106.28 Inequalities involving the inradius and altitudes of a triangle

Given a triangle ΔABC with area Δ, inradius r and altitudes h_a, h_b, h_c, there is an elegant inequality

$$9r \leqslant h_a + h_b + h_c \tag{1}$$

which follows from summing three variants of the area formula $ah_a = 2\Delta r = (a + b + c)$, to obtain

$$\frac{h_a}{r} + \frac{h_b}{r} + \frac{h_c}{r} = (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right).$$

By the arithmetic-harmonic mean inequality, this is at least 9, and so (1) follows. Note that there is equality if, and only if, $a = b = c$, which is when ΔABC is equilateral.

A natural question is whether we can find an upper bound for $h_a + h_b + h_c$ in (1). In this Note we show that it is possible to do so.

Theorem

If $a \leqslant b \leqslant c$,

$$h_a + h_b + h_c \leqslant 9r + 2r\left(\sqrt{\frac{a}{c}} - \sqrt{\frac{c}{a}}\right)^2 \tag{2}$$

with equality if, and only if, $a = b$ or $b = c$. Furthermore, the 2 on the right-hand side is the best possible constant.

Proof

Using $\dfrac{h_a}{r} = \dfrac{a + b + c}{a}$ and so on, we have

$$\begin{aligned}
& h_a + h_b + h_c \leqslant 9r + 2r\left(\sqrt{\frac{a}{c}} - \sqrt{\frac{c}{a}}\right)^2 \\
\Leftrightarrow\ & (a + b + c)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \leqslant 9 + 2\left(\frac{a}{c} - 2 + \frac{c}{a}\right) \\
\Leftrightarrow\ & \frac{a}{b} + \frac{a}{c} + \frac{b}{a} + \frac{b}{c} + \frac{c}{a} + \frac{c}{b} \leqslant 2 + 2\left(\frac{a}{c} + \frac{c}{a}\right) \\
\Leftrightarrow\ & \frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} \leqslant \frac{(a + c)^2}{ac} \\
\Leftrightarrow\ & (b^2 + ac)(a + c) \leqslant b(a + c)^2 \\
\Leftrightarrow\ & b^2 + ac \leqslant b(a + c) \\
\Leftrightarrow\ & (b - a)(b - c) \leqslant 0
\end{aligned}$$

which is true since $a \leqslant b \leqslant c$. Equality holds when $a = b$ or $b = c$.

Now we show that 2 is the best possible constant.

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Assume that, when $a \leqslant b \leqslant c$, the inequality

$$h_a + h_b + h_c \leqslant 9r + ar\left(\sqrt{\frac{a}{c}} - \sqrt{\frac{c}{a}}\right)^2$$

holds.

Let $a = b = 1$ and $c = x$, where $1 \leqslant x < 2$ (to satisfy the triangle inequality). Then, using $\frac{h_a}{r} = \frac{a + b + c}{a}$ and so on, the inequality is equivalent to

$$3 + \frac{2}{x} + 2(x + 1) \leqslant 9 + \alpha\left(x - 2 + \frac{1}{x}\right)$$

$$\Leftrightarrow (2 - \alpha)\left(x + \frac{1}{x}\right) \leqslant 2(2 - \alpha)$$

$$\Leftrightarrow (2 - \alpha)\frac{(x - 1)^2}{x} \leqslant 0.$$

Since this is true for all $1 \leqslant x < 2$, we have $\alpha \geqslant 2$, and so 2 is the best possible constant.

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106.29 An improvement on the Garfunkel-Bankoff inequality*

In a triangle ABC, the semi-perimeter, circumradius and inradius are denoted by s, R and r respectively. In [1] Garfunkel proposed the following inequality as an open problem

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geqslant 2 - 8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}. \qquad (1)$$

This was first proved by Bankoff in [2], and is known as the Garfunkel-Bankoff inequality. It has received considerable attention from researchers in the field of geometrical inequalities and has motivated a number of papers providing various generalisations and analogue, such as [3] and the references in it.

In this Note, we give a sharpened version of (1), which appears as a corollary to a theorem. The proof of the theorem relies on

$$s^2 \leqslant 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R(R - 2r)}, \qquad (2)$$

which is described in [4, 5.10] as 'the fundamental inequality of a triangle'.

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