Macroeconomic Dynamics, **1**, 1997, 740–769. Printed in the United States of America.

VINTAGE ARTICLE

SELF-FULFILLING PROPHECIES AND THE BUSINESS CYCLE

ROGER E.A. FARMER
University of California at Los Angeles

MICHAEL WOODFORD
Princeton University

We demonstrate that multiple stationary rational-expectations equilibria exist in a version of Lucas's island economy. The existence of these equilibria follows from the fact that there is an indeterminate set of monetary equilibria in the two-period overlapping-generations model. We show how to construct stationary rational-expectations equilibria by randomizing over the set of nonstationary monetary equilibria. In some of our equilibria, a positively sloped Phillips curve exists even though our economy contains no signal-extraction problem as in the original Lucas paper. Our equilibria are indexed by beliefs and are examples of the existence of sunspot equilibria in which allocations may differ across states of nature for which preferences, technology, and endowments are identical. Our technique for constructing stationary sunspot equilibria should prove useful in a wide class of models in which an indeterminate stationary equilibrium exists.

Keywords: Indeterminacy, Sunspots, Dynamic General Equilibrium, Rational Expectations

1. PREAMBLE

We wrote this piece in 1984, when Farmer was an assistant professor at the University of Pennsylvania and Woodford was visiting Penn, having recently completed his doctorate at the Massachusetts Institute of Technology. At that time, the idea of a sunspot equilibrium was much discussed at the University of Pennsylvania, but was regarded by most macroeconomists as an esoteric branch of general equilibrium theory of little obvious importance for macroeconomics. The examples of sunspot equilibria that were known at the time were highly stylized and involved, for example, only a finite number of random states.

The contribution of our paper was to exploit the indeterminacy of equilibrium, a phenomenon that is common in overlapping-generations models, to construct sunspot equilibria in a version of the model introduced by Robert Lucas (1972). The technique that we describe for constructing sunspot equilibria has proven to be

This paper was originally circulated in 1984 as CARESS working paper 84-12. Address correspondence to: Roger E.A. Farmer, Department of Economics, University of California at Los Angeles, 405 Hilgard Avenue, Los Angeles, CA 90024, USA; e-mail: rfarmer@econ.ucla.edu.

 1365-1005/97 $9.00 + .10

widely applicable to intertemporal general equilibrium models with indeterminacy, and has since been exploited in a variety of different environments, including many models with infinite-lived agents, some of which are quite similar to the kind of stochastic growth model used in real-business-cycle theory.

The equilibria constructed using this method are described by stationary probability distributions that fulfill all of the desiderata of rational choice in a stationary environment, but in which the beliefs of agents are no longer uniquely determined by economic fundamentals. In the context of our example, we show that there is a continuum of alternative stationary rational-expectations equilibria, and that some of these equilibria have Keynesian features, in the sense that quantities rather than prices respond to monetary shocks. We do not take up the question of the quantitative similarity of the predicted stationary fluctuations to the properties of actual aggregate data, although our methods would allow such questions to be addressed, as they have been more recently, in the context of more sophisticated models.

Our purpose in publishing the article here is to provide a convenient English-language source. Although the paper is widely cited and a Spanish translation was published in 1987, it has never been published in English.

2. INTRODUCTION

In this paper, we demonstrate the existence of many stationary rational-expectations equilibria in a variant of the model considered by Lucas (1972). Several authors [e.g., Taylor (1977), Shiller (1978), Futia (1981), Gourieroux et al. (1982), Sargent and Wallace (1984)] have observed that linear rational expectations models may possess a continuum of stationary equilibria, but the hope has been expressed that the multiplicity is possible only because these models are not derived rigorously from the optimizing behavior of agents [e.g., Taylor (1977, pp. 1378, 1383); Gourieroux et al. (1982, pp. 424–25)]. However, a growing literature shows that this is not the case. For example, it is known that continua of nonstationary perfect-foresight equilibria may exist in overlapping-generations models [e.g., Calvo (1978), Wallace (1980)], that a large number of periodic perfect-foresight equilibria may exist in such models [Grandmont (1985)], that a large number of stochastic rational-expectations equilibria may exist in these models, even when there is nothing random about preferences, endowments, or technology [Shell (1977), Peck (1984)], and indeed that such models may possess many stationary rational expectations equilibria [Azariadis (1981), Azariadis and Guesnerie (1982, 1986)]. This suggests that the results of Hansen and Sargent (1980), that rational-expectations equilibrium is unique for a class of economies in which a single agent solves a linear-quadratic optimization problem, is critically dependent upon the assumption that the agent is infinite-lived.

Our paper demonstrates a method for constructing large classes of stationary rational-expectations equilibria in exact optimizing models. We hope that this technique will clarify the importance of the aforementioned developments for business-cycle theory. Nonstationary equilibria may not be considered interesting models of the business cycle both because it is the repetitive character of these phenomena that has attracted most attention and because, arguably, the rational-expectations hypothesis is most plausible in a stationary context. The multiple stationary equilibria discussed by Azariadis, Guesnerie, and Grandmont avoid this objection, but the techniques used by these authors allow them to display only stationary equilibria of very special kinds—deterministic cycles on the one hand and two-state Markov processes on the other. The stationary equilibria that we display, by contrast, are of the stochastic autoregressive form, which is characteristic of econometric work on the business cycle. Our results have several important implications for current discussions of rational expectations in macroeconomic models.

A first implication concerns the discussion of the neutrality of money in the Lucas (1972) paper. A single stationary rational-expectations equilibrium is discussed in that paper, and it has the property that, in the absence of informational asymmetries, monetary shocks (stochastic interest payments on money balances) would have no effect on output. Hence, informational asymmetries are proposed as an explanation of the Phillips-curve correlation between inflation and output. In our model, by contrast, it is shown that even in the absence of any imperfect information, almost all of the large class of stationary equilibria display a nonvertical Phillips-curve relationship. In some of these equilibria, output is positively correlated with inflation; in others, the correlation is negative. In both cases, the correlation exists in a stationary state because of the self-fulfilling character of expectations.

A second implication concerns the incidence of an inflation tax used as a means of financing government expenditure. It often is supposed that alternative methods of government finance can result in different allocations across generations of the burden of paying for a given government expenditure. However, our model, shows that, even given a particular method of government finance (e.g., an inflation tax), the integenerational allocation of the burden may be indeterminate.

A final implication concerns the long-standing debate over the possibility of sticky nominal wages. Previous authors have assumed that predetermined nominal wages indicate that agents must commit themselves to particular terms of employment before the realization of an aggregate demand shock. We show, instead, that there may exist a stationary rational-expectations equilibrium in which nominal wages are predetermined, even if all labor is traded in a competitive spot market after all stochastic shocks have become common knowledge. This result, like the others, results from the existence in our model of self-fulfilling price expectations, and hence may be seen as a demonstration that the possibly self-fulfilling nature of expectations noted by Cass and Shell (1983) should be taken seriously by macroeconomists.

3. MODEL

Our economy consists of a sequence of overlapping generations, each of which lives for two consecutive periods. All individuals in all generations are assumed to have the same preferences, and so, without loss of generality, we refer to a single representative agent from each generation. We refer to the generation that lives in periods t and $t+1$ as generation t. There is a single perishable consumption good each period, produced at constant returns to scale through the labor of the young agents in that period. During the first period of life, each member of generation t chooses to supply n_t units of labor, which yield n_t units of output. It is assumed that individuals wish to consume only during the second period of life.[1] Let the consumption by each member of generation t be c_{t+1}. Preferences are described by the utility function

$$u(n_t, c_{t+1}) = c_{t+1} - \tfrac{1}{2}n_t^2 \tag{1}$$

for the members of each generation. As is discussed in the Appendix, our results do not depend on this special utility function, which we choose for computational convenience.

In addition to labor output, there is one other good: fiat money, issued by the government. This money enters the economy in two ways. The government purchases some of the consumption good each period, in the amount g_t in period t, and pays for these purchases by issuing new money.[2] The government also makes beginning-of-period money transfers to the members of the older generation in each period, in a quantity proportional to the pretransfer holdings of each. If we let m_t denote the money supply per member of generation t at the end of period t, then the posttransfer balances held by each member of that generation at the beginning of period $t+1$ are $m_t x_{t+1}$. Because the consumption good is perishable, all savings are achieved by holding fiat money. It is assumed that no individuals care about their descendants, so that there is no inheritance; fiat money holdings at the beginning of the second period of life are entirely spent on consumption during that period.

Our model simplifies Lucas's model in two important respects. First, we assume that all exchange takes place in a single competitive spot market each period; there is only one "island." Second, we assume that all individuals know the value of the government policy variables g_t and x_t in period t; there is no asymmetric information. We discard these features of the Lucas model, because we wish to show that the Lucas mechanism, involving misunderstanding of the significance of price-level fluctuations on a given island, is not necessary in order for a short-run Phillips-curve relation to be consistent with rational expectations.

Let P_t be the money price of the consumption good in period t. Then, young individuals in period t supply labor n_t so as to maximize the expected value of (1), subject to the budget constraints

$$m_t < n_t P_t, \tag{2}$$

$$P_{t+1} c_{t+1} < m_t x_{t+1}. \tag{3}$$

In equilibrium, conditions (2) and (3) will hold with equality, and n_t will satisfy the first-order condition[3]

https://doi.org/10.1017/S1365100597005051 Published online by Cambridge University Press

$$n_t = P_t E_t \left[\frac{x_{t+1}}{P_{t+1}} \right], \tag{4}$$

where E_t denotes expectation conditional upon information available to all individuals in period t. The national income accounting identity in period t will be

$$c_t + g_t = n_t. \tag{5}$$

Substituting (2), (3), and (5) into (4) yields

$$n_t^2 = E_t[n_{t+1} - g_{t+1}], \tag{6}$$

which must hold each period in any rational-expectations equilibrium.

We assume that the government policy variables x_t and g_t are chosen randomly each period, and that these variables are both independent and identically distributed (i.i.d.) across periods. That is, there is a cumulative distribution function $F(x)$ such that x_t is drawn independently from this distribution each period, and a cumulative distribution function $H(g)$ such that g_t is drawn independently from this distribution each period, and the stochastic processes for x and g are independent of each other. We also assume that $F(x)$ has bounded support $[a, b]$, where $a > 0$ that $H(g)$ has bounded support $[c, d]$ where $c > 0$ and both distributions are common knowledge.

We then define stationary rational-expectations equilibrium in the following manner:

DEFINITION. *An equilibrium price function is a continuous positive function $\phi(n, x, g)$ bounded and bounded away from zero, defined for n in some bounded nonnegative interval, $x \in [a, b]$ and $g \in [c, d]$ such that*

(i) *$P_{t+1} = m_t \phi(n_t, x_{t+1}, g_{t+1})$ is a market clearing price in period $t + 1$, given individuals' expectations that future prices will continue to obey this rule; that is,*

$$n^2 = \iint \frac{x}{\phi(n, x, g)}\, dF(x)\, dH(g). \tag{7}$$

(ii) *There exists an invariant distribution for n, for the stochastic process for n implied by the price function ϕ; that is, there exists a cumulative distribution function, $\pi(n)$ with bounded support $[e, f]$ where $e > 0$ such that*

$$\pi(n') = \iiint 1_{n'} \left[\frac{x}{\phi(n, x, g)} + g \right] dF(x)\, dH(g)\, d\pi(n), \tag{8}$$

where $1_{n'}$ is the indicator function for the set $[e, n']$.[4] *Equation (7) states that the stationary stochastic process for n implied by the price function ϕ, that is,*

$$n_{t+1} = \frac{x_{t+1}}{\phi(n_t, x_{t+1}, g_{t+1})} + g_{t+1}, \tag{9}$$

always satisfies condition (6). Equation (8) states that if $\pi(n)$ describes the probability distribution for n_t in any period, the derived probability distribution for n_{t+1} using equation (9) also will be $\pi(n)$.

https://doi.org/10.1017/S1365100597005051 Published online by Cambridge University Press

This definition is more general than that considered by Lucas, and it accounts for the multiplicity of stationary rational-expectations equilibria that we find. Hence, the generalization demands some discussion. Lucas allows only for price functions of the form $\phi(n, x) = \phi(x)$, and demands only that (7) be satisfied. The existence of an invariant distribution for n is guaranteed in that case, because it is derived from the distribution $F(x)$ using the relation $n = x/\phi(x)$. It is our contention that any predetermined variable might reasonably be included as an argument of the equilibrium price function. It is this restriction that allows Lucas to derive his classical neutrality of money theorem, namely, that in the absence of a signal extraction problem, the unique rational-expectations equilibrium is of the form $P_{t+1} = m_{t+1}/\bar{n}$, where $\bar{n}$ is a constant rate of labor supply (the natural rate), so that even the short-run Phillips curve must be vertical.

Like Lucas, we restrict our attention to *stationary* rational-expectations equilibria. By this, we mean not only that the equilibrium price function does not depend on time, but also that there exist invariant distributions for all real variables in the model. This restriction is reasonable, because there is no good argument why individuals should have rational expectations except in such a stochastic stationary state.

It is customary, for example, in the literature on linear rational-expectations models, to consider as equilibria those linear functions

$$P_t = \sum_{j=0}^{\infty} a_j e_{t-j},$$

where P_t is the endogenous variable in the model and e_t is the innovation in period t of the exogenous variable, satisfying the square-summability condition

$$\sum_{j=0}^{\infty} |a_j|^2 < \infty$$

and the relevant equilibrium condition. [See, e.g., Lucas (1975), Futia (1981).] Attention is not restricted to functions in which the endogenous variable depends only on the current value of the exogenous shock; instead, it may depend on the entire history of shocks. On the other hand, one usually restricts one's attention to stationary solutions, both in the sense that the coefficients a_j do not depend upon time, and in the sense that there exists an invariant distribution with finite variance for the endogenous variable (this is the reason for the square-summability condition). It is our supposition that Lucas considers only price functions of the form $\phi(n, x) = \phi(x)$ as a way of ensuring that his solution will be stationary in this sense. We therefore regard our generalization of the definition as a natural one. It is shown below that not all equilibria satisfying our more general definition are of the Lucas type.

The considerations just mentioned give us no reason to restrict our attention to price functions of the form $\phi(n_t, x_{t+1}, g_{t+1})$. This amounts to an assumption that the past history of government policy affects current prices only through the quantity of output in the previous period. More general functional forms might reasonably be considered. However, we show below that even when we restrict our attention to this class of functions, an extremely large number of stationary rational-expectations equilibria exist. Nor have we any reason to require that the invariant distribution $\pi(n)$ has bounded support. However, certain proofs having to do with the existence of and the convergence to the invariant distribution are simpler in this case, and we show below that equilibria in which $\pi(n)$ has bounded support do exist in the case of the model considered here.

4. STOCHASTIC INTEREST PAYMENTS

In this section, we consider a particular case of the model set out above, that in which real government expenditure is equal to a constant, $g > 0$. The only stochastic shocks then are stochastic interest payments on money, as in Lucas's paper. The case that we treat in this section thus differs from Lucas's model in only one respect: We assume $g > 0$, rather that $g = 0$. This is an important stipulation.

If one sets $x = 1$ for all time (no interest payments), so that there is nothing stochastic in the model at all, and looks at the set of perfect-foresight equilibria, one finds an important difference between the cases $g = 0$ and $g > 0$. In the case of no government expenditure, the model is an example of what Gale (1973) calls "the Samuelson case." As is well known, there are two stationary equilibria in such a model, a monetary steady state in which labor supply (and hence real balances) each period equals $\bar{n}$ (see Figure 1A), and a nonmonetary steady state in which money is not valued and no labor is supplied in any period. In addition, there is a one-parameter family of perfect-foresight equilibria in which money is initially valued, but asymptotically comes to have zero value. In these equilibria the labor supply approaches zero as the rate of inflation accelerates. (See Figure 1A for the construction of a member of this class.) In the case that $g > 0$, by contrast (see Figure 1B), there are two monetary steady states. This is because of the familiar proposition that there are two rates of money creation that will support a given level of inflation-financed government expenditure [Bailey (1956), Sargent and Wallace (1984)].[5] As in the $g = 0$ case, one steady state is determinate and the other is indeterminate[6]; that is, there is a one-parameter family of perfect-foresight equilibria converging asymptotically to the high-inflation steady state (corresponding to labor supply $\underline{n}$ in Figure 1B), whereas the only perfect-foresight equilibrium converging to the low-inflation steady state (labor supply $\bar{n}$) is the equilibrium that begins with that rate of inflation. The large family of stationary rational-expectations equilibria that we find in the case of stochastic government policy are all in the vicinity of the indeterminate monetary steady state (in a sense to be displayed shortly); and the existence of an indeterminate monetary steady state is crucial to the method we use to construct our examples of multiple stationary rational-expectations equilibria.[7] This is the reason for the assumption that $g > 0$.

We assume in this section that the stochastic interest factor, x_t, is i.i.d., being drawn each period from a distribution $F(x)$ with bounded support $[a, b]$ where

https://doi.org/10.1017/S1365100597005051 Published online by Cambridge University Press

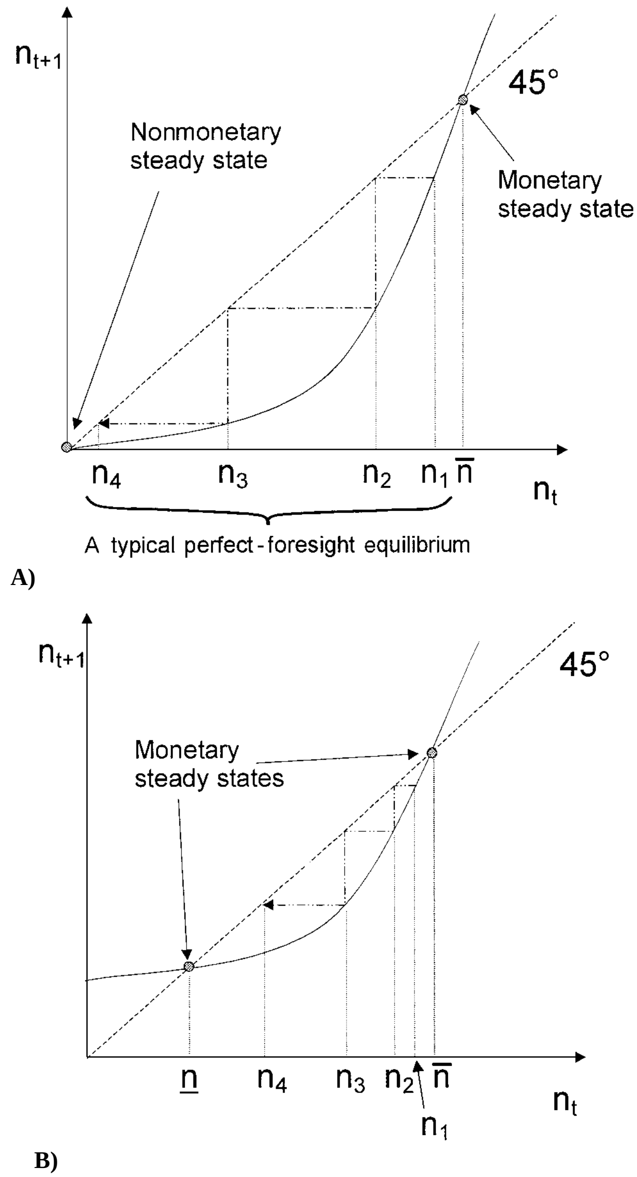


FIGURE 1. Monetary and nonmonetary steady states.

https://doi.org/10.1017/S1365100597005051 Published online by Cambridge University Press

$a < 1 < b$, and with mean 1. It is evident from considerations of neutrality that there is no loss of generality in the assumption that $E(x) = 1$. In this variant of the Lucas model, with our generalized definition of equilibrium, a very large number of stationary rational-expectations equilibria exist. As a preliminary indication of the richness of possibilities, consider the following one-parameter family of equilibria.[8]

THEOREM 1. *In the case that government expenditure g is constant each period, and satisfies $0 < g < \frac{1}{4}$, each member of the one-parameter family of price functions*

$$\phi(n, x; \lambda) = \frac{x}{[1 + \lambda(x - 1)]n^2} \tag{10}$$

for λ in the interval

$$\max\left\{-\frac{1/(4-g)}{(1-a)g}, -\frac{1}{b-1}\right\} < \lambda < \min\left\{\frac{\frac{1}{4}-g}{(b-1)g}, \frac{1}{1-a}\right\} \tag{11}$$

corresponds to a stationary rational-expectations equilibrium.

The family of solutions referred to in Theorem 1 is constructed by taking combinations of two particular rational-expectations equilibria to which we are able to attach economic significance. One of these solutions we refer to as a predetermined-quantity solution and the other we refer to as a predetermined-price solution. We first describe the predetermined-quantity solution. For convenience, we restate the fundamental equilibrium condition (6):

$$n_t^2 = E_t[n_{t+1} - g].$$

It is also convenient to remember the relationship between employment and end-of-period real balances:

$$n_{t+1} - g = (x_{t+1} m_t)/P_{t+1}. \tag{12}$$

Note that an obvious candidate for a solution to (6) is given by

$$n_t^2 = n_{t+1} - g. \tag{13}$$

We refer to any solution of this form as a predetermined-quantity solution because it has the property that n_{t+1} is independent of current realizations of x_{t+1}. If n_t obeys (13), then from (12) it follows that prices are described by

$$P_{t+1} = (x_{t+1} m_t)\big/n_t^2. \tag{14}$$

It follows from our definition of equilibrium that any stationary value of n that satisfies (13) will generate a rational-expectations equilibrium price function of the form

$$\phi(n, x) = x/n^2. \tag{15}$$

https://doi.org/10.1017/S1365100597005051 Published online by Cambridge University Press

Equation (13) has two stationary solutions that correspond to the nonstochastic equilibria $\underline{n}$ and $\overline{n}$ depicted in Figure 1B. There are, therefore, two predetermined-quantity equilibria, both of which are such that current monetary noise has no real effects. It is the low-inflation equilibrium, $n = \overline{n}$ that is isolated by Lucas as the unique rational-expectations equilibrium in his model. The Lucas equilibrium is unique because if $g = 0$, then in the other steady state, $n = \underline{n} = 0$, and money has no value.

Predetermined-quantity solutions, however, are not the only candidates for equilibrium. Combining equations (6) and (12), it follows that

$$n_t^2 = E_t[(x_{t+1}m_t)/P_{t+1}]. \quad \textbf{(16)}$$

Consider solutions in which P_{t+1} is independent of x_{t+1}.[9] In this case, (16) implies

$$n_t^2 = (m_t/P_{t+1})E_t[x_{t+1}] = (m_t/P_{t+1}),$$

and so, the equilibrium price function must be of the form

$$\phi(n, x) = (1/n^2). \quad \textbf{(17)}$$

In an equilibrium such as this, output would follow the stochastic difference equation,

$$n_{t+1} = x_{t+1}n_t^2 + g. \quad \textbf{(18)}$$

In the basic Lucas model, equation (18) is unstable in the neighborhood of the unique monetary stationary state. If g is positive, however, there may exist a neighborhood of the high-inflation steady state to which solutions of (18) converge (locally). This fact allows us to prove that there exists an invariant probability distribution for output.

The family of solutions that is described in Theorem 1 is constructed by taking linear combinations with weight λ of the predetermined-quantity solution and the predetermined-price solution. We prove Theorem 1 below.

Proof of Theorem 1. Note that, under the assumptions on a, b, and g made above, the interval (11) is nonempty and includes $\lambda = 0$ (the case of pure neutrality of money). Substitution of (10) into (7) reveals that the equilibrium condition is satisfied by every price function in family (10). It remains to show that, under the stochastic dynamics defined by (9), there is an invariant distribution $\pi(n)$ for the labor-supply variable. The result that we need in order to show this is the following:

LEMMA 1. *Consider the Markov process for random variable $\{n_t\}$ defined by the difference equation*

$$n_{t+1} = h(n_t, x_{t+1}),$$

where x_{t+1} is an i.i.d. r.v., and h is a continuous function of n, for n in some bounded interval $[e, f]$, and all values of x. Suppose furthermost that $h(n, x) \in [e, f]$, for any $n \in [e, f]$, and for any x. Then there exists an invariant distribution $\pi(n)$ for random variable $\{n_t\}$ with support $[e, f]$.

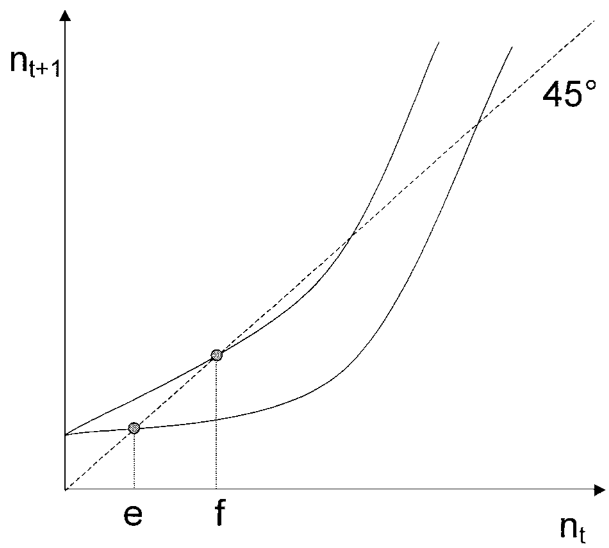


Figure 2. Geometry of the construction of the interval $[e, f]$.

For proof of this lemma, see Futia (1982).[10] Substituting (10) into (9) then yields

$$n_{t+1} = [1 + \lambda(x_{t+1} - 1)]n_t^2 + g. \quad \textbf{(19)}$$

It is clear that this function $h(n, x)$ maps a bounded interval $[e, f]$ into itself. Adopting the notation

$$u \equiv \max_{a \le x \le b} [1 + \lambda(x - 1)],$$

$$v \equiv \min_{a \le x \le b} [1 + \lambda(x - 1)],$$

it is obvious that (11) guarantees

$$0 < v < u < (1/4g).$$

It then follows that the interval $[e, f]$ is given by

$$e = \frac{2g}{1 + \sqrt{1 - 4vg}}, \quad \textbf{(20)}$$

$$f = \frac{2g}{1 + \sqrt{1 - 4ug}}. \quad \textbf{(21)}$$

(The geometry is displayed in Figure 2.) Hence, by Lemma 1, an invariant distribution $\pi(n)$ exists. ■

Theorem 1 guarantees the existence of a continuum of stationary rational-expectations equilibria, indexed by values of λ, in interval (11). One member

https://doi.org/10.1017/S1365100597005051 Published online by Cambridge University Press

of this family, corresponding to $\lambda = 0$, is the predetermined-quantity solution discussed above. In this case, the interval $[e, f]$ collapses to the single point $\underline{n}$ in Figure 1B. Hence, in the stochastic stationary state, labor supply is always equal to $\underline{n}$ (the natural rate) and monetary shocks have no effect on allocations.

On the other hand, in the case that $\lambda = 1$, it is instead P_{t+1} that is predetermined. Therefore, a monetary shock in period $t + 1$ has *no effect* on the price level in that period, but it does have real effects: The labor supply in period $t + 1$ is given by equation (18) so that an unexpectedly large increase in the money supply results in an unexpectedly large output in the period in which it occurs. (Hence the realized value of x_{t+1} affects both m_{t+1} and n_{t+1}, and through them future price levels, beginning with P_{t+1}.) In general, an increase in the money supply may result in a one-for-one increase in the current price level (the $\lambda = 0$ case), a more than one-for-one increase (the $\lambda < 0$ case), partial increase (the cases $0 < \lambda < 1$), no change at all in the current price level (the $\lambda = 1$ case), or even a decrease in the current price level (the $\lambda > 1$ cases). In Lucas's paper, only the first type of response is consistent with rational expectations, but we have shown that stationary rational-expectations equilibria that exhibit each of these types of response can exist.

Note that Theorem 1 requires no assumptions about the distribution $F(x)$, except that it has bounded support $[a, b]$. Under quite weak assumptions on the distribution of stochastic interest payments, we can prove an even stronger result about the invariant distribution $\pi(n)$.

THEOREM 2. *Suppose that, in addition to the assumptions of Theorem* 1, *the cumulative probability* $F(x)$ *is strictly monotonic in* x; *that is,* $F(x_1) > F(x_2)$ *if* $a < x_2 < x_1 < b$. *Then, the invariant distribution* $\pi(n)$ *with support* $[e, f]$ *is unique. Furthermore, there exist positive constants* M *and* ε *such that*

$$|P^t(n', n) - \pi(n)| < \frac{M}{(1+\varepsilon)^t}$$

for any $n', n \in [e, f]$, *and all* $t > 1$, *where*

$$P^t(n', n) \equiv \text{Prob}[n_t < n|_{n_0} = n'].$$

The additional assumption rules out the existence of subintervals of $[a, b]$ in which x occurs with zero probability. This guarantees not only uniqueness of the invariant probability distribution $\pi(n)$ but that rational expectations about the distribution of n at some future date converge at a geometric rate to the distribution $\pi(n)$ as the date is moved further into the future, regardless of the current value of n. This convergence property is characteristic of familiar examples of stationary rational-expectations equilibrium, and might be thought by some to be a necessary part of a reasonable definition of stationary rational-expectations equilibrium.

The proof of Theorem 2 depends on the following lemma, proved in Futia (1982).[11]

LEMMA 2. *The invariant distribution* $\pi(n)$ *in Lemma* 1 *is unique if and only if there is a point* $n^* \in [e, f]$ *such that for any open interval* $(n_1, n_2) \subset [e, f]$

containing n^*, and any $n' \in [e, f]$, there exists an integer t such that $P^t(n', n_2) > P^t(n', n_1)$. The convergence property stated in Theorem 2 holds if there is a point $n^* \in [e, f]$ such that for any open interval $(n_1, n_2) \subset [e, f]$ containing n^* any $n' \in [e, f]$, and any integer, $k > 1$, there exists an integer t such that $P^{kt}(n', n_2) > P^{kt}(n', n_1)$.

It is not clear that the additional assumption made in Theorem 2 is necessary; we do not know of a counterexample in which that assumption is not made. However, the proof of Theorem 2 is particularly simple when the additional assumption is made.

Proof of Theorem 2. Let

$$h_1(n) \equiv \min_{a \le x \le b} h(n, x),$$

$$h_2(n) \equiv \max_{a \le x \le b} h(n, x)$$

for $e < n < f$. It follows immediately from the strict monotonicity of $F(x)$, and the fact that in our example $h(n, x)$ is a continuous function of x, that $P(n', n)$ is strictly monotonic in n, for n in the interval $h_1(n') < n < h_2(n')$. Similarly, it follows from the expression

$$P^t(n', n) \iint_{h(\tilde{n}, x) \le n} dP^{T-1}(n', \tilde{n})\, dF(x)$$

that if $P^{t-1}(n', n)$ is strictly monotonic in n, for n in the interval $h_1^{t-1}(n') < n < h_2^{t-1}(n')$, that $P^t(n', n)$ will be strictly montonic in n, for n in the interval $h_1^t(n') < n < h_2^t(n')$. It follows, by induction, that $P^t(n', n)$ is strictly monotonic in n, for n in the interval $h_1^t(n') < n < h_2^t(n')$. Now note that the function $h(n, x)$ given in (19) has the property that $h_1(n) < n$ for all $e < n \le f$ and $h_2(n) > n$ for all $e \le n < f$. It follows from this that $h_1^t(n)$ converges uniformly to the constant function e as $t \to \infty$, and that $h_2^t(n)$ converges uniformly to the constant function f as $t \to \infty$. Hence, there is a finite T such that

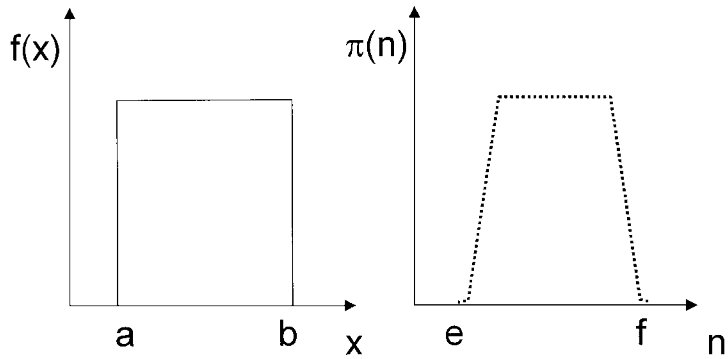
$$\max_{e \le n \le f} h_1^T(n) < \min_{e \le n \le f} h_2^T(n).$$

Let n^* lie in the open interval

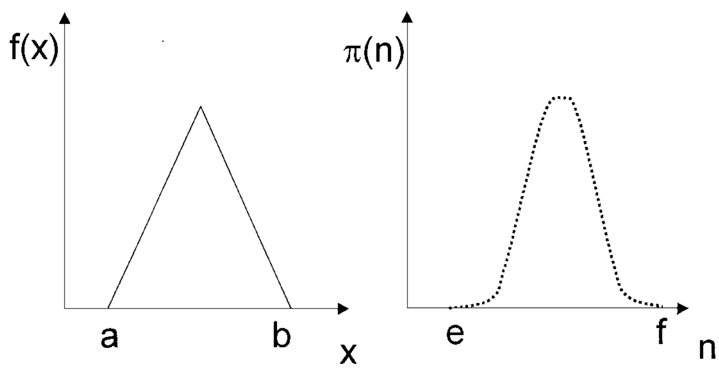
$$\max_{e \le n \le f} h_1^T(n) < n^* < \min_{e \le n \le f} h_2^T(n).$$

Then, $P^T(n', n)$ is monotonically increasing in n for n in a certain neighborhood of n^* for any $n' \in [e, f]$ for any $t < T$. Then, for any open interval (n_1, n_2) containing n^*, any n', and any integer $k > 1$, there exists an integer t such that $P^{kt}(n', n_2) > P^{kt}(n', n_1)$. Theorem 2 then follows from Lemma 2. ∎

Theorem 2 allows us to compute the invariant distribution $\pi(n)$, and likewise the invariant distributions of all other real variables, by simply integrating the



A) The case of a uniform shock distribution



B) The case of a triangular shock distribution

Figure 3. Invariant distributions for n corresponding to different distributions of the monetary shock.

Markov dynamics (9). Invariant distributions $\pi(n)$, corresponding to two different distributions of monetary shocks $F(x)$ calculated in this way are displayed in Figures 3A and 3B.

5. PHILLIPS CURVES

One purpose of the original paper by Lucas was to explain the observed correlation between employment and inflation in a model in which all agents displayed rational choice. In Lucas's paper, in the absence of uncertainty about the distribution of agents across islands, output is always constant regardless of monetary shocks; even the short-run Phillips curve is vertical. Hence, he finds it necessary to introduce informational asymmetries, to produce an explanation of the observed Phillips-curve relation. However, in almost all of the stationary rational-expectations equilibria displayed in Theorem 1, an econometrician making observations upon the stochastic stationary state would find a nonvertical Phillips curve,

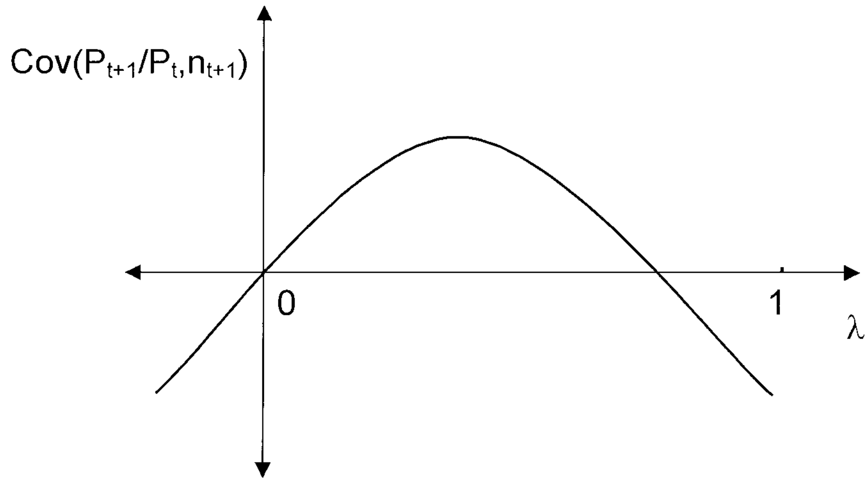


FIGURE 4. Covariance of P_{t+1}/P_t with n_t for different values of λ.

even though there is no informational asymmetry, and all money growth occurs through interest payments on money balances.

In any of the equilibria described in Theorem 1, the current rate of inflation, in period $t+1$, is equal to

$$\frac{P_{t+1}}{P_t} = \frac{x_{t+1}}{[1+\lambda(x_{t+1}-1)]n_t},$$

while the current labor supply, as a function of n_t and x_{t+1}, is equal to

$$n_{t+1} = [1+\lambda(x_{t+1}-1)]n_t^2 + g.$$

Because the distribution of x_{t+1} is independent of the value of n_t, the covariance of these two quantities, in the stochastic stationary state, is

$$\operatorname{cov}\left(\frac{P_{t+1}}{P_t}, n_{t+1}\right) = E(n) - E\left[\frac{x}{1+\lambda(x-1)}\right] E\left(\frac{1}{n}\right) E(n^2), \qquad \textbf{(22)}$$

where $E(n)$, $E(1/n)$, and $E(n^2)$ are evaluated using the invariant distribution $\pi(n)$, and $E[x/(1-\lambda(x-1))]$ is evaluated using the distribution $F(x)$. The general features of the variation of this covariance with variations in λ are illustrated by Figure 4.

It can be shown that[12] the family of rational-expectations equilibria allowed by (11) necessarily includes open-interval λ values for which the conventional Phillips curve would be observed (small positive values of λ) and an open interval of λ values for which a Phillips curve of the opposite slope would be observed (small negative values of λ), as well as one equilibrium for which the Phillips curve would be vertical ($\lambda = 0$). If the interval (11) includes sufficiently large positive

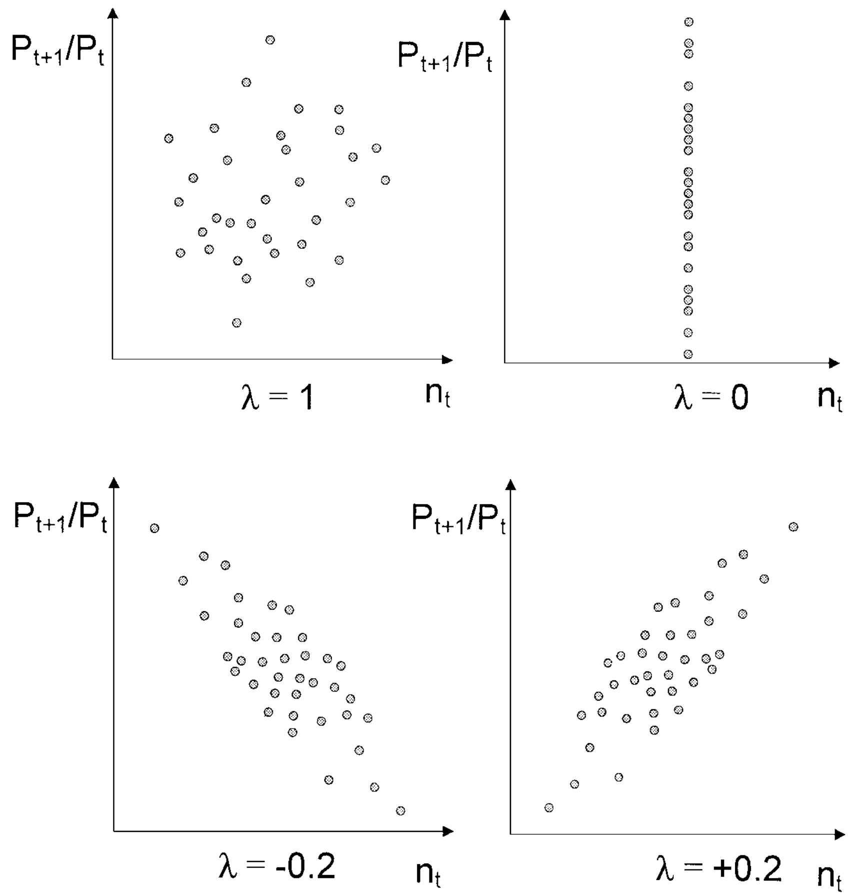


FIGURE 5. Phillips curve for different values of λ.

values of λ, it also will include at least one other equilibrium for which there is zero covariance between output and inflation (although money is not neutral in this equilibrium) and another interval of λ values for which the Phillips curve has an anticonventional slope. The various possible types of observed Phillips curve are illustrated in Figure 5, which shows scatter plots of inflation and output values obtained by sampling from the invariant joint distribution of these variables, for several different stationary rational-expectations equilibria of the same economy.[13]

Of course, this analysis does not challenge the main policy conclusion from the Lucas analysis, namely, that the mere observation of a correlation between inflation and output in a given stochastic stationary state does not mean that a stable output-inflation relation exists that could be exploited through systematic policy. In our equilibria, as in Lucas's, an increase in the mean of the distribution of x [by drawing x's from the previous distribution $F(x)$ and then multiplying

https://doi.org/10.1017/S1365100597005051 Published online by Cambridge University Press

each by a scalar $a > 1$] would have no effect on allocations. That is to say, for each equilibrium that exists when $E(x) = 1$, there exists a corresponding equilibrium in the $E(x) = a$ case, in which the stationary distributions of all real quantities are the same, but all gross rates of inflation are multiplied by a. On the other hand, it is not so easy to say, in our case, that upon the establishment of the new regime (multiplication by a), there should be no real effects. It would be consistent with rational-expectations equilibrium if the economy were to adjust to the corresponding equilibrium in which only rates of inflation were affected, but it would also be consistent with rational-expectations equilibrium if it were to adjust to some other of the set of possible equilibria when $E(x) = a$. This is a problem arising from the multiplicity of stationary equilibria, which Lucas does not have to consider. It is discussed further in Section 8.

6. STOCHASTIC GOVERNMENT EXPENDITURE

Our model as set out in Section 3 allows for both stochastic government expenditures and random interest payments; in this section we look at another special case, in which only g is stochastic. The reason for looking at this class of models is to illustrate the implications of our findings for some traditional questions that relate to the incidence of taxation. We show that the incidence of the inflation tax in this model is indeterminate and that beliefs independently influence the incidence of taxes. First, we set out an analogue of the one-parameter family of equilibria that we considered in Section 4 for the case in which g is the only random variable. As stated in Section 3, we assume that g_t is i.i.d., being drawn each period from a distribution $H(g)$ with bounded support $[c, d]$, where $0 < c < 1$. For a stationary rational-expectations equilibrium to exist, we also must assume that $\bar{g} < \frac{1}{4}$, where $\bar{g}$ is the mean of the distribution $H(g)$. Among the very large number of stationary rational-expectations equilibria that exist when $\bar{g} < \frac{1}{4}$, we exhibit the following one-parameter family:

THEOREM 3. *In the case that $x = 1$ each period, and g_t is stochastic with mean $\bar{g} < \frac{1}{4}$, each member of the one-parameter family of price functions,*

$$\phi(n, g, \lambda) = \frac{1}{n^2 + (1-\lambda)(\bar{g} - g)} \tag{23}$$

for λ in the interval

$$\frac{\frac{1}{4} - \bar{g}}{\bar{g} - c} < \lambda < \frac{\frac{1}{4} - \bar{g}}{d - \bar{g}} \tag{24}$$

corresponds to a stationary rational-expectations equilibrium.

Proof of Theorem 3. Note that the interval (24) has a nonempty interior that includes λ. Substitution of (23) into (7) reveals that the equilibrium condition is

https://doi.org/10.1017/S1365100597005051 Published online by Cambridge University Press

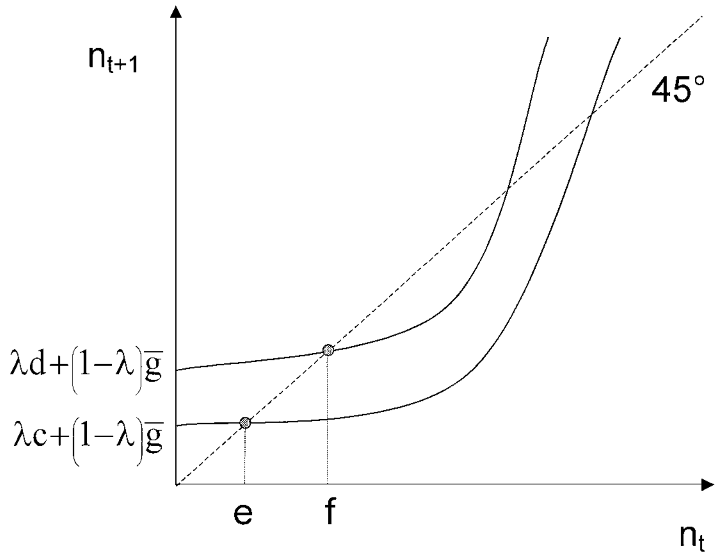


FIGURE 6. Geometry of the construction of the interval $[e,\ f]$ for the case of random government spending.

satisfied. The existence of an invariant distribution $\pi(n)$ is verified using Lemma 1, as in the proof of Theorem 1. The invariant interval is given by

$$e = \frac{1 - \sqrt{1 - 4\overline{g} + 4\lambda(\overline{g} - c)}}{2},$$

$$f = \frac{1 - \sqrt{1 - 4\overline{g} + 4\lambda(d - \overline{g})}}{2}.$$

(The geometry of this is shown in Figure 6.) The function

$$h(n, g) = n^2 + \lambda g + (1 - \lambda)\overline{g}$$

maps $[e,\ f]$ into itself for all $g \in [c, d]$, and is continuous in n for all $g \in [c, d]$, and so, an invariant distribution $\pi(n)$ exists and the proof of Theorem 3 is complete. ■

Paralleling the proof of Theorem 2, one also can prove uniqueness and convergence in this case, assuming that the cumulative distribution function $H(g)$ is strictly monotonic for $g \in [c, d]$.

Different equilibria in this one-parameter family exhibit qualitatively different behavior. For example, different quantities are predetermined in different equilibria. In the $\lambda = 0$ equilibrium, the interval $[e,\ f]$ collapses to a single point, so that in the stochastic stationary state, n_t is constant. In this equilibrium,

$$c_t = n - g_t,$$

so that consumption by the old in a given period is not known until the level of government expenditure during that period is announced. In the $\lambda = 1$ equilibrium (which exists as long as $d < \frac{1}{4}$), by contrast,

$$n_{t+1} = n_t^2 + g_{t+1},$$

so that n_{t+1} varies with current government expenditure, whereas

$$c_{t+1} = n_t^2,$$

so that c_t is predetermined. Furthermore,

$$P_{t+1} = m_t / n_t^2,$$

so that the price level also is predetermined in this equilibrium. Money creation by the government has no effect on the current period's price, but, by affecting n_{t+1} and m_{t+1}, it affects future prices.

This implies that questions having to do with the incidence of fiscal policy have no determinate answer in such a model. In the case of an unexpectedly large government expenditure in a given period, whose consumption must be decreased? In the $\lambda = 0$ equilibrium, it is achieved entirely through an unexpected reduction in the consumption of the old; the consumption of leisure by the young is unaffected. In the $\lambda = 1$ equilibrium, it is achieved entirely through a reduction in the consumption of leisure by the young. Individuals can be completely confident as to the value of their savings in old age, because unexpected money creation by the government has no effect on the purchasing power of the money they hold. For intermediate values of λ, the effects are intermediate between these extremes. For $\lambda < 0$, unexpectedly high government expenditure actually results in increased consumption of leisure by the young, whereas for $\lambda > 1$, unexpectedly high government expenditure results in increased consumption by the old. These different possible effects have nothing to do with alternative methods of government finance. In each case, the same expenditure is financed in the same way (money creation); the different effects are due to differences in what individuals expect. And all of these expectations are consistent with stationary rational-expectations equilibrium.

7. PREDETERMINED NOMINAL WAGES

Many recent explanations of the business cycle [e.g., Phelps and Taylor (1977), Fischer (1977a,b)] assume that nominal wages are slow to adjust in response to shocks in aggregate demand. A large body of literature has sought to provide a rationalization of this assumption consistent with the theory of rational choice. It often has been assumed that sticky wages can only result from a model in which agents commit themselves to an employment contract before the realization of the aggregate demand shocks. To date, however, this research program has been unsuccessful. Models that assume that contracts involve state-invariant nominal

https://doi.org/10.1017/S1365100597005051 Published online by Cambridge University Press

wages [Gray (1976), Fischer (1977a,b)] are inconsistent with rational choice [Barro (1977)]; models that are consistent with rational behavior [Baily (1974), Azariadis (1975), Grossman and Hart (1981)] sometimes predict real-wage rigidity, but never predetermined nominal wages.

By contrast, in our model (whenever government policy satisfies the bound $bd \leq \frac{1}{4}$), there is a solution in which the nominal wage is independent of current realizations of stochastic shocks, despite the facts that no agents commit themselves prior to realization of these shocks and that these realizations are public knowledge. In the model presented here, the price level is also the nominal wage, and so, the predetermined-price solution *is* a predetermined nominal-wage solution. This might be thought to be an uninteresting example because of the model's failure to distinguish between wages and prices. However, we know of examples with more complicated technologies in which the real wage is variable and yet there exists a stationary rational-expectations equilibrium in which the nominal wage is predetermined.[14]

Using our methods, it is possible to construct examples in which workers and firms write labor contracts prior to the realization of aggregate-demand shocks and to embed these contracts in a full general-equilibrium model. Robust examples exist in which one of the set of possible rational-expectations equilibria has the property that nominal wages are state independent in the optimal contract. However, the insurance motive for contracting prior to realization has nothing to do with this result, which depends on the self-fulfilling nature of expectations over future price distributions.

8. A MORE GENERAL CLASS OF EQUILIBRIA

In fact, the one-parameter families of equilibria exhibited above do not come close to characterizing the complete set of stationary rational-expectations equilibria in these models. For example, in the case of the model considered in Section 4, the following larger class of equilibria exists:

THEOREM 4. *In the case that government expenditure* g *is constant each period, and satisfies* $0 < g < \frac{1}{4}$, *any price function of the form*

$$\phi(n, x) = \frac{x}{n^2\psi(x)} \int_a^b \psi(x)\, dF(x), \tag{25}$$

where $\psi(x)$ *is a continuous function of* x *satisfying the bounds*

$$4g < \psi(x) < 1 \tag{26}$$

for all $x \in [a,b]$, *corresponds to a stationary rational-expectations equilibrium. Furthermore, if the cumulative probability* $F(x)$ *is strictly monotonic in* x, *then the invariant distribution* $\pi(x)$ *associated with the stochastic stationary state is unique and has the convergence property stated in Theorem 2.*

https://doi.org/10.1017/S1365100597005051 Published online by Cambridge University Press

Proof of Theorem 4. The proof of Theorem 4 follows that of Theorems 1 and 2. Substitution of (25) into (7) reveals that the equilibrium condition is satisfied by any price function of this form. Adopting the notation

$$u \equiv \max_{a \le x \le b} \frac{\psi(x)}{E(\psi)},$$

$$v \equiv \min_{a \le x \le b} \frac{\psi(x)}{E(\psi)},$$

where the expectation is taken with respect to the distribution $F(x)$, it is clear that the bounds (26) ensure that

$$0 < v < u < (1/4g).$$

It then follows that an invariant interval $[e, f]$ exists, given by (20) and (21) as before. Hence, Lemma 1, an invariant-distribution $\pi(n)$, exists. In the proof of Theorem 2, the only properties of $h(n, x)$ used were the property that $h(n, x)$ was continuous in x, that

$$\min_{a \le x \le b} h(n, x) \le n \quad \text{for} \quad n \in [e, f]$$

with equality only for $n = e$, and that

$$\max_{a \le x \le b} h(n, x) \ge n \quad \text{for} \quad n \in [e, f]$$

with equality only for $n = f$. All these properties hold for the price functions described in Theorem 4, and so, the proof goes through as before. ■

Theorem 4 does not come close to being a complete characterization of the set of stationary rational-expectations equilibria. It should be apparent that any price function of the much more general form

$$\phi(n, x) = \frac{x}{n^2 \psi(n, x)} \int_a^b \psi(n, x')\, dF(x') \tag{27}$$

is acceptable, as long as there exists an invariant interval $[e, f]$. It also should be clear from the argument thus far that such an interval will exist for any function of form (27) which is sufficiently close to a function described in Theorem 4; that is, for any function of form (27) in which $\psi(n, x)$ varies sufficiently little with variations in n. Furthermore, even if it were possible to give a complete characterization of the set of price functions satisfying the definition of equilibrium given in Section 3, this really would be only a restricted subset of the set of stationary rational-expectations equilibria, because, as discussed previously, one might equally well include other predetermined quantities as arguments in the price function.

What Theorem 4 does establish is that the set of stationary rational-expectations equilibria is not a simple finite-dimensional continuum as in the case of the multiple stationary equilibria previously considered for linear rational-expectations models; instead, the set includes a set of equilibria associated with the elements of an open set in an infinite-dimensional function space.[15] Although we draw no conclusions from this characterization, it does emphasize the fact that in addition to there being a very large number of equilibria, there are a very large number of equilibria arbitrarily close to any one equilibrium in the set. In the case of any price function of form (25), there is an infinite number of independent directions in which it is possible to slightly vary the function and still have a function of this class.

This local indeterminacy of the equilibrium price function poses a grave difficulty for many popular interpretations of the rational-expectations hypothesis. It often is claimed that the hypothesis denies any role for arbitrary subjective factors in dynamic economic analysis by rendering the expectations of rational agents completely endogenous. In the case of an economic model such as that considered here, it cannot be said that expectations are endogenously determined by fundamental considerations (preferences, endowments, transactions technology, policy regime) alone.

Furthermore, many discussions of the implications of rational expectations are concerned with the effects of a change in policy regime, and assume that such a change would cause an endogenous change in expectations to a new equilibrium. This endogenous change typically is supposed to be both unique and immediate. In certain cases it is argued that the unique, immediate response will undo the attempted intervention, so that equilibrium allocations are unaffected by the change in policy regime. Such arguments cannot be made if rational-expectations equilibrium is indeterminate. For example, in the case of the model considered here, a shift in policy regime that does not alter the distribution $H(g)$, but changes $F(x)$ so that $E(x) > 1$, cannot be said to necessarily have no effect on allocations, even if one supposes that a new stationary rational-expectations equilibrium is immediately reached. Although there will exist an equilibrium in which all real quantities are unchanged, there will exist arbitrarily close to it a very large number of other equilibria in which expectations would be equally rational.

What is more, the assumption that a new rational-expectations equilibrium is immediately reached is very difficult to sustain in the case of indeterminacy. Even supposing that all agents know the correct model of their economy, the expectations that are rational for each of them to hold depend upon what the other agents expect. Even in a model with a unique rational-expectations equilibrium, rational choice does not explain how a decentralized economy will arrive at such an equilibrium. [See, e.g., Phelps (1983).] But the problem is much worse if rational-expectations equilibrium is indeterminate, for then agents cannot solve their coordination problem by each solving the model for its rational-expectations equilibrium and expecting the others all to be computing the same solution. Complete awareness on the part of each agent of the theoretically possible set of equilibria would give none of them any way of guessing which equilibrium the others expected to occur. Hence it is possible that coordination of expectations could occur only after a period of disequilibrium adjustment; it is not clear what rules of inference agents should use during such a period, and so it is not even clear that eventual coordination would result.

9. CONCLUSION

We show that, in an exact general-equilibrium model, a very large set of stationary-rational-expectations equilibria exists. This result poses serious problems for a common interpretation of rational expectations, according to which rationality uniquely determines expectations in terms of market fundamentals. We also show that, contrary to widespread belief, certain business-cycle phenomena (in particular the Phillips curve and nominal-wage stickiness) are consistent with rational expectations and exact optimizing behavior in the absence of informational imperfections.

Our results demonstrate that the rational-expectations method introduced by Lucas (1972) is incomplete. One interpretation might be that expectations may be influenced by non-economic factors, such as social norms, or "animal spirits," which cannot be captured in a purely economic model. Another approach would be to postulate a learning mechanism, as in Bray (1982) or Blume and Easley (1982), under which one might hope that only a small number of rational-expectations equilibria would be stable.

Another question suggested by our results is under what circumstances rational-expectations equilibrium is unique. Indeed, for a given specification of preferences and technologies, one might compare alternative policy regimes. If certain policy regimes are characterized by indeterminacy, and others are not, then one might advocate one of the latter polices. It will not always be the case that laissez-faire is one of the regimes in which equilibrium is determinate. For example, in the overlapping-generations model considered by Azariadis (1981) and Azariadis and Guesnerie (1982, 1986), multiple stationary rational-expectations equilibria exist under laissez-faire monetary policy. Grandmont (1985) gives an example of an activist monetary policy that renders the rational-expectations equilibrium unique in that model. This provides a possible justification for stabilization policy even in the case of models that possess a Pareto-optimal rational-expectations equilibrium under laissez-faire.

NOTES

1. This assumption is not necessary for the existence of the multiplicity of rational-expectations equilibria considered herein. It is made simply for computational convenience. The model presented in this section also can be reinterpreted as one in which there is inelastic supply of labor in the first period of life, and consumption in both periods of life, with quadratic utility of consumption in the first period and linear utility in the second. The interpretation in terms of elastic labor supply is used here to allow us to discuss Phillips-curve relations in Section 5.

2. In this respect our model generalizes Lucas's model. In the subsequent discussion in Section 3, we consider the effects on prices and allocations of two different kinds of fluctuations in government policy: the stochastic interest payments considered by Lucas, and stochastic government expenditure.

3. It would be realistic to assume an upper bound $\bar{n}$ on feasible labor supply per capita. However, because all equilibria discussed in this paper are such that n_t is bounded, it is possible to assume that the upper bound $\bar{n}$ is sufficiently high as never to be binding.

https://doi.org/10.1017/S1365100597005051 Published online by Cambridge University Press

4. That is,

$$1_{n'} \equiv \begin{cases} 1 & \text{if} \quad e \le n \le n' \\ 0 & \text{if} \quad n > n' \end{cases}.$$

5. This assumes that g is not too high; specifically, that $g < \frac{1}{4}$. If the desired level of inflation-financed government expenditure is too high, there is no rate of money creation that can provide that amount of revenue.

6. This was noted by Black (1974). For a discussion of conditions under which indeterminacy of perfect foresight equilibria may occur in overlapping generations models, see Kehoe and Levine (1985), Muller and Woodford (1983).

7. It is easily seen from elementary considerations of neutrality that rational-expectations equilibria other than the Lucas solution can exist, in a model in which the only stochastic policy involves stochastic interest payments on money balances, only if in the corresponding model without interest payments there exist sunspot equilibria, i.e., rational-expectations equilibria in which prices and allocations are affected by random variables that have no effect on preferences, endowments, technology, or government policy. [See, e.g., Shell (1977), Cass and Shell (1983), Azariadis (1981), Azariadis and Guesnerie (1982, 1986).] We conjecture that stationary sunspot equilibria can exist arbitrarily close to a nonstochastic steady state of such a model only if the nonstochastic steady state is indeterminate. This will be the subject of a future paper. It would then follow that, in models with stochastic interest payments, stationary rational-expectations equilibria exist arbitrarily close to a Lucas solution only if, in the corresponding model without interest payments, the Lucas solution corresponds to an indeterminate monetary steady state.

8. The reference to a one-parameter family of equilibria should not suggest that what we find here is qualitatively the same as the multiplicity of perfect-foresight equilibria in the nonstochastic model, illustrated in Figure 1A, where there are only two stationary equilibria. Here, there is a one-parameter family of stationary equilibria, and, as subsequently shown, there are many more than this. There is, however, a more subtle relationship between the multiplicity of perfect-foresight equilibria in the nonstochastic model and the multiplicity of stationary rational-expectations equilibria in the stochastic model, as discussed in note 7.

9. Costas Azariadis first showed us that the Lucas model contains rational-expectations equilibria in which prices are predetermined; however, his examples were all nonstationary. His suggestion that we attempt to generate stationary equilibria of this kind provided the initial stimulus for the work reported here.

10. In Futia's terminology, the Markov process of Lemma 1 defines a Markov operator T on the Banach space of bounded functions on the interval $[e, f]$ as

$$Ty(n) \equiv \int y(h(n,x))\, dF(x).$$

It follows from Futia's Theorem 4.6 that T is a weakly compact operator, and hence quasi compact. Because $h(n, x)$ maps the interval $[e, f]$ into itself for each x, T is a stable operator (Futia's Definition 2.1), and so it is equi continuous (Futia's Theorem 3.3). It then follows from Futia's Theorem 2.9 that an invariant distribution $\pi(x)$ exists.

11. Our Lemma 2 combines theorems 2.12, 3.6, and 3.7 of Futia (1982). Futia's Theorem 3.7 states that there exist a continuous linear operator V and constants M and ε such that

$$|T^t - V| \le \frac{M}{(1+\varepsilon)^t}$$

for all t, where the operator norm is defined by

$$|T| \equiv \max_{|f| \le 1} |Tf|.$$

https://doi.org/10.1017/S1365100597005051 Published online by Cambridge University Press

However, because $P(n', n) = T^t l_n(n')$,

$$|P^t(n', n) - V l_n(n')| \leq |T^t - V||l_n|.$$

Furthermore, it follows from Futia's Theorem 3.10 that $V l_n n'$ does not depend on n', and hence is equal to $\pi(n)$. Therefore, the convergence property stated in our Theorem 2 is equivalent to that stated in Futia's Theorem 3.7.

12. Because $1/n$ and n^2 are both convex functions of n, Jensen's inequality tells us that

$$E(n^2) > E(n)^2,$$
$$E(1/n) > 1/E(n)$$

with equality, in each case, only if $\text{var}(n) = 0$. The function $x/[1 + \lambda(x - 1)]$ is a concave function of x for $0 < \lambda < 1$ and a convex function of x for $\lambda < 0$ or $\lambda > 1$ and a linear function for $\lambda = 0$ or $\lambda = 1$. Hence, Jensen's inequality tells us that $E\{x/[1 + \lambda(x - 1)]\}$ is less than one for $0 < \lambda < 1$, greater than one for $\lambda < 0$ or $\lambda > 1$, and exactly equal to one for $\lambda = 0$ or $\lambda = 1$. Substitution of these inequalities into (22) tells us that

$$\text{Cov}(P_{t+1}/P_t, n_{t+1}) < 0$$

for $\lambda < 0$ or $\lambda > 1$ and that the covariance is exactly zero when $\lambda = 0$. It is also possible to show that the covariance is positive for small positive values of λ. It follows from (12) and (21) that

$$(f - e) = 0(\lambda).$$

Hence,

$$\begin{aligned} E(n^2)E(1/n) - E(n) &= E\{[n^2 - E(n^2)][(1/n) - E(1/n)]\} \\ &< (f^2 - e^2)\left(\frac{1}{e} - \frac{1}{f}\right) \\ &= \frac{(f-e)^2(f+e)}{ef} \end{aligned}$$

implies that

$$E(n^2)E(1/n) - E(n) = 0(\lambda^2).$$

Simple differentiation indicates that

$$\lim_{\lambda \to 0} \frac{1}{\lambda}\left\{1 - E\left[\frac{x}{1 + \lambda(x-1)}\right]\right\} = \text{var}(x).$$

Therefore,

$$\begin{aligned} \lim_{\lambda \to 0}(1/\lambda)\text{cov}(P_{t+1}/P_t, n_{t+1}) &= \lim_{\lambda \to 0}(1/\lambda)\{E(n) - E[(1/n)E(n^2)]\} \\ &\quad + \lim_{\lambda \to 0}(1/\lambda)\left(E(1/n)E(n^2)\left\{1 - E\left[\frac{x}{1+\lambda(x-1)}\right]\right\}\right) \\ &= \underline{n}\,\text{var}(x) > 0. \end{aligned}$$

Hence, there must exist an open neighborhood $(0, \bar{\lambda})$, such that for $\lambda \in (0, \bar{\lambda})$, the covariance is positive.

Certain features of the curve shown in Figure 4, such as the property that the convergence is zero for only two values ($\lambda = 0$ or $\lambda = \bar{\lambda}$), have not been proved to necessarily hold. They can, however,

https://doi.org/10.1017/S1365100597005051 Published online by Cambridge University Press

be established in the limiting case of a sufficiently small value for var(x). In this case, one can use the following approximations:

$$\text{var}(n)\frac{\lambda^2 \underline{n}^4}{1-4\underline{n}^2}\text{var}(x)+o(\text{var}(x))$$

$$E\left[\frac{x}{1+\lambda(x-1)}\right]=1-\lambda(1-\lambda)\text{var}(x)+o(\text{var}(x))$$

$$E(1/n)=(1/\underline{n}^3)\text{var}(n)+o(\text{var}(n))$$

to obtain

$$\text{cov}\left(\frac{P_{t+1}}{P_t}, n_{t+1}\right)=\underline{n}\lambda\left[1-\lambda\left(\frac{1-2\underline{n}^2}{1-4\underline{n}^2}\right)\right]\text{var}(x)+o(\text{var}(x)).$$

Hence, in the case of sufficiently small variability of the stochastic interest factor, the covariance with λ is in the manner shown in Figure 4, and

$$\bar{\lambda}=\frac{1-4\underline{n}^2}{1-2\underline{n}^2}+o(\text{var}(x)).$$

13. The plots in Figure 5 are all generated for an economy in which $g = 0.125$ and $F(x)$ is the convolution of five uniform distributions.

14. Geanakoplos and Polemarchakis (1983) make a similar suggestion in the context of perfect-foresight equilibrium. Our example shows that it is possible for a predetermined nominal wage to exist in a stationary rational-expectations equilibrium, and makes it clear that the existence of a predetermined-nominal-wage solution to a model of this sort has nothing to do with failure of markets to clear at any time.

15. Adopting the norm

$$|\psi| = \sup_{a\le x\le b} |\psi(x)|$$

so as to make the space of bounded continuous functions on the interval $[a, b]$ a metric space, then the set described by (26) is the intersection of a closed ball of radius 1 with its center at the origin, and closed ball of radius 1–4g with its center at the function $\psi(x = 1)$. There is not a one-to-one correspondence between this set of functions and the stationary rational-expectations equilibria described in Theorem 4, because scalar multiplication of $\psi(x)$ does not change $\phi(n, x)$. This consideration, however, only reduces the number of independent directions of variation in $\psi(x)$ that correspond to distinct equilibria by one.

REFERENCES

Azariadis, C. (1975) Implicit contracts and underemployment equilibria. *Journal of Political Economy* 83, 1183–1202.

Azariadis, C. (1981) Self fulfilling prophecies. *Journal of Economic Theory* 25, 380–396.

Azariadis, C., & R. Guesnerie (1982) Prophéties créatrices et persistence des theories. *Revue Economique* 33, 787–805.

Azariadis, C., & R. Guesnerie (1986) Sunspots and cycles. *Review of Economic Studies* 53, 725–736.

Bailey, M.J. (1956) The welfare cost of inflationary finance. *Journal of Political Economy* 64, 93–110.

Baily, M.N. (1974) Wages and employment under uncertain demand. *Review of Economic Studies* 41, 37–50.

Barro, R.J. (1977) Long-term contracting, sticky prices, and monetary policy. *Journal of Monetary Economics* 3, 305–316.

Black, F. (1974) Uniqueness of the price level in monetary growth models with rational expectations. *Journal of Economic Theory* 7, 53–65.

Blume, L.E., & D. Easley (1982) Learning to be rational. *Journal of Economic Theory* 26, 340–351.

Bray, M. (1982) Learning, estimation, and the stability of rational expectations. *Journal of Economic Theory* 26, 318–339.

Calvo, G. (1978) On the indeterminacy of interest rates and wages with perfect foresight. *Journal of Economic Theory* 19, 321–337.

Cass, D. & K. Shell (1983) Do sunspots matter? *Journal of Political Economy* 91, 193–227.

Fischer, S. (1977a) Wage indexation and macroeconomic stability. In K. Brunner & A. Meltzer (eds.), *Stabilization of the Domestic and International Economy*. Carnegie–Rochester Conference Series, Vol. 5. Amsterdam: North-Holland.

Fischer, S. (1977b) Long-term contracts, rational expectations and the optimal money supply rule. *Journal of Political Economy* 85, 191–205.

Futia, C.A. (1981) Rational expectations in stationary linear models. *Econometrica* 49, 171–92.

Futia, C.A. (1982) Invariant distributions and the limiting behavior of Markovian economic models. *Econometrica* 50, 377–408.

Gale, D. (1973) Pure exchange equilibrium of dynamic economic models. *Journal of Economic Theory* 6, 12–36.

Geanakoplos, J.D. & H.M. Polemarchakis (1983) Walrasian Indeterminacy and Dynamic Macroeconomic Equilibrium: The Case of Certainty. Manuscript, Yale University.

Gourieroux, C., Laffont, J.J., & A. Monfort (1982) Rational expectations in dynamic linear models: Analysis of the solutions. *Econometrica* 50, 409–426.

Grandmont, JM. (1985) On endogenous competitive business cycles. *Econometrica* 53, 995–1046.

Gray, J.A. (1976) Wage indexation: A macroeconomic approach. *Journal of Monetary Economics* 2, 221–235.

Grossman, S.J., & O. Hart (1981) Implicit contracts, moral hazard, and unemployment. *American Economic Review* 71, 301–307.

Hansen, L.P., & T.J. Sargent (1980) Formulating and estimating linear rational expectations models. *Journal of Economic Dynamics and Control* 2, 7–46.

Kehoe, T.J. & D.K. Levine (1985) Comparative statics and perfect foresight in infinite horizon economics. *Econometrica* 53, 433–453.

Lucas, R.E., Jr. (1972) Expectations and the neutrality of money *Journal of Economic Theory* 4, 103–124.

Lucas, R.E., Jr. (1975) An equilibrium model of the business cycle. *Journal of Political Economy* 83, 1113–1144.

Muller, W.J. & M. Woodford (1983) Stationary Overlapping Generations Economies with Production and Infinite-Lived Consumers: II. Determinacy of Equilibrium. MIT working paper 326, Massachusetts Institute of Technology.

Peck, J. (1984) On the Existence of Sunspot Equilibria in an Overlapping Generations Model. Manuscript, University of Pennsylvania.

Phelps, E.S. (1983) The Trouble with "Rational Expectations" and the Problem of Inflation Stabilization. In R. Frydman and E.S. Phelps (eds.), *Individual Forecasting and Aggregate Outcomes*, pp. 31–41. Cambridge, England: Cambridge Univ. Press.

Phelps, E.S. & J.B. Taylor (1977) Stabilizing power of monetary policy under rational price expectations. *Journal of Political Economy* 85, 163–190.

Sargent, T.J. & N. Wallace (1984) Exploding Inflation. Manuscript, Yale University.

Shell, K. (1977) Monnaie et allocation intertemporelle. CNRS Séminaire d'Econométrie de M. Edmond Malinvaud. Paris, November 1977.

Shiller, R.J. (1978) Rational expectations and the dynamic structure of macroeconomic models. *Journal of Monetary Economics* 4, 1–44.

Taylor, J.B. (1977) Conditions for unique solutions to stochastic macroeconomic models with rational expectations. *Econometrica* 45, 1377–1385.

Wallace, N. (1980) The Overlapping Generations Model of Fiat Money. In J.H. Kareken & N. Wallace (eds.), *Models of Monetary Economics*, pp. 49–82. Federal Reserve Bank of Minneapolis.

APPENDIX

Although the utility function assumed in Section 3 is quite specific, the results are not dependent upon this at all. Let (1) be replaced by any function of the form

$$u(n_t, c_{t+1}) = v(c_{t+1}) - w(n_t), \qquad \textbf{(A.1)}$$

where v is a concave, increasing function of c, w is a convex increasing function of n, and both are twice continuously differentiable. Then, (4) becomes

$$w'(n_t) = P_t E_t[v'(c_{t+1})(x_{t+1}/P_{t+1})],$$

and (6) becomes

$$W(n_t) = E_t[V(n_{t+1} - g_{t+1})], \qquad \textbf{(A.2)}$$

where

$$V(c) \equiv c\, v'(c),$$
$$W(n) \equiv n\, w'(c).$$

For preference (A.1) to permit the existence of a monetary equilibrium in the absence of government policy ($g = 0$, $x = 1$), it is well known that the real rate of return in the autarchic equilibrium must be negative; that is, one must have $w'(0) < v'(0)$ or, equivalently, $W'(0) < V'(0)$. We therefore make this assumption. This turns out to be the only feature of the utility function assumed in (1) that is needed to guarantee the existence of a very large number of stationary rational-expectations equilibria.

Consider the proof of Theorem 1 in Section 4. It follows from the definition of $V(c)$ that $V(0) = 0$, $V'(0) > 0$, so that $V(c)$ has an inverse, V^{-1}, for c in some interval $[0, \bar{c}]$. Then, corresponding to the family of price functions (10), we have in the general case

$$\phi(n, x; \lambda) = \frac{x}{V^{-1}([1 + \lambda(x-1)]W(n))}. \qquad \textbf{(A.3)}$$

Note that any price function of the form (A.3) will satisfy (A.2). For a function of the form (A.3) to be an equilibrium price function, one further property must be verified. There must exist an interval $[e, f]$, such that for any $n \in [e, f]$, and any $x \in [a, b]$, $[1+\lambda(x-1)]W(n) \in [0, \bar{c}]$ and $V^{-1}([1 + \lambda(x-1)]W(n)) + g \in [e, f]$. Consider the case $\lambda = 0$. In this case the invariant distribution will be one with all mass at a single point, $\underline{n}$, satisfying

$$V(\underline{n} - g) = W(\underline{n}).$$

Because of the assumptions made in the preceding paragraph, $V(0) = W(0) = 0$, and, $V'(0) > W'(0) > 0$, so that, for sufficiently small positive g, there must exist a solution

https://doi.org/10.1017/S1365100597005051 Published online by Cambridge University Press

$\underline{n} > 0$; and for g sufficiently small, $\underline{n}$ can be made arbitrarily small. Thus for g sufficiently small, there exists a solution such that $0 < \underline{n} < \bar{c} + g$ and such that $V'(\underline{n} - g) > W'(\underline{n})$. Let g be a small positive value such that both of these are true. Now consider a small nonzero value for λ (positive or negative). If λ is small enough, then there will exist a neighborhood N of $\underline{n}$ such that for any $n \in N$, and any $x \in [a, b]$, $[1 + \lambda(x - 1)]W(n) \in [0, \bar{c}]$, and

$$V'(V^{-1}([1 + \lambda(x - 1)]W(n))) > [1 + \lambda(x - 1)]W'(n). \tag{A.4}$$

Let $\tilde{n}(y)$ be the unique $\tilde{n} \in N$ such that $V(\tilde{n} - g) = yW(\tilde{n})$; by the implicit function theorem, such an $\tilde{n}(y)$ exists for y in some neighborhood of 1. Choose λ so small that $1 + \lambda(a - 1)$ and $1 + \lambda(b - 1)$ are both in this neighborhood. Then let the interval $[e, f]$ be given by $e = \tilde{n}(v)$, $f = \tilde{n}(u)$, where u and v are defined as a Section 4. Inequality (A.4) then guarantees that for any $x \in [a, b]$, $V^{-1}([1 + \lambda(x - 1)]W(n))$ is a monotonically increasing function of n with slope less than one, over the entire interval $[e, f]$; hence, by a construction like that in Figure 2, $V^{-1}([1 - \lambda(x - 1)]W(n)) + g \in [e, f]$ for all $n \in [e, f]$. There must exist an interval of λ values, including zero, for which the above construction is possible; for any λ in this interval, (A3) is an equilibrium price function. Hence, a one-parameter family of stationary rational-expectations equilibria exists, as in Theorem 1. By similar adaptations of the proofs, analogs of Theorems 2, 3, and 4 may likewise be proved for the more general utility function (A.1).

It is easily verified that the one-parameter family of equilibria constructed above includes an interval of values of λ for which there is a positive-slope Phillips curve, an interval of values of λ for which there is a negative-slope Phillips curve, and one equilibrium (corresponding to $\lambda = 0$) for which the Phillips curve is exactly vertical. Because for this family of equilibria one has

$$\frac{P_{t+1}}{P_t} = \frac{n_t x_{t+1}}{V^{-1}([1 + \lambda(x_{t+1} - 1)]W(n_t))},$$

$$n_{t+1} = V^{-1}([1 + \lambda(x_{t+1} - 1)]W(n_t)) + g,$$

it follows that

$$\operatorname{cov}\left(\frac{P_{t+1}}{P_t}, n_{t+1}\right) = E(n) - E\left(\frac{nx}{V^{-1}([1 + \lambda(x - 1)]W(n))}\right) E(V^{-1}([1 + \lambda(x - 1)]W(n_t)),$$

where the expectations are taken over the distributions $\pi(n)$ and $F(x)$. By the methods used in the text, it may be shown that for small λ,

$$E(V^{-1}) = V^{-1}(W(\underline{n})) + 0(\lambda^2),$$

$$E\left(\frac{nx}{V^{-1}}\right) = \frac{n}{V^{-1}(W(\underline{n}))} - \lambda \frac{\underline{n}W(\underline{n})}{V^{-1}(W(\underline{n}))V'(V^{-1}(W(\underline{n})))}$$

so that

$$\operatorname{cov}\left(\frac{P_{t+1}}{P_t}, n_{t+1}\right) = \lambda \frac{nx}{V^{-1}(W(\underline{n}))V'(V^{-1}(W(\underline{n})))}$$

$$= \lambda \left(\frac{\underline{n}}{\underline{n} - g}\right) \frac{V(\underline{n} - g)}{V'(\underline{n} - g)} \operatorname{var}(x) + 0(\lambda^2).$$

https://doi.org/10.1017/S1365100597005051 Published online by Cambridge University Press

Because $\underline{n} - g > 0$, and $V(\underline{n} - g) > 0$, and $V'(\underline{n} - g) > 0$ from the construction described above, it follows that $\text{cov}(P_{t+1}/P_t, n_{t+1}) > 0$ for sufficiently small $\lambda < 0$. Hence equilibria exist with Phillips curves of both kinds.

It also can be shown that, if the x and g variables remain within certain bounds, a stationary rational-expectations equilibrium exists in which prices (and hence nominal wages) are predetermined. In general, this equilibrium will not be a member of the above one-parameter family. It is constructed in the following manner. If a predetermined price equilibrium is to exist, the price function $\phi(n, x, g)$ must depend upon n alone. It follows from (A.2) that any such price function must satisfy

$$W(n) = E\left\{V\left[\frac{x}{\phi(n)}\right]\right\},$$

where the expectations are taken over the distributions $F(x)$. Writing $q(n)$ for $1/\phi(n)$, one sees that $E[V(qx)]$ is a continuously differentiable function of q, that $q = 0$ is a solution to $W(0) = E[V(qx)]$, and that

$$\frac{d}{dq}E[V(qx)]|_{q=0} = V'(0) > 0.$$

Then, by the implicit function theorem, there exists for sufficiently small nonnegative n, a unique function $q(n)$ satisfying $W(n) = E[V(qx)]$ and $q(0) = 0$. This function is continuously differentiable, positive for $n > 0$, and satisfies $0 < q'(0) < 1$, because $0 < W'(0) < V'(0)$. The price function $\phi(n) = q(n)^{-1}$ then will describe a stationary rational-expectations equilibrium characterized by predetermined prices, if an invariant distribution for n exists with support entirely inside the interval on which $\phi(n)$ is defined. The evaluation of n_t over time in this equilibrium is given by

$$n_{t+1} = q(n_t)x_{t+1} + g_{t+1}.$$

Suppose that b [the upper bound of the support of $F(x)$] satisfies $bW'(0) < V(0)$. Then $bq(n) < n$, for all $n > 0$ sufficiently small, and $bq(0) = 0$. It follows that there exists a $\hat{g} > 0$ such that, for any $0 < g < \hat{g}$, there exists an $n^*(g)$ such that $bq(n) + g > n$ for all $0 < n^*(g)$, $bq(n^*(g)) + g = n^*(g) > 0$, belongs to the interval on which $q(n)$ is defined. One also will have $bq(n) + g < n^*(g)$ for all $0 < n < n^*(g)$. Suppose, furthermore, that d [the upper bound of the support of $H(g)$] is no greater than $\hat{g}$. Then, for any $x \in [a, b]$, any $g \in [c, d]$, and $n \in [0, n^*(d)]$, $q(n) + g \in [0, n^*(d)]$ so, by Lemma 1, an invariant distribution $\pi(n)$ exists with support $[0, n^*(d)]$. Hence a stationary rational-expectations equilibrium exists with predetermined prices.

There is no reason to suppose that either the additive separability, the absence of first-period consumption, or the assumption of homogeneous preferences, reflected in (A.1) is necessary for any of these results. As discussed in note 7, the only property that seems to be essential for the sort of construction illustrated in this paper is the indeterminacy of perfect-foresight equilibrium in the corresponding certainty model, and none of those special features of (A.1) is necessary for indeterminacy to exist in the case of a small positive value of g.