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Estimates for sums and gaps of eigenvalues of Laplacians on measure spaces

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For Laplacians defined by measures on a bounded domain in $\mathbb{R}^n$, we prove analogues of the classical eigenvalue estimates for the standard Laplacian: lower bound of sums of eigenvalues by Li and Yau, and gaps of consecutive eigenvalues by Payne, Pólya and Weinberger. This work is motivated by the study of spectral gaps for Laplacians on fractals.

Keywords: Eigenvalue estimate; Fractal; Measure; Laplacian

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1. Introduction

One of the anomalous behaviours of Laplacians on fractals is the existence of spectral gaps, i.e., $\overline{\lim}_{k\to\infty} \lambda_{k+1}/\lambda_k > 1$, where λ_k are the eigenvalues. This is not possible for the standard Laplacian on bounded domains in $\mathbb{R}^n$ or on compact Riemannian manifolds. In fact, according to the Weyl law, on a compact connected oriented n-dimensional Riemannian manifold M,

$$(\lambda_k)^{n/2} \sim \frac{(2\pi)^n k}{B_n \operatorname{vol}(M)} \quad \text{as } k \to \infty,$$

where vol(M) is the volume of M, and B_n is the volume of the unit ball in $\mathbb{R}^n$ (see, e.g., [**5**]). Consequently, $\lim_{k\to\infty} \lambda_{k+1}/\lambda_k = 1$. Strichartz [**36**] showed that the existence of spectral gaps implies better convergence of Fourier series. Rigorous proofs for the existence of spectral gaps have been obtained for only a limited number of fractals, such as the Sierpiński gasket and the Vicsek set (see [**9**, **19**, **38**]). For Laplacians defined by most self-similar measures, especially those with overlaps, it is not clear whether spectral gaps exist. This is the main motivation of the present paper. This paper is also a continuation of the work by the authors [**8**] and by Pinasco and Scarola [**32**] on estimating the first eigenvalue of Laplacians with respect to fractal measures.

To describe some classical results, let Ω be a bounded domain on $\mathbb{R}^n$ and let λ_k be the k-th Dirichlet eigenvalue. Li and Yau [**24**] obtained the following lower estimate for the sum of the first k eigenvalues

$$\sum_{i=1}^{k} \lambda_i \geqslant \frac{nC_n}{n+2} k^{(n+2)/n} \operatorname{vol}(\Omega)^{-2/n}, \tag{1.1}$$

where vol(Ω) denotes the volume of Ω, and $C_n = (2\pi)^2 B_n^{-2/n}$.

An upper estimate was obtained by Kröger [**23**]. The results of Li-Yau and Kröger have been extended to homogeneous Riemannian manifolds by Strichartz [**35**].

For the gaps between consecutive eigenvalues, Payne, Pólya and Weinberger [**31**] (see also [**33**]) proved the following estimate for the gaps between two consecutive eigenvalues:

$$\lambda_{k+1} - \lambda_k \leqslant \frac{4\sum_{i=1}^{k} \lambda_i}{nk}. \tag{1.2}$$

The goal of this paper is to prove analogues of (1.1) and (1.2) for Laplacians defined by a measure μ. Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset of $\mathbb{R}^n$ and μ be a positive finite Borel measure with $\mu(\Omega) > 0$ and with support being contained in $\overline{\Omega}$. Under suitable conditions (see § 2), μ defines a Dirichlet Laplacian $-\Delta_\mu$; moreover, there exists an orthonormal basis $\{\varphi_n\}$ consisting of eigenfunctions of $-\Delta_\mu$ and the eigenvalues λ_n satisfy $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, with $\lim_{n\to\infty} \lambda_n = \infty$. We remark that if μ is the restriction of Lebesgue measure to Ω, then Δ_μ is the classical Dirichlet Laplacian.

We first prove an analogue of the classical lower estimate of the sum of eigenvalues of the standard Laplacian obtained by Li and Yau [**24**]. We let $L^2_\mu(\Omega)$ denote the Hilbert space of square-integrable functions with respect to μ. For $u \in L^2_\mu(\Omega)$, if there is no confusion of what Ω is, we let

$$\|u\|_\mu = \left(\int_\Omega |u|^2 \, d\mu\right)^{1/2}.$$

If μ is the restriction of Lebesgue measure to Ω, we denote the corresponding L^2-space and norm respectively by $L^2(\Omega)$ and $\|\cdot\|$.

THEOREM 1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, μ be a positive finite Borel measure on Ω with* $\operatorname{supp}(\mu) \subseteq \overline{\Omega}$, *$-\Delta_\mu$ be the Dirichlet Laplacian defined by μ*

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described in § 2, λ_k *be the* k*-th eigenvalue of* $-\Delta_\mu$*, and* φ_k *be the corresponding* $L^2_\mu(\Omega)$*-normalized eigenfunction. Then*

$$\sum_{j=1}^k \lambda_j \geqslant \frac{n}{n+2}\left(\sum_{j=1}^k \|\varphi_j\|^2\right)^{(n+2)/n}\left(B_n \sup_{z\in\mathbb{R}^n}\sum_{j=1}^k |\hat{\varphi}_j(z)|^2\right)^{-2/n}$$
$$\geqslant \frac{nC_n}{n+2}\left(\sum_{j=1}^k \|\varphi_j\|^2\right)\mathrm{vol}(\Omega)^{-2/n},$$

where $C_n = (2\pi)^2 B_n^{-2/n}$ *as in* (1.1).

Finally, we generalize the classical theorem by Payne, Pólya and Weinberger [**31**] on the gaps between two consecutive eigenvalues.

Theorem 1.2. *Assume the hypotheses of theorem* 1.1 *and assume in addition that the domain of the Dirichlet form* $\mathcal{E}(\cdot,\cdot)$ *in* (2.2) *is* $H^1_0(\Omega)$*. Then for all* $k \geqslant 1$,

$$\lambda_{k+1} - \lambda_k \leqslant \frac{4\sum_{i=1}^k \lambda_i}{n\sum_{i=1}^k \|\varphi_i\|^2}\cdot\frac{f(\mu;\varphi_1,\ldots,\varphi_k)}{g(\mu;\varphi_1,\ldots,\varphi_k)}, \tag{1.3}$$

where

$$f(\mu;\varphi_1,\ldots,\varphi_k) := \frac{1}{n}\int_\Omega |x|^2\left(\sum_{i=1}^k \varphi_i^2\right)\mathrm{d}x$$
$$-\frac{2}{n}\sum_{i,\ell=1}^k\sum_{\alpha=1}^n\left(\int_\Omega x_\alpha\varphi_i\varphi_\ell\,d\mu\right)\int_\Omega x_\alpha\varphi_i\varphi_\ell\,\mathrm{d}x$$
$$+\frac{1}{n}\sum_{i,j,\ell=1}^k\sum_{\alpha=1}^n\left(\int_\Omega x_\alpha\varphi_i\varphi_j\,\mathrm{d}\mu\right)\left(\int_\Omega x_\alpha\varphi_i\varphi_\ell\,\mathrm{d}\mu\right)\int_\Omega \varphi_j\varphi_\ell\,\mathrm{d}x$$

and

$$g(\mu;\varphi_1,\ldots,\varphi_k) := \frac{1}{n}\int_\Omega |x|^2\left(\sum_{i=1}^k \varphi_i^2\right)\mathrm{d}\mu - \frac{1}{n}\sum_{i,\ell=1}^k\sum_{\alpha=1}^n\left(\int_\Omega x_\alpha\varphi_i\varphi_\ell\,\mathrm{d}\mu\right)^2.$$

Remark 1.1. In the case μ is Lebesgue measure, $f(\mu;\varphi_1,\ldots,\varphi_k) = g(\mu;\varphi_1,\ldots,\varphi_k)$, and consequently the inequality in theorem 1.2 reduces to

$$\lambda_{k+1} - \lambda_k \leqslant \frac{4\sum_{i=1}^k \lambda_i}{nk},$$

which coincides with the classical Payne, Pólya and Weinberger inequality (see [**31**, **33**]).

Remark 1.2. We note that in (1.3), $\|\varphi_j\| > 0$ for all i. In fact, if $\nabla\varphi_j = 0$, then, in view of the Poincaré inequality for measures [see (2.1)], we would get $\|\varphi_j\|_\mu = 0$, a contradiction.

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REMARK 1.3. If $\Omega = (a, b)$ and $\mathrm{supp}(\mu) = [a, b]$, it is proved in [4] that the domain of the Dirichlet form $\mathcal{E}$ is equal to $H_0^1(\Omega)$. The same holds in higher dimensions if μ is equivalent to Lebesgue measure [30].

Both theorems 1.1 and 1.2 involve the sum $\sum_{i=1}^k \|\varphi_i\|^2$, which suggests that it is necessary to study the eigenfunctions. We are not able to obtain a good estimate for this sum. Properties of eigenfunctions in one-dimension, especially when the support of μ is an interval, have been studied in [3]). In § 3, we focus on the case when the support of μ is not an interval, such as a Cantor-type measure.

This paper is organized as follows. In § 2, we recall the definition and some elementary properties of the Dirichlet Laplacian Δ_μ defined on a domain by a measure μ. In § 3 we prove the min-max principle for $-\Delta_\mu$ and some properties of the eigenfunctions in one-dimension. Theorem 1.1 is proved in § 4. § 5 is devoted to the proof of theorem 1.2. Finally in § 6 we state some comments and open questions.

2. Preliminaries

For convenience, we summarize the definition of the Dirichlet Laplacian with respect to a measure μ; details can be found in [20]. Let $\Omega \subset \mathbb{R}^n$ be a bounded open subset and μ be a positive finite Borel measure with $\mathrm{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$. We further suppose μ satisfies the following Poincaré inequality (PI) for measures: There exists a constant $C > 0$ such that

$$\int_\Omega |u|^2 \,\mathrm{d}\mu \leqslant C \int_\Omega |\nabla u|^2 \,\mathrm{d}x \quad \text{for all } u \in C_c^\infty(\Omega). \tag{2.1}$$

Notice that (PI) cannot be immediately extended to $H_0^1(\Omega)$ functions. For example, let $\Omega = (0, 1) \subseteq \mathbb{R}$ and μ be the standard Cantor measure, which is supported on the Cantor set. For any $u \in H_0^1(\Omega)$, if we increase the value of u on the Cantor set, $\int_0^1 |\nabla u|^2 \,\mathrm{d}x$ remains unchanged but $\int_0^1 |u|^2 \,\mathrm{d}\mu$ can be increased within the same equivalence class of u without bound and hence the inequality cannot hold. However, the following is true. (PI) implies that each equivalence class $u \in H_0^1(\Omega)$ contains a unique (in $L^2_\mu(\Omega)$ sense) member $\bar{u}$ that belongs to $L^2_\mu(\Omega)$ and satisfies both conditions below:

(1) There exists a sequence $\{u_n\}$ in $C_c^\infty(\Omega)$ such that $u_n \to \bar{u}$ in $H_0^1(\Omega)$ and $u_n \to \bar{u}$ in $L^2_\mu(\Omega)$;

(2) $\bar{u}$ satisfies the inequality in (2.1).

We call $\bar{u}$ the $L^2_\mu(\Omega)$-*representative* of u. Consider our Cantor set example above. For $u \in H_0^1(\Omega)$, let $\{u_n\} \subseteq C_c^\infty(\Omega)$ be a sequence convergent to u and hence Cauchy in $H_0^1(\Omega)$. By (PI), $\{u_n\}$ is Cauchy and hence convergent in $L^2_\mu(\Omega)$. Then $\bar{u}$ is the function obtained by redefining u on the Cantor set to be the $L^2_\mu(\Omega)$ limit of u_n.

Assume (PI) holds and define a mapping $\iota : H_0^1(\Omega) \to L^2_\mu(\Omega)$ by

$$\iota(u) = \bar{u}.$$

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ι is a bounded linear operator, but not necessarily injective. Consider the subspace $\mathcal{N}$ of $H_0^1(\Omega)$ defined as

$$\mathcal{N} := \{u \in H_0^1(\Omega) : \|\iota(u)\|_\mu = 0\}.$$

Now let $\mathcal{N}^\perp$ be the orthogonal complement of $\mathcal{N}$ in $H_0^1(\Omega)$. Then $\iota : \mathcal{N}^\perp \to L^2_\mu(\Omega)$ is injective. Throughout the rest of this paper, unless explicitly stated otherwise, we will use the $L^2_\mu(\Omega)$-representative $\bar{u}$ of u and denote it simply by u.

Consider a nonnegative bilinear form $\mathcal{E}(\cdot,\cdot)$ in $L^2_\mu(\Omega)$ given by

$$\mathcal{E}(u,v) := \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \tag{2.2}$$

with *domain* $\mathrm{dom}(\mathcal{E}) = \mathcal{N}^\perp$, or more precisely, $\iota(\mathcal{N}^\perp)$. (PI) implies that $(\mathcal{E}, \mathrm{dom}(\mathcal{E}))$ is a closed quadratic form on $L^2_\mu(\Omega)$. Hence, there exists a nonnegative self-adjoint operator $-\Delta_\mu$ in $L^2_\mu(\Omega)$ such that $\mathrm{dom}(\mathcal{E}) = \mathrm{dom}((-\Delta_\mu)^{1/2})$ and

$$\mathcal{E}(u,v) = \Big((-\Delta_\mu)^{1/2}u, (-\Delta_\mu)^{1/2}v\Big)_\mu \quad \text{for all } u, v \in \mathrm{dom}(\mathcal{E}),$$

(see [7]), where throughout this paper, $(\cdot,\cdot)_\mu$ denotes the inner product in $L^2_\mu(\Omega)$. We call Δ_μ the *(Dirichlet) Laplacian with respect to* μ. It follows that $u \in \mathrm{dom}(\Delta_\mu)$ and $-\Delta_\mu u = f$ if and only if $-\Delta u = f \, \mathrm{d}\mu$ in the sense of distribution: for all $\varphi \in C_c^\infty(\Omega)$, $\int_\Omega \nabla u \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega f\varphi \, \mathrm{d}\mu$ (see [20, proposition 2.2]). A real number $\lambda \in \mathbb{R}$ is a (Dirichlet) eigenvalue of $-\Delta_\mu$ with eigenfunction f if for all $\varphi \in C_c^\infty(\Omega)$,

$$\int_\Omega \nabla f \cdot \nabla \varphi \, \mathrm{d}x = \lambda \int_\Omega f\varphi \, \mathrm{d}\mu. \tag{2.3}$$

From [20, theorem 1.2], when μ satisfies (PI), there exists an orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ of $L^2_\mu(\Omega)$ consisting of (Dirichlet) eigenfunctions of $-\Delta_\mu$. The eigenvalues $\{\lambda_n\}_{n=1}^\infty$ satisfy $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$. Moreover, if $\dim(\mathrm{dom}\,\mathcal{E}) = \infty$, then $\lim_{n\to\infty} \lambda_n = \infty$. We have the following characterizations of $\mathrm{dom}\,\mathcal{E}$ and $\mathrm{dom}\,(-\Delta_\mu)$:

$$\mathrm{dom}\,\mathcal{E} = \mathcal{N}^\perp = \left\{ \sum_{n=1}^\infty c_n\varphi_n : \sum_{n=1}^\infty c_n^2\lambda_n < \infty \right\},$$
$$\mathrm{dom}\,(-\Delta_\mu) = \left\{ \sum_{n=1}^\infty c_n\varphi_n : \sum_{n=1}^\infty c_n^2\lambda_n^2 < \infty \right\}.$$

The Laplacian Δ_μ can be used to describe various physical phenomena on a domain Ω with an inhomogeneous mass distribution. For example, an inhomogeneous vibrating string or membrane with mass distribution μ and satisfying the Dirichlet boundary condition is governed by a wave equation of the form $u_{tt} = c^2\Delta_\mu u$. Similarly, heat conduction in such a domain can be described by a heat equation of the form $u_t = k\Delta_\mu u$.

Classically, in one dimension, the operator Δ_μ has been studied quite extensively. Kac and Kreĭn [21] studied the spectrum of Δ_μ as well as the associated spectral function, i.e., a nonnegative increasing function having jumps at each eigenvalue of Δ_μ. In [**21**], the support of the measure is allowed to be noncompact and the measure is allowed to be infinite, and for these cases, criteria for the spectrum to be positive and discrete are obtained by making use of a generalized Fourier transform mapping the L^2-space defined by μ to an L^2-space defined by the spectral measure. The operator Δ_μ can be defined equivalently by Volterra–Stieltjes integral equations (see [**2**, **22**]). Feller studied the operator Δ_μ in connection with diffusion processes [**10**, **11**]. Spectral asymptotics of Δ_μ was studied by McKean and Ray [**25**].

More recently, the operator Δ_μ has been studied extensively in connection with fractal measures by authors including Fujita, Solomyak, Verbitsky, Naimark, Freiberg, Lobus, Zähle, Bird *et al.*, Hu *et al.*, Andrews *et al.*, Gu *et al.*, Tang, Xie and the authors (see [**1**, **3**, **4**, **6**, **8**, **12**–**18**, **20**, **26**–**30**, **34**, **37**] and the references therein). Many of these papers study the spectral asymptotics of Δ_μ, while others study the associated wave, heat and Schrödinger equations. We point out that the operators in some of Freiberg and Zähle's work are more general. More precisely, one may regard Δ_μ formally as $d^2/(\mathrm{d}\mu\,\mathrm{d}x)$, since the Dirichlet form defining Δ_μ [see (2.2)] is an integral with respect to Lebesgue measure $\mathrm{d}x$. Freiberg and Zähle studied more general operators of the form $d^2/(\mathrm{d}\mu\,\mathrm{d}\nu)$, where ν is a Borel measure without point mass.

To state a sufficient condition for (PI), we recall that the *lower L^∞-dimension* of a measure μ is defined by

$$\underline{\dim}_\infty(\mu) = \liminf_{\delta\to 0^+} \frac{\ln(\sup_x \mu(B_\delta(x)))}{\ln \delta},$$

where the supremum is taken over all $x \in \mathrm{supp}(\mu)$.

Theorem 2.1. ([**20**, theorems 1.1 and 1.2]) *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set and μ be a positive finite Borel measure on $\mathbb{R}^n$ with $\mathrm{supp}(\mu) \subseteq \overline{\Omega}$ and $\mu(\Omega) > 0$. Assume $\underline{\dim}_\infty(\mu) > n-2$.*

(a) *(PI) holds. In particular, if $n = 1$, or $n = 2$ and μ is upper s-regular with $s > 0$, or μ is absolutely continuous with bounded density, then (PI) holds.*

(b) *The set of eigenvalues of $-\Delta_\mu$ is contained in $(0,\infty)$ and has no accumulation point. Hence $-\Delta_\mu$ has a positive smallest eigenvalue λ_1^μ.*

3. Min-max principle and properties of eigenfunctions

Let Ω be a bounded domain in $\mathbb{R}^n$ and μ be a positive finite Borel measure with $\mathrm{supp}(\mu) \subseteq \overline{\Omega}$. In this section we extend the variational principle for the principal eigenvalue and Courant's min-max principle for the k-th eigenvalue to the Laplacians Δ_μ. This will be needed in the proof of theorem 1.2. We introduce some additional notation that will be needed in the proof of the theorem. For any subset $S \subseteq \mathrm{dom}\,\mathcal{E}$, let $\langle S\rangle$ be the vector subspace of $\mathrm{dom}\,\mathcal{E}$ spanned by S, and let $S^\perp$ be the orthogonal complement of S in $\mathrm{dom}\,\mathcal{E}$ with respect to the inner product in $H_0^1(\Omega)$.

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THEOREM 3.1 Min-max principle. *Let Δ_μ be the Dirichlet Laplacian defined on a bounded domain $\Omega \subseteq \mathbb{R}^n$ and let $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ be the eigenvalues. Then for $k = 1, 2, \ldots$, the k-th eigenvalue satisfies*

$$\lambda_k = \max_{S \in \Sigma_{k-1}} \min \big\{ \mathcal{E}(u,u) : u \in S^{\perp}, \|u\|_\mu = 1 \big\}, \tag{3.1}$$

where Σ_{k-1} is the collection of all $(k-1)$-dimensional subspaces of $\operatorname{dom}\mathcal{E}$. In particular, for $k = 1$ we have the variational principle for the principal eigenvalue:

$$\lambda_1 = \min \big\{ \mathcal{E}(u,u) : u \in \operatorname{dom}\mathcal{E}, \|u\|_\mu = 1 \big\}. \tag{3.2}$$

Proof. *Step 1.* Let $\{\varphi_k\} \subseteq L^2_\mu(\Omega)$ be an orthonormal basis of $L^2_\mu(\Omega)$ with $\{\varphi_k\} \subset \operatorname{dom}\mathcal{E}$ satisfying

$$\begin{cases} -\Delta_\mu \varphi_k = \lambda_k \varphi_k & \text{in } \Omega \\ \varphi_k = 0 & \text{in } \partial\Omega \end{cases} \tag{3.3}$$

for $k = 1, 2, \ldots$. Hence

$$\mathcal{E}(\varphi_k, \varphi_l) = (\lambda_k \varphi_k, \varphi_l)_\mu = \lambda_k \delta_{kl}, \tag{3.4}$$

where δ_{kl} is the Kronecker delta. If $u \in \operatorname{dom}\mathcal{E}$ and $\|u\|_\mu = 1$, then we can write

$$u = \sum_{k=1}^{\infty} d_k \varphi_k \quad \text{in } L^2_\mu(\Omega), \tag{3.5}$$

where $d_k = (u, \varphi_k)_\mu$, and the equality holds in the sense that $\|u - \sum_{k=1}^{N} d_k \varphi_k\|_\mu \to 0$ as $N \to \infty$. Moreover,

$$\sum_{k=1}^{\infty} d_k^2 = \|u\|_\mu^2 = 1. \tag{3.6}$$

Step 2. Equation (3.4) implies that $\{\varphi_k/\lambda_k^{1/2}\}_{k=1}^{\infty} \subseteq \operatorname{dom}\mathcal{E}$ is an orthonormal set with respect to the inner product $\mathcal{E}(\cdot,\cdot)$. We claim that $\{\varphi_k/\lambda_k^{1/2}\}_{k=1}^{\infty}$ is an orthonormal basis of $\operatorname{dom}\mathcal{E}$ with respect to $\mathcal{E}(\cdot,\cdot)$. To see this, we let $u \in \operatorname{dom}\mathcal{E}$ such that $\mathcal{E}(\varphi_k, u) = 0$ for all $k \geqslant 1$. Then $(\lambda_k \varphi_k, u)_\mu = 0$ for all $k \geqslant 1$ and hence $(\varphi_k, u)_\mu = 0$ for all $k \geqslant 1$. Thus, $u = 0$ μ-a.e. on Ω, which implies that $u = 0$ in $\operatorname{dom}\mathcal{E}$ (in the H^1_0-norm), since $\iota : \mathcal{N}^{\perp} \to L^2_\mu(\Omega)$ is an injection. Thus, for all $u \in \operatorname{dom}\mathcal{E}$,

$$u = \sum_{k=1}^{\infty} a_k \frac{\varphi_k}{\lambda_k^{1/2}}, \tag{3.7}$$

where $a_k = \mathcal{E}(u, \varphi_k/\lambda_k^{1/2})$. Observe that

$$a_k = \mathcal{E}\left(\frac{\varphi_k}{\lambda_k^{1/2}}, u\right) = \frac{1}{\lambda_k^{1/2}} (\lambda_k \varphi_k, u)_\mu = \frac{1}{\lambda_k^{1/2}} \left(\lambda_k \varphi_k,\ \sum_{\ell=1}^{\infty} d_\ell \varphi_\ell\right)_\mu = \lambda_k^{1/2} d_k.$$

Substituting this into (3.7), we get

$$u = \sum_{k=1}^{\infty} (\lambda_k^{1/2} d_k) \frac{\varphi_k}{\lambda_k^{1/2}} = \sum_{k=1}^{\infty} d_k \varphi_k \quad \text{in dom}\,\mathcal{E}. \tag{3.8}$$

Combining (3.4), (3.6), (3.8), we get

$$\mathcal{E}(u,u) = \mathcal{E}\bigg(\sum_{k=1}^{\infty} d_k \varphi_k,\ \sum_{k=1}^{\infty} d_k \varphi_k \bigg) = \sum_{k=1}^{\infty} d_k^2 \lambda_k \geqslant \lambda_1.$$

Since $\mathcal{E}(\varphi_1, \varphi_1) = \lambda_1$, (3.2) follows.

Step 3. Let $\{\varphi_k\}_{k=1}^{\infty} \subseteq L^2_\mu(\Omega)$ be as in Step 1. Let $v \in \text{dom}\,\mathcal{E}$ with $\|v\|_\mu = 1$. Write $v = \sum_{i=1}^{\infty} c_i \varphi_i$, where the equality holds in both $L^2_\mu(\Omega)$ and $\text{dom}\,\mathcal{E}$. As $\mathcal{E}(v,v) = \sum_{i=1}^{\infty} \lambda_i c_i^2$,

$$\begin{aligned} \lambda_k &= \min\bigg\{ \sum_{i=1}^{\infty} \lambda_i c_i^2 : c_1 = \cdots = c_{k-1} = 0,\ \sum_{i=1}^{\infty} c_i^2 = 1 \bigg\} \\ &= \min\big\{ \mathcal{E}(v,v) : v \in \langle \varphi_1, \ldots, \varphi_{k-1} \rangle^{\perp},\ \|v\|_\mu = 1 \big\}. \end{aligned} \tag{3.9}$$

We claim that this is equal to

$$\max_{S \in \Sigma_{k-1}} \min\big\{ \mathcal{E}(v,v) : v \in S^{\perp}, \|v\|_\mu = 1 \big\}.$$

To prove this, let $S \in \Sigma_{k-1}$ and let $S = \langle v_1, \ldots, v_{k-1} \rangle$ with $v_\ell = \sum_{i=1}^{\infty} d_{\ell i} \varphi_i$ for $\ell = 1, \ldots, k-1$. Consider the following two cases.

Case 1. $\det(d_{\ell i})_{\ell,i=1}^{k-1} = 0$.

In this case, there exist $c_1, \ldots, c_{k-1}$, not all zero, such that

$$\sum_{i=1}^{k-1} c_i^2 = 1 \quad \text{and} \quad \sum_{i=1}^{k-1} d_{\ell i} c_i = 0, \quad \ell = 1, \ldots, k-1. \tag{3.10}$$

Let $\tilde{v} := \sum_{i=1}^{k-1} c_i \varphi_i$. Then by (3.10), $\tilde{v} \in S^{\perp}$ and $\|\tilde{v}\|_\mu = 1$. Using these and (3.9), we get

$$\min_{v \in S^{\perp}, \|v\|_\mu = 1} \mathcal{E}(v,v) \leqslant \mathcal{E}(\tilde{v}, \tilde{v}) = \sum_{i=1}^{k-1} \lambda_i c_i^2 \leqslant \lambda_k = \min_{v \in \langle \varphi_1, \ldots, \varphi_{k-1} \rangle^{\perp}, \|v\|_\mu = 1} \mathcal{E}(v,v).$$

Case 2. $\det(d_{\ell i})_{\ell,i=1}^{k-1} \neq 0$.

Let $v \in \langle \varphi_1, \ldots, \varphi_{k-1} \rangle^{\perp}$ with $\|v\|_\mu = 1$, i.e., $v = \sum_{i=k}^{\infty} c_i \varphi_i$ with $\sum_{i=k}^{\infty} c_i^2 = 1$ and $\mathcal{E}(v,v) = \sum_{i=k}^{\infty} \lambda_i c_i^2$. We will find $v' \in S^{\perp}$ with $\|v'\|_\mu = 1$ such that $\mathcal{E}(v', v') \leqslant \mathcal{E}(v,v)$.

We claim that there exists $\tilde{v} = \sum_{i=1}^{\infty} c_i \varphi_i$ (i.e., $\tilde{v}$ has the same φ_i components as v for $i \geqslant k$) such that $(\tilde{v}, v_\ell)_\mu = 0$ for all $\ell = 1, \ldots, k-1$. To see this notice that

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the condition $(\tilde{v}, v_\ell)_\mu = 0$ implies that

$$\sum_{i=1}^{k-1} d_{\ell i} c_i = -\sum_{i=k}^{\infty} d_{\ell i} c_i, \quad \ell = 1, \ldots, k-1.$$

Since $\det(d_{\ell i})_{\ell,i=1}^{k-1} \neq 0$, a solution $c_1, \ldots, c_{k-1}$ exists (possibly all 0). This proves the claim. Let $v' := \tilde{v}/\|\tilde{v}\|_\mu$. Note that

$$\|\tilde{v}\|_\mu^2 = \sum_{i=1}^{k-1} c_i^2 + \sum_{i=k}^{\infty} c_i^2 = \sum_{i=1}^{k-1} c_i^2 + 1,$$

and

$$\sum_{j=1}^{k-1} \lambda_j c_j^2 \leqslant \left(\sum_{j=1}^{k-1} c_j^2\right) \lambda_k = \left(\sum_{j=1}^{k-1} c_j^2\right)\left(\sum_{j=k}^{\infty} c_j^2\right) \lambda_k \leqslant \left(\sum_{j=1}^{k-1} c_j^2\right)\left(\sum_{j=k}^{\infty} \lambda_j c_j^2\right).$$

Thus,

$$\begin{aligned} \mathcal{E}(v', v') &= \frac{1}{\|\tilde{v}\|_\mu^2} \mathcal{E}(\tilde{v}, \tilde{v}) = \frac{1}{1 + \sum_{i=1}^{k-1} c_i^2} \sum_{j=1}^{\infty} \lambda_j c_j^2 \\ &= \frac{1}{1 + \sum_{i=1}^{k-1} c_i^2} \left(\sum_{j=1}^{k-1} \lambda_j c_j^2 + \sum_{j=k}^{\infty} \lambda_j c_j^2\right) \\ &\leqslant \sum_{j=k}^{\infty} \lambda_j c_j^2 = \mathcal{E}(v, v). \end{aligned}$$

It follows that

$$\min_{v' \in S^\perp, \|v'\|_\mu = 1} \mathcal{E}(v', v') \leqslant \min_{v \in \langle \varphi_1, \ldots, \varphi_k \rangle^\perp, \|v\|_\mu = 1} \mathcal{E}(v, v).$$

This completes the proof. □

The following proposition establishes some properties of eigenfunctions in one-dimension, some of them being specific for measures on bounded domains in $\mathbb{R}$. Additional properties of eigenfunctions can be found in [**3**]. Let $\mathcal{L}^1(E)$ be the one-dimensional Lebesgue measure of a subset $E \subseteq \mathbb{R}$.

Proposition 3.2. *Let $\Omega \subset \mathbb{R}$ be a bounded open interval, μ be a positive finite Borel measure on Ω with $\operatorname{supp}(\mu) \subseteq \overline{\Omega}, \Delta_\mu$ be the Dirichlet Laplacian with respect to μ, and $\varphi \in H_0^1(\Omega)$ be an eigenfunction of $-\Delta_\mu$, i.e., there exists $\lambda \in \mathbb{R}$ such that $-\Delta_\mu \varphi = \lambda \varphi$. Then*

(a) $\varphi \in C^{0,1/2}(\Omega)$;

(b) *φ is linear over any component of $\Omega \setminus \operatorname{supp}(\mu)$;*

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(c) *if* $\mathcal{L}^1(\operatorname{supp}(\mu)) = 0$, *then* $\varphi \notin C^2(\Omega)$, *and in fact*, φ' *is not absolutely continuous (with respect to Lebesgue measure)*;

(d) *eigenfunctions corresponding to the first eigenvalue do not change sign*;

(e) *the first eigenvalue is simple.*

Proof.

(a) It follows directly from Sobolev's embedding theorem that $H_0^1(\Omega) \hookrightarrow C^{0,1/2}(\Omega)$.

(b) Consider a component (a, b) of $\Omega \setminus \operatorname{supp}(\mu)$. For all $v \in C_c^\infty(a, b) \subseteq C_c^\infty(\Omega)$,

$$\int_\Omega \varphi' v' \,\mathrm{d}x = \lambda \int_\Omega \varphi v \,\mathrm{d}\mu = 0,$$

and hence it also holds for continuous piecewise linear v with $\operatorname{supp}(v) \subset (a, b)$. Note that $\varphi'|_{(a,b)} \in L^2(a, b) \subset L^1(a, b)$. Let $\delta > 0$ and

$$a < x_1 - \delta < x_1 < x_1 + \delta < x_2 - \delta < x_2 < x_2 + \delta < b.$$

Let $v \in C(a, b)$ that is equal to 0 on $(a, x_1 - \delta) \cup (x_2 + \delta, b)$, equal to 1 on $(x_1 + \delta, x_2 - \delta)$, and linear over $(x_1 - \delta, x_1 + \delta)$ and $(x_2 - \delta, x_2 + \delta)$. Then

$$\begin{aligned} 0 = \int_a^b \varphi' v' \,\mathrm{d}y &= \int_{x_1-\delta}^{x_1+\delta} \varphi'(y) v'(y) \,\mathrm{d}y + \int_{x_2-\delta}^{x_2+\delta} \varphi'(y) v'(y) \,\mathrm{d}y \\ &= \frac{1}{2\delta} \int_{x_1-\delta}^{x_1+\delta} \varphi'(y) \,\mathrm{d}y - \frac{1}{2\delta} \int_{x_2-\delta}^{x_2+\delta} \varphi'(y) \,\mathrm{d}y. \end{aligned}$$

By the Lebesgue differentiation theorem, for Lebesgue a.e. $x \in (a, b)$, $\varphi'(x) = c$, a constant. As $\varphi \in H_0^1(\Omega)$ is absolutely continuous, for all $x \in (a, b)$,

$$\varphi(x) = \varphi(a) + \int_a^x \varphi'(y) \,\mathrm{d}y = \varphi(a) + c(x - a),$$

completing the proof of (b).

(c) By part (b), $\varphi'' = 0$ Lebesgue a.e. Hence, if φ' is absolutely continuous, then for Lebesgue a.e. $a, b \in \Omega$,

$$\varphi'(a) - \varphi'(b) = \int_a^b \varphi''(y) \,\mathrm{d}y = 0.$$

Thus, φ' is a constant. Since $\varphi \in H_0^1(\Omega)$, we conclude that $\varphi \equiv 0$, contradicting that φ is an eigenfunction.

(d) By [**3**, proposition 3.4], eigenfunctions corresponding to the first eigenvalue are concave or convex. As they vanish at the end points, they do not change sign.

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(e) Let φ_1 and φ_2 be normalized eigenfunctions corresponding to the first eigenvalue of $-\Delta_\mu$. Then $\varphi_1, \varphi_2 \in C^{0,1/2}(\Omega) \subset C(\overline{\Omega})$. If $\varphi_1 \equiv \varphi_2$ on $\operatorname{supp}(\mu)$, then by linearity on components of $\Omega \setminus \operatorname{supp}(\mu)$ and continuity, $\varphi_1 \equiv \varphi_2$ on Ω. Thus $\varphi_1 \not\equiv \varphi_2$ if and only if $\varphi_1 \not\equiv \varphi_2$ on $\operatorname{supp}(\mu)$.
Suppose that φ_1 and φ_2 are of the same sign, say positive, and $\varphi_1 \not\equiv \varphi_2$. If $\varphi_1 \geqslant \varphi_2$, then $\varphi_1 > \varphi_2$ on some subset $E \subset \Omega$ with $\mu(E) > 0$. Hence

$$1 = \int_\Omega |\varphi_1|^2 \, \mathrm{d}\mu > \int_\Omega |\varphi_2|^2 \, \mathrm{d}\mu = 1,$$

a contradiction. Thus, there exist $x_1, x_2 \in \Omega$ such that $\varphi_1(x_1) > \varphi_2(x_1)$ and $\varphi_1(x_2) < \varphi_2(x_2)$. Now $\varphi = \varphi_1 - \varphi_2$ is an eigenfunction with $\varphi(x_1) > 0$ and $\varphi(x_2) < 0$, contradicting (d). □

4. Lower estimate of sums of eigenvalues

We will use the lemma from [**24**] which says that if f is a real-valued function defined on $\mathbb{R}^n$ with $0 \leqslant f \leqslant M_1$, and

$$\int_{\mathbb{R}^n} |z|^2 f(z) \, \mathrm{d}z \leqslant M_2,$$

then

$$\int_{\mathbb{R}^n} f(z) \, \mathrm{d}z \leqslant (M_1 B_n)^{2/(n+2)} M_2^{n/(n+2)} \left(\frac{n+2}{n} \right)^{n/(n+2)}, \tag{4.1}$$

where we recall that B_n is the volume of the unit ball in $\mathbb{R}^n$.

Proof of theorem 1.1. Let

$$\Phi(x,y) = \sum_{j=1}^k \varphi_j(x)\varphi_j(y), \ x, y \in \Omega \quad \text{and} \quad f(z) := \int_\Omega |\hat{\Phi}(z,y)|^2 \, \mathrm{d}\mu(y), \ z \in \mathbb{R}^n, \tag{4.2}$$

where

$$\hat{\Phi}(z,y) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \Phi(x,y) \, \mathrm{e}^{-ix\cdot z} \, \mathrm{d}x$$

is the Fourier transform of Φ at z and each φ_j is extended to $\mathbb{R}^n$ by setting it equal to 0 on $\mathbb{R}^d \setminus \Omega$. Then, using the linearity of the Fourier transform, we have

$$\begin{aligned} f(z) &= \int_\Omega \Big| \sum_{j=1}^k \hat{\varphi}_j(z)\varphi_j(y) \Big|^2 \, \mathrm{d}\mu(y) \\ &= \int_\Omega \sum_{j,\ell=1}^k \hat{\varphi}_j(z)\varphi_j(y)\overline{\hat{\varphi}_\ell(z)}\varphi_\ell(y) \, \mathrm{d}\mu(y) \\ &= \sum_{j=1}^k \big| \hat{\varphi}_j(z) \big|^2. \end{aligned} \tag{4.3}$$

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Let $M_1 := \sup_{z\in\mathbb{R}^n} \sum_{j=1}^{k} |\hat{\varphi}_j(z)|^2$. Then it follows that for all $z \in \mathbb{R}^n$,

$$0 \leqslant f(z) \leqslant M_1 \leqslant (2\pi)^{-n} \sum_{j=1}^{k} \|\varphi_j\|_{L^1(\Omega)}^2 \leqslant (2\pi)^{-n}\mathrm{vol}(\Omega) \sum_{j=1}^{k} \|\varphi_j\|^2. \tag{4.4}$$

Also,

$$\begin{aligned}
\int_{\mathbb{R}^n} |z|^2 f(z)\,\mathrm{d}z &= \int_\Omega \int_{\mathbb{R}^n} |z|^2 |\hat{\Phi}(z,y)|^2\,\mathrm{d}z\,\mathrm{d}\mu(y) \\
&= \int_\Omega \int_{\mathbb{R}^n} |\widehat{\nabla_z \Phi}(z,y)|^2\,\mathrm{d}z\,\mathrm{d}\mu(y) \\
&= \int_\Omega \int_\Omega |\nabla_x \Phi(x,y)|^2\,\mathrm{d}x\,\mathrm{d}\mu(y) \quad \text{(Plancherel's theorem)} \\
&= \int_\Omega \int_\Omega \left(\sum_{j=1}^{k} \lambda_j \varphi_j(x)\varphi_j(y) \right) \left(\sum_{\ell=1}^{k} \varphi_\ell(x)\varphi_\ell(y) \right) \mathrm{d}\mu(x)\,\mathrm{d}\mu(y) \\
&= \sum_{j=1}^{k} \lambda_j =: M_2,
\end{aligned} \tag{4.5}$$

where the fourth equality follows from (2.3). Using (4.3) followed by the Plancherel theorem, we get

$$\int_{\mathbb{R}^n} f(z)\,\mathrm{d}z = \sum_{j=1}^{k} \|\hat{\varphi}_j(z)\|^2 = \sum_{j=1}^{k} \|\varphi_j\|^2. \tag{4.6}$$

By applying [**24**, lemma 1] (see (4.1)) to the function f in (4.2), and using (4.5) and (4.6), we get

$$\begin{aligned}
\sum_{j=1}^{k} \|\varphi_j\|^2 = \int_{\mathbb{R}^n} f(z)\,\mathrm{d}z &\leqslant (M_1 B_n)^{2/(n+2)} M_2^{n/(n+2)} \left(\frac{n+2}{n} \right)^{n/(n+2)} \\
&= \left(B_n \sup_{z\in\mathbb{R}^n} \sum_{j=1}^{k} |\hat{\varphi}_j(z)|^2 \right)^{2/(n+2)} \left(\sum_{j=1}^{k} \lambda_j \right)^{n/(n+2)} \left(\frac{n+2}{n} \right)^{n/(n+2)}.
\end{aligned}$$

Thus, using (4.4), we get

$$\begin{aligned}
\sum_{j=1}^{k} \lambda_j &\geqslant \left(\sum_{j=1}^{k} \|\varphi_j\|^2 \right)^{(n+2)/n} \left(B_n \sup_{z\in\mathbb{R}^n} \sum_{j=1}^{k} |\hat{\varphi}_j(z)|^2 \right)^{-2/n} \left(\frac{n}{n+2} \right) \\
&\geqslant \left(\sum_{j=1}^{k} \|\varphi_j\|^2 \right) \left((2\pi)^{-n}\mathrm{vol}(\Omega) B_n \right)^{-2/n} \left(\frac{n}{n+2} \right),
\end{aligned}$$

which completes the proof. □

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5. Upper estimate of gaps of eigenvalues

This section is devoted to generalizing the estimate of Payne, Pólya and Weinberger in (1.2) to Laplacians with respect to measures. We use the same notation as in theorem 1.1.

Proof of theorem 1.2. By the min-max principle (theorem 3.1),

$$\lambda_{k+1} = \inf\left\{ \frac{\int_\Omega |\nabla v|^2 \,\mathrm{d}x}{\int_\Omega v^2 \,\mathrm{d}\mu} : \int_\Omega v\varphi_i \,\mathrm{d}\mu = 0,\ i = 1, \ldots, k,\ v \in \operatorname{dom}\mathcal{E} \right\}. \tag{5.1}$$

For $i = 1, \ldots, k$, choose test functions $v_i = g\varphi_i - \sum_{j=1}^{k} a_{ij}\varphi_j$, where the a_{ij} are determined below and g is some polynomial function that will be chosen later. As $\operatorname{dom}\mathcal{E}$ can be identified with the entire $H_0^1(\Omega)$, we have $v_i \in \operatorname{dom}\mathcal{E}$. We assume $(v_i, \varphi_\ell)_\mu = 0$ for $\ell = 1, \ldots, k$. Then

$$0 = \int_\Omega g\varphi_i\varphi_\ell \,\mathrm{d}\mu - \sum_{j=1}^{k} a_{ij} \int_\Omega \varphi_j\varphi_\ell \,\mathrm{d}\mu = \int_\Omega g\varphi_i\varphi_\ell \,\mathrm{d}\mu - a_{i\ell}.$$

This determines $a_{i\ell}$ and shows that $a_{i\ell} = a_{\ell i}$. We also have

$$\int_\Omega v_i^2 \,\mathrm{d}\mu = \int_\Omega \left(g\varphi_i v_i - \sum_{j=1}^{k} a_{ij}\varphi_j v_i \right) \mathrm{d}\mu = \int_\Omega g\varphi_i v_i \,\mathrm{d}\mu \tag{5.2}$$

and

$$\begin{aligned} \int_\Omega |\nabla v_i|^2 \,\mathrm{d}x &= \int_\Omega \nabla\left(g\varphi_i - \sum_{j=1}^{k} a_{ij}\varphi_j \right) \cdot \nabla v_i \,\mathrm{d}x \\ &= \int_\Omega \nabla(g\varphi_i)\nabla v_i \,\mathrm{d}x - \sum_{j=1}^{k} a_{ij} \int_\Omega \nabla\varphi_j \cdot \nabla v_i \,\mathrm{d}x \\ &= \int_\Omega \left(\varphi_i\nabla g + g\nabla\varphi_i \right) \cdot \nabla v_i \,\mathrm{d}x + 0 \\ &= \int_\Omega \varphi_i\nabla g \cdot \nabla v_i \,\mathrm{d}x + \int_\Omega g\nabla\varphi_i \cdot \nabla v_i \,\mathrm{d}x. \end{aligned}$$

Note that

$$\lambda_i \int_\Omega \varphi_i g v_i \,\mathrm{d}\mu = \int_\Omega \nabla\varphi_i \cdot \nabla(gv_i) \,\mathrm{d}x = \int_\Omega \nabla\varphi_i \cdot \left(g\nabla v_i + v_i\nabla g \right) \mathrm{d}x$$

and hence

$$\int_\Omega g\nabla\varphi_i \cdot \nabla v_i = \lambda_i \int_\Omega \varphi_i g v_i \,\mathrm{d}\mu - \int_\Omega v_i\nabla\varphi_i \cdot \nabla g \,\mathrm{d}x.$$

Combining the above expressions, we get

$$\int_\Omega |\nabla v_i|^2 \,\mathrm{d}x = \int_\Omega \varphi_i\nabla g \cdot \nabla v_i \,\mathrm{d}x + \lambda_i \int_\Omega \varphi_i g v_i \,\mathrm{d}\mu - \int_\Omega v_i\nabla\varphi_i \cdot \nabla g \,\mathrm{d}x. \tag{5.3}$$

The first term on the right-hand side can be expressed as

$$\int_\Omega \varphi_i \nabla g \cdot \nabla v_i \,\mathrm{d}x = -\int_\Omega \operatorname{div}(\varphi_i \nabla g) v_i \,\mathrm{d}x = -\int_\Omega v_i \nabla \varphi_i \cdot \nabla g \,\mathrm{d}x - \int_\Omega \varphi_i v_i \Delta g \,\mathrm{d}x.$$

Hence

$$\int_\Omega |\nabla v_i|^2 \,\mathrm{d}x = -\int_\Omega \varphi_i v_i \Delta g \,\mathrm{d}x - 2\int_\Omega v_i \nabla \varphi_i \cdot \nabla g \,\mathrm{d}x + \lambda_i \int_\Omega g \varphi_i v_i \,\mathrm{d}\mu. \tag{5.4}$$

Consider the second term on the right-hand side of (5.3). By using the definition of v_i and the symmetry of a_{ij}, we have

$$\begin{aligned} -2\sum_{i=1}^k \int_\Omega v_i \nabla g \cdot \nabla \varphi_i \,\mathrm{d}x &= -2\sum_{i=1}^k \int_\Omega g \nabla g \cdot \varphi_i \nabla \varphi_i \,\mathrm{d}x + 2\sum_{i,j=1}^k a_{ij} \int_\Omega \varphi_j \nabla \varphi_i \cdot \nabla g \,\mathrm{d}x \\ &= -\frac{1}{2}\sum_{i=1}^k \int_\Omega \nabla g^2 \cdot \nabla \varphi_i^2 \,\mathrm{d}x + \sum_{i,j=1}^k a_{ij} \int_\Omega \nabla(\varphi_i \varphi_j) \cdot \nabla g \,\mathrm{d}x \\ &= \frac{1}{2}\sum_{i=1}^k \int_\Omega \varphi_i^2 \Delta g^2 \,\mathrm{d}x - \sum_{i,j=1}^k a_{ij} \int_\Omega \varphi_i \varphi_j \Delta g \,\mathrm{d}x. \end{aligned} \tag{5.5}$$

Combining (5.2), (5.4), (5.5), and using the definition of v_i again, we get

$$\begin{aligned} \sum_{i=1}^k \int_\Omega |\nabla v_i|^2 \,\mathrm{d}x &= -\sum_{i=1}^k \int_\Omega v_i \varphi_i \Delta g \,\mathrm{d}x + \frac{1}{2}\sum_{i=1}^k \int_\Omega \varphi_i^2 \Delta g^2 \,\mathrm{d}x \\ &\quad - \sum_{i,j=1}^k a_{ij} \int_\Omega \varphi_i \varphi_j \Delta g \,\mathrm{d}x + \sum_{i=1}^k \lambda_i \int_\Omega g \varphi_i v_i \,\mathrm{d}\mu \\ &= -\sum_{i=1}^k \int_\Omega \varphi_i^2 g \Delta g \,\mathrm{d}x + \sum_{i,j=1}^k a_{ij} \int_\Omega \varphi_i \varphi_j \Delta g \,\mathrm{d}x \\ &\quad + \frac{1}{2}\sum_{i=1}^k \int_\Omega \varphi_i^2 \Delta g^2 \,\mathrm{d}x \\ &\quad - \sum_{i,j=1}^k a_{ij} \int_\Omega \varphi_i \varphi_j \Delta g \,\mathrm{d}x + \sum_{i=1}^k \lambda_i \int_\Omega g \varphi_i v_i \,\mathrm{d}\mu \\ &= \sum_{i=1}^k \int_\Omega \varphi_i^2 |\nabla g|^2 \,\mathrm{d}x + \sum_{i=1}^k \lambda_i \int_\Omega v_i^2 \,\mathrm{d}\mu \\ &\leqslant \sum_{i=1}^k \int_\Omega \varphi_i^2 |\nabla g|^2 \,\mathrm{d}x + \lambda_k \sum_{i=1}^k \int_\Omega v_i^2 \,\mathrm{d}\mu. \end{aligned}$$

For $i = 1, \ldots, k$, (5.1) implies that

$$\lambda_{k+1} \int_\Omega v_i^2 \,\mathrm{d}\mu \leqslant \int_\Omega |\nabla v_i|^2 \,\mathrm{d}x.$$

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Hence

$$\lambda_{k+1} - \lambda_k \leqslant \frac{\sum_{i=1}^k \int_\Omega \varphi_i^2 |\nabla g|^2 \,\mathrm{d}x}{\sum_{i=1}^k \int_\Omega v_i^2 \,\mathrm{d}\mu}.$$

Now take $g = g_a(x) = \sum_{\beta=1}^n a_\beta x_\beta$ with $\sum_{\beta=1}^n a_\beta^2 = 1$. Then $\Delta g = 0$ and $|\nabla g| = 1$. It follows that

$$\lambda_{k+1} - \lambda_k \leqslant \frac{\sum_{i=1}^k \int_\Omega \varphi_i^2 \,\mathrm{d}x}{\sum_{i=1}^k \int_\Omega v_{ia}^2 \,\mathrm{d}\mu} = \frac{\sum_{i=1}^k \|\varphi_i\|^2}{\sum_{i=1}^k \int_\Omega v_{ia}^2 \,\mathrm{d}\mu}, \tag{5.6}$$

where $v_{ia} = g_a\varphi_i - \sum_{j=1}^k a_{ij}\varphi_j$. Using (5.5) and the facts that $\Delta g^2 = 2$ and $\Delta g = 0$, we get

$$\begin{aligned}\sum_{i=1}^k \|\varphi_i\|^2 = \sum_{i=1}^k \int_\Omega \varphi_i^2 \,\mathrm{d}x &= -2\sum_{i=1}^k \int_\Omega v_{ia}(\nabla g_a \cdot \nabla \varphi_i) \,\mathrm{d}x \\ &= -2\sum_{i=1}^k \int_\Omega v_{ia}\left(\sum_{\beta=1}^n a_\beta \frac{\partial \varphi_i}{\partial x_\beta}\right) \mathrm{d}x.\end{aligned}$$

Let $\mathrm{d}S$ be the normalized uniform measure on S^{n-1} so that $\int_{S^{n-1}} \mathrm{d}S = 1$. Then

$$\begin{aligned}\frac{1}{2}\sum_{i=1}^k \|\varphi_i\|^2 &= -\sum_{i=1}^k \int_{S^{n-1}} \int_\Omega v_{ia}\left(\sum_{\beta=1}^n a_\beta \frac{\partial \varphi_i}{\partial x_\beta}\right) \mathrm{d}x \,\mathrm{d}S \\ &= -\int_{S^{n-1}} \int_\Omega \left(\sum_{i=1}^k v_{ia}\left(\sum_{\beta=1}^n a_\beta \frac{\partial \varphi_i}{\partial x_\beta}\right)\right) \mathrm{d}x \,\mathrm{d}S \\ &\leqslant \left(\int_\Omega \int_{S^{n-1}} \sum_{i=1}^k v_{ia}^2 \,\mathrm{d}S \,\mathrm{d}x\right)^{1/2} \\ &\quad \times \left(\int_\Omega \int_{S^{n-1}} \sum_{i=1}^k \left(\sum_{\beta=1}^n a_\beta \frac{\partial \varphi_i}{\partial x_\beta}\right)^2 \mathrm{d}S \,\mathrm{d}x\right)^{1/2}.\end{aligned}$$

Obviously, for $\alpha, \beta \in \{1, \ldots, n\}$, we have $\int_{S^{n-1}} a_\alpha a_\beta \, dS = \delta_{\alpha\beta}/n$. Hence

$$\begin{aligned}\int_\Omega \int_{S^{n-1}} \sum_{i=1}^k \left(\sum_{\beta=1}^n a_\beta \frac{\partial \varphi_i}{\partial x_\beta}\right)^2 \mathrm{d}S \,\mathrm{d}x &= \sum_{i=1}^k \int_\Omega \frac{1}{n} \sum_{\beta=1}^n \left(\frac{\partial \varphi_i}{\partial x_\beta}\right)^2 \mathrm{d}x \\ &= \sum_{i=1}^k \frac{1}{n} \int_\Omega |\nabla \varphi_i|^2 \,\mathrm{d}x \\ &= \frac{1}{n}\sum_{i=1}^k \lambda_i \int_\Omega \varphi_i^2 \,\mathrm{d}\mu = \frac{1}{n}\sum_{i=1}^k \lambda_i.\end{aligned}$$

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It follows that

$$\frac{1}{4}\left(\sum_{i=1}^{k}\|\varphi_i\|^2\right)^2 \leqslant \frac{1}{n}\left(\sum_{i=1}^{k}\lambda_i\right)\int_\Omega\int_{S^{n-1}}\sum_{i=1}^{k} v_{ia}^2\,\mathrm{d}S\,\mathrm{d}x. \tag{5.7}$$

Also, from (5.6) we get

$$(\lambda_{k+1}-\lambda_k)\sum_{i=1}^{k}\int_{S^{n-1}}\int_\Omega v_{ia}^2\,\mathrm{d}\mu\,\mathrm{d}S \leqslant \sum_{i=1}^{k}\|\varphi_i\|^2. \tag{5.8}$$

Combining (5.7) and (5.8) yields

$$\begin{aligned}\lambda_{k+1}-\lambda_k &\leqslant \frac{\sum_{i=1}^{k}\|\varphi_i\|^2}{\sum_{i=1}^{k}\int_\Omega\int_{S^{n-1}} v_{ia}^2\,\mathrm{d}S\,\mathrm{d}x}\cdot\frac{\sum_{i=1}^{k}\int_\Omega\int_{S^{n-1}} v_{ia}^2\,\mathrm{d}S\,\mathrm{d}x}{\sum_{i=1}^{k}\int_\Omega\int_{S^{n-1}} v_{ia}^2\,\mathrm{d}S\,\mathrm{d}\mu}\\ &\leqslant \frac{4\sum_{i=1}^{k}\lambda_i}{n\sum_{i=1}^{k}\|\varphi_i\|^2}\cdot\frac{\sum_{i=1}^{k}\int_{S^{n-1}}\int_\Omega v_{ia}^2\,\mathrm{d}x\,\mathrm{d}S}{\sum_{i=1}^{k}\int_{S^{n-1}}\int_\Omega v_{ia}^2\,d\mu\,\mathrm{d}S}\end{aligned} \tag{5.9}$$

To compute the integrals in (5.9), we first note that

$$\begin{aligned}\int_\Omega v_{ia}^2\,\mathrm{d}x &= \int_\Omega\left(g_a\varphi_i-\sum_{j=1}^{k}a_{ij}\varphi_j\right)\left(g_a\varphi_i-\sum_{\ell=1}^{k}a_{i\ell}\varphi_\ell\right)\mathrm{d}x\\ &= \int_\Omega g_a^2\varphi_i^2\,\mathrm{d}x - 2\int_\Omega g_a\varphi_i\sum_{\ell=1}^{k}a_{i\ell}\varphi_\ell\,\mathrm{d}x + \sum_{j,\ell=1}^{k}\int_\Omega a_{ij}a_{i\ell}\varphi_j\varphi_\ell\,\mathrm{d}x\\ &= \mathrm{(I)} - 2\mathrm{(II)} + \mathrm{(III)}.\end{aligned}$$

$$\mathrm{(I)} = \int_\Omega\left(\sum_{\alpha=1}^{n}a_\alpha x_\alpha\right)\left(\sum_{\beta=1}^{n}a_\beta x_\beta\right)\varphi_i^2\,\mathrm{d}x = \sum_{\alpha,\beta=1}^{n}a_\alpha a_\beta\int_\Omega x_\alpha x_\beta\varphi_i^2\,\mathrm{d}x.$$

Notice that

$$a_{ij} = \int_\Omega g_a\varphi_i\varphi_j\,\mathrm{d}\mu = \sum_{\alpha=1}^{n}a_\alpha\int_\Omega x_\alpha\varphi_i\varphi_j\,\mathrm{d}\mu. \tag{5.10}$$

Using the definition of g, followed by (5.10), we get

$$\begin{aligned}\mathrm{(II)} &= \int_\Omega\left(\sum_{\alpha=1}^{n}a_\alpha x_\alpha\right)\varphi_i\sum_{\ell=1}^{k}a_{i\ell}\varphi_\ell\,\mathrm{d}x = \sum_{\alpha=1}^{n}\sum_{\ell=1}^{k}a_\alpha a_{i\ell}\int_\Omega x_\alpha\varphi_i\varphi_\ell\,\mathrm{d}x\\ &= \sum_{\ell=1}^{k}\sum_{\alpha,\beta=1}^{n}a_\alpha a_\beta\left(\int_\Omega x_\beta\varphi_i\varphi_\ell\,\mathrm{d}\mu\right)\left(\int_\Omega x_\alpha\varphi_i\varphi_\ell\,\mathrm{d}x\right).\end{aligned}$$

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$$\begin{aligned}(\mathrm{III}) &= \sum_{j,\ell=1}^{k}\left(\sum_{\alpha=1}^{n} a_\alpha \int_\Omega x_\alpha\varphi_i\varphi_j\,\mathrm{d}\mu\right)\left(\sum_{\beta=1}^{n} a_\beta \int_\Omega x_\beta\varphi_i\varphi_\ell\,\mathrm{d}\mu\right)\int_\Omega \varphi_j\varphi_\ell\,\mathrm{d}x\\ &= \sum_{j,\ell=1}^{k}\sum_{\alpha,\beta=1}^{n} a_\alpha a_\beta\left(\int_\Omega x_\alpha\varphi_i\varphi_j\,\mathrm{d}\mu\right)\left(\int_\Omega x_\beta\varphi_i\varphi_\ell\,\mathrm{d}\mu\right)\int_\Omega \varphi_j\varphi_\ell\,\mathrm{d}x\end{aligned}$$

$$\begin{aligned}\int_{S^{n-1}}(\mathrm{I})\,\mathrm{d}S &= \sum_{\alpha,\beta=1}^{n}\left(\int_{S^{n-1}} a_\alpha a_\beta\,\mathrm{d}S\right)\int_\Omega x_\alpha x_\beta\varphi_i^2\,\mathrm{d}x = \frac{1}{n}\sum_{\alpha=1}^{n}\int_\Omega x_\alpha^2\varphi_i^2\,\mathrm{d}x\\ &= \frac{1}{n}\int_\Omega |x|^2\varphi_i^2\,\mathrm{d}x.\end{aligned}$$

$$\begin{aligned}\int_{S^{n-1}}(\mathrm{II})\,\mathrm{d}S &= \sum_{\ell=1}^{k}\sum_{\alpha,\beta=1}^{n}\frac{\delta_{\alpha\beta}}{n}\left(\int_\Omega x_\beta\varphi_i\varphi_\ell\,\mathrm{d}\mu\right)\left(\int_\Omega x_\alpha\varphi_i\varphi_\ell\,\mathrm{d}x\right)\\ &= \frac{1}{n}\sum_{\ell=1}^{k}\sum_{\alpha=1}^{n}\left(\int_\Omega x_\alpha\varphi_i\varphi_\ell\,d\mu\right)\left(\int_\Omega x_\alpha\varphi_i\varphi_\ell\,\mathrm{d}x\right)\end{aligned}$$

$$\begin{aligned}\int_{S^{n-1}}(\mathrm{III})\,\mathrm{d}S &= \sum_{j,\ell=1}^{k}\sum_{\alpha,\beta=1}^{n}\frac{\delta_{\alpha\beta}}{n}\left(\int_\Omega x_\alpha\varphi_i\varphi_j\,\mathrm{d}\mu\right)\left(\int_\Omega x_\beta\varphi_i\varphi_\ell\,\mathrm{d}\mu\right)\int_\Omega\varphi_j\varphi_\ell\,\mathrm{d}x\\ &= \frac{1}{n}\sum_{j,\ell=1}^{k}\sum_{\alpha=1}^{n}\left(\int_\Omega x_\alpha\varphi_i\varphi_j\,\mathrm{d}\mu\right)\left(\int_\Omega x_\alpha\varphi_i\varphi_\ell\,\mathrm{d}\mu\right)\int_\Omega\varphi_j\varphi_\ell\,\mathrm{d}x.\end{aligned}$$

Combining the above integrals we get

$$\begin{aligned}\int_{S^{n-1}}\int_\Omega v_{ia}^2\,\mathrm{d}x\,\mathrm{d}S &= \frac{1}{n}\int_\Omega |x|^2\varphi_i^2\,\mathrm{d}x - \frac{2}{n}\sum_{\ell=1}^{k}\sum_{\alpha=1}^{n}\left(\int_\Omega x_\alpha\varphi_i\varphi_\ell\,\mathrm{d}\mu\right)\int_\Omega x_\alpha\varphi_i\varphi_\ell\,\mathrm{d}x\\ &\quad + \frac{1}{n}\sum_{j,\ell=1}^{k}\sum_{\alpha=1}^{n}\left(\int_\Omega x_\alpha\varphi_i\varphi_j\,d\mu\right)\left(\int_\Omega x_\alpha\varphi_i\varphi_\ell\,\mathrm{d}\mu\right)\int_\Omega\varphi_j\varphi_\ell\,\mathrm{d}x.\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{i=1}^{k}\int_{S^{n-1}}\int_\Omega v_{ia}^2\,\mathrm{d}x\,\mathrm{d}S\\ &\quad= \frac{1}{n}\int_\Omega |x|^2\left(\sum_{i=1}^{k}\varphi_i^2\right)\mathrm{d}x - \frac{2}{n}\sum_{i,\ell=1}^{k}\sum_{\alpha=1}^{n}\left(\int_\Omega x_\alpha\varphi_i\varphi_\ell\,\mathrm{d}\mu\right)\int_\Omega x_\alpha\varphi_i\varphi_\ell\,\mathrm{d}x\\ &\qquad + \frac{1}{n}\sum_{i,j,\ell=1}^{k}\sum_{\alpha=1}^{n}\left(\int_\Omega x_\alpha\varphi_i\varphi_j\,\mathrm{d}\mu\right)\left(\int_\Omega x_\alpha\varphi_i\varphi_\ell\,\mathrm{d}\mu\right)\int_\Omega\varphi_j\varphi_\ell\,\mathrm{d}x. \qquad (5.11)\end{aligned}$$

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A similar calculation gives

$$\sum_{i=1}^{k}\int_{S^{n-1}}\int_{\Omega} v_{ia}^2\,\mathrm{d}\mu\,\mathrm{d}S = \frac{1}{n}\int_{\Omega}|x|^2\left(\sum_{i=1}^{k}\varphi_i^2\right)\mathrm{d}\mu - \frac{1}{n}\sum_{i,\ell=1}^{k}\sum_{\alpha=1}^{n}\left(\int_{\Omega} x_\alpha\varphi_i\varphi_\ell\,\mathrm{d}\mu\right)^2. \tag{5.12}$$

Combining (5.9), (5.11) and (5.12) completes the proof of the theorem. □

6. Comments and open problems

In view of theorems 1.1 and 1.2, it is of interest to estimate the bound of the norm $\|\varphi_i\|$ of the eigenfunctions φ_i that satisfy $\|\varphi_i\|_\mu = 1$. It is also of interest to characterize measures μ that satisfy the condition $\mathrm{dom}\,\mathcal{E} = H_0^1(\Omega)$ in theorem 1.2. An upper estimate for the sum of eigenvalues was obtained by Kröger [**23**]. Let $\mathrm{dist}(x, \partial\Omega)$ denote the distance from a point $x \in \Omega$ to the boundary of Ω. Let $\Omega_r = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < 1/r\}$ and B be a unit ball in $\mathbb{R}^n$. Kröger proved that if there exists a constant $C_\Omega^{(0)}$ such that $\mathrm{vol}(\Omega_r) \leqslant (C_\Omega^{(0)}/r)\mathrm{vol}(\Omega)^{(n-2)n}$ for every $r > \mathrm{vol}(\Omega)^{-1/n}$, then for every $k \geqslant (C_\Omega^{(0)})^n$,

$$\sum_{j=1}^{k}\lambda_j \leqslant (2\pi)^2\frac{n}{n+2}\big(\mathrm{vol}(\Omega)\mathrm{vol}(B)\big)^{-2/n}\big(k^{(n+2)/n} + C_n^{(1)}C_\Omega^{(1)}k^{(n+1)/n}\big), \tag{6.1}$$

where $C_n^{(1)}$ is a constant depending only on the dimension n. It is of interest to generalize this result to Laplacians with respect to measures.

The spectral asymptotics of Laplacians defined on domains by fractal measures have been investigated and obtained by a number of authors (see [**14**, **17**, **25**–**27**, **29**] and the references therein). It is of interest to find examples among these measure for which spectral gaps exist.

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