PROCEEDINGS
OF THE
Cambridge Philosophical Society.

A case of distinction between Fourier integrals and Fourier series.

By MARGARET ELEANOR GRIMSHAW, Newnham College. (Communicated by Mr S. POLLARD.)

[*Received* 12 March, *read* 2 May, 1927.]

1. A Fourier integral is said to be of *finite type** if its generating function vanishes for all sufficiently large values of $|x|$. Because the coefficient functions are defined by integrals over a finite range, the behaviour of such a Fourier integral usually resembles closely that of the corresponding series.

It may be shewn that when $k \geqslant 1$ the necessary and sufficient conditions for the summability (C, k) of a Denjoy-Fourier integral of finite type are exactly the same as those for the corresponding series. When $k < 1$ this is no longer the case, and the integral may be summable while the series is not. Indeed, it is possible to give an example of a function whose series is summable (C, k) for $0 \leqslant k \leqslant \mathbf{k} < 1$ almost nowhere in the interval $(-\pi, \pi)$, while the integral is summable for all $k > 0$ everywhere in the interval.

This result is due to the fact that for the coefficients of a Denjoy-Fourier series the most that can generally be stated is

$$a_n = o(n), \quad b_n = o(n),$$

so that the series is, in general, at most summable $(C, 1)$, whereas the corresponding integral is not necessarily so restricted by the behaviour of the coefficients.

To make matters quite definite, let $f(x)$ be zero outside $(-\pi, \pi)$ and integrable in the general Denjoy sense in $(-\pi, \pi)$. Then $f(x)$ has a Fourier series

$$(1{\cdot}1) \qquad \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx),$$

where

$$(1{\cdot}2) \qquad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos nt\,dt, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin nt\,dt \qquad (n = 0, 1, 2, \ldots),$$

and it has a Fourier integral of finite type

$$(1{\cdot}3) \qquad \int_0^{\infty} (a_s \cos sx + b_s \sin sx)\,ds,$$

where

$$(1{\cdot}4) \qquad a_s = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos st\,dt, \quad b_s = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin st\,dt \quad (s \geqslant 0).$$

* S. Pollard, *Proc. Camb. Phil. Soc.*, 23 (1926), 373—382.

For a particular value of $k \geqslant 1$ the summability (C, k) at a point x in $-\pi < x < \pi$ of either the integral or the series implies that of the other, and the sums are the same. For other values of k the conditions differ in quite a marked degree.

2. In a consideration of the summability of the integral (1·3) we are led to investigate the behaviour of

$$(2{\cdot}1) \qquad I_k(\omega) = \int_0^\omega \left(1 - \frac{s}{\omega}\right)^k (a_s \cos sx + b_s \sin sx)\, ds$$

as $\omega \to \infty$.

The corresponding expression for the series (1·1) is

$$(2{\cdot}2) \qquad \sigma_k(\omega) = \tfrac{1}{2} a_0 + \sum_{1 \leqslant n < \omega} \left(1 - \frac{n}{\omega}\right)^k (a_n \cos nx + b_n \sin nx).$$

The final form of the criteria for summability of the integral depends essentially upon the possibility of changing the order of integration in a repeated integral, and is in some respects easier to obtain than the corresponding form for the series.

It will be noticed that $I_k(\omega)$ does not exist, in general, unless $k > -1$. In what follows this will be assumed throughout, and other restrictions on k will be introduced as required.

For the purpose of comparison we consider a generating function $f(x)$ which vanishes outside the interval $(-\pi, \pi)$. It will be evident, however, that the results of paragraphs 3 and 5 are quite general, and can be established in exactly the same way for a function which is integrable in the general Denjoy sense in any finite interval (p, q) and vanishes outside this interval.

3. Theorem I. *If $k > -1$, then*

$$(3{\cdot}1) \qquad I_k(\omega) = \frac{\omega}{\pi} \int_{-\pi-x}^{\pi-x} f(x+t)\, \gamma(\omega t)\, dt,$$

where

$$(3{\cdot}2) \qquad \gamma(t) = \int_0^1 (1-u)^k \cos tu\, du.$$

It immediately appears that

$$\omega\gamma(\omega t) = \int_0^\omega \left(1 - \frac{u}{\omega}\right)^k \cos tu\, du$$

so that (3·1) is equivalent to

$$I_k(\omega) = \frac{1}{\pi} \int_{-\pi-x}^{\pi-x} dt f(x+t) \int_0^\omega \left(1 - \frac{s}{\omega}\right)^k \cos ts\, ds.$$

Moreover, by (1·4) and (2·1),

$$I_k(\omega) = \frac{1}{\pi}\int_0^\omega ds \left(1 - \frac{s}{\omega}\right)^k \int_{-\pi}^{\pi} f(t) \cos s\,(t - x)\, dt$$

$$(3\cdot3) \qquad = \frac{1}{\pi}\int_0^\omega ds \left(1 - \frac{s}{\omega}\right)^k \int_{-\pi-x}^{\pi-x} f(x+t) \cos st\,dt.$$

Thus the result to be established is equivalent to the possibility of inverting the order of integration in (3·3).

Write $$g(x) = f(x) - \tfrac{1}{2}a_0,$$

so that, by (1·2), $$\int_{-\pi}^{\pi} g(t)\, dt = 0.$$

If $$G(x) = \int_{-\pi}^{x} g(t)\, dt,$$

then $$G(-\pi) = G(\pi) = 0.$$

By integration by parts, which is justified since $\cos st$ is of bounded variation in any finite interval,

$$\int_{-\pi-x}^{\pi-x} g(x+t) \cos st\,dt = \Big[G(x+t)\cos st\Big]_{-\pi-x}^{\pi-x} + s\int_{-\pi-x}^{\pi-x} G(x+t)\sin st\,dt$$

$$= s\int_{-\pi-x}^{\pi-x} G(x+t)\sin st\,dt.$$

Thus (3·3) becomes

$$I_k(\omega) = \frac{a_0}{2\pi}\int_0^\omega ds\left(1 - \frac{s}{\omega}\right)^k \int_{-\pi-x}^{\pi-x} \cos st\,dt + \frac{1}{\pi}\int_0^\omega ds\,s\left(1 - \frac{s}{\omega}\right)^k \int_{-\pi-x}^{\pi-x} G(x+t)\sin st\,dt.$$

Now $G(x+t)$ is continuous, and so $|G(x+t)|$ has a finite upper bound M in $(-\pi - x, \pi - x)$. If we replace the integrands in the above integrals by their absolute values, the result is not greater than

$$a_0\int_0^\omega \left(1 - \frac{s}{\omega}\right)^k ds + 2\omega M \int_0^\omega \left(1 - \frac{s}{\omega}\right)^k ds < \infty$$

since $k > -1$.

https://doi.org/10.1017/S0305004100015541 Published online by Cambridge University Press

By de la Vallée Poussin's theorem we may change the order of integration and write

$$I_k(\omega) = \frac{a_0}{2\pi}\int_{-\pi-x}^{\pi-x} dt \int_0^\omega \left(1-\frac{s}{\omega}\right)^k \cos ts\,ds$$
$$+\frac{1}{\pi}\int_{-\pi-x}^{\pi-x} dt\, G(x+t)\int_0^\omega s\left(1-\frac{s}{\omega}\right)^k \sin ts\,ds.$$

Now

$$\int_{-\pi-x}^{\pi-x} dt g(x+t)\int_0^\omega \left(1-\frac{s}{\omega}\right)^k \cos ts\,ds$$
$$= \left[G(x+t)\int_0^\omega \left(1-\frac{s}{\omega}\right)^k \cos ts\,ds\right]_{-\pi-x}^{\pi-x}$$
$$+\int_{-\pi-x}^{\pi-x} dt\, G(x+t)\int_0^\omega s\left(1-\frac{s}{\omega}\right)^k \sin ts\,ds$$
$$= \int_{-\pi-x}^{\pi-x} dt\, G(x+t)\int_0^\omega s\left(1-\frac{s}{\omega}\right)^k \sin ts\,ds,$$

the integration by parts being allowed since

$$\int_0^\infty \left(1-\frac{s}{\omega}\right)^k \cos ts\,ds$$

regarded as a function of t is of bounded variation in any finite interval.

Thus

$$I_k(\omega) = \frac{1}{\pi}\int_{-\pi-x}^{\pi-x} dt\,\{\tfrac{1}{2}a_0 + g(x+t)\}\int_0^\omega \left(1-\frac{s}{\omega}\right)^k \cos ts\,ds$$
$$= \frac{\omega}{\pi}\int_{-\pi-x}^{\pi-x} f(x+t)\,\gamma(\omega t)\,dt,$$

which establishes the theorem.

4. For $k > -1$, $$\int_0^1 (1-u)^k\,du$$

is absolutely convergent.

Also $$\cos tu = \sum_{n=0}^\infty (-)^n \frac{(ut)^{2n}}{(2n)!},$$

the series being, for each t, uniformly convergent in u in $(0, 1)$.

Thus by (3·2)

$$\gamma(t) = \sum_{n=0}^{\infty} (-)^n \frac{t^{2n}}{(2n)!} \int_0^1 (1-u)^k u^{2n}\, du$$

$$= \sum_{n=0}^{\infty} (-)^n \frac{t^{2n}}{(2n)!} \frac{\Gamma(k+1)\Gamma(2n+1)}{\Gamma(2n+k+2)}$$

$$= \frac{\Gamma(k+1)}{t^{k+1}} \sum_{n=0}^{\infty} (-)^n \frac{t^{2n+k+1}}{\Gamma(2n+k+2)}$$

$$= \frac{\Gamma(k+1)}{t^{k+1}} C_{k+1}(t),$$

where
$$C_p(t) = \sum_{n=0}^{\infty} (-)^n \frac{t^{p+2n}}{\Gamma(p+2n+1)}.$$

This last function has been investigated in detail* and many of its properties are well known.

For $t \geqslant 0$ and $0 \leqslant p \leqslant 2$, $C_p(t)$ is bounded, while for $p > 2$

$$C_p(t) \sim \frac{t^{p-2}}{\Gamma(p+1)}$$

as $t \to \infty$.

It follows that

(4·1)
$$|\gamma(t)| \leqslant \frac{A}{|t|^{k+1}} \quad (-1 < k \leqslant 1)$$

(4·2)
$$\gamma(t) \sim \frac{k}{t^2} \quad (k \geqslant 1),$$

where here, and in subsequent contexts, A is constant relative to the function considered.

Now $C_p(t)$ is defined as above for $p > -1$, and it is clear that

$$C_p(t) = \frac{t^p}{\Gamma(p+1)} - C_{p+2}(t).$$

Thus $C_p(t)$ is bounded for $t \geqslant 1$ and $-1 < p \leqslant 2$.

Moreover
$$\frac{d}{dt} C_{p+1}(t) = C_p(t),$$

and so
$$\frac{1}{\Gamma(k+1)} \gamma'(t) = -\frac{k+1}{t} \frac{C_{k+1}(t)}{t^{k+1}} + \frac{1}{t} \frac{C_k(t)}{t^k};$$

whence, for $t \geqslant 1$,

(4·3)
$$|\gamma'(t)| \leqslant \frac{A}{|t|^{1+\alpha}} \quad (k > -1)$$

where $\alpha = \min(k, 2)$.

* W. H. Young, *Quart. Journal of Maths.*, 43 (1912), 161—177.

https://doi.org/10.1017/S0305004100015541 Published online by Cambridge University Press

For small values of t, $\gamma(t)$ and all its differential coefficients are bounded, since

$$\gamma^{(r)}(t) = \int_0^1 (1-u)^k u^r \cos^{(r)} tu\, du \quad (r = 0, 1, 2, \ldots),$$

and therefore

$$|\gamma^{(r)}(t)| \leqslant \int_0^1 (1-u)^k u^r\, du = B(k+1, r+1).$$

This may be written

(4·4) $$|\gamma^{(r)}(t)| \leqslant A.$$

From (4·3) and (4·4) it immediately appears that

$$\int_0^\infty |\gamma'(t)|\, dt < \infty$$

provided that $k > 0$. That is, $\gamma(t)$ is of bounded variation in $(0, \infty)$ if $k > 0$.

It is also evident from (4·4) that $\gamma(t)$ is of bounded variation in any finite range for all $k > -1$.

5. When $k > -1$ the limits of $I_k(\omega)$ are identical with those of

$$\frac{\omega}{\pi} \int_{-\pi-x}^{\pi-x} f(x+t)\, \gamma(\omega t)\, dt.$$

If now η is any positive number less than both $\pi - x$ and $\pi + x$,

(5·1) $$\int_\eta^{\pi-x} f(x+t)\, \omega\gamma(\omega t)\, dt = \Big[F(x+t)\, \omega\gamma(\omega t)\Big]_\eta^{\pi-x} - \int_\eta^{\pi-x} F(x+t)\, \omega^2 \gamma'(\omega t)\, dt,$$

where $F(t)$ is an indefinite integral of $f(t)$, the integration by parts being justified since $\gamma(t)$ is of bounded variation in any finite range.

In the range of integration

$$|\omega\gamma(\omega t)| \leqslant \frac{A}{\omega^k \eta^{k+1}} \quad (-1 < k \leqslant 1)$$

$$\leqslant \frac{A}{\omega\eta^2} \qquad (k \geqslant 1)$$

by (4·1) and (4·2).

Take now $k > 0$, and consider the right-hand side of (5·1). The term between limits tends to zero with $\omega\gamma(\omega t)$ as $\omega \to \infty$.

For the integral, if (λ, μ) be any interval in $(\eta, \pi - x)$,

$$\int_\lambda^\mu \omega^2 \gamma'(\omega t)\, dt = \Big[\omega\gamma(\omega t)\Big]_\lambda^\mu \to 0,$$

so that, since $F(x+t)$ is continuous, by the Riemann-Lebesgue convergence theorem

$$\int_\eta^{\pi - x} F(x+t)\, \omega^2 \gamma'(\omega t)\, dt \to 0$$

as $\omega \to \infty$.

Thus $$\int_\eta^{\pi - x} f(x+t)\, \omega\gamma(\omega t)\, dt \to 0.$$

In like manner

$$\int_{-\pi - x}^{-\eta} f(x+t)\, \omega\gamma(\omega t)\, dt \to 0,$$

and the limits of $\Psi_k(\omega)$, for $k > 0$, are the same as those of

$$\frac{\omega}{\pi}\int_{-\eta}^{\eta} f(x+t)\, \omega\gamma(\omega t)\, dt.$$

It is known that

(5·2) $$\frac{1}{\pi}\int_0^\infty \gamma(t)\, dt = \tfrac{1}{2}\,*.$$

Thus, as $\omega \to \infty$, $$\frac{\omega}{\pi}\int_0^\eta \gamma(\omega t)\, dt \to \tfrac{1}{2}.$$

If then we write

$$\phi(t) = f(x+t) + f(x-t) - 2s$$

the necessary and sufficient condition for summability (C, k) of the integral (1·3) to the sum s at x takes the form

$$\frac{\omega}{\pi}\int_0^\eta \phi(t)\, \gamma(\omega t)\, dt \to 0.$$

By what appears above the limits of this last integral are independent of η, and the condition may be written

(5·3) $$\lim_{\eta \to 0}\ \overline{\lim_{\omega \to \infty}} \left| \omega \int_0^\eta \phi(t)\, \gamma(\omega t)\, dt \right| = 0 \quad (k > 0).$$

Write now $$\Phi(t) = \int_0^t \phi(u)\, du.$$

* W. H. Young, *Quart. Journal of Maths.*, 43 (1912), 166, 170.

Integration by parts gives

$$\omega \int_0^\eta \phi(t)\,\gamma(\omega t)\,dt = \Big[\omega\Phi(t)\,\gamma(\omega t)\Big]_0^\eta - \omega^2 \int_0^\eta \Phi(t)\,\gamma'(\omega t)\,dt$$

$$= \omega\Phi(\eta)\,\gamma(\omega\eta) - \omega^2 \int_0^\eta \Phi(t)\,\gamma'(\omega t)\,dt.$$

By (4·1) and (4·2), since $k > 0$,

$$\omega\gamma(\omega\eta) \to 0 \text{ as } \omega \to \infty.$$

Hence another form of the necessary and sufficient condition of summability to s at x is

$$\lim_{\eta\to 0} \overline{\lim_{\omega\to\infty}} \left| \omega^2 \int_0^\eta \Phi(t)\,\gamma'(\omega t)\,dt \right| = 0 \quad (k > 0). \tag{5·4}$$

6. For the purposes of investigation of the Fourier series of $f(x)$ we define a function $f_1(x)$ which is periodic in 2π, and coincides with $f(x)$ in the interval $(-\pi, \pi)$. Clearly the two functions have, relative to $(-\pi, \pi)$, the same Fourier series.

We proceed to examine the behaviour of $\sigma_k(\omega)$, the partial Rieszian sum of this series, as $\omega \to \infty$.

Write $$A_n(x) = a_n \cos nx + b_n \sin nx.$$

Then by (2·2) $$\sigma_k(\omega) = \tfrac{1}{2}a_0 + \sum_{1\leq n<\omega} \left(1 - \frac{n}{\omega}\right)^k A_n(x).$$

THEOREM II. *If* $k > 0$,

$$\sigma_k(\omega) - \tfrac{1}{2}a_0 = \frac{1}{\pi}\int_0^\infty \psi_1\left(\frac{t}{\omega}\right)\gamma(t)\,dt, \tag{6·1}$$

where $$\psi_1(t) = f_1(x+t) + f_1(x-t) - a_0.$$

Then, since $$\frac{1}{\pi}\int_0^\infty \gamma(t)\,dt = \tfrac{1}{2},$$

$\sigma_k(\omega)$ will have the limit s, for $k > 0$, if, and only if,

$$\int_0^\infty \phi_1\left(\frac{t}{\omega}\right)\gamma(t)\,dt \to 0,$$

or, what is equivalent,

$$\omega \int_0^\infty \phi_1(t)\,\gamma(\omega t)\,dt \to 0,$$

where $$\phi_1(t) = f_1(x+t) + f_1(x-t) - 2s.$$

https://doi.org/10.1017/S0305004100015541 Published online by Cambridge University Press

7. It is readily verified that $\psi_1(t)$ is periodic in 2π, even, and such that

$$\int_{-\pi}^{\pi} \psi_1(t)\,dt = 0,$$

so that
$$\Psi_1(t) = \int_0^t \psi_1(u)\,du$$

is a continuous function, periodic in 2π.

Now let $v(t)$ be any function of bounded variation in $(0, \infty)$ which tends to zero as $t \to \infty$. Then

$$\begin{aligned}\int_0^\infty \psi_1(t)\,v(t)\,dt &= \lim_{\lambda\to\infty}\int_0^\lambda \psi_1(t)\,v(t)\,dt \\ &= \lim_{\lambda\to\infty}\left\{\Big[\Psi_1(t)\,v(t)\Big]_0^\lambda - \int_0^\lambda \Psi_1(t)\,dv(t)\right\} \\ &= -\int_0^\infty \Psi_1(t)\,dv(t),\end{aligned}$$

since $\Psi_1(t)$, being continuous and periodic, is bounded in $(0, \infty)$.

If $S_n(t)$ is the nth partial Cesàro sum of the Fourier series of $\Psi_1(t)$,

$$S_n(t) \to \Psi_1(t)$$

uniformly in $(0, \infty)$ by the above property of $\Psi_1(t)$, and it follows immediately that

$$(7{\cdot}1)\quad \lim_{n\to\infty}\int_0^\infty - S_n(t)\,dv(t) = -\int_0^\infty \Psi_1(t)\,dv(t) = \int_0^\infty \psi_1(t)\,v(t)\,dt.$$

The Denjoy-Fourier series of $\psi_1(t)$ proves to be

$$\sum_{n=1}^\infty c_n \cos nt$$

where
$$c_n = 2A_n(x),$$

and it may be verified that

$$(7{\cdot}2)\qquad \int_0^\infty - S_n(t)\,dv(t) = \sum_{m=1}^{n-1}\left(1 - \frac{m}{n}\right) c_m\, v_m,$$

where
$$v_m = \lim_{\lambda\to\infty}\int_0^\lambda v(t)\cos mt\,dt.$$

This limit clearly exists.

https://doi.org/10.1017/S0305004100015541 Published online by Cambridge University Press

8. By (4·1) and (4·2), $\gamma(t) \to 0$ as $t \to \infty$. It is moreover of bounded variation in $(0, \infty)$ for $k > 0$. Thus we may replace $v(t)$ by $\frac{\omega}{\pi}\gamma(\omega t)$, and the coefficients are found to be

$$v_m = \frac{\omega\Gamma(k+1)}{\pi}\int_0^\infty \frac{C_{k+1}(\omega t)}{(\omega t)^{k+1}}\cos mt\,dt$$

$$= \frac{\Gamma(k+1)}{\pi}\int_0^\infty \frac{C_{k+1}(t)}{t^{k+1}}\cos\frac{m}{\omega}t\,dt$$

$$= \frac{1}{2}\left(1-\frac{m}{\omega}\right)^k \quad (m \leqslant \omega),$$

$$= 0 \qquad (m \geqslant \omega)$$

for $k > 0$*.

The series $\sum_{m=1}^\infty c_m v_m$

is therefore actually finite, and its sum by any positive Cesàro mean coincides with the actual sum. In view of (7·1), for the particular function $v(t)$ we have chosen, (7·2) now becomes

$$\sum_{m=1}^\infty c_m v_m = \int_0^\infty \psi_1(t)\, v(t)\, dt$$

or

$$\sum_{1 \leqslant n < \omega}\left(1-\frac{n}{\omega}\right)^k A_n(x) = \frac{\omega}{\pi}\int_0^\infty \psi_1(t)\,\gamma(\omega t)\,dt$$

$$= \frac{1}{\pi}\int_0^\infty \psi_1\left(\frac{t}{\omega}\right)\gamma(t)\,dt.$$

Hence (6·1) holds if $k > 0$, and Theorem II is proved.

The result established may be expressed as follows.

For $k > 0$, the necessary and sufficient condition that the Denjoy-Fourier series (1·1) should be summable (C, k) to s at a point x in $(-\pi, \pi)$ is that

(8·1) $$\omega\int_0^\infty \phi_1(t)\,\gamma(\omega t)\,dt \to 0$$

as $\omega \to \infty$.

9. Write (8·1) in the equivalent form

$$\frac{\omega}{\pi}\int_0^\infty \psi_1(t)\,\gamma(\omega t)\,dt \to s - \tfrac{1}{2}a_0,$$

or, using (7·1), since $\gamma(t)$ is of bounded variation in $(0, \infty)$ when $k > 0$,

$$-\frac{\omega^2}{\pi}\int_0^\infty \Psi_1(t)\,\gamma'(\omega t)\,dt \to s - \tfrac{1}{2}a_0 \qquad (k > 0).$$

* W. H. Young, *Quart. Journal of Maths.*, 43 (1912), 166.

If η is any fixed positive number, $\omega > \frac{1}{\eta}$, and M is the upper bound of $|\Psi_1(t)|$,

$$\left|\omega^2 \int_\eta^\infty \Psi_1(t)\,\gamma'(\omega t)\,dt\right| \leqslant M\omega^2 \int_\eta^\infty \frac{A}{(\omega t)^{\alpha+1}}\,dt \quad (\alpha = \min\{k, 2\})$$

$$= \frac{MA}{\omega^{\alpha-1}} \int_\eta^\infty \frac{dt}{t^{\alpha+1}}$$

$$\to 0 \text{ when } k > 1.$$

If $k=1$, the above shews that we may choose X such that

$$\left|\omega^2 \int_X^\infty \Psi_1(t)\,\gamma'(\omega t)\,dt\right| < \epsilon,$$

where ϵ is any previously assigned positive number.

And then, with the argument used in § 5, it may be shewn by Lebesgue's theorem that

$$\left|\omega^2 \int_\eta^X \Psi_1(t)\,\gamma'(\omega t)\,dt\right| \to 0$$

as $\omega \to \infty$.

Thus, when $k \geqslant 1$, the condition (8·1) becomes

$$-\frac{\omega^2}{\pi} \int_0^\eta \Psi_1(t)\,\gamma'(\omega t)\,dt \to s - \tfrac{1}{2}a_0$$

for fixed positive η.

Moreover,

$$\int_0^\infty t\gamma'(t)\,dt = \lim_{\lambda\to\infty} \int_0^\lambda t\gamma'(t)\,dt$$

$$= \lim_{\lambda\to\infty} \left\{\Big[t\gamma(t)\Big]_0^\lambda - \int_0^\lambda \gamma(t)\,dt\right\}$$

$$= -\frac{\pi}{2} \quad (k > 0),$$

by (4·1), (4·2) and (5·2), and the final form of the condition for summability of the series is

$$\text{(9·1)} \qquad \lim_{\eta\to 0} \overline{\lim_{\omega\to\infty}} \left|\omega^2 \int_0^\eta \Phi_1(t)\,\gamma'(\omega t)\,dt\right| = 0 \quad (k \geqslant 1).$$

Now $\phi_1(t)$ coincides with $\phi(t)$ for $x+t, x-t$ in $(-\pi, \pi)$. Thus (9·1) is equivalent to

$$\text{(9·2)} \qquad \lim_{\eta\to 0} \overline{\lim_{\omega\to\infty}} \left|\omega^2 \int_0^\eta \Phi(t)\,\gamma'(\omega t)\,dt\right| = 0 \quad (k \geqslant 1).$$

10. It is convenient at this stage to collect the results established. (5·3) and (5·4), (8·1) and (9·2) shew that for $k \geqslant 1$ the necessary and sufficient condition that the Denjoy-Fourier series and the Denjoy-Fourier integral of $f(x)$ shall be summable (C, k) to s at x is

$$\lim_{\eta \to 0} \overline{\lim_{\omega \to \infty}} \left| \omega^2 \int_0^\eta \Phi(t) \gamma'(\omega t)\, dt \right| = 0. \tag{10·1}$$

If $0 < k < 1$ the necessary and sufficient condition of summability of the integral is still (10·1), or it may be written in the equivalent form

$$\lim_{\eta \to 0} \overline{\lim_{\omega \to \infty}} \left| \omega \int_0^\eta \phi(t) \gamma(\omega t)\, dt \right| = 0, \tag{10·2}$$

whereas the condition for the series can in general take no more simple a form than

$$\lim_{\omega \to \infty} \omega \int_0^\infty \phi_1(t) \gamma(\omega t)\, dt = 0. \tag{10·3}$$

That (10·3) cannot always be reduced to (10·2), i.e. that the series and the integral are not summable (C, k) in the same way for $k < 1$, is best shewn by an example.

11. If the series

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is summable (C, k), where $k > 0$, at all points of a set of positive measure, then, since at these points

$$\frac{a_n \cos nx + b_n \sin nx}{n^k} \to 0,$$

it follows that $$\frac{a_n}{n^k} \to 0, \quad \frac{b_n}{n^k} \to 0$$

as $n \to \infty$ *.

Thus, if $\frac{a_n}{n^k}$, $\frac{b_n}{n^k}$ do not converge to zero,

$$\frac{a_n \cos nx + b_n \sin nx}{n^k}$$

can only converge to zero at points of a set of measure zero. In other words, unless

$$a_n = o(n^k), \quad b_n = o(n^k)$$

the series $$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

is summable (C, k) almost nowhere in any interval.

* E. W. Hobson, *Theory of functions of a real variable*, Vol. 2, 682.

12. Titchmarsh has shewn* that if $\lambda(n)$ be positive and decrease steadily to zero as $n \to \infty$, then however slowly $\lambda(n) \to 0$ there is a function whose Fourier coefficients are such that

$$a_n \neq o\{n\lambda(n)\}, \quad b_n \neq o\{n\lambda(n)\}.$$

To construct such a function we proceed as follows.

Let $T(x)$ vanish outside the interval $(-\pi, \pi)$. In the interval write

$$T(x) = p^2 n_p \lambda(n_p) \sin n_p x \quad \left(\frac{\pi}{p+1} < x \leqslant \frac{\pi}{p}\right),$$

where $p = 1, 2, 3, \ldots$ and n_p is an integer, depending on p, which tends to ∞ so rapidly that $n_p > 2n_{p-1}$ and $\Sigma p^2 \lambda(n_p)$ is convergent,

$$T(0) = 0,$$
$$T(-x) = -T(x).$$

$T(x)$ has a single point of non-summability in $(-\pi, \pi)$, namely the origin. It is Denjoy integrable in $(-\pi, \pi)$.

As Titchmarsh shews,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} T(x) \sin nx\, dx \neq o\{n\lambda(n)\}.$$

Thus, for any $\mathbf{k}$, $0 < \mathbf{k} < 1$, we can so arrange $\lambda(n)$ as to make

$$b_n \neq o(n^{\mathbf{k}}),$$

and the Denjoy-Fourier series of $T(x)$ will be summable (C, k), $0 \leqslant k \leqslant \mathbf{k}$, almost nowhere in the interval $(-\pi, \pi)$. The origin is an exceptional point where the series vanishes.

On the other hand, if $k > 0$, the Denjoy-Fourier integral of $T(x)$ is summable (C, k) everywhere in the interval. For it is easily seen that, unless $x = 0$, by a classical argument, since $T(x)$ is absolutely integrable in any interval to which the origin is exterior,

$$\lim_{\eta \to 0} \overline{\lim_{\omega \to \infty}} \left| \omega \int_0^{\eta} (T(x+t) + T(x-t) - 2s)\, \gamma(\omega t)\, dt \right| = 0 \quad (k > 0),$$

where $$s = \frac{T(x+0) + T(x-0)}{2},$$

and at the origin the integral is convergent to zero.

* E. C. Titchmarsh, *Proc. Lond. Math. Soc.*, 22 (1924), Records XXV.