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On the validity of the Picard algorithm for nonlinear parabolic equations

Marco Cannone
Laboratoire d'Analyse et de Mathématiques Appliquées,
Université de Marne-la-Vallée, Cité Descartes-5, bd Descartes,
Champs-sur-Marne, 77454 Marne-la-Vallée Cedex 2, France
(cannone@math.univ-mlv.fr)

Grzegorz Karch
Instytut Matematyczny, Uniwersytet Wrocławski, pl. Grunwaldzki 2/4,
50-384 Wrocław, Poland and
Institute of Mathematics, Polish Academy of Sciences, PO Box 21,
ul. Sniadeckich 8, 00-956 Warszawa 10, Poland.
(karch@math.uni.wroc.pl)

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We discuss some ill-posedness results for solutions arising from the Picard iterations algorithm (i.e. the Banach fixed-point theorem) in the case of the nonlinear heat equation, the viscous Hamilton–Jacobi equation, the convection–diffusion equation and the incompressible Navier–Stokes system.

1. Introduction

Suppose that we try to solve the Cauchy problem for the semilinear evolution equation

$$u_t = \Delta u + F(u), \quad u(0) = u_0, \tag{1.1}$$

where $u = u(x,t)$, $x \in \mathbb{R}^n$, $t > 0$. The usual procedure is as follows. First, we convert problem (1.1) into the integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)F(u(\tau))\,\mathrm{d}\tau, \tag{1.2}$$

with the heat semigroup $S(t)$ given as the convolution with the Gauss–Weierstrass kernel $G(x,t) = (4\pi t)^{-n/2}\exp(-|x|^2/(4t))$. Next, we look for a Banach space $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and for a closed subset $\mathcal{M}_T \subset C([0,T],\mathcal{X})$ such that the right-hand side of equation (1.2) forms the contraction on $\mathcal{M}_T$ (usually, either for sufficiently small $T > 0$ and arbitrary large u_0 or for arbitrary large T and sufficiently small u_0). Finally, the Banach fixed-point theorem gives a solution $u \in \mathcal{M}_T$ of the integral equation (1.2). Moreover, this fixed point can be obtained as the limit of the Picard iterations

$$u_0(t) = S(t)u_0, \qquad u_{k+1}(t) = S(t)u_0 + \int_0^t S(t-\tau)F(u_k(\tau))\,\mathrm{d}\tau \quad \text{for } k = 1,2,3,\dots.$$

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The goal of this paper is to present a very simple method of showing that this procedure may fail if we consider the Cauchy problem (1.1) in some subcritical spaces. The scaling properties of problem (1.1) and of the norm $\|\cdot\|_{\mathcal{X}}$ play an important role in our reasoning.

Let us be more precise. We shall present a method of finding a sequence of initial data $\{u_0^N\}_{N=1}^{\infty}$, obtained as the rescaling of one function $u_0^N(x) = N^{\beta}u_0(Nx)$ with suitably chosen $\beta \in \mathbb{R}$, such that

$$\sup_{0\leqslant t\leqslant T} \|S(t)u_0^N\|_{\mathcal{X}} \leqslant C\|u_0\|_{\mathcal{X}} \tag{1.3}$$

for a constant C independent of N and u_0. However,

$$\sup_{0\leqslant t\leqslant T} \left\| \int_0^t S(t-\tau)F(S(\tau)u_0^N)\,\mathrm{d}\tau \right\|_{\mathcal{X}} \to \infty \quad \text{as } N \to \infty \tag{1.4}$$

for every $T > 0$.

Note that these two conditions imply that there is no estimate of the Picard iterations u_k which would imply the convergence of the sequence $\{u_k\}_{k=1}^{\infty}$. In particular, (1.4) shows that the nonlinear operator defined by the right-hand side of (1.2) cannot be a contraction on any bounded subset of $C([0,T],\mathcal{X})$. In fact, this nonlinear operator does not even preserve bounded sets in $L^{\infty}((0,T),\mathcal{X})$.

Definition 1.1. We shall say that the Picard algorithm for equation (1.2) (or the problem (1.1)) fails to hold in $C([0,T],\mathcal{X})$ for $u_0 \in \mathcal{X}$ if there exists $\beta \in \mathbb{R}$ such that (1.3) and (1.4) are satisfied for the sequence of the rescaled initial data $u_0^N(x) = N^{\beta}u_0(Nx)$.

Below, we shall use the Lebesgue space $L^p(\mathbb{R}^n)$, $1 \leqslant p \leqslant \infty$, with the usual norm $\|\cdot\|_p$, as the model example of the space $\mathcal{X}$. Note, however, that our method requires a scaling property of a norm only. Hence, our results can be rewritten directly in the case of the homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^n)$ (see the recent paper by Molinet *et al*. [18]), the Lorentz spaces $L^{p,q}(\mathbb{R}^n)$, the homogeneous Morrey spaces $\dot{M}^{p,q}(\mathbb{R}^n)$, etc. We find the range of p for which the Picard algorithm fails to hold in $L^p(\mathbb{R}^n)$ if applied in the case of the nonlinear heat equation, the viscous Hamilton–Jacobi equation, the convection–diffusion equation and the incompressible Navier–Stokes system.

Finally, let us also recall that similar ideas have appeared in the work by Tzvetkov [20], where the Korteweg–de Vries equation was considered in the homogeneous Sobolev spaces $\dot{H}^s(\mathbb{R}^n)$ and in papers by Molinet *et al*. [16, 17] on the Benjamin–Ono and the Kadomtsev–Petviashvili equations, respectively.

2. Nonlinear heat equation

In this section, we illustrate the ideas described in § 1, using the Cauchy problem for the nonlinear heat equation

$$u_t = \Delta u + a|u|^{q-1}u, \quad x \in \mathbb{R}^n,\ t > 0, \tag{2.1}$$

$$u(x,0) = u_0(x), \tag{2.2}$$

where $a \in \mathbb{R} \setminus \{0\}$ is a constant and $q > 1$. We immediately convert problem (2.1), (2.2) into the equivalent integral equation

$$u(t) = S(t)u_0 + a \int_0^t S(t-\tau)(|u|^{q-1}u)(\tau)\,\mathrm{d}\tau. \tag{2.3}$$

The following lemma plays the crucial role in the proof of the our main result.

LEMMA 2.1. *For every $q > 1$, $p \in [1,\infty]$, $u_0 \in L^{pq}(\mathbb{R}^n)$ and $T > 0$, the quantity*

$$\sup_{0 \leqslant t \leqslant T} \left\| \int_0^t S(t-\tau)(|S(\tau)u_0|^{q-1}S(\tau)u_0)\,\mathrm{d}\tau \right\|_p \tag{2.4}$$

is well defined and finite.

Proof. It follows from the well-known estimates of the heat semigroup that

$$\|S(t)u_0\|_r \leqslant Ct^{-(n/2)(1/p-1/r)}\|u_0\|_p \tag{2.5}$$

for every $1 \leqslant p \leqslant r \leqslant \infty$, all $t > 0$, and $C = C(p,r)$ independent of t and u_0. Hence, a direct computation of the L^p-norm in (2.4) combined with (2.5) (recall that $C(p,p) = 1$) gives

$$\begin{aligned} \left\| \int_0^t S(t-\tau)(|S(\tau)u_0|^{q-1}S(\tau)u_0)\,\mathrm{d}\tau \right\|_p &\leqslant \int_0^t \|S(\tau)u_0\|_{pq}^q\,\mathrm{d}\tau \\ &\leqslant \int_0^t \|u_0\|_{pq}^q\,\mathrm{d}\tau \\ &= T\|u_0\|_{pq}^q, \end{aligned}$$

for all $t \in [0,T]$. □

We are now in a position to prove the main theorem of this section.

THEOREM 2.2. *Let $q > 1$ and assume that $1 \leqslant p < \frac{1}{2}n(q-1)$. For every $u_0 \in L^p(\mathbb{R}^n) \cap L^{pq}(\mathbb{R}^n)$, the Picard algorithm fails to hold for problem (2.1), (2.2) in the space $C([0,T], L^p(\mathbb{R}^n))$ for each $T > 0$.*

Proof. Take $u_0 \in L^p(\mathbb{R}^n) \cap L^{pq}(\mathbb{R}^n)$ from lemma 2.1 as the initial datum. Define the sequence $u_0^N(x) = N^{n/p}u_0(Nx)$ for $N = 1, 2, \dots$. It follows from estimates (2.5) of the heat semigroup and from the scaling property of the L^p-norm that

$$\sup_{0 \leqslant t \leqslant T} \|S(t)u_0^N\|_p \leqslant \|u_0^N\|_p = \|u_0\|_p. \tag{2.6}$$

Hence, the first estimate (1.3) required by definition 1.1 is proven.

A direct calculation based on the self-similar form of the heat kernel $G(x,t) = t^{-n/2}G(x/\sqrt{t}, 1)$ and on the change of variables gives

$$S(t)u_0^N(x) = N^{n/p}[S(N^2t)u_0](Nx).$$

Consequently, a similar reasoning leads to

$$\begin{aligned} &S(t-\tau)[|S(\tau)u_0^N|^{q-1}(S(\tau)u_0^N)](x) \\ &\qquad = N^{nq/p}S(N^2(t-\tau))[|S(N^2\tau)u_0|^{q-1}(S(N^2\tau)u_0)](Nx). \end{aligned} \tag{2.7}$$

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Now, the second condition (1.4) required by definition 1.1 results from (2.7) because, by the change of variables, we obtain the following two equalities:

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}&\left\|\int_0^t S(t-\tau)[|S(\tau)u_0^N|^{q-1}(S(\tau)u_0^N)]\,\mathrm{d}\tau\right\|_p\\&=\sup_{0\leqslant t\leqslant T}N^{nq/p-n/p}\left\|\int_0^t S(N^2(t-\tau))[|S(N^2\tau)u_0|^{q-1}(S(N^2\tau)u_0)]\,\mathrm{d}\tau\right\|_p\\&=N^{(nq/p)-(n/p)-2}\sup_{0\leqslant t\leqslant T}\left\|\int_0^{N^2t}S(N^2t-s)[|S(s)u_0|^{q-1}(S(s)u_0)]\,\mathrm{d}\tau\right\|_p.\end{aligned}\tag{2.8}$$

Note that

$$N^{(nq/p)-(n/p)-2}\to\infty\quad\text{as } N\to\infty$$

because the inequality $(nq/p)-(n/p)-2>0$ is equivalent to $p<\frac{1}{2}n(q-1)$. Moreover, the second factor on the right-hand side is finite and positive by lemma 2.1. □

The supremum in (2.4) in lemma 2.1 may increase to infinity as $T\nearrow\infty$. The goal of the next lemma is to show that this is not the case for some $u_0\in\mathcal{S}(\mathbb{R}^n)$.

LEMMA 2.3. *For every $p\in[1,\infty]$ and $q>1$, there exists $u_0\in\mathcal{S}(\mathbb{R}^n)$ such that*

$$0<\sup_{t>0}\left\|\int_0^t S(t-\tau)(|S(\tau)u_0|^{q-1}S(\tau)u_0)\,\mathrm{d}\tau\right\|_p<\infty.\tag{2.9}$$

Proof. We begin with the remark that, given $\kappa>0$, there exists non-trivial $u_0\in\mathcal{S}(\mathbb{R}^n)$ such that

$$\|S(t)u_0\|_r\leqslant C(1+t)^{-\kappa}\tag{2.10}$$

for every $r\in[1,\infty]$, all $t\geqslant 0$ and C independent of t. Indeed, it suffices to take $u_0\in\mathcal{S}(\mathbb{R}^n)$ satisfying $\hat{u}_0(\xi)=0$ for $|\xi|\leqslant 1$. Then, all moments of u_0 disappear:

$$\int_{\mathbb{R}^n}x^\alpha u_0(x)\,\mathrm{d}x=0\quad\text{for every multi-index }\alpha,$$

where, as usual, $\alpha=(\alpha_1,\dots,\alpha_n)$, $\alpha_i\in\{0,1,2,\dots\}$ and $x^\alpha=x_1^{\alpha_1}\cdots x_n^{\alpha_n}$.

Next, one should use the asymptotic expansion of solutions to the heat equation proved in [5],

$$\begin{aligned}\left\|S(t)u_0-\sum_{|\alpha|\leqslant k}\frac{(-1)^{|\alpha|}}{\alpha!}\left(\int_{\mathbb{R}^n}x^\alpha u_0(x)\,\mathrm{d}x\right)\frac{\partial^{|\alpha|}}{\partial x^\alpha}G(t)\right\|_r&\\\leqslant C(k,n,r)t^{-(n/2)(1-1/r)-(k+1)/2}\||x|^{k+1}u_0\|_1.&\end{aligned}\tag{2.11}$$

Since $\|S(t)u_0\|_r\leqslant\|u_0\|_r$, using expansion (2.11) with sufficiently large k, we obtain inequality (2.10).

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Now, we show (2.9) in the most direct way, using properties of the heat semigroup:

$$\begin{aligned}\left\| \int_0^t S(t-\tau)(|S(\tau)u_0|^{q-1}S(\tau)u_0)\,\mathrm{d}\tau \right\|_p &\leqslant \int_0^t \||S(\tau)u_0|^{q-1}S(\tau)u_0\|_p\,\mathrm{d}\tau \\ &= \int_0^t \|S(\tau)u_0\|_{qp}^q\,\mathrm{d}\tau \\ &\leqslant C\int_0^\infty (1+\tau)^{-q\kappa}\,\mathrm{d}\tau \\ &< \infty\end{aligned}$$

by (2.10) with $r = qp$ and $\kappa > 1/q$. □

REMARK 2.4. Now, it is clear that the bounds in theorem 2.2 imposed on the exponent p are optimal because, by lemma 2.3, there exist initial data such that the second factor on the right-hand site of (2.8) tends (as $N \to \infty$) towards

$$\sup_{t\geqslant 0}\left\| \int_0^t S(t-\tau)[|S(\tau)u_0|^{q-1}(S(\tau)u_0)]\,\mathrm{d}\tau \right\|_p,$$

which is finite and positive.

REMARK 2.5. An analogous result in the case of Sobolev spaces $H^s(\mathbb{R})$ for the one-dimensional problem (2.1), (2.2) was obtained (using a different method) by Molinet *et al*. [18]. They show that the Picard algorithm fails for suitably chosen $u_0 \in H^s(\mathbb{R})$ (like those in lemma 2.3) for $s < -1$ if $q = 2$ and for $s < 1/2 - 2/(q-1)$ otherwise.

Let us look at theorem 2.2 from the point of view of what is already known about problem (2.1), (2.2).

The critical quantity $\frac{1}{2}n(q-1)$ appeared in the papers by Fujita [7,8], who studied classical solutions to (2.1) and (2.2). His results are as follows. If $\frac{1}{2}n(q-1) < 1$, then no non-negative global-in-time solutions exist for any non-trivial initial data. If $\frac{1}{2}n(q-1) > 1$, then global solutions do exist for any non-negative initial datum dominated by a sufficiently small Gaussian.

In the context of L^p-spaces, Weissler [21] proved that, for every $u_0 \in L^p(\mathbb{R}^n)$ with $p > \frac{1}{2}n(q-1)$, there exist $T > 0$ and $u \in C([0,T], L^p(\mathbb{R}^n))$ satisfying (2.1), (2.2) in a suitable sense (analogous results in the limit case $p = \frac{1}{2}n(q-1)$ are also given in [21]). Moreover, the solution is obtained via the Picard iteration scheme in a suitable subspace of $C([0,T], L^p(\mathbb{R}^n))$.

On the other hand, if $1 \leqslant p < \frac{1}{2}n(q-1)$, there exists $u_0 \in L^p(\mathbb{R}^n)$ with $u_0 \geqslant 0$ such that no solution $u \in C([0,T], L^p(\mathbb{R}^n))$ exists on any non-trivial interval $[0,T]$ (see [21], for details). Finally, let us also recall the result by Haraux and Weissler [10], who constructed a solution $\psi = \psi(x,t)$ to equation (2.1) such that

$$\lim_{t\to 0}\|\psi(\cdot,t)\|_p = 0$$

when $1 < \frac{1}{2}n(q-1) < q+1$ and $1 \leqslant p < \frac{1}{2}n(q-1)$. Hence, the uniqueness fails in the space $C([0,T], L^p(\mathbb{R}^n))$ in this range of p and q.

3. Viscous Hamilton–Jacobi equations

Let us now apply the reasoning used in the previous section to the Cauchy problem for the viscous Hamilton–Jacobi equation

$$u_t = \Delta u + a|\nabla u|^q, \tag{3.1}$$

$$u(x,0) = u_0(x), \tag{3.2}$$

with constants $a \in \mathbb{R} \setminus \{0\}$ and $q > 1$.

LEMMA 3.1. *For every $q > 1$, $p \in [1,\infty]$, u_0 such that $\nabla u_0 \in L^{pq}(\mathbb{R}^n)$, and $T > 0$, the quantity*

$$\sup_{0\leqslant t\leqslant T} \left\| \int_0^t S(t-\tau)|\nabla S(\tau)u_0|^q \,\mathrm{d}\tau \right\|_p$$

is well defined and finite.

Proof. Here, the reasoning is similar to that in the proof of lemma 2.1 and we omit it. □

The next lemma is the direct counterpart of lemma 2.3.

LEMMA 3.2. *For every $p \in [1,\infty]$ and $q > 1$ there exists $u_0 \in \mathcal{S}(\mathbb{R}^n)$ such that*

$$0 < \sup_{t\geqslant 0} \left\| \int_0^t S(t-\tau)|\nabla S(\tau)u_0|^q \,\mathrm{d}\tau \right\|_p < \infty.$$

Proof. It suffices to repeat arguments used in the proof of lemma 2.3 because, given $\kappa > 0$, there exists a non-trivial $u_0 \in \mathcal{S}(\mathbb{R}^n)$ such that

$$\|\nabla S(t)u_0\|_r \leqslant C(1+t)^{-\kappa} \tag{3.3}$$

for every $r \in [1,\infty]$, all $t \geqslant 0$, and a constant C. For the proof of inequality (3.3), we choose u_0 from lemma 2.3 and we apply the expansion (2.11) with the Gauss–Weierstrass kernel $G(x,t)$ replaced by $\nabla G(x,t)$ (see [5] for details). Other details are completely analogous to those from the proof of lemma 2.3. □

THEOREM 3.3. *If $1 < q < 2$, we assume that $1 \leqslant p < n(q-1)/(2-q)$. Let $p \in [1,\infty)$ for $q = 2$, and $p \in [1,\infty]$ for $q > 2$. For every $u_0 \in L^p(\mathbb{R}^n)$ such that $\nabla u_0 \in L^{pq}(\mathbb{R}^n)$, the Picard algorithm for problem (3.1), (3.2) fails to hold in the space $C([0,T], L^p(\mathbb{R}^n))$ for each $T > 0$.*

Proof. As in the proof of theorem 2.2 we take the initial datum from lemma 3.1 and we consider the sequence $u_0^N(x) = N^{n/p}u_0(Nx)$ with $N \in \{1,2,3,\dots\}$. It follows from (2.6) that $\sup_{0\leqslant t\leqslant T} \|S(\cdot)u_0^N\|_p \leqslant \|u_0\|_p$. Using the self-similar form of the Gauss–Weierstrass kernel, we obtain

$$S(t-\tau)|\nabla S(\tau)u_0^N|^q(x) = N^{q(n/p+1)}S(N^2(t-\tau))|\nabla S(N^2\tau)u_0|^q(Nx).$$

Next, by the change of variables, we have

$$\sup_{0\leqslant t\leqslant T}\left\| \int_0^t S(t-\tau)|\nabla S(\tau)u_0^N|^q(x)\,\mathrm{d}\tau\right\|_p$$
$$= N^{q(n/p+1)-n/p-2}\sup_{0\leqslant t\leqslant T}\left\|\int_0^{N^2t} S(N^2t-s)|\nabla S(N^2s)u_0|^q(Nx)\,\mathrm{d}s\right\|_p.$$

Finally, observe that, for $1<q<2$, the inequality $q(n/p+1)-n/p-2>0$ is equivalent to $p<n(q-1)/(2-q)$. Moreover, $q(n/p+1)-n/p-2>0$ for any $p\in[1,\infty)$ if $q=2$, and for all $p\in[1,\infty]$ if $q>2$ (here, we put $n/p=0$ for $p=\infty$). □

Again, we can observe the perfect agreement of theorem 3.3 with the existing knowledge on problem (3.1), (3.2) studied in the Lebesgue spaces. Here, the paper by Ben-Artzi *et al*. [1] contains the most recent and the most general results. First of all, using the Picard algorithm, they prove the well posedness of problem (3.1), (3.2) in $L^p(\mathbb{R}^n)$ provided $1\leqslant q<2$ and $p\geqslant n(q-1)/(2-q)$. In the case $a>0$ and $u_0\geqslant 0$, the existence fails in all L^p-spaces when $q\geqslant 2$. When $q<2$, it is shown in [1] that both the existence and the uniqueness fail if $1\leqslant p<n(q-1)/(2-q)$. We refer the reader to [1] for other results on problem (3.1), (3.2) studied in $L^p(\mathbb{R}^n)$.

4. Convection–diffusion equations

Here, we apply our method to the Cauchy problem for the convection–diffusion equation

$$u_t = \Delta u + a\cdot\nabla(|u|^q),\quad x\in\mathbb{R}^n,\ t>0, \tag{4.1}$$
$$u(x,0)=u_0(x), \tag{4.2}$$

where $a\in\mathbb{R}^n\setminus\{0\}$ is a constat vector and $q>1$. For simplicity of notation, we choose the nonlinearity in equation (4.1) to be of the form $f(u)=a|u|^q$; note, however, that we can consider any sufficiently regular function $f(u)=(f_1(u),\dots,f_n(u))$ which is homogeneous of degree q (e.g. any linear combination of $|u|^q$ and $|u|^{q-1}u$). First, we need counterparts of lemmas 2.1 and 2.3.

LEMMA 4.1. *For every $q>1$, $p\in[1,\infty]$, $u_0\in L^{pq}(\mathbb{R}^n)$ and $T>0$, the quantity*

$$\sup_{0\leqslant t\leqslant T}\left\|\int_0^t a\cdot\nabla S(t-\tau)|S(\tau)u_0|^q\,\mathrm{d}\tau\right\|_p$$

is well defined and finite.

Proof. The proof is analogous to the proof of lemma 2.1 using the estimates of the heat semigroup

$$\|\nabla S(t)u_0\|_r\leqslant Ct^{-(n/2)(1/p-1/r)-1/2}\|u_0\|_p \tag{4.3}$$

for every $1\leqslant p\leqslant r\leqslant\infty$, all $t>0$, and $C=C(p,r)$ independent of t and u_0. Now, the counterpart of the main estimate form the proof of lemma 2.1 has the following form:

$$\left\| \int_0^t a \cdot \nabla S(t-\tau)|S(\tau)u_0|^q \,\mathrm{d}\tau \right\|_p \leqslant C \int_0^t (t-\tau)^{-1/2} \|S(\tau)u_0\|_{pq}^q \,\mathrm{d}\tau \leqslant CT^{1/2}\|u_0\|_{pq}^q \quad \text{for all } t \in [0,T].$$

□

Lemma 4.2. *For every $p \in [1,\infty]$ and $q > 1$ there exists $u_0 \in \mathcal{S}(\mathbb{R}^n)$ such that*

$$0 < \sup_{t>0} \left\| \int_0^t a \cdot \nabla S(t-\tau)|S(\tau)u_0|^q \,\mathrm{d}\tau \right\|_p < \infty.$$

Proof. Here, the reasoning is completely analogous to that used in the proofs of lemmas 2.3 and 3.2, so we omit the details. □

Theorem 4.3. *Let $q > 1$ and assume that $1 \leqslant p < n(q-1)$. For every $u_0 \in L^p(\mathbb{R}^n) \cap L^{pq}(\mathbb{R}^n)$, the Picard algorithm fails to hold for problem (4.1), (4.2) in the space $C([0,T], L^p(\mathbb{R}^n))$ for every $T > 0$.*

Proof. In order to apply the reasoning from the proofs of theorems 2.2 and 3.3, it suffices to take $u_0 \in L^p(\mathbb{R}^n) \cap L^{pq}(\mathbb{R}^n)$ and the sequence $u_0^N(x) = N^{n/p}u_0(Nx)$. Next, one should note the following two equalities:

$$\nabla S(t-\tau)|S(\tau)u_0^N|^q(x) = N^{(nq/p)+1}\nabla S(N(t-\tau))|S(N^2\tau)u_0|^q(Nx)$$

and their consequence

$$\sup_{0\leqslant t\leqslant T} \left\| \int_0^t a \cdot \nabla S(t-\tau)|S(\tau)u_0^N|^q \,\mathrm{d}\tau \right\|_p = N^{(nq/p)+1-(n/p)-2} \sup_{0\leqslant t\leqslant T} \left\| \int_0^{N^2 t} a \cdot \nabla S(N^2 t - s)|S(s)u_0|^q \,\mathrm{d}s \right\|_p.$$

Finally, $nq/p + 1 - n/p - 2 > 0$ if and only if $p < n(q-1)$. □

It is a completely standard reasoning to show that problem (4.1), (4.2) is well posed in $C([0,T], L^p(\mathbb{R}^n))$ for $p \geqslant n(q-1)$. Several results in this direction were proved by Giga [9], with a further extension by Ribaud [19]. To the best knowledge of the authors, the only result on the non-well-posedness of problem (4.1), (4.2) was obtained by Dix [4], who proved that the uniqueness for the Burgers equation,

$$u_t - u_{xx} + uu_x = 0,$$

fails in the Sobolev spaces $H^s(\mathbb{R})$ for $s < -\frac{1}{2}$.

Remark 4.4. In particular, theorem 4.3 gives the failure of the Picard algorithm in $L^1(\mathbb{R}^n)$ for any $u_0 \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$. However, it is well known that the Cauchy problem (4.1), (4.2) is well posed in $C([0,\infty), L^1(\mathbb{R}^n))$ (see, for example, [6]). The proof of this fact uses essentially several additional properties of the problem (4.1), (4.2), such as the maximum principle and the conservation in time of the integral $\int_{\mathbb{R}^n} u(x,t)\,\mathrm{d}x$.

5. Navier–Stokes system

The Navier–Stokes equations, describing the evolution of the velocity field $u = u(x,t)$ and the scalar pressure $p = p(x,t)$ in the whole $\mathbb{R}^n$ are given by

$$u_t - \Delta u + (u \cdot \nabla)u + \nabla p = 0, \quad x \in \mathbb{R}^n,\ t > 0, \tag{5.1}$$

$$\nabla \cdot u = 0, \tag{5.2}$$

$$u(x,0) = u_0(x). \tag{5.3}$$

Let us recall the projection $\mathbb{P}$ of $L^2(\mathbb{R}^n)^n$ onto the subspace

$$L^2_\sigma(\mathbb{R}^n)^n \equiv \mathbb{P}[L^2(\mathbb{R}^n)^n]$$

of solenoidal vector fields (i.e. those characterized by the divergence condition (5.2)). It is known that $\mathbb{P}$ is a pseudodifferential operator of order 0. In fact, it can be written as a combination of the Riesz transforms R_j with symbols $\xi_j/|\xi|$,

$$\mathbb{P}(v_1, \dots, v_n) = (v_1 - R_1\omega, \dots, v_n - R_n\omega),$$

where $\omega = R_1 v_1 + \dots + R_n v_n$. This explicit formula for $\mathbb{P}$ allows us to define this operator on $L^p(\mathbb{R}^n)^n$ for every $1 < p < \infty$.

Using this projection, one can remove the pressure from the model (5.1)–(5.3) and obtain an equivalent Cauchy problem

$$u_t - \Delta u + \mathbb{P}\nabla \cdot (u \otimes u) = 0, \quad x \in \mathbb{R}^n,\ t > 0, \tag{5.4}$$

$$u(0) = u_0. \tag{5.5}$$

We study solutions to problem (5.4), (5.5) rewritten as the integral equation

$$u(t) = S(t)u_0 - \int_0^t S(t-\tau)\mathbb{P}\nabla \cdot (u \otimes u)(\tau)\,\mathrm{d}\tau. \tag{5.6}$$

LEMMA 5.1. *For every $p \in [1,\infty]$, $u_0 \in L^{2p}(\mathbb{R}^n)^n$ and $T > 0$, the quantity*

$$\sup_{0 \leqslant t \leqslant T} \left\| \int_0^t S(t-\tau)\mathbb{P}\nabla \cdot (S(\tau)u_0 \otimes S(\tau)u_0)\,\mathrm{d}\tau \right\|_p$$

is well defined and finite.

Proof. Here, we proceed as in the proof of lemma 4.1. Indeed, it is well known (see, for example, [13]) that the operator $S(t)\mathbb{P}\nabla$ is realized as the convolution with the Oseen kernel $\mathcal{K} = \mathcal{K}(x,t)$, which is a bounded and integrable function in x such that $\|K(\cdot,t)\|_1 \leqslant Ct^{-1/2}$ for all $t > 0$. Hence, using the Young inequality for the convolution, we find that

$$\left\| \int_0^t S(t-\tau)\mathbb{P}\nabla \cdot (S(\tau)u_0 \otimes S(\tau)u_0)\,\mathrm{d}\tau \right\|_p \leqslant C\int_0^t (t-\tau)^{-1/2}\|S(\tau)u_0\|_{2p}^2\,\mathrm{d}\tau \leqslant CT^{1/2}\|u_0\|_{2p}^2 \quad \text{for all } t \in [0,T].$$

□

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Lemma 5.2. *For every $p \in (1,\infty)$ there exists a vector field $u_0 \in \mathcal{S}(\mathbb{R}^n)^n$ satisfying $\nabla \cdot u_0 = 0$ such that*

$$0 < \sup_{t \geqslant 0} \left\| \int_0^t S(t-\tau)\mathbb{P}\nabla \cdot (S(\tau)u_0 \otimes S(\tau)u_0)\,\mathrm{d}\tau \right\|_p < \infty \quad \textit{for every } T > 0.$$

Proof. First, consider non-trivial $\varphi \in \mathcal{S}(\mathbb{R}^n)$ satisfying $\hat{\varphi}(\xi) = 0$ for $|\xi| \leqslant 1$. Next, define $\hat{u}_0(\xi) = (\xi_1\hat{\varphi}(\xi), -\xi_2\hat{\varphi}(\xi), 0, \dots, 0)$, which guarantees that $\nabla \cdot u_0 = 0$. As in the proofs of lemmas 2.3 and 3.2, we show that, given $\kappa > 0$, there exists $C > 0$ such that

$$\|S(t)u_0\|_r \leqslant C(1+t)^{-\kappa} \quad \text{and} \quad \|\nabla S(t)u_0\|_r \leqslant C(1+t)^{-\kappa} \tag{5.7}$$

for every $r \in [1,\infty]$, all $t \geqslant 0$ and a constant C. Since the projection $\mathbb{P}$ (as the combination of Riesz transforms) is bounded on $L^p(\mathbb{R}^n)^n$, $1 < p < \infty$, we obtain

$$\begin{aligned} \|S(t-\tau)\mathbb{P}\nabla \cdot (S(\tau)u_0 \otimes S(\tau)u_0)\|_p &\leqslant C\|\nabla \cdot (S(\tau)u_0 \otimes S(\tau)u_0)\|_p \\ &\leqslant C\|\nabla S(\tau)u_0\|_{2p}^{1/2}\|S(\tau)u_0\|_{2p}^{1/2} \\ &\leqslant C(1+\tau)^{-\kappa}, \end{aligned}$$

by (5.7) with $r = 2p$. We now choose $\kappa > 1$ to complete the proof. □

Theorem 5.3. *For every $1 < p < n$ and*

$$u_0 \in L^p_\sigma(\mathbb{R}^n)^n \cap L^{2p}_\sigma(\mathbb{R}^n)^n,$$

the Picard algorithm fails to hold for the Navier–Stokes system (5.4), (5.5) in the space $C([0,T], L^p_\sigma(\mathbb{R}^n)^n)$ for every $T > 0$.

Proof. For $u_0 \in L^p_\sigma(\mathbb{R}^n)^n \cap L^{2p}_\sigma(\mathbb{R}^n)^n$ and the sequence $u_0^N(x) = N^{n/p}u_0(Nx)$, we obtain

$$\begin{aligned} &S(t-\tau)\mathbb{P}\nabla \cdot (S(\tau)u_0^N \otimes S(\tau)u_0^N)(x) \\ &\qquad = N^{(2n/p)+1}S(N^2(t-\tau))\mathbb{P}\nabla \cdot (S(N^2\tau)u_0 \otimes S(N^2\tau)u_0)(Nx). \end{aligned}$$

Here, we have used the fact that $\mathbb{P}$ is the pseudo-differential operator of order 0. Consequently,

$$\begin{aligned} &\sup_{0 \leqslant t \leqslant T} \left\| \int_0^t S(t-\tau)\mathbb{P}\nabla \cdot (S(\tau)u_0^N \otimes S(\tau)u_0^N)\,\mathrm{d}\tau \right\|_p \\ &\qquad = N^{(2n/p)+1-(n/p)-2} \sup_{0 \leqslant t \leqslant T} \left\| \int_0^{N^2 t} S(N^2t - s)\mathbb{P}\nabla \cdot (S(s)u_0 \otimes S(s)u_0)\,\mathrm{d}s \right\|_p. \end{aligned}$$

Finally, the inequality

$$\frac{2n}{p} + 1 - \frac{n}{p} - 2 > 0$$

is equivalent to $p < n$. □

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Given $u_0 \in L^p_\sigma(\mathbb{R}^n)^n$ and a neighbourhood $V_{u_0} \subset L^p_\sigma(\mathbb{R}^n)^n$ of u_0, we define the flow-map $u_0 \mapsto u(u_0)$ from V_{u_0} to $C([0,T], L^p_\sigma(\mathbb{R}^n)^n)$. If the Cauchy problem (5.1)–(5.3) is well posed in $L^p_\sigma(\mathbb{R}^n)^n$, this flow-map is well defined. The continuous dependence on initial conditions means the continuity of the flow-map. Moreover, it is relatively easy to prove that the solutions obtained via the Picard algorithm for $p > n$ (see, for example, [2, 3, 11–13]) depend analytically on initial data.

In the subcritical case, the regularity of the flow-map changes drastically.

THEOREM 5.4. *There is no application of class C^2 at the point $u_0 \equiv 0$ that associates a (mild or weak) solution $u \in C([0,T); L^p_\sigma(\mathbb{R}^n)^n)$, $p < n$, for the system (5.4), (5.5) to the corresponding initial datum $u_0 \in L^p_\sigma(\mathbb{R}^n)^n$.*

This result was originally obtained by Y. Meyer and announced at the Conference in honour of Jacques-Louis Lions held in Paris in 1998. The full proof will appear in detail in [15]. Note that $p = 2 < n = 3$ corresponds to the most interesting case of weak solutions by Leray [14]. In particular:

(1) There is no application of class C^2 that associates Leray's weak solution $u \in L^\infty((0,T); L^2_\sigma(\mathbb{R}^3)^3)$ with the initial datum $u_0 \in L^2_\sigma(\mathbb{R}^3)^3$.

(2) If a mild solution exists in the subcritical case ($2 \leqslant p < 3$), it does not arise from the Picard algorithm.

The proof of theorem 5.4 is the simple consequence of theorem 5.3 and is based on a contradiction argument. Briefly stated, it is assumed that for the initial data λu_0, the solution $u_\lambda(x,t)$, whose existence is supposed in theorem 5.4, could be written in the form $\lambda u^{(1)}(x,t) + \lambda^2 u^{(2)}(x,t) + o(\lambda^2)$, where '$o$' corresponds to the norm $L^\infty([0,T); L^p_\sigma(\mathbb{R}^n)^n)$ and $\lambda \to 0$. We then observe that $u^{(1)}$ and $u^{(2)}$ are equal to the Picard iterations $u_0(t)$ and $u_1(t)$. Hence, theorem 5.3 implies that the second-order Taylor expansion of $u_\lambda(x,t)$ as the function of λ is impossible. This is a standard argument (see, for example, [15, 18, 20]). Hence, we omit the other details.

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