Bull. Aust. Math. Soc. **104** (2021), 295–301
doi:10.1017/S0004972720001318

THE CYCLIC GRAPH OF A Z-GROUP

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(Received 23 September 2020; accepted 9 October 2020; first published online 14 December 2020)

Abstract

For a group G, we define a graph $\Delta(G)$ by letting $G^{\#} = G \setminus \{1\}$ be the set of vertices and by drawing an edge between distinct elements $x, y \in G^{\#}$ if and only if the subgroup $\langle x, y\rangle$ is cyclic. Recall that a Z-group is a group where every Sylow subgroup is cyclic. In this short note, we investigate $\Delta(G)$ for a Z-group G.

2020 *Mathematics subject classification*: primary 20F16; secondary 05C25.

Keywords and phrases: cyclic graph, enhanced power graph, Z-group.

1. Introduction

The groups under consideration in this note are finite. Let G be a group and define a graph $\Delta(G)$ associated with G as follows. Take $G^{\#} = G \setminus \{1\}$ as the vertex set. Then draw an edge between distinct vertices $x, y \in G^{\#}$ if and only if the subgroup $\langle x, y\rangle$ is cyclic. We shall refer to $\Delta(G)$ as the *cyclic graph* of G, although we note that the graph $\Delta(G)$ has also been called the *deleted enhanced power graph*. See, for example, [2]. The *enhanced power graph* includes the identity element as a vertex and so the enhanced power graph of a group is always connected. A brief investigation of this graph was undertaken in [1].

The cyclic graph of a group G was investigated in [4, 5]. In those papers, classification results were obtained under the assumption that the connected components of $\Delta(G)$ were complete graphs. In our previous paper [3], we studied the cyclic graph of a direct product.

Next, we mention another graph that can be attached to a group. Let G be a nonabelian group. The *commuting graph* of G, denoted by $\Gamma(G)$, is the graph whose vertices are the noncentral elements of G and whose edges connect distinct vertices x and y if and only if $xy = yx$. The commuting graph of a finite solvable group with trivial centre was classified in [6].

Recall that a group is called a *Z-group* if every Sylow subgroup is cyclic. Observe that a Frobenius complement of odd order is a Z-group and so is any group of

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https://doi.org/10.1017/S0004972720001318 Published online by Cambridge University Press

square-free order. Our focus in this short note is the graph $\Delta(G)$ for a Z-group G. We have been able to characterise the disconnectedness of $\Delta(G)$.

THEOREM 1.1. *Let G be a Z-group. Then $\Delta(G)$ is disconnected if and only if G is a Frobenius group.*

If the graph $\Delta(G)$ is connected for a Z-group G, then a diameter bound follows.

THEOREM 1.2. *If G is a Z-group and $\Delta(G)$ is connected, then* $\mathrm{diam}(\Delta(G)) \leq 4$.

The next result describes a relationship between the graph $\Delta(G)$ and the subgroup $\mathbf{Z}(G)$ for a Z-group G.

THEOREM 1.3. *If G is a Z-group, then* $\mathrm{diam}(\Delta(G)) \leq 2$ *if and only if* $\mathbf{Z}(G) \neq \{1\}$.

Following [2], a vertex z in $\Delta(G)$ is called a *dominating vertex* if z is adjacent to every vertex in $\Delta(G) \setminus \{z\}$. The terms *complete vertex*, *cone vertex* and *universal vertex* have also been used as synonyms for a dominating vertex. If the graph $\Delta(G)$ has a dominating vertex, we shall say that $\Delta(G)$ is *dominatable*. In the proof of the previous theorem, we end up establishing the existence of a dominating vertex. We point out a necessary and sufficient condition for a dominating vertex in $\Delta(G)$ to exist, which answers a request in [2] for a characterisation of a group with a dominatable cyclic graph.

THEOREM 1.4. *Let G be a group, $g \in G$ and $\pi = \pi(o(g))$. Write $g = \prod_{p\in\pi} g_p$, where each g_p is a p-element for $p \in \pi$ and $g_p g_q = g_q g_p$ for all $p, q \in \pi$. Then g is a dominating vertex for $\Delta(G)$ if and only if, for each $p \in \pi$, a Sylow p-subgroup P of G is cyclic or generalised quaternion and $\langle g_p\rangle \leq P \cap \mathbf{Z}(G)$.*

As a corollary, we offer a generalisation of Theorem 3.2 in [2].

COROLLARY 1.5. *For a nilpotent group G, the graph $\Delta(G)$ is dominatable if and only if G has a cyclic or generalised quaternion Sylow subgroup.*

Let G be a Z-group and let $x, y \in G^{\#}$ be distinct. If x is adjacent to y in $\Delta(G)$, then $xy = yx$. In fact, the converse is true too. So, in particular, if $\mathbf{Z}(G) = \{1\}$, then $\Gamma(G)$ and $\Delta(G)$ are the same graph. In light of the previous results, we obtain the following corollary concerning the commuting graph of a Z-group with trivial centre.

COROLLARY 1.6. *If G is a Z-group with $\mathbf{Z}(G) = \{1\}$ and G is not a Frobenius group, then $\Gamma(G)$, the commuting graph of G, is connected with diameter* 3 *or* 4.

2. Notation and preliminaries

Let G be a group and let $x, y \in G$. We write $x \approx y$ to indicate that the subgroup $\langle x, y\rangle$ is cyclic. If n is a positive integer, then $\pi(n)$ denotes the set of prime divisors of n. For a group G, set $\pi(G) = \pi(|G|)$. Fix a set of prime numbers π. An element $x \in G$ is called a *π-element* if every prime divisor of $o(x)$ is a member of π. If every

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prime divisor of $o(x)$ lies outside of π, then x is called a *π'-element*. In the case where $\pi = \{p\}$, we use the terms p-element and p'-element. The set of prime numbers is denoted by $\mathbb{P}$.

Let G be a group. Notice that if $x, y \in G^{\#}$ are commuting elements with coprime orders, then $x \approx y$. This fact gives us a useful way to build paths in $\Delta(G)$. We also mention that conjugation preserves adjacency in $\Delta(G)$: specifically, if $x, y \in G^{\#}$ with $x \approx y$, then $x^g \approx y^g$ for each $g \in G$.

A graph related to the cyclic graph is the *commuting graph*, which is defined as follows. Let G be a nonabelian group. The commuting graph $\Gamma(G)$ is the graph whose vertices are the noncentral elements of G and whose edges connect distinct nonidentity elements x and y if and only if $xy = yx$. Taking the noncentral elements of G as the vertices for $\Gamma(G)$ is fairly standard, although variations on the vertex set do exist. If G is a Z-group with a trivial centre, then the vertex set of $\Gamma(G)$ is the same as the vertex set of $\Delta(G)$. In fact, the edge sets are the same too; the following lemma also appears as a part of Theorem 30 in [1].

LEMMA 2.1. *If G is a Z-group with $\mathbf{Z}(G) = \{1\}$, then $\Delta(G) = \Gamma(G)$.*

PROOF. If $x, y \in G$ with $x \approx y$, then clearly $xy = yx$. But notice that if $xy = yx$, then $\langle x, y\rangle$ is an abelian Z-group, which is therefore cyclic. Hence, $x \approx y$. □

Next, we make a few remarks about Z-groups. Many properties of Z-groups are known. For example, if G is a Z-group, then G is p-nilpotent for the smallest prime divisor p of $|G|$. We also know that Z-groups are solvable. The specific results that we need in this paper are encapsulated in the following theorem.

THEOREM 2.2 [7, Theorem 10.26]. *If G is a Z-group, then the derived subgroup G' is cyclic and the factor group G/G' is cyclic. Moreover, G' is a Hall subgroup of G.*

Finally, we need to make an observation about Frobenius groups. Recall that a group G is a *Frobenius group* if G has a nontrivial proper subgroup H such that $H \cap H^g = \{1\}$ for each $g \in G \setminus H$. The subgroup H is called a *Frobenius complement*. Now, let G be a Frobenius group with Frobenius complement H. Frobenius groups are centreless and so $\Gamma(G)$ and $\Delta(G)$ have the same vertex set. In particular, $\Delta(G)$ is a spanning subgraph of $\Gamma(G)$. Because $\mathbf{C}_G(h) \leq H$ for each $h \in H^{\#}$, the graph $\Gamma(G)$ is disconnected. (This fact appears as Lemma 3.1 in [6].) Hence, $\Delta(G)$ is disconnected as well.

3. Main results

Our first theorem provides a necessary and sufficient condition for the cyclic graph of a Z-group G to be disconnected. Additionally, a diameter bound of $\Delta(G)$ is available under the assumption that $\Delta(G)$ is connected.

THEOREM 3.1. *Let G be a Z-group. Then $\Delta(G)$ is disconnected if and only if G is a Frobenius group. Moreover, if $\Delta(G)$ is connected, then* $\operatorname{diam}(\Delta(G)) \leq 4$.

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PROOF. Frobenius groups have disconnected cyclic graphs. To prove the converse, assume that G is not a Frobenius group. We shall establish the connectedness of $\Delta(G)$.

Abelian Z-groups are cyclic and so we may assume that G is nonabelian. Hence, $\{1\} < G' < G$. If $\mathbf{C}_G(g) \leq G'$ for each $g \in (G')^\#$, then G is a Frobenius group with kernel G', contrary to our hypothesis. Hence, there exists some $g_0 \in (G')^\#$ with $\mathbf{C}_G(g_0) \nleq G'$. Let H be a complement for G' in G. Fix $x \in \mathbf{C}_G(g_0) \setminus G'$ and write $x = yh$ for $y \in G'$ and $h \in H$. Then $g_0^{yh} = g_0^x = g_0$ and so $g_0^{h^{-1}} = g_0^y = g_0$. It follows that $h \in \mathbf{C}_H(g_0)$.

Now, let $g \in G^\#$. If $\pi(o(g)) \cap \pi(G') \neq \emptyset$, then let $p \in \pi(o(g)) \cap \pi(G')$. For a suitable integer n, $o(g^n) = p$; hence, $g^n \in G'$ and $g \approx g^n \approx g_0$. Otherwise, $\pi(o(g)) \cap \pi(G') = \emptyset$ and $g \in H^a$ for some $a \in G'$. Note that $h^a \approx g_0^a = g_0$ as $h \approx g_0$ and conjugation preserves adjacency. Hence, $g \approx h^a \approx g_0$. The result follows. □

The group `SmallGroup(60,7)` furnishes an example of a Z-group with connected cyclic graph of diameter 4 and so the bound in the previous theorem is sharp. The cyclic graph for `SmallGroup(60,7)` is displayed in Figure 1. We mention a few more examples. The group `SmallGroup(210,2)` is a Z-group with connected cyclic graph of diameter 3. The cyclic graph for `SmallGroup(210,2)` is displayed in Figure 2. Finally, `SmallGroup(60,3)` provides an example of a Z-group with connected cyclic graph of diameter 2. The cyclic graph for `SmallGroup(60,3)` is displayed in Figure 3. These three graphs were computed using GAP [9] and displayed using Mathematica.

The next theorem highlights a connection between the subgroup $\mathbf{Z}(G)$ and the graph $\Delta(G)$ for a Z-group G.

THEOREM 3.2. *If G is a Z-group, then* $\operatorname{diam}(\Delta(G)) \leq 2$ *if and only if* $\mathbf{Z}(G) \neq \{1\}$.

PROOF. Assume that $\mathbf{Z}(G) \neq \{1\}$. Fix $z \in \mathbf{Z}(G)^\#$ with $o(z) = p$, a prime. Since $\langle z\rangle$ is a normal p-subgroup of G and every Sylow p-subgroup of G is cyclic, $\langle z\rangle$ is the unique subgroup of G with order p. If $g \in G^\#$ and p divides $o(g)$, then $\langle z\rangle \leq \langle g\rangle$. Hence, $g \approx z$.

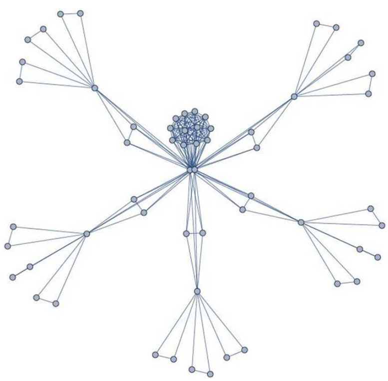

FIGURE 1. Cyclic graph of `SmallGroup(60,7)`.

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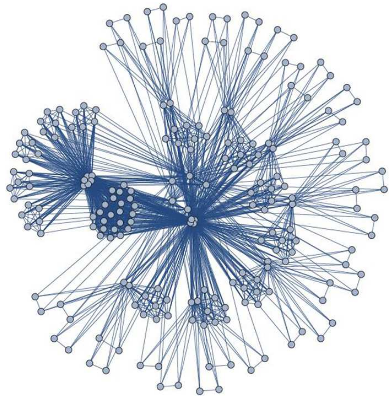

FIGURE 2. Cyclic graph of `SmallGroup(210,2)`.

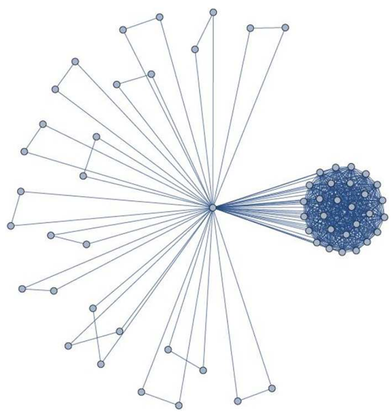

FIGURE 3. Cyclic graph of `SmallGroup(60,3)`.

Otherwise, $o(g)$ is a p'-number and so g and z are commuting elements with coprime orders. Again, $g \approx z$.

Assume that $\mathrm{diam}(\Delta(G)) \leq 2$. Let H be a complement of G'. Set $G' = \langle x \rangle$. As $G/G' \cong H$, the subgroup H is cyclic. Set $H = \langle h \rangle$. If $x \approx h$, then G is abelian and so $\mathbf{Z}(G) = G \neq \{1\}$. Otherwise, $x \approx z \approx h$ for some $z \in G^{\#}$. Now, $G' = \langle x \rangle \leq \mathbf{C}_G(z)$ and $H = \langle h \rangle \leq \mathbf{C}_G(z)$. It follows that $G = G'H \leq \mathbf{C}_G(z)$. Hence, $z \in \mathbf{Z}(G)^{\#}$. □

Let G be a group and recall that a vertex z in $\Delta(G)$ is called a *dominating vertex* if $z \approx g$ for each $g \in \Delta(G) \setminus \{z\}$. A dominating vertex appears in the previous proof and the following theorem highlights a necessary and sufficient condition for such a vertex to exist.

THEOREM 3.3. *The cyclic graph of a group G has a dominating vertex if and only if G has a unique subgroup of order p for some prime p and this subgroup is central.*

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PROOF. Let c be a dominating vertex of $\Delta(G)$. For suitable integer t, $o(c^t) = p \in \mathbb{P}$. For each $g \in \Delta(G) \setminus \{c^t\}$,

$$\langle c^t, g\rangle \leq \langle c, g\rangle$$

and so c^t is a dominating vertex as well. Note that $\langle c^t\rangle$ is a central subgroup of prime order. Suppose that $\langle y\rangle$ has order p. The subgroup $\langle c^t, y\rangle$ is cyclic and therefore has a unique subgroup of order p. Hence, $\langle c^t\rangle = \langle y\rangle$.

Conversely, suppose that $\langle z\rangle$ is a central subgroup of order $p \in \mathbb{P}$ and, further, that $\langle z\rangle$ is the *unique* subgroup of order p. If $g \in G$ is a p'-element, then $z \approx g$ since z and g are commuting elements with coprime orders. If p divides $o(g)$, then $|\langle g^t\rangle| = p$ for a suitable integer t. Our uniqueness hypothesis forces $\langle z\rangle = \langle g^t\rangle$. Again, $z \approx g$. The element z is a dominating vertex. □

The relationship between the existence of a dominating vertex for the cyclic graph of a group and the Sylow subgroup structure of the group can be developed a bit further. Let G be a group, $g \in G$ and $\pi = \pi(o(g))$. Using Theorem 5.1.5 in [8], write $g = \prod_{p\in\pi} g_p$, where each g_p is a p-element for $p \in \pi$ and $g_p g_q = g_q g_p$ for all $p, q \in \pi$. Then g is a dominating vertex for $\Delta(G)$ if and only if, for each $p \in \pi$, a Sylow p-subgroup P of G is cyclic or generalised quaternion and $\langle g_p\rangle \leq P \cap \mathbf{Z}(G)$. We remark that this result strengthens Theorem 3.3 and has essentially the same proof.

Bera and Bhuniya [2] showed that if G is abelian, then $\Delta(G)$ is dominatable if and only if G has a cyclic Sylow subgroup. We generalise this result.

COROLLARY 3.4. *If G is a nilpotent group, then $\Delta(G)$ is dominatable if and only if G has a cyclic or generalised quaternion Sylow subgroup.*

PROOF. If $\Delta(G)$ has a dominating vertex, then, by Theorem 3.3, G has a unique subgroup $\langle x\rangle$ of prime order, say p, that is contained in $\mathbf{Z}(G)$. It is easy to check that if P is the Sylow p-subgroup of G, then $\langle x\rangle$ is the unique subgroup of P of order p; hence, P is cyclic or generalised quaternion.

Conversely, suppose that G has a Sylow p-subgroup P that is cyclic or generalised quaternion. Let $\langle z\rangle$ be the unique subgroup of G of order p. Let $g \in G^{\#}$. If $o(g)$ is a p'-number, then $z \approx g$ as z and g are therefore commuting elements with coprime orders. If p divides $o(g)$, then $|\langle g^s\rangle| = p$ for a suitable integer s. Hence, $\langle z\rangle = \langle g^s\rangle$ and so z is a power of g. Again, $z \approx g$. We conclude that z is a dominating vertex. □

As mentioned previously, if G is a Z-group with $\mathbf{Z}(G) = \{1\}$, then $\Gamma(G) = \Delta(G)$. We now obtain information about the commuting graph $\Gamma(G)$ of a Z-group G with trivial centre.

COROLLARY 3.5. *Let G be a Z-group with $\mathbf{Z}(G) = \{1\}$. If G is not a Frobenius group, then $\Gamma(G)$ is connected with* $\mathrm{diam}(\Gamma(G)) \in \{3, 4\}$.

PROOF. By Lemma 2.1, $\Gamma(G) = \Delta(G)$. Since G is not a Frobenius group, Theorem 3.1 yields that $\Gamma(G)$ is connected. Theorem 3.2 gives us that $\mathrm{diam}(\Gamma(G)) \geq 3$. Finally, an application of Theorem 3.1 implies that $\mathrm{diam}(\Gamma(G))$ is either 3 or 4. □

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Acknowledgement

This research was conducted during an REU at Kent State University. The first, third, fourth and fifth authors thank the faculty and staff at Kent State University for their hospitality.

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