2} - 2ce^{-ct^2} + 2ct^{-1}e^{-ct^2} + 2t^{-3}e^{-ct^2}$$

for $t \in (0, \infty)$. Now since

$$4c^2t + 2ct^{-1} \ge 2\sqrt{8c^3} > 2c,$$

considering $c > 1/2\sqrt{2}$, we can see that $F''_{a,b,c}(t) > 0$ on $(0, \infty)$. This shows that for any $\alpha > 0$, $F_{a,b,c}(t) \in \tilde{\mathcal{F}}_\alpha$ is a convex function. Considering the third derivative of a function $F_{a,b,c}(t) \in \tilde{\mathcal{F}}_\alpha$, we have

$$F'''_{a,b,c}(t) = -8c^3t^3e^{-ct^2} - 8c^3t^2e^{-ct^2} + 12c^2te^{-ct^2} - 6ct^{-2}e^{-ct^2} - 6t^{-4}e^{-ct^2}$$

for $t \in (0, \infty)$. Then we have

$$\lim_{t\to\infty} \left(\frac{F'''_{a,b,c}(t)}{F''_{a,b,c}(t)} - \sqrt{2\pi F''_{a,b,c}(t)} \right) = \lim_{t\to\infty} \frac{-8c^3t^3e^{-ct^2}}{4c^2t^2e^{-ct^2}} = -\infty.$$

This means that $F_{a,b,c}(t) \in \tilde{\mathcal{F}}_\alpha$ satisfies equation (4.1). To see that the supporting lines of a function $F_{a,b,c}(t) \in \tilde{\mathcal{F}}_\alpha$ intersect the vertical axis in a bounded domain in $[0, \infty)$, write the function as

$$F_{a,b,c}(t) = at + b + e^{-ct^2} + t^{-1}e^{-ct^2}.$$

Its graph on $(0, \infty)$ is a strictly convex smooth curve with asymptotes $y = at + b$ and $t = 0$. □

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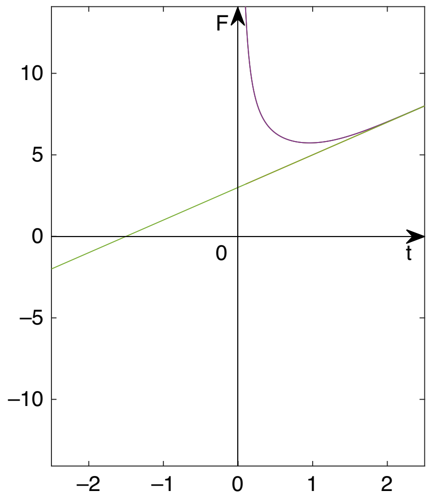

FIGURE 1. Graph of $F_{2,3,1}(t)$.

In Figure 1, we provide the readers with the graph of the function

$$F_{2,3,1}(t) = \frac{2t^2 + 3t + te^{-t^2} + e^{-t^2}}{t}$$

on $(0, \infty)$.

This means that any function in the family $\tilde{\mathcal{F}}_\alpha$ cannot be fitted by any generic Hölder potential on any shift space globally, by Theorem 4.1. In the following, we exclude the possibility that they can be fitted by *non-generic* Hölder potentials on shift spaces with finite symbols. One can show that [**PP**, Ch. 4] if ϕ is non-generic, then there exists a continuous function $u : X \to \mathbb{R}$, $c_\phi \in \mathbb{R}$, and a locally constant potential $\tilde{\phi} : X \to \mathbb{R}$, such that

$$\phi(x) = u \circ \sigma(x) - u(x) + c_\phi + \tilde{\phi}(x). \tag{4.2}$$

PROPOSITION 4.3. *For any $\alpha > 0$ and any $F_{a,b,c}(t) \in \tilde{\mathcal{F}}_\alpha$ with $a, b > 0$, $c > (1/2\sqrt{2})$, there does not exist a shift space with finite symbols with a non-generic Hölder potential ϕ satisfying*

$$P(t\phi) = F(t)$$

on (α, ∞).

Proof. Note that for a non-generic Hölder potential ϕ on a shift space with finite symbols, according to equation (4.2), we have

$$P(t\phi) = tc_\phi + P(t\tilde{\phi}),$$

where $\tilde{\phi}$ is some locally constant potential. By the explicit formula (see for example [**Wal1**, p. 214]) for the pressure functions of locally constant potentials on shift spaces with finite

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symbols, we see that any $F_{a,b,c}(t)$ cannot be fitted by pressure of any non-generic Hölder potential ϕ globally. □

Equipped with all the above results, Theorem 1.7 follows instantly from Propositions 4.2 and 4.3.

5. *Local fitting of prescribed germs via pressures of locally constant potentials*

In this section, we deal with the local fitting of analytic functions by the pressures of Hölder potentials, especially the pressures of piecewise constant ones. First, we borrow some notion originating from analytic continuation.

Definition 5.1. A germ of level $\iota(\iota \in \mathbb{N} \cup \{\infty\})$ at t_* is the formal power series

$$g(t) = a_0 + a_1(t - t_*) + \frac{a_2}{2!}(t - t_*)^2 + \frac{a_3}{3!}(t - t_*)^3 + \cdots + \frac{a_\iota}{\iota!}(t - t_*)^\iota$$

for some $(a_0, a_1, \ldots, a_\iota) \in \mathbb{R}^{\iota+1}$.

The convergent radius (the superior of values $\delta \geq 0$ on $[t_* - \delta, t_* + \delta]$ such that the germ converges) of the power series is called the *radius* of the germ. Any finite-level germ admits infinite radius while an infinite-level germ may admit some finite radius. We are only interested in germs of radius $\delta > 0$. The following problem will be our concern in this section.

Problem 5.2. For a germ

$$g(t) = a_0 + a_1(t - t_*) + \frac{a_2}{2!}(t - t_*)^2 + \cdots + \frac{a_\iota}{\iota!}(t - t_*)^\iota$$

of level ι $(\iota \in \mathbb{N} \cup \{\infty\})$ at t_* with some strictly positive radius, does there exist some Hölder potential ϕ on some shift space with finite symbols and some $\delta > 0$, such that

$$P(t\phi) = g(t) + O((t - t_*)^{\iota+1})$$

on $[t_* - \delta, t_* + \delta]$?

We assume $O((t - t_*)^\infty) = 0$ in Problem 5.2. The question can still be understood in Katok's flexibility program in the class of symbolic dynamical systems, or even in some smooth systems. Obvious conditions to guarantee a positive answer to the problem are equation (1.5) and

$$a_2 > 0 \tag{5.1}$$

in the case $\iota \geq 2$. The condition in equation (5.1) guarantees convexity of the germ (in some neighbourhood of t_*), while equation (1.5) guarantees the supporting lines of the germ intersect the vertical axis in a bounded set in $[0, \infty)$ (also in some neighbourhood of t_*). We are especially interested in its answer when the Hölder potential in Problem 5.2 is required to be a locally constant one. We have seen the importance of the family of locally constant potentials in approximating convex analytic functions in Corollary 1.4. In fact, Corollary 1.4 has some interesting interpretation in approximation theory [**Tim**, Ch. I] when we consider the explicit expressions of the pressures of locally constant potentials on the shift space. For $n \in \mathbb{N}$, recall that

$$\Lambda_n = \{1, 2, \ldots, n\}.$$

LEMMA 5.3. *For an integer $k \geq 0$, consider some locally constant potential*

$$\phi(x) = c_{x_{-k}x_{-k+1}\cdots x_0\cdots x_{k-1}x_k}$$

for $x = \cdots x_{-1}x_0x_1 \cdots \in [x_{-k} \cdots x_k]$ on the shift space $\Lambda_n^{\mathbb{Z}}$, we have

$$P(t\phi) = \log \sum_{(x_{-k},\ldots,x_k)\in\Lambda_n^{2k+1}} e^{tc_{x_{-k}\cdots x_k}}$$

for any $t \in (-\infty, \infty)$.

Proof. This follows from [**Wal1**, Theorem 9.6] by some direct calculations through Definition 2.1 of the pressure. □

Now combining Corollary 1.4 and Lemma 5.3, we have the following result.

COROLLARY 5.4. *Let $F(t)$ be a convex Lipschitz function on (α, ∞) for some $\alpha > 0$, such that its supporting lines intersect the vertical axis in $[\underline{\gamma}, \overline{\gamma}]$ with $0 \leq \underline{\gamma} \leq \overline{\gamma} < \infty$. Then there exists some $K \in \mathbb{N}$ and some sequences of constants*

$$\{c_{n,j}\}_{j=1}^{K^n},$$

such that

$$\lim_{n\to\infty} \log \sum_{j=1}^{K^n} e^{tc_{n,j}} = F(t) \tag{5.2}$$

for any $t \in (\alpha, \infty)$.

Proof. Take $K = \#\Lambda$ for the symbolic set in the proof of Corollary 1.4, then the locally constant potential $\phi_n(x) = \phi_{n,-}(x)$ admits K^n constant values on corresponding level-n cylinder sets. Denote these values by $\{c_{n,j}\}_{j=1}^{K^n}$ for $n \in \mathbb{N}$. According to Lemma 5.3,

$$P(t\phi_{n,-}) = \log \sum_{j=1}^{K^n} e^{tc_{n,j}}$$

for any $n \geq 1$. This gives equation (5.2) by virtue of equation (1.3). □

Corollary 5.4 indicates that logarithm of the finite sums of the exponential maps in the family $\{e^{tc}\}_{c\in\mathbb{R}}$ are dense in the space of certain convex Lipschitz maps on (α, ∞). The above approximation is uniform with respect to t in a bounded set. This makes the family $\{e^{tc}\}_{c\in\mathbb{R}}$ (family of locally constant potentials) important in detecting the properties of certain convex Lipschitz maps (among continuous or Hölder potentials).

From now on, we turn our attention to Problem 5.2, but with restriction to locally constant potentials. We focus on locally constant potentials defined on the level-0 cylinder sets, whose theory is equivalent to those defined on the deeper cylinder sets, where the

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symbols are replaced by words (recoding). On the shift space $\Lambda_n^{\mathbb{Z}}$ with $n \geq 2$, consider the locally constant potential

$$\phi(x) = z_{x_0}$$

for $x = \cdots x_{-1}x_0x_1 \cdots \in [x_0]$, where $\{z_i\}_{1\leq i\leq n}$ are all constants. Let

$$Q_0(t, z_1, z_2, \ldots, z_n) = \sum_{i=1}^{n} e^{tz_i},$$

so

$$P(t\phi) = \log Q_0(t, z_1, \ldots, z_n)$$

by Lemma 5.3. Let

$$Q_1(t, z_1, z_2, \ldots, z_n) = \sum_{i=1}^{n} z_i e^{tz_i}$$

and

$$Q_2(t, z_1, z_2, \ldots, z_n) = \sum_{1\leq i<j\leq n} (z_i - z_j)^2 e^{t(z_i+z_j)}.$$

Through some elementary calculations, one can check that

$$P'(t\phi) = \frac{dP(t\phi)}{dt} = \frac{Q_1(t, z_1, \ldots, z_n)}{Q_0(t, z_1, \ldots, z_n)},$$

while

$$P''(t\phi) = \frac{d^2P(t\phi)}{dt^2} = \frac{Q_2(t, z_1, \ldots, z_n)}{Q_0^2(t, z_1, \ldots, z_n)}. \tag{5.3}$$

Let

$$R_2(t, z_1, z_2, \ldots, z_n) = \sum_{i=1}^{n} z_i^2 e^{tz_i},$$

one can check that

$$Q_2(t, z_1, \ldots, z_n) = Q_0(t, z_1, \ldots, z_n)R_2(t, z_1, \ldots, z_n) - Q_1^2(t, z_1, \ldots, z_n).$$

In the following, we will often fix $t = t_* > 0$, so we will frequently write

$$Q_0(t_*, z_1, z_2, \ldots, z_n) = Q_0(z_1, z_2, \ldots, z_n)$$

with t_* omitted for convenience. Similar notation apply to other terms above. Let

$$Q_0(z_1, \ldots, z_n) = \sum_{i=1}^{n} e^{t_* z_i} = e^{a_0}, \tag{5.4}$$

$$Q_1(z_1, \ldots, z_n) = \sum_{i=1}^{n} z_i e^{t_* z_i} = a_1 e^{a_0} \tag{5.5}$$

be two equations with unknowns $\{z_1, z_2, \ldots, z_n\}$ for fixed $t_* > 0$, $(a_0, a_1) \in \mathbb{R}^2$ and some $n \geq 2$. Let

$$\Gamma^n_{5.4} = \{(z_1, z_2, \ldots, z_n) \in \mathbb{R}^n : z_1, z_2, \ldots, z_n \text{ satisfy equation (5.4)}\}$$

and

$$\Gamma^n_{5.5} = \{(z_1, z_2, \ldots, z_n) \in \mathbb{R}^n : z_1, z_2, \ldots, z_n \text{ satisfy equation (5.5)}\}.$$

They are both $n-1$ dimensional smooth hypersurfaces. We first present readers with the following result on fitting an analytic function

$$a_0 + a_1(t - t_*) + O((t - t_*)^2)$$

with t_*, a_0, a_1 subject to equation (1.5) around some fixed $t_* > 0$ by pressures of locally constant potentials on general shift spaces.

THEOREM 5.5. *Let $t_* > 0$, $(a_0, a_1) \in \mathbb{R}^2$, $n \geq 2$ satisfying equation (1.5) and*

$$\frac{a_0 - \log n}{t_*} < a_1. \tag{5.6}$$

Then there exists some $\delta_n > 0$ and some sequence $\{r_{i,n}\}_{i=1}^n \subset \mathbb{R}$, such that the locally constant potential

$$\phi(x) = r_{x_0}$$

for $x = \cdots x_{-1}x_0x_1 \cdots \in [x_0]$ on the full shift space $\Lambda_n^{\mathbb{Z}}$ satisfies

$$P(t\phi) = a_0 + a_1(t - t_*) + O((t - t_*)^2)$$

on $[t_* - \delta_n, t_* + \delta_n]$.

Proof. In fact, it suffices for us to show that the system of equations

$$\begin{cases} \text{equation (5.4)}, \\ \text{equation (5.5)}, \end{cases}$$

with unknowns $\{z_1, z_2, \ldots, z_n\}$ admits a solution under conditions of the theorem. Without loss of generality, we assume

$$z_1 \leq z_2 \leq \cdots \leq z_n. \tag{5.7}$$

Under this assumption, it is easy to see that

$$\frac{a_0 - \log n}{t_*} \leq z_n < \frac{a_0}{t_*}.$$

Now we estimate the values of $Q_1(z_1, \ldots, z_n)$ with z_n approaching the terminals. When z_n approaches the right terminal from below, we have

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$$\lim_{(z_1,z_2,\ldots,z_n)\in\Gamma^n_{5.4},\ z_n\nearrow a_0/t_*} Q_1(z_1,\ldots,z_n)=\frac{a_0}{t_*}e^{a_0}>a_1e^{a_0}$$

by virtue of equation (1.5). When z_n approaches the left terminal from above, we have

$$\lim_{(z_1,z_2,\ldots,z_n)\in\Gamma^n_{5.4},\ z_n\searrow a_0/t_*} Q_1(z_1,\ldots,z_n)=\frac{a_0-\log n}{t_*}e^{a_0}<a_1e^{a_0}$$

by virtue of equation (5.6). Since $\Gamma^n_{5.4}$ is a smooth hypersurface, by the mean value theorem, there exists some $(r_{1,n},r_{2,n},\ldots,r_{n,n})\in\Gamma^n_{5.4}$ satisfying equations (5.4) and (5.5) simultaneously. At last, for $x=\cdots x_{-1}x_0x_1\cdots\in[x_0]$ on the full shift space $\Lambda_n^{\mathbb{Z}}$, let

$$\phi(x)=r_{x_0,n}$$

be the locally constant potential. As $P(t\phi)$ is analytic, there exists some $\delta_n>0$ such that

$$P(t\phi)=a_0+a_1(t-t_*)+O((t-t_*)^2)$$

for $t\in[t_*-\delta_n,t_*+\delta_n]$. □

Remark 5.6. The core step in the proof of Theorem 5.5 is in fact finding the extremes of the function $Q_1(z_1,\ldots,z_n)$ subject to equations (5.4), (1.5) and (5.6). One can detect the points of extremes by the Karush–Kuhn–Tucker (KKT) conditions [**Kar**, **KT**], which generalises the method of Lagrange multipliers by allowing inequality constraints.

Care must be taken that the $\{r_{i,n}\}_{i=1}^n$ all depend on n. Theorem 5.5 induces the following interesting flexibility result on fitting certain analytic functions locally by pressures of locally constant potentials on general shift spaces.

COROLLARY 5.7. *Let $t_*>0$ and $(a_0,a_1)\in\mathbb{R}^2$ satisfy equation (1.5). Then there exists some $N\in\mathbb{N}$, such that for any $n\geq N$, there exist some $\delta_n>0$ and some sequence $\{r_{i,n}\}_{i=1}^n\subset\mathbb{R}$, such that the locally constant potential*

$$\phi(x)=r_{x_0,n}$$

for $x=\cdots x_{-1}x_0x_1\cdots\in[x_0]$ on the full shift space $\Lambda_n^{\mathbb{Z}}$ satisfies

$$P(t\phi)=a_0+a_1(t-t_*)+O((t-t_*)^2)$$

on $[t_*-\delta_n,t_*+\delta_n]$.

Proof. Under conditions of the corollary, for the given values t_*,a_0,a_1 satisfying equation (1.5), choose $N\in\mathbb{N}$ large enough such that

$$\frac{a_0-\log N}{t_*}<a_1.$$

This means that for any $n>N$, the condition in equation (5.6) is satisfied for t_*,a_0,a_1,n. Then the conclusion follows from Theorem 5.5. □

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Note that on some particular symbolic spaces, Theorems 5.5 and 5.7 may be trivial. For example, for given $(t_*, a_0, a_1) \in \mathbb{R}^3$ without any constraints, by choosing $\beta = e^{a_0 - t_* a_1}$, consider the constant potential

$$\phi(x) = a_1$$

on the β-shift space with symbols $\{0, 1, \ldots, \lfloor \beta \rfloor\}$. It is easy to see that

$$P(t\phi) = a_0 - t_* a_1 + a_1 t = a_0 + a_1(t - t_*)$$

on $(-\infty, \infty)$. However, our results guarantee conclusions on general shift spaces beyond these specific ones.

From now on, we go towards the proof of Theorem 1.9. For fixed $t_* > 0$, $(a_0, a_1) \in \mathbb{R}^2$ and $n \geq 3$, let

$$\begin{aligned}\Gamma^n_{5.4,5.5} &= \Gamma^n_{5.4} \cap \Gamma^n_{5.5}\\ &= \{(z_1, z_2, \ldots, z_n) \in \mathbb{R}^n : z_1, z_2, \ldots, z_n \text{ satisfy equations (5.4) and (5.5)}\}.\end{aligned}$$

We describe some topological properties of the set $\Gamma^n_{5.4,5.5}$ in the following result.

LEMMA 5.8. *For fixed $t_* > 0$, $(a_0, a_1) \in \mathbb{R}^2$ subject to equation (1.5) and $n \geq 3$, in the case $\Gamma^n_{5.4,5.5} \neq \emptyset$ and $a_1 \neq (a_0 - \log n/t_*)$, it is a compact $(n-2)$-dimension smooth manifold.*

Proof. The Jacobian of the functions $Q_0(z_1, \ldots, z_n) - e^{a_0}$ and $Q_1(z_1, \ldots, z_n) - a_1 e^{a_0}$ with respect to $z_1, z_2, \ldots, z_n$ is

$$J = \begin{pmatrix} t_* e^{t_* z_1} & t_* e^{t_* z_2} & \cdots & t_* e^{t_* z_n} \\ e^{t_* z_1} + t_* z_1 e^{t_* z_1} & e^{t_* z_2} + t_* z_2 e^{t_* z_2} & \cdots & e^{t_* z_n} + t_* z_n e^{t_* z_n} \end{pmatrix}.$$

Its rank is strictly less than 2 if and only if

$$z_1 = z_2 = \cdots = z_n.$$

Since $a_1 \neq (a_0 - \log n/t_*)$, this is excluded from points in $\Gamma_{5.4,5.5}$. By the implicit function theorem [**Lan**, Theorem 5.9], if $\Gamma^n_{5.4,5.5}$ is not empty, it is an $(n-2)$-dimension smooth manifold locally. The gradient of the function $Q_0(z_1, \ldots, z_n) - e^{a_0}$ is

$$\nabla(Q_0(z_1, \ldots, z_n) - e^{a_0}) = (t_* e^{t_* z_1}, t_* e^{t_* z_2}, \ldots, t_* e^{t_* z_n}),$$

whose individual components will always be strictly positive. The gradient of the function $Q_1(z_1, \ldots, z_n) - a_1 e^{a_0}$ is

$$\nabla(Q_1(z_1, \ldots, z_n) - a_1 e^{a_0}) = (e^{t_* z_1} + t_* z_1 e^{t_* z_1}, e^{t_* z_2} + t_* z_2 e^{t_* z_2}, \ldots, e^{t_* z_n} + t_* z_n e^{t_* z_n}),$$

with the ith individual component vanishing if and only if $z_i = -(1/t_*)$ for $1 \leq i \leq n$. So $\Gamma^n_{5.4}$ and $\Gamma^n_{5.5}$ cannot be tangent to each other. Moreover, note that

$$e^{t_* z_i} + t_* z_i e^{t_* z_i} > 0$$

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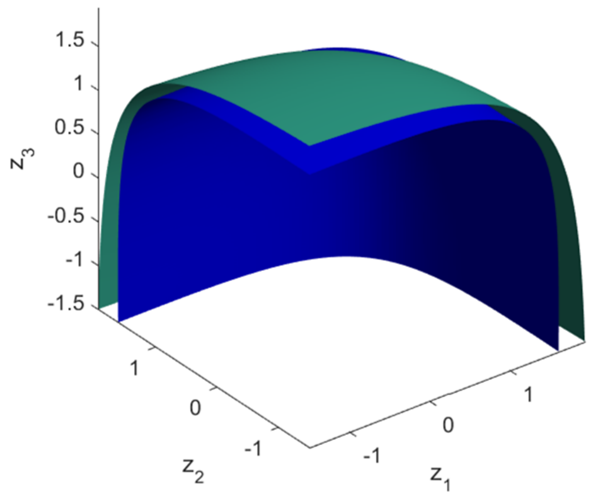

FIGURE 2. $\Gamma^3_{5.4,1,2,1}$ (lighter) and $\Gamma^3_{5.5,1,2,1}$ (darker).

if $z_i > -(1/t_*)$, while

$$e^{t_* z_i} + t_* z_i e^{t_* z_i} < 0$$

if $z_i < -(1/t_*)$ for any $1 \leq i \leq n$. These force the intersection of zeros of the two functions $Q_0(z_1, \ldots, z_n) - e^{a_0}$ and $Q_1(z_1, \ldots, z_n) - a_1 e^{a_0}$ to be connected if the intersection is not empty. This implies $\Gamma_{5.4,5.5}$ is a manifold globally in the case of being non-empty. Here, $\Gamma^n_{5.4,5.5}$ is compact since it is a bounded set. □

Let

$$\Gamma^3_{5.4,1,2,1} = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1, z_2, z_3 \text{ satisfy } e^{z_1} + e^{z_2} + e^{z_3} = e^2\}$$

and

$$\Gamma^3_{5.5,1,2,1} = \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1, z_2, z_3 \text{ satisfy } z_1 e^{z_1} + z_2 e^{z_2} + z_3 e^{z_3} = e^2\}$$

be the corresponding surfaces with $t_* = 1, a_0 = 2, a_1 = 1$. Figure 2 depicts parts of the two 2-dimension surfaces, whose intersection will be a 1-dimension smooth curve.

Equipped with all the above results, now we are ready to prove Theorem 1.9.

Proof of Theorem 1.9. First, for the given $t_* > 0$ and $(a_0, a_1) \in \mathbb{R}^2$ satisfying equation (1.5), if n is large enough, $\Gamma^n_{5.4,5.5}$ is not empty according to Corollary 5.7. So $\Gamma^n_{5.4,5.5}$ is a compact $(n-2)$-dimension smooth manifold for n large enough. We recall here the R_2 and Q_2 defined after Corollary 5.4. In the following, we always assume n is large enough. Now let

$$m_{t_*,a_0,a_1,n} = \min\left\{\frac{R_2(t_*, z_1, z_2, \ldots, z_n)}{e^{a_0}} - a_1^2 : (z_1, z_2, \ldots, z_n) \in \Gamma^n_{5.4,5.5}\right\},$$

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while

$$M_{t_*,a_0,a_1,n} = \max\left\{\frac{R_2(t_*, z_1, z_2, \ldots, z_n)}{e^{a_0}} - a_1^2 : (z_1, z_2, \ldots, z_n) \in \Gamma^n_{5.4,5.5}\right\}. \tag{5.8}$$

For any $m_{t_*,a_0,a_1,n} \le a_2 \le M_{t_*,a_0,a_1,n}$, since $\Gamma^n_{5.4,5.5}$ is a smooth manifold, there exist $\{c_{i,n}\}_{i=1}^n \subset \mathbb{R}$, such that $(c_{1,n}, c_{2,n}, \ldots, c_{n,n})$ satisfies equations (5.4), (5.5) and

$$a_2 = \frac{Q_2(t_*, c_{1,n}, \ldots, c_{n,n})}{Q_0^2(t_*, c_{1,n}, \ldots, c_{n,n})} = \frac{R_2(t_*, c_{1,n}, \ldots, c_{n,n})}{e^{a_0}} - a_1^2 \tag{5.9}$$

simultaneously. Now let

$$\phi(x) = c_{x_0,n}$$

for $x = \cdots x_{-1}x_0x_1 \cdots \in [x_0]$ on the full shift space $\Lambda_n^{\mathbb{Z}}$. It is a locally constant potential. According to equations (5.3) and (5.9), we have

$$P''(t_*\phi) = \frac{Q_2(t_*, c_{1,n}, \ldots, c_{n,n})}{Q_0^2(t_*, c_{1,n}, \ldots, c_{n,n})} = a_2. \tag{5.10}$$

Since $(c_{1,n}, c_{2,n}, \ldots, c_{n,n})$ satisfies equations (5.4) and (5.5), we have

$$P(t_*\phi) = \frac{Q_2(t_*, c_{1,n}, \ldots, c_{n,n})}{Q_0^2(t_*, c_{1,n}, \ldots, c_{n,n})} = a_0, \tag{5.11}$$

while

$$P'(t_*\phi) = \frac{Q_2(t_*, c_{1,n}, \ldots, c_{n,n})}{Q_0^2(t_*, c_{1,n}, \ldots, c_{n,n})} = a_1. \tag{5.12}$$

Note that $P(t\phi)$ is analytic with respect to t on (α, ∞) for any $\alpha > 0$, so there exists some $\delta_n > 0$, such that equation (1.6) holds on $[t_* - \delta_n, t_* + \delta_n]$, by equations (5.10), (5.11) and (5.12). □

In the following, we illustrate some dependent relationship between

$$\{m_{t_*,a_0,a_1,n}, M_{t_*,a_0,a_1,n}\}_{n\in\mathbb{N}}$$

and some particular t_*, a_0, a_1, n satisfying equation (1.5). There should be some universal relationship between them, while we hope the following observations will provide some hints. The first one is that it is possible for $m_{t_*,a_0,a_1,n} = 0$ for some t_*, a_0, a_1, n.

PROPOSITION 5.9. *Let $t_* > 0$ and $(a_0, a_1) \in \mathbb{R}^2$ satisfy equation (1.5). Then $m_{t_*,a_0,a_1,n} = 0$ for $n \ge 2$ if and only if*

$$a_1 = \frac{a_0 - \log n}{t_*}. \tag{5.13}$$

Proof. Note that $m_{t_*,a_0,a_1,n} = 0$ is equivalent to say that there exists some locally constant potential ϕ on $\Lambda_n^{\mathbb{Z}}$ such that $P''(t_*\phi) = 0$ according to Theorem 1.9. By [**PP**, Proposition 4.12], this happens if and only if ϕ is a constant potential on $\Lambda_n^{\mathbb{Z}}$. In this case, we have

$$\phi(x) = \frac{a_0 - \log n}{t_*}$$

for any $x \in \Lambda_n^{\mathbb{Z}}$, which implies equation (5.13). □

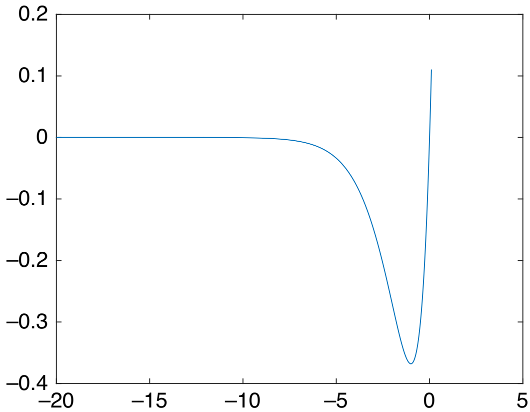

FIGURE 3. Graph of $\varsigma(z) = ze^z$.

This result does not tell things about the sequence

$$\{m_{t_*,a_0,a_1,n}\}_{n\in\mathbb{N}\text{ large enough}}$$

for given t_*, a_0, a_1, since equation (5.13) will never be true for any n large enough for fixed t_*, a_0, a_1. The following result describes a limiting behaviour of the sequence

$$\{M_{t_*,a_0,a_1,n}\}_{n\in\mathbb{N}\text{ large enough}}$$

for $t_* = 1$, $a_0 = 2$, $a_1 = 1$.

PROPOSITION 5.10. *Let $t_* = 1$, $a_0 = 2$, $a_1 = 1$, in symbols of Theorem 1.9, we have*

$$\lim_{n\to\infty} M_{1,2,1,n} = \infty. \tag{5.14}$$

To justify Proposition 5.10, we first illustrate some basic properties about the function ze^{t_*z} for $t_* > 0$.

LEMMA 5.11. *For $t_* > 0$, ze^{t_*z} is strictly decreasing on $(-\infty, -(1/t_*))$, strictly increasing on $(-(1/t_*), \infty)$, while it attains its minimum $-(1/t_*)e^{-1}$ at $z = -(1/t_*)$. It admits one and only one inflection in $(-\infty, -(1/t_*))$.*

Proof. One can check these conclusions by some direct computations on the first and second derivatives of the function ze^{t_*z}. □

In Figure 3, we depict the graph of $\varsigma(z) = ze^z$.

Proof of Proposition 5.10. Since we are considering the limit behaviour of $M_{1,2,1,n}$, we always assume n is large enough throughout the proof. Now consider the following two equations:

$$(n-1)e^{z_a} + e^{z_b} = e^2 \tag{5.15}$$

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and

$$(n-1)z_a e^{z_a} + z_b e^{z_b} = e^2 \tag{5.16}$$

with unknowns z_a, z_b. Let

$$\Gamma_{5.15} = \{(z_a, z_b) \in \mathbb{R}^2 : z_a, z_b \text{ satisfy equation (5.15)}\}$$

and

$$\Gamma_{5.16} = \{(z_a, z_b) \in \mathbb{R}^2 : z_a, z_b \text{ satisfy equation (5.16)}\}.$$

We describe the graphs of $\Gamma_{5.15}$ and $\Gamma_{5.16}$ separately in the following. Here, $\Gamma_{5.15}$ is a one-dimensional smooth curve with two asymptotes $z_a = 2 - \log(n-1)$ and $z_b = 2$. It is strictly decreasing when we consider the curve as the graph of the function

$$z_b = \log(e^2 - (n-1)e^{z_a})$$

for $z_a \in (-\infty, 2 - \log(n-1))$. Here, $\Gamma_{5.16}$ is also a one-dimensional smooth curve with two asymptotes $z_a = \varsigma^{-1}(e^2/(n-1))$ and $z_b = \varsigma^{-1}(e^2)$. When we consider the $\Gamma_{5.16}$ as the graph of the function

$$z_b = \eta(z_a)$$

as the implicit function induced by equation (5.16), it is strictly increasing for $z_a \in (-\infty, -1)$, strictly decreasing for $z_a \in (-1, \varsigma^{-1}(e^2/n-1))$, with its maximum $\varsigma^{-1}(e^2 + (n-1)e^{-1})$ attained at $z_a = -1$. Let $\varsigma_l^{-1}(-(e^2/(n-1)))$ be the smaller one of the two intersections of $z_b = 2$ and $\Gamma_{5.16}$, then $\Gamma_{5.15}$ and $\Gamma_{5.16}$ must intersect at some unique point $c_{a,n} \in (-\infty, \varsigma_l^{-1}(-(e^2/(n-1))))$. Obviously,

$$\lim_{n\to\infty} c_{a,n} = -\infty$$

since $\lim_{n\to\infty} \varsigma_l^{-1}(-(e^2/(n-1))) = -\infty$. Now we analyse the order of $c_{a,n}$ with respect to n as $n \to \infty$. Let

$$z_{a,n} = -\log n - \log\log n + \log 1 - 1.$$

One can check that

$$\lim_{n\to\infty, z_b\to 2}((n-1)e^{z_{a,n}} + e^{z_b}) = e^2,$$

while

$$\lim_{n\to\infty, z_b\to 2}((n-1)z_{a,n}e^{z_{a,n}} + z_b e^{z_b}) = e^2.$$

These imply that

$$c_{a,n} = -\log n - \log\log n + o(\log\log n).$$

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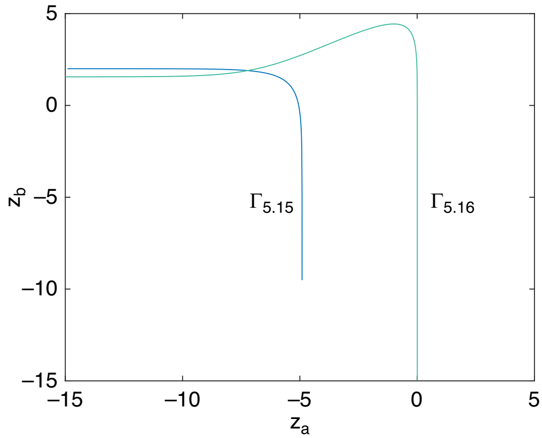

FIGURE 4. $\Gamma_{5.15}$ and $\Gamma_{5.16}$.

Note that $(c_{a,n}, c_{a,n}, \ldots, c_{a,n}, \eta(c_{a,n})) \in \Gamma^n_{5.4,5.5}$ for $t_* = 1$, $a_0 = 2$, $a_1 = 1$. Now

$$\begin{aligned} &R_2(c_{a,n}, c_{a,n}, \ldots, c_{a,n}, \eta(c_{a,n})) \\ &\quad = (n-1)c_{a,n}^2 e^{c_{a,n}} + (\eta(c_{a,n}))^2 e^{\eta(c_{a,n})} \\ &\quad = (n-1)(-\log n - \log\log n + o(\log\log n))^2 e^{-\log n - \log\log n + o(\log\log n)} + 4e^2 + o(1) \\ &\quad = \log n + o(\log n), \end{aligned}$$

from which it is easy to see that

$$\lim_{n\to\infty} R_2(c_{a,n}, c_{a,n}, \ldots, c_{a,n}, \eta(c_{a,n})) = \infty.$$

This forces

$$\lim_{n\to\infty} M_{1,2,1,n} = \infty,$$

by equation (5.8). □

We provide the readers with the curves $\Gamma_{5.15}$ and $\Gamma_{5.16}$ in Figure 4. Obviously, some more general conclusions are available if one considers variations of the parameters t_*, a_0, a_1 in Proposition 5.10. In the proof of Proposition 5.10, we analysed the asymptotic behaviour of $c_{a,n}$ as $n \to \infty$. To illustrate this, we provide the readers with some solutions $\{c_{a,n}\}_{n\in\mathbb{N}}$ and $\{\eta(c_{a,n})\}_{n\in\mathbb{N}}$ in Table 3, from which one can see the order of decay and increase of the sequences with respect to n clearly.

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TABLE 3. $\{c_{a,n}\}_{n\in\mathbb{N}}$ and $\{\eta(c_{a,n})\}_{n\in\mathbb{N}}$.

n	$c_{a,n}$	$\eta(c_{a,n})$
10	−1.85995393917976537809966863644493	1.76340424775818606363428125209881
10^2	−4.62785299403019471574581803056766	1.85809069285605051409608751804338
10^3	−7.22789233650463543039196714750352	1.89657082100674548171296990663334
10^4	−9.75292792230419581894019401286774	1.91807103892852590821383963667555
10^5	−12.2342618412217854056518768558282	1.93194942038187967171518665253066
10^6	−14.6866894851123831962533508855528	1.94170104203817613268248858594343
10^7	−17.1184755091303384193214492191776	1.94895071801313634311296014177922
10^8	−19.5347377367521112496707411765774	1.95456281336907363919131411297777
10^9	−21.9388778842818978934220874285999	1.95904178330801938860687035806662
10^{10}	−24.3332775923466023382637503500222	1.96270276204691539554889598453337
10^{11}	−26.7196721724613718137359326288994	1.96575318147295953788541814562218
10^{12}	−29.0993666702574352619822748618111	1.96833537071115737384924651308007
10^{13}	−31.4733681675710306244561991538449	1.97055035049694776128554517683838
10^{14}	−33.8424706272695953266115358589551	1.97247186852169295822061150290334
10^{15}	−36.2073114123875113940739342289922	1.97415505835468270468553440071226
10^{16}	−38.5684101551989518363378968228881	1.97564198636943790477268372057
10^{17}	−40.9261962228690589891740116163114	1.97696532087300889047496195999928
10^{18}	−43.2810285842129478778122580929991	1.97815082717036133653890807506992
10^{19}	−45.6332104756234277296479388698556	1.97921910564590125340629767477555
10^{20}	−47.9830004233533897413289905575776	1.98018682848468513796104731788004
10^{21}	−50.3306206600083322718206943068339	1.98106763637152920203698625574229
10^{22}	−52.6762636430828551946718030537442	1.98187279967720320796422608006619
10^{23}	−55.0200971682915928490668881764554	1.98261171341330189445969360813992
10^{24}	−57.3622684270770609225783790632446	1.98329227284679498173122091156553
10^{25}	−59.7029072601601322013517238564661	1.98392116211029610842221055234008
10^{26}	−62.0421287914470745386168658260992	1.98450407848856010431861758015229
10^{27}	−64.3800355790304705539785776162448	1.98504590853717112814043429887332
10^{28}	−66.7167193860027551269636196137768	1.98555086770573578849210724716882
10^{29}	−69.0522626491377148819225744497662	1.98602261200888203562928803213885
10^{30}	−71.3867397053852773269628202490444	1.98646432807353401817747846681339
10^{31}	−73.7202178226698188293210417966	1.98687880630254565103989961610088
10^{32}	−76.0527580713762577249568062292201	1.98726850074176252126365880612333
10^{33}	−78.3844160652407074976063450343229	1.98763557839113706498947899068331
10^{34}	−80.7152425944901268288082382972991	1.98798196007258891805580423887717
10^{35}	−83.0452841695382972012692286950551	1.98830935449689269334189863929556
10^{36}	−85.3745834900110932049269107730442	1.98861928681623228551949946425995
10^{37}	−87.7031798510995004084418852428221	1.98891312267786628697106908984993
10^{38}	−90.0311094970450125539792496901771	1.98919208858587552125889114441223
10^{39}	−92.3584059298155212306222549142338	1.98945728921648319614991573263111
10^{40}	−94.6851001796304398868176781699888	1.98970972220645834855497416145556

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