

# Foliated hyperbolicity and foliations with hyperbolic leaves

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*Abstract.* Given a lamination in a compact space and a laminated vector field  $X$  which is hyperbolic when restricted to the leaves of the lamination, we distinguish a class of  $X$ -invariant probabilities that describe the behavior of almost every  $X$ -orbit in every leaf, which we call Gibbs  $u$ -states. We apply this to the case of foliations in compact manifolds having leaves with negative curvature, using the foliated hyperbolic vector field on the unit tangent bundle to the foliation generating the leaf geodesics. When the Lyapunov exponents of such ergodic Gibbs  $u$ -states are negative, it is an SRB measure (having a positive Lebesgue basin of attraction). When the foliation is by hyperbolic leaves, this class of probabilities coincide with the classical harmonic measures introduced by Garnett. Furthermore, if the foliation is transversally conformal and does not admit a transverse invariant measure we show that there are finitely many ergodic Gibbs  $u$ -states, each supported in one minimal set of the foliation, each having negative Lyapunov exponents, and the union of their basins of attraction has full Lebesgue measure. The leaf geodesics emanating from a point have a proportion whose asymptotic statistics are described by each of these ergodic Gibbs  $u$ -states, giving rise to continuous visibility functions of the attractors. Reversing time, by considering  $-X$ , we obtain the existence of the same number of repellers of the foliated geodesic flow having the same harmonic measures as projections to  $M$ . In the case of only one attractor, we obtain a north to south pole dynamics.

## 1. Introduction

1.1. *Foliated hyperbolicity versus partial hyperbolicity.* Hyperbolicity is a fundamental tool for understanding the chaotic behavior of dynamical systems, even for non-hyperbolic

systems. The aim of this paper is to show that ideas from partially hyperbolic dynamics may be useful in the study of foliations, in particular when the leaves carry a negatively curved, or better a hyperbolic, Riemannian metric. The foliated geodesic flow is then hyperbolic along the leaves. However, we make no assumption on the normal direction of the foliation, so that the foliated geodesic flow does not need to be partially hyperbolic: the directions along the leaves are hyperbolic, while the normal directions may be anything.

This behavior resembles the notion of the *partially hyperbolic diffeomorphism*  $f$ , where the tangent bundle of the whole manifold splits into a sum of three  $Df$ -invariant bundles  $E^s \oplus E^c \oplus E^u$ , the stable and unstable bundles being uniformly contracted and expanded, respectively. In the center bundle, no expansion or contraction is required: the only requirement is that the center bundle is dominated by the stable and the unstable ones, meaning that no contraction in the center bundle is stronger than the contraction in the stable one and no expansion is stronger than the expansion in the unstable bundle. Partial hyperbolicity implies the existence of stable and unstable foliations, tangent to  $E^s$  and  $E^u$ , respectively. Furthermore, a fundamental tool for the ergodic theory of partially hyperbolic dynamics is the fact that these foliations are absolutely continuous with respect to Lebesgue measure, assuming that the diffeomorphism  $f$  is  $C^2$ , and distinguishing certain probabilities, called Gibbs  $u$ -states.

The foliated geodesic flow of a foliation by leaves with negative curvature is not necessarily partially hyperbolic: the hyperbolic behavior in the leaves does not dominate the normal behavior, that is, the holonomy along the geodesics. However, this lack of domination is substituted by the *a priori* existence of a foliation carrying the flow, allowing one to recover the existence of the stable and unstable foliations tangent to the hyperbolic stable and unstable manifolds, their absolute continuity with respect to Lebesgue, as well as distinguishing the corresponding Gibbs  $u$ -states. Thus we will define the notion of *foliated hyperbolicity*.

**1.2. Foliated hyperbolicity.** In §2 we consider a lamination  $\mathcal{F}$  in the compact space  $M$  and an  $\mathcal{F}$ -hyperbolic vector field  $X$ : a  $C^r$ ,  $r \geq 1$ , vector field in each leaf that varies continuously for the  $C^r$ -topology and is uniformly hyperbolic when restricted to any leaf of  $\mathcal{F}$  (Definition 2.5). The behavior of an  $\mathcal{F}$ -hyperbolic vector field is very close to that of a continuous family of Anosov vector fields, and §2 states classical properties of the hyperbolic dynamics which can be straightforwardly generalized to  $\mathcal{F}$ -hyperbolic vector fields. Thus the leaves of  $\mathcal{F}$  are foliated by stable and unstable foliations  $(\mathcal{W}_{\mathcal{F},X}^s, \mathcal{W}_{\mathcal{F},X}^u)$  whose leaves are  $C^r$  and vary continuously for the  $C^r$ -topology (Theorem 2.6). The central unstable foliation  $\mathcal{W}_{\mathcal{F},X}^{cu}$  is generated by  $\mathcal{W}_{\mathcal{F},X}^u$  and  $X$ , its leaves are contained in the leaves of  $\mathcal{F}$ , and since for backward time  $X_t$  is contracting, it is formed of  $X$ -orbits which share the same asymptotic past.

For  $X$  an  $\mathcal{F}$ -hyperbolic vector field, we introduce in Definition 3.5 the Gibbs  $u$ -states of  $(\mathcal{F}, X)$ , which are  $X$ -invariant probability measures on  $M$  which are absolutely continuous along the leaves of the unstable foliation of  $X$  in  $\mathcal{F}$  and have a prescribed positive density in each unstable leaf (coming from the distant past). In a similar way to the partially hyperbolic setting (see [6, Theorem 11.16]), we prove in §3 that the statistical behavior of most of the orbits of  $X$  is described by the Gibbs  $u$ -states.

PROPOSITION 1. *Let  $(M, \mathcal{F})$  be a compact metrizable space with a  $C^r$ -lamination and  $X$  an  $\mathcal{F}$ -hyperbolic vector field. Then there exists a set  $E \subset M$  intersecting every leaf of the center-unstable foliation  $\mathcal{W}_{\mathcal{F}, X}^{cu}$  of  $X$  in  $\mathcal{F}$  on a full Lebesgue measure subset, such that for any  $p \in E$ , every accumulation measure  $\nu$  of*

$$\nu_{T,p} := X_*(\cdot, p)|_{[0,T]} \frac{dt}{T}$$

with  $T \rightarrow \infty$  is a Gibbs  $u$ -state.

Furthermore, if  $\mathcal{F}$  is a  $C^1$  foliation (that is, if the transverse structure of  $\mathcal{F}$  is a  $C^1$  manifold), the linear holonomy cocycle over the flow of  $X$  is well defined on the normal bundle to  $\mathcal{F}$ . Oseledets's theorem can then be applied to every  $X$ -invariant ergodic probability, defining the normal Lyapunov exponents of the measure. These describe the exponential rate of growth of the holonomy of the foliation along most of the  $X$ -orbits contained in the support of the measure.

When the vector field is not just  $C^2$  in the leaves, but  $C^2$  in the ambient manifold, Pesin theory can be applied to fully describe the ergodic attractors of the flow. Recall that an  $X$ -invariant measure  $\mu$  is an SRB measure if its basin of attraction has positive Lebesgue measure. In §5 we prove the following result.

PROPOSITION 2. *Let  $M$  be a compact manifold with a smooth foliation  $\mathcal{F}$  and  $X$  an  $\mathcal{F}$ -hyperbolic vector field which is  $C^2$  on  $M$ .*

- (1) *If  $\mu$  is an ergodic Gibbs  $u$ -state for  $X$  with normal Lyapunov exponents which are all negative, then  $\mu$  is an SRB measure.*
- (2) *If for every ergodic Gibbs  $u$ -state for  $X$  the normal Lyapunov exponents are all negative, then there are only a finite number of ergodic Gibbs  $u$ -states, each being an SRB measure, and the union of their basins of attraction has full Lebesgue measure.*

1.3. *Laminations by negatively curved leaves.* If one considers laminations  $\mathcal{F}$  by leaves with a  $C^3$ -Riemannian metric along the leaves of  $\mathcal{F}$  with leafwise strictly negative curvature, the laminated geodesic flow is a flow in the unit tangent bundle  $T^1\mathcal{F}$  of the lamination, which is uniformly  $C^2$  and hyperbolic in the leaves. Therefore the behavior of most of the foliated geodesics is described by the Gibbs  $u$ -states of the geodesic flow.

COROLLARY 1. *Let  $M$  be a compact Riemannian manifold with a foliation  $\mathcal{F}$ , such that the metric is  $C^3$  and its restriction to every leaf has negative sectional curvature. Denote by  $X$  the generator of the foliated geodesic flow on  $T^1\mathcal{F}$ .*

- (1) *Let  $\mu$  be an ergodic Gibbs  $u$ -state for  $X$  with normal Lyapunov exponents which are all negative. Then  $\mu$  is an SRB measure for the foliated geodesic flow.*
- (2) *If all ergodic Gibbs  $u$ -states for  $X$  have all their normal Lyapunov exponents negative, then there are only a finite number of ergodic Gibbs  $u$ -states, each being an SRB measure, and the union of their basins of attraction has full Lebesgue measure in  $T^1\mathcal{F}$ .*

It is natural to consider the projection  $p_*$  on the manifold of these Gibbs  $u$ -states defined in the unit tangent bundle of the foliation. In this way one obtains a well-defined class of measures absolutely continuous with respect to Lebesgue measure inside the leaves, and with strictly positive density (so that their support consists of entire leaves; see Theorem 4.1).

For case 2 in Corollary 1, we define in §6 the *visibility function*  $f_i : M \rightarrow [0, 1]$  which associates to a point  $p$  of  $M$  the spherical Lebesgue measure in  $T_p^1(M)$  of the set of directions at  $p$  that correspond to geodesics through  $p$  whose forward statistic tends to the ergodic Gibbs  $u$ -state  $\mu_i$ . We show the following result.

**THEOREM 1.** *Let  $M$  be a compact Riemannian manifold with a foliation  $\mathcal{F}$ , such that the metric is  $C^3$  and its restriction to every leaf has negative sectional curvature. Denote by  $X$  the generator of the foliated geodesic flow on  $T^1\mathcal{F}$ . If all ergodic Gibbs  $u$ -states for  $X$  have all their normal Lyapunov exponents negative, the visibility function  $f_i$  is continuous in  $M$  and the value of  $f_i$  is 1 on the support of the measure  $p_*\mu_i$ .*

That is, the visibility functions, arising naturally from the geodesic flow, succeed in identifying each minimal set of  $\mathcal{F}$  as well as the set of leaves accumulating on it. Furthermore, they give, for each point  $x \in M$ , a quantitative measure of the directions along which the leaf through  $x$  is attracted to each minimal set.

If the laminations we consider are not just by negatively curved leaves but by hyperbolic leaves (that is, constant  $-1$  sectional curvatures) then the projection of Gibbs  $u$ -states to  $M$  is precisely the harmonic measure introduced by Garnett in [15]. In this case the visibility functions are *harmonic*, in the sense that their foliated Laplacian is zero.

We consider in §7 smooth foliations by hyperbolic leaves which are transversely conformal. In this setting, Deroin and Kleptsyn in [12] proved that, if the foliation does not admit any transverse invariant measure, then the foliation has finitely minimal sets, each of them carries exactly one harmonic measure, and the normal Lyapunov exponent of each of these measures is negative for the Brownian motion in the leaves. We prove that the normal Lyapunov exponent of the corresponding Gibbs  $u$ -state coincides with that of Brownian motion, and therefore is negative, giving the following theorem.

**THEOREM 2.** *Let  $M$  be a compact manifold with a transversely conformal foliation  $\mathcal{F}$  of class  $C^1$ , with no transverse invariant measure, endowed with a leafwise  $C^3$  hyperbolic metric and denote by  $X$  the generator of the foliated geodesic flow on the unit tangent bundle  $T^1\mathcal{F}$  to the foliation.*

- (1) *Every ergodic Gibbs  $u$ -state for  $X$  has negative normal Lyapunov exponent.*
- (2) *There are only a finite number of ergodic Gibbs  $u$ -states, each of them is an SRB measure, and the union of their basins of attraction has full Lebesgue measure on  $T^1\mathcal{F}$ .*

Our initial interest in such foliations comes from the study of holomorphic foliations on complex surfaces. In some cases, one knows that the leaves are conformally hyperbolic surfaces but the hyperbolic metric can *a priori* depend only continuously on the leaves [10]. This explains why some of our statements do not require transversal smoothness.

These results may be seen as generalization of [7] where the first two authors proved the existence of a unique Gibbs  $u$ -state for foliations obtained as suspension of the fundamental group of a Riemann surface over finite-volume hyperbolic surfaces, leading to the same result for some Riccati equations.

In §§2–4 we deal with laminations, in §§5 and 6 we deal with foliations and in the latter section the foliation has leaves of negative curvature. In §7 we deal with transversely conformal foliations with hyperbolic leaves.

2. Laminated hyperbolicity

2.1. Basic definition of foliations and laminations.

*Definition 2.1.* A lamination or foliated space  $(M, \mathcal{F})$  is a metrizable space  $M$  together with an open covering  $\{U_\alpha\}$  and homeomorphisms  $\varphi_\alpha : \mathbb{R}^n \times T_\alpha \rightarrow U_\alpha$  such that:

- $T_\alpha$  is a locally compact space;
- whenever  $U_\alpha \cap U_\beta \neq \emptyset$ , the change of coordinates  $\varphi_\beta^{-1} \circ \varphi_\alpha$  has the form  $\varphi_\beta^{-1} \circ \varphi_\alpha(x, t) = (g_{\alpha\beta}(x, t), h_{\alpha\beta}(t))$ ;
- for fixed  $t$  in  $T_\alpha$ , the map  $g_{\alpha\beta}$  is differentiable.

A lamination is a *foliation* (of class  $C^r$ ) if  $T_\alpha = \mathbb{R}^m$  and the maps  $\varphi_\beta^{-1} \circ \varphi_\alpha$  are differentiable (of class  $C^r$ ), in which case the space  $M$  is a manifold of dimension  $n + m$ .

The  $U_\alpha \simeq \mathbb{R}^n \times T_\alpha$  are called *foliated charts*. As usual, in the notation for a foliated chart we will omit the homeomorphism  $\varphi_\alpha$ . In a foliated chart, the sets of the form  $\mathbb{R}^n \times \{t\}$  are called *plaques*. Maximal connected manifolds that can be obtained by gluing plaques via the  $g_{\alpha\beta}$  are called *leaves*.

*Definition 2.2.* A Riemannian metric on a lamination  $(M, \mathcal{F})$  (of class  $C^r$ ) is a choice of Riemannian metric on each leaf that, when seen in a foliated chart  $\mathbb{R}^n \times T_\alpha$ , has a continuous variation (in the  $C^r$ -topology) with respect to  $t \in T_\alpha$ . It induces on every leaf a distance function  $d_{\mathcal{F}}$ .

In a foliation, a Riemannian metric can be obtained by restricting to the leaves a Riemannian metric in the ambient manifold  $M$ . In a lamination, Riemannian metrics can be obtained by defining them locally and using partitions of unity.

*Definition 2.3.* Given a lamination  $(M, \mathcal{F})$  with a Riemannian metric, its tangent bundle  $T\mathcal{F}$  is the lamination whose leaves are the tangent spaces to the leaves of  $\mathcal{F}$ , where the laminated structure is defined in the natural way. The unit tangent bundle  $T^1\mathcal{F} \subset T\mathcal{F}$  is formed by those vectors of norm 1.

2.2. Definition of laminated hyperbolicity. For the rest of §2, let  $M$  be a compact metric space endowed with a lamination  $\mathcal{F}$ , a Riemannian metric and  $X$  a continuous vector field on  $M$  tangent to the plaques of  $\mathcal{F}$ .

*Definition 2.4.* The vector field will be called *uniformly  $C^r$  in the plaques* if its expression in any laminated chart of  $\mathcal{F}$  induces a  $C^r$  vector field in each plaque of  $\mathcal{F}$ , varying continuously with the plaque for the  $C^r$ -topology.

*Example 1.* Let  $K \subset \mathbb{R}$  be the compact set  $\{0\} \cup \{1/n\}_{n \in \mathbb{N}^*}$  and  $M = S^1 \times K$  (where  $S^1$  is the circle  $\mathbb{R}/\mathbb{Z}$ ) trivially laminated by the circles  $S^1 \times \{t\}$ . We consider the vector field  $X$  tangent to the leaves and whose expression in  $S^1 \times \{1/n\}$  is

$$X\left(t, \frac{1}{n}\right) = \frac{1}{n} \sin(2\pi nt) \frac{\partial}{\partial t},$$

with  $X$  vanishing on  $S^1 \times \{0\}$ . Then  $X$  is a continuous vector field, smooth in every leaf but not uniformly  $C^1$  in the plaques.

Assume that  $X$  is uniformly  $C^r$  in the plaques. Let  $\{X_t\}_{t \in \mathbb{R}}$  denote the flow of  $X$ . For every  $t$ ,  $X_t$  is a homeomorphism of  $M$  which is uniformly  $C^r$  in the plaques. This allows us to consider the differential  $D_{\mathcal{F}}(X_t)$  along the plaques, which defines a map on the tangent bundle to the lamination  $T\mathcal{F}$  which is uniformly  $C^{r-1}$  along the plaques.

*Definition 2.5.* Let  $M$  be a compact metric space,  $\mathcal{F}$  a  $C^r$ -lamination,  $r \geq 1$  on  $M$  with a Riemannian metric on  $T\mathcal{F}$ . Let  $X$  be a non-zero vector field tangent to  $\mathcal{F}$  which is uniformly  $C^1$  in the plaques. We say that  $X$  is *laminated hyperbolic*, or  *$\mathcal{F}$ -hyperbolic* for short, if  $T\mathcal{F}$  splits into  $D_{\mathcal{F}}(X_t)$ -invariant bundles

$$T\mathcal{F} = E^s \oplus \mathbb{R}X \oplus E^u$$

such that there exist  $T_0 > 0$  and  $\lambda > 1$  such that, for every  $x \in M$  and any vectors  $u \in E^s(x)$  and  $v \in E^u(x)$  and any  $T \geq T_0$ , one has

$$\|D_{\mathcal{F}}(X_T)(u)\| \leq \lambda^{-T} \|u\| \quad \text{and} \quad \|D_{\mathcal{F}}(X_T)(v)\| \leq \lambda^T \|v\|.$$

**2.3. Stable and unstable laminations.** Exactly as in the usual hyperbolic case, the stable and unstable bundles of a laminated hyperbolic vector field are always continuous.

Given any  $x \in M$ , the  $\mathcal{F}$ -stable manifold of  $x$   $W_{\mathcal{F}}^s(x)$  is the set of points  $y$  in the leaf  $\mathcal{F}_x$  through  $x$  such that  $\lim_{t \rightarrow +\infty} d_{\mathcal{F}}(X_t(x), X_t(y)) = 0$ . In the same way, the  $\mathcal{F}$ -unstable manifold  $W_{\mathcal{F}}^u(x)$  is the set of  $z \in \mathcal{F}_x$  such that  $\lim_{t \rightarrow -\infty} d_{\mathcal{F}}(X_t(x), X_t(z)) = 0$ .

The local laminated stable and unstable manifolds of radius  $\delta$ , denoted by  $W_{\mathcal{F},\delta}^s(x)$  and  $W_{\mathcal{F},\delta}^u(x)$ , are the subsets of the laminated stable and unstable manifolds whose positive (respectively, negative) iterates remain at distance (in the leaf) less than  $\delta$ .

The proof of the existence of stable and unstable foliations [21] can be directly adapted to this case, leading to the following statement.

**THEOREM 2.6.** *Let  $(M, \mathcal{F})$  be a  $C^r$ -lamination on a compact space and  $X$  an  $\mathcal{F}$ -hyperbolic vector field which is uniformly  $C^r$  along the plaques,  $r \geq 1$ . Then there exists  $\delta > 0$  such that the local laminated stable manifold  $W_{\mathcal{F},\delta}^s(x)$  is a  $C^r$ -disk centered at  $x$  in  $\mathcal{F}_x$  of the same dimension as  $E^s$  and tangent to  $E^s$ . The family of disks  $\{W_{\mathcal{F},\delta}^s(x)\}_{x \in M}$  is continuous for the  $C^r$ -topology. The laminated local unstable manifolds  $\{W_{\mathcal{F},\delta}^u(x)\}_{x \in M}$  form a family of  $C^r$ -disks tangent to  $E^u$  and varying continuously for the  $C^r$ -topology. The global stable manifolds satisfy*

$$W_{\mathcal{F},X}^s(x) = \bigcup_{t>0} X_{-t} W_{\mathcal{F},\delta}^s(X_t(x)).$$

*The global  $\mathcal{F}$ -stable manifolds are  $C^r$ -immersed manifolds which fit into a  $C^0$ -lamination of  $M$  by  $C^r$ -plaques, sublaminating the plaques of  $\mathcal{F}$  and varying  $C^r$ -continuously:  $\mathcal{W}_{\mathcal{F},X}^s$  and  $\mathcal{W}_{\mathcal{F},X}^u$ .*

*Sketch of proof.* The classical proof of the existence of invariant manifolds in the theory of hyperbolic diffeomorphisms involves considering the space of  $C^1$ -disk families  $\{D_x^u\}_{x \in M}$ ,

of a given radius and tangent to an invariant cone field around the unstable bundle (in particular, transverse to the stable bundle), and depending continuously on  $x$  for the  $C^1$ -topology. Then one considers the natural action of the diffeomorphism on one such family of disks: the disks become larger (since the dynamics expands the unstable direction) and one shrinks them in order to recover a family of disks with the same properties. The transverse contraction implies that the action of the space of all such families is contracting, leading to a unique fixed family of disks which is the announced family of local unstable manifolds. This family depends continuously on the dynamics for the  $C^1$ -topology. Applying the same argument to the differential of the diffeomorphism, one gets that the invariant manifolds are indeed  $C^r$ , and that they depend continuously on the point and on the diffeomorphism for the  $C^r$ -topology.

Here we consider the families of disks  $D_x^u$  contained in the leaves, of a given radius, and tangent to a cone field in the leaves around the unstable bundle, and depending continuously on  $x \in M$  for the  $C^1$ -topology. One defines the action on the set of such families of disks exactly as in the classical manner. This action is still a contraction, leading to a unique fixed point which is the announced family of local unstable manifolds. The higher regularity is obtained with the same trick. □

### 3. Gibbs $u$ -states for $\mathcal{F}$ -hyperbolic vector fields

In this section we consider a compact lamination  $(M, \mathcal{F})$  endowed with an  $\mathcal{F}$ -hyperbolic vector field  $X$  which is uniformly leafwise  $C^2$  and a Riemannian metric.

3.1. *Distortion control.* As explained in the previous section, the foliated unstable leaves form a foliation by  $C^2$  leaves depending  $C^2$ -continuously from the point. Hence, the restriction of the flow  $X_T$  is also uniformly  $C^2$ . Furthermore, for  $T$  large enough,  $X_{-T}$  induces a uniform contraction in the  $\mathcal{F}$ -unstable leaves. In particular, given two points  $x, y$  in the same  $\mathcal{F}$ -unstable leaf, the foliated distance  $d_{\mathcal{F}}(X_{-nT}(x), X_{-nT}(y))$  is decreasing exponentially (at rate  $\lambda^{-T}$  so that the sum  $\sum_0^{+\infty} d_{\mathcal{F}}(X_{-nT}(x), X_{-nT}(y))$  converges and is bounded proportionally to  $d_{\mathcal{F}}(x, y)$ ). Furthermore, the function  $\log \text{Det} D^u X_{-T}$  is uniformly  $C^1$  along the  $\mathcal{F}$ -unstable leaves, where the *unstable derivative*  $D^u X_{-T}$  is the restriction of the derivative  $D_{\mathcal{F}} X_{-T}$  to the unstable direction  $E^u$ .

One deduces the following foliated version of the classical distortion lemma.

LEMMA 3.1. *There exist  $C > 0$  such that, for every  $x, y \in M$  in the same  $\mathcal{F}$ -unstable leaf and for every  $T > 0$ , one has*

$$|\log|\text{Det}D^u X_{-T}(x)| - \log|\text{Det}D^u X_{-T}(y)|| < C d_{\mathcal{F}}(x, y)$$

*Proof.*

$$\begin{aligned} & \left| \log|\text{Det}D^u X_{-n}(x)| - \log|\text{Det}D^u X_{-n}(y)| \right| \\ & \leq \sum_0^{n-1} \left| \log|\text{Det}D^u X_{-1}(X_{-i}(x))| - \log|\text{Det}D^u X_{-1}(X_{-i}(y))| \right| \\ & \leq C_0 \sum_0^{n-1} d_{\mathcal{F}}(X_{-i}(x), X_{-i}(y)) \end{aligned}$$

where  $C_0$  is a bound for the derivative of  $\log |\text{Det}D^u X_{-1}|$  along the  $\mathcal{F}$ -unstable leaves, and the constant  $C$  is given by  $C = C_0 \cdot \sum_0^{+\infty} \lambda^{-i}$ . □

3.2. *Absolute continuity of the  $\mathcal{F}$ -stable and unstable foliations.* When  $X$  is uniformly  $C^2$  in the leaves, an important consequence of the distortion control is the absolute continuity in the leaves of  $\mathcal{F}$  of the  $\mathcal{F}$ -stable and  $\mathcal{F}$ -unstable foliations.

*Definition 3.2.*

- (1) Let  $M$  be a manifold and  $\mathcal{G}$  be a foliation on  $M$ . One says that  $\mathcal{G}$  is *absolutely continuous* with respect to Lebesgue measure in  $M$  if, given any foliated chart  $U$  of  $\mathcal{G}$  and  $\Sigma$  a cross-section intersecting the plaques  $P_x$  of  $\mathcal{G}$  in  $U$  in exactly one point, one has the following property. Given any measurable set  $A$  in  $U$  such that  $A$  has positive Lebesgue measure, consider the set  $\Sigma_A \subset \Sigma$  of points  $x$  in  $\Sigma$  so that the plaque  $P_x$  through  $x$  cuts  $A$  in a positive  $P_x$ -Lebesgue measure set. Then  $\Sigma_A$  has positive measure for the Lebesgue measure of  $\Sigma$ .
- (2) Assume that  $\mathcal{F}$  is a lamination or a foliation of  $M$  and that  $\mathcal{G}$  is a subfoliation of  $\mathcal{F}$ . One says that  $\mathcal{G}$  is *absolutely continuous with respect to Lebesgue measure in the leaves of  $\mathcal{F}$*  if, for any leaf  $\mathcal{L}$  of  $\mathcal{F}$ , the restriction of  $\mathcal{G}$  to  $\mathcal{L}$  is a foliation which is absolutely continuous with respect to Lebesgue measure in  $\mathcal{L}$ .

**THEOREM 3.3.** *When  $X$  is uniformly  $C^2$  in the leaves, the  $\mathcal{F}$ -stable and  $\mathcal{F}$ -unstable foliated invariant manifolds form foliations  $\mathcal{W}^s$  and  $\mathcal{W}^u$  of  $M$  which are absolutely continuous with respect to Lebesgue measure in the leaves of  $\mathcal{F}$ .*

The proof is identical to that for hyperbolic systems; see [9].

3.3. *Definition of Gibbs  $u$ -states.* The distortion control provides a map which associates to a pair  $(x, y)$  of points in the same leaf of  $\mathcal{W}^u$  the limit

$$\psi^u(x, y) = \lim_{T \rightarrow +\infty} \frac{\text{Det}D^u X_{-T}(y)}{\text{Det}D^u X_{-T}(x)} > 0.$$

(See [6, Theorem 11.8].) Furthermore, this map is uniformly Lipschitz along the unstable leaves for the distance  $d_{\mathcal{W}^u}$  along the unstable leaves. Finally, the map  $\psi^u$  depends continuously on the unstable leaves in the following sense. Let  $U$  be a foliated chart of the  $\mathcal{F}$ -unstable foliation. Then the map  $\psi^u$  is continuous on the set of pairs  $(x, y) \in U^2$  such that  $x$  and  $y$  belong to the same plaque.

Write  $\psi_x^u = \psi^u(x, \cdot)$ . Notice that, for  $y \in \mathcal{W}_x^u$ , one has

$$\psi_y^u = \psi_y^u(x) \cdot \psi_x^u.$$

Given a compact domain  $D$  in an unstable leaf, we denote by  $\mathcal{M}_D$  the probability measure on  $D$  whose density with respect to the unstable volume  $dm^u$  is proportional to  $\psi_x^u$  for  $x \in D$ . In other words, given  $x \in D$ , one gets

$$\mathcal{M}_D = \frac{\psi_x^u}{\int_D \psi_x^u dm^u} \cdot (dm^u|_D).$$

*Remark 3.4.* If  $U \subset V$  are two domains contained in an unstable leaf, then  $\mathcal{M}_U = 1/\mathcal{M}_V(U)\mathcal{M}_V|_U$ . The probability measure  $\mathcal{M}_D$  is the limit of the image by  $X_T$  of the normalized volume on  $X_{-T}(D)$ :

$$\mathcal{M}_D = \lim_{T \rightarrow +\infty} (X_T)_* \left( \frac{1}{m^u(X_{-T}(D))} (dm^u|_{X_{-T}(D)}) \right).$$

*Definition 3.5.* Let  $\mu$  be a probability on  $M$ . One says that  $\mu$  is a *Gibbs  $u$ -state* for the  $(M, \mathcal{F})$ -hyperbolic vector field  $X$  if:

- (1)  $\mu$  is invariant by the flow of  $X$ ;
- (2) for any foliated chart of the unstable foliation, the disintegration of  $\mu$  along the plaques induces the probability measure  $\mathcal{M}_P$  on  $\mu$ -almost every plaque  $P$ .

Let us state more precisely the second condition in the definition. Let  $\varphi: U \rightarrow V \times \Sigma \subset \mathbb{R}^k \times \Sigma$  be a foliated chart of the unstable foliation  $\{W_{\mathcal{F}}^u(x)\}_{x \in M}$ . Rokhlin’s disintegration theorem [20] asserts that there exist a probability measure  $\theta$  on  $\Sigma$  and a family of probabilities  $\{\mu_p\}_{p \in \Sigma}$  such that  $\mu_p(V \times \{p\}) = 1$  for  $\theta$ -almost every  $p \in \Sigma$  such that, for any continuous function  $h$  with support on  $V \times \Sigma$ , one has

$$\int_M h d\mu = \int_{\Sigma} \left( \int_{U \times \{p\}} h d\mu_p \right) d\theta(p).$$

$\mu$  is a Gibbs  $u$ -state if and only if  $\mu_p = \mathcal{M}_{V \times \{p\}}$  for  $\theta$  almost every  $p \in \Sigma$ . In other words,

$$\int_M h d\mu = \int_{\Sigma} \left( \int_{U \times \{p\}} h d\mathcal{M}_{V \times \{p\}} \right) d\theta(p).$$

### 3.4. Existence and properties of Gibbs $u$ -states.

**PROPOSITION 3.6.** *For any  $\mathcal{F}$ -unstable leaf  $\mathcal{L}^u$  and for any measurable set  $E$  contained in  $\mathcal{L}^u$  having a positive  $\mathcal{L}^u$ -Lebesgue measure  $0 < m^u(E) < +\infty$ , every accumulation point of the sequence of probability measures*

$$\mu_T := \frac{1}{T} \int_0^T X_{t*} \left( \frac{m_E^u}{m_E^u(E)} \right) dt, \quad T \rightarrow \infty,$$

*is a Gibbs  $u$ -state.*

*Proof.* As explained in the proof of [6, Lemma 11.12], we can assume without loss of generality that  $E$  is a domain inside an  $\mathcal{F}$ -unstable leaf. Then the same arguments as in the proof of Pesin and Sinai’s theorem [6, Theorem 11.8] yield the desired result.  $\square$

*Remark 3.7.* The conclusion of Proposition 3.6 extends to  $C^2$ -disks contained in a leaf  $\mathcal{L}$  of  $\mathcal{F}$  and transverse to the  $\mathcal{F}$ -center-stable foliation  $\mathcal{W}$ . In order to see this, just project such a disk onto an unstable domain in some unstable leaf, along the  $\mathcal{F}$ -center-stable foliation. The disk and its projection have the same average.

**PROPOSITION 3.8.** *The set  $\mathcal{G}_{\text{Gibbs}}^u$  of Gibbs  $u$ -states is a non-empty closed convex subset of the space of  $X$ -invariant probabilities. Given a Gibbs  $u$ -state, its ergodic components are Gibbs  $u$ -states.*

*Proof.* The fact that  $\mathcal{G}_{\text{Gibbs}}^u$  is non-empty follows from Proposition 3.6. The fact that  $\mathcal{G}_{\text{Gibbs}}^u$  is convex is straightforward. The fact that  $\mathcal{G}_{\text{Gibbs}}^u$  is closed comes from the fact that for any Gibbs  $u$ -state its disintegration along the leaves has prescribed density functions  $\psi^u$ , which depend continuously on the leaves.

Finally, let us show that the ergodic components of a Gibbs  $u$ -state  $\mu$  are Gibbs  $u$ -states. Recall that a point  $x$  is *regular* if its positive and negative average along the orbit are defined and are equal for every continuous function, and they correspond to the integral with respect to an ergodic measure. In other words, the probabilities

$$\mu_T(x) = \frac{1}{T} \int dt |_{X_{[0,T]}(x)} \quad \text{and} \quad \mu_{-T}(x) = \frac{1}{T} \int dt |_{X_{[-T,0]}(x)}$$

converge to the same ergodic probability measure  $\mu_x$ . The set of regular points has full  $\mu$  measure. Given a measure  $\nu$ , one says that a point  $x$  is regular for  $\nu$  if it is regular and if  $\mu_x = \nu$ .

The ergodic decomposition of  $\mu$  is given as follows. We consider the probability measure on the space of  $X$ -invariant ergodic probabilities  $\mathcal{P}(X)$  such that the measure of a subset  $\mathcal{P}$  of  $\mathcal{P}(X)$  is the  $\mu$  measure of the union of the regular points for the elements of  $\mathcal{P}$ . We can intersect this set of points with any measurable set of  $\mu$  measure 1.

As  $\mu$  is a Gibbs  $u$ -state, there is a set  $\mathcal{E}$  of full  $\mu$  measure of regular points for which the intersection of  $\mathcal{E}$  with the unstable leaf has total Lebesgue measure. Note that for  $x, y$  regular in the same unstable leaf, one has  $\mu_x = \mu_y$ . Hence the set of regular points for any probability in the ergodic decomposition of  $\mu$  contains a subsets of full Lebesgue measure for the corresponding unstable leaf.

That is, for any ergodic component  $\nu$  of  $\mu$ , there is at least one unstable leaf  $\mathcal{L}^u$  in which almost every point is regular for  $\nu$ . That is the limit of  $\mu_T(x)$  for almost every  $x \in \mathcal{L}^u$  is  $\nu$ . One deduces that the limit of Lebesgue measure restricted to a disk in  $\mathcal{L}^u$  is  $\nu$ . But such a limit is a Gibbs  $u$ -state according to Proposition 3.6. So  $\nu$  is a Gibbs  $u$ -state, concluding the proof. □

*Definition 3.9.* An  $X$ -invariant probability measure  $\mu$  on  $M$  is *absolutely continuous in the unstable direction* if for every laminated box  $U = V_{\mathcal{F}} \times \Sigma$  of  $\mathcal{W}^u$  such that  $\mu(U) > 0$  the conditional measure of  $\mu|_U$  with respect to the partition into unstable plaques  $\{W_{\mathcal{F}}^u \times \{y\} : y \in \Sigma\}$  is absolutely continuous with respect to Lebesgue measure along the corresponding plaque, for  $\mu$ -almost every plaque.

As a consequence of Propositions 3.6 and 3.8, one gets the following result.

**COROLLARY 3.10.** *If an  $X$ -invariant probability  $\mu$  is absolutely continuous in the unstable direction, then it is a Gibbs  $u$ -state.*

**3.5. Almost every orbit accumulates on Gibbs  $u$ -states.** We will now prove Proposition 1, which says that the  $X$ -orbit of Lebesgue almost every  $x \in M$  distributes according to a Gibbs  $u$ -state.

*Proof of Proposition 1.* The proof follows the lines of argument of the proof of the similar statement [6, Theorem 11.16] where one is assuming partial hyperbolicity. In the present case, we do not have the hypothesis of an invariant transversal bundle to the  $\mathcal{F}$ -unstable

foliation  $\mathcal{W}^u$  for which the flow  $X$  is expanding less than in  $\mathcal{W}^{cu}$ . What we do have is the foliation  $\mathcal{F}$ .

We first fix an unstable domain  $D$ , that is, a disk contained in an  $\mathcal{F}$ -unstable leaf  $\mathcal{L}$ . We want to show that, for almost every point  $x \in D$ , every accumulation point of the measures  $\mu_T(x) = 1/T dt|_{X_{[0,T]}(x)}$  is a Gibbs  $u$ -state. That is, the mass of  $\mu_T(x)$ , for large  $T$ , is well distributed along the unstable leaves according to the densities  $\psi_x^u$ . We will now formalize this idea.

Let  $U \subset M$  be a  $\mathcal{W}^u$ -foliated chart endowed with coordinates so that  $U = B \times Z$  where the  $B \times \{z\}$ ,  $z \in Z$ , are unstable disks; one chooses this foliated chart so that it is also a flow box for  $X$ : the set  $U$  is of the form  $X_{[0,1]}(B \times Y)$ , where  $\Sigma = B \times Y$  is a cross-section of  $X$ . Furthermore, we require that the unstable disks  $B \times \{y\}$ ,  $y \in Y$  are unstable disks of the same volume.

In order to understand the measure  $\mu_T(x)$  in  $U$  we have to understand the returns of the  $X$ -orbit of  $x$  in  $U$ . Notice that each component of a return map of the disk  $D$  on  $\Sigma$  is at constant time and is contained in a plaque  $B \times \{y\}$ : the flow preserves the  $\mathcal{F}$ -unstable leaves,  $D$  is contained in a  $\mathcal{F}$ -unstable leaf, and  $\Sigma$  is foliated by  $\mathcal{F}$ -unstable plaques.

One considers a disk  $A \subset B$  far from the boundary of  $B$ :  $A$  is a subdisk centered at the same point as  $B$  but with a radius 10 times smaller. Up to shrinking  $Y$ , one may assume that, for every  $y \in Y$ , the  $A \times \{y\}$  have the same volume  $m_{B \times \{y\}}(A \times \{y\})$  as  $A$ :

$$m_{B \times \{y\}}(A \times \{y\}) = m_B(A).$$

The proof involves showing that, for almost every point  $x$  of  $D$ , every accumulation point of the measures  $\mu_T(x) = 1/T dt|_{X_{[0,T]}(x)}$  gives  $A$  a weight smaller than  $K m_B(A)$  where  $K$  is a constant independent of  $A$ . That is expressed in the following proposition which is [6, Proposition 11.17].

Always following [6], let  $K_0 = (m_B(A)/m_B(B))C$ , where  $C$  is the distortion control bound given in Lemma 3.1.

PROPOSITION 3.11. *Let  $D$  be any unstable domain contained in a leaf of  $\mathcal{W}^u$ . There exists  $c > 0$  such that, given any subdisk  $A$  far from the boundary of  $B$ , there exists  $n_0$  such that, for  $T \geq n_0$ ,*

$$m_D\left(\left\{x \in D : \mu_T(x)X_{[0,1]}(A \times Y) \geq 10K_0 \frac{m_B(A)}{m_B(B)}\right\}\right) \leq e^{-cT}.$$

In order to prove Proposition 3.11 one first shows the following lemma.

LEMMA 3.12. *Let  $1 \leq k \leq n$  and  $0 \leq n_1 < \dots < n_k < n$  be fixed. Then the  $m_D$  measure of the set of points  $x \in D$  which hit  $A \times Y$  at times  $t_1, \dots, t_k$  such that each  $t_i \in [n_i, n_i + 1]$ , is bounded by  $(K_0(m_B(A)/m_B(\bar{B})))^k$ .*

Proposition 3.11 is deduced from Lemma 3.12 by using Stirling’s formula exactly as in [6, Lemma 11.20]. Let us just give the idea of the proof of Lemma 3.12 [6, Lemma 11.19].

*Proof.* First assume (even if it is not realistic) that the returns of the plaques  $A \times \{y\}$  to  $A \times Y$  are Markovian. This means that whenever  $X_t(A \times \{y\}) \cap (A \times Y) \neq \emptyset$ , it consists of a plaque of  $A \times Y$ . Then the distortion lemma implies that for each sequence  $t_1, \dots, t_k$

of return times of a plaque  $A \times \{y\}$  to  $A \times Y$ , the proportion of points having a return in  $A \times Y$  at time  $t_{i+1}$  will be exactly  $\mathcal{M}_D(A)$  times the proportion of points for which  $t_{i+1} - t_i$  is a return time to  $A \times Y$ , leading to the estimates claimed.

However, the returns are not Markovian and we need to deal with returns of  $A \times \{y\}$  which are only partially crossing  $A \times Y$ . However, these partial returns correspond to returns of points very close (exponentially close) to the boundary of  $A$ , because of the uniform dilation of  $A$  by the dynamics. [6, Lemma 11.19] shows that, replacing  $X_t$  by  $X_{t'}$  for some larger  $t'$  so that the expansion will be very large in comparison to the ratio of the radius of  $B$  and  $A$ , one can slightly increase  $A$  and  $B$  in  $\tilde{A} \subset B$  and  $\tilde{B}$  so that, for every return of  $A$  in  $B$ , the return of  $\tilde{B}$  covers the whole of  $\tilde{B}$ . This allows us to neglect the effect of the non-complete return. □

Proposition 3.11, together with a Borel–Cantelli argument, proves that, given any disk  $A \subset B$  far from the boundary of  $B$ , there exists a full  $m_D$ -measure subset of points  $x \in D$  such that

$$\mu_T(x)(X_{[0,1]}(A \times Y)) \leq 10K_0 \frac{m_B(A)}{m_B(B)}$$

for all but at most a finite-length set of  $T \in [0, \infty)$ . Consequently, the same upper bound holds for  $\mu(X_{[0,1]}(A \times Y))$ , for any accumulation point  $\mu$  of the measures  $\mu_T$ . Applying the conclusion to a countable generating family of disks far from the boundary of  $B$ , and then taking the intersection of the corresponding full measure subset of  $D$ , we get a full  $m_D$ -measure subset of points  $x \in D$  for which

$$\mu(X_{[0,1]}(A \times Y)) \leq 10K_0 \frac{m_B(A)}{m_B(B)}$$

for any measurable set  $A$  far from the boundary of  $B$ .

To complete the proof, cover the manifold by foliated boxes  $X_{[0,1]}(B \times Y)$  such that the foliated boxes obtained by replacing each  $X_{[0,1]}(B \times Y)$  by  $X_{[0,1]}(A \times Y)$ —where  $A$  is, as before, far from the boundary of  $B$ —still cover  $M$ . If  $\mu(X_{[0,1]}(B \times Y)) = 0$  there is nothing to prove. Otherwise, by the previous paragraph,

$$\frac{\mu(X_{[0,1]}(A \times Y))}{\mu(X_{[0,1]}(B \times Y))} \leq \frac{10K_0}{\mu(X_{[0,1]}(B \times Y))} \frac{m_B(A)}{m_B(B)}$$

for every disk  $A \subset B$  far from the boundary. This implies that  $\mu$  is a Gibbs  $u$ -state, as claimed. □

#### 4. Laminations by leaves of negative curvature

4.1. *Gibbs  $u$ -states.* In this section, we consider laminations  $\mathcal{F}$  whose leaves have dimension  $n$ , admit a metric with strictly negative curvature, and the metric depends continuously on the plaques for the  $C^r$ -topology,  $r \geq 3$ . We denote by  $(T^1\mathcal{F}, \tilde{\mathcal{F}})$  the unit tangent bundle to the foliation, which is itself a space laminated by  $(2n - 1)$ -dimensional leaves, each leaf being the unit tangent bundle of a leaf of  $(M, \mathcal{F})$  which carries a natural laminated Riemannian metric coming from the Riemannian metric on  $M$  and the unit spherical metric on the tangent vectors to the lamination. Let us call  $X$  the vector field on  $T^1\mathcal{F}$  generating the laminated geodesic flow  $X_t, \pi : T^1\mathcal{F} \rightarrow M$  the canonical projection,

and  $\pi_*$  the map it induces between measures on  $T^1\mathcal{F}$  and measures on  $M$ . The fact that the metric tensor varies from leaf to leaf in a continuous way with respect to the  $C^r$ -topology implies that the flow  $X_t$ , which has continuous dependence on the derivatives of the metric up to order 2, is a continuous flow on the space  $T^1\mathcal{F}$ .

Then the foliated geodesic flow defined on  $T^1\mathcal{F}$  is uniformly  $C^2$  in the leaves and is foliated hyperbolic. Then Proposition 1 implies that, at every point  $x \in M$  and for Lebesgue almost every direction  $v \in T_x^1\mathcal{F}$ , every accumulation point of the probabilities associated to the positive orbit of the geodesic flow is a Gibbs  $u$ -state.

The set of probabilities  $\mathcal{G}_{\text{Gibbs}}^u$  on  $T^1\mathcal{F}$  which are Gibbs  $u$ -states for  $X$  is a closed convex subset of the set of probability measures invariant under the geodesic flow. For every Gibbs  $u$ -state, its decomposition in ergodic components is through ergodic Gibbs  $u$ -states (Proposition 3.8).

As a direct consequence, the projections on  $M$  of these Gibbs  $u$ -states are the probability measures which describe the asymptotic behavior of almost all geodesics in each plaque. In the case of hyperbolic leaves, these measures coincide with the harmonic measures [11]. However, in the case of variable curvature, these measures are *a priori* no longer harmonic measures. These measures form a new class of measures which are naturally associated to the lamination and are of interest for understanding the dynamics of the lamination  $\mathcal{F}$ . They have also been studied by Alvarez in [1–3], where their relations with other classes of measures (e.g. harmonic measures) are clearly explained.

**THEOREM 4.1.** *Let  $(M, \mathcal{F})$  be a  $C^3$ -lamination with a plaque Riemannian metric of negative sectional curvature and let  $\mu$  be a Gibbs  $u$ -state of the geodesic flow on  $T^1\mathcal{F}$ . Let  $\pi : T^1\mathcal{F} \rightarrow M$  be the natural projection. Then  $\nu := \pi_*\mu$  is a probability measure on  $M$  whose disintegration  $\nu_k$  along the plaques of  $\mathcal{F}$  satisfies the following statements.*

- *The  $\{\nu_k\}$  are absolutely continuous with respect to the Lebesgue measure of the plaques of  $\mathcal{F}$ .*
- *The density of  $\nu_k$  with respect to Lebesgue measure on the plaques is strictly positive on the whole plaque, and its logarithm is uniformly Lipschitz along the plaques on the whole manifold.*

*In particular, the support of  $\nu$  is  $\mathcal{F}$  invariant (it consists of entire leaves).*

*Proof.* Let  $\mathcal{L}_x, x \in M$ , be a leaf of  $\mathcal{F}$  and consider a direction  $v \in T_x^1\mathcal{L}_x$ . Consider now the center-stable manifold  $W^{cu}(v)$  of  $v$  for the geodesic flow. It consists of all the vectors in  $T_x^1\mathcal{L}_x$  whose corresponding geodesic tends to the same point of the boundary at infinity of the universal cover  $\tilde{\mathcal{L}}_x$ . We fix a lift  $\tilde{x}$  of  $x$  on  $\tilde{\mathcal{L}}_x$ .

$W^{cu}(v)$  is either diffeomorphic to a Euclidian space (if  $v$  does not belong to the stable manifold of a closed orbit) or to a cylinder. We denote by  $\tilde{W}^{cu}(v)$  its universal cover and we fix a lift  $\tilde{v}$  of  $v$ . The natural projection of  $\tilde{W}^{cu}(v)$  on  $\tilde{\mathcal{L}}_x$  sending  $\tilde{v}$  on  $\tilde{x}$  is a diffeomorphism. Hence we endow every center-unstable plaque with the lifted metric.

The disintegration of  $\mu$  along center-stable plaques has a prescribed positive density whose logarithm is uniformly Lipschitz along the plaques. From the invariance of the metric by the geodesic flow and the fact that the action of the geodesic flow on the center-stable plaque has a bounded Jacobian, one deduces that the disintegration of  $\mu$  on center-stable plaques has a logarithm that is uniformly Lipschitz with a prescribed constant.

If a family of positive functions has a logarithm which is Lipschitz for a given constant, then its sum or average has the same property. One deduces that the projection of  $\mu$  on  $\tilde{\mathcal{L}}_x$  and therefore on  $\mathcal{L}_x$  has a density with respect to the Lebesgue measure whose logarithm is uniformly Lipschitz. As a consequence, the density is bounded away from 0 and  $+\infty$  in all compact domains.  $\square$

**4.2. Laminations by hyperbolic leaves.** We will now take  $M$  to be a compact metrizable space with a lamination  $\mathcal{F}$  by hyperbolic manifolds of dimension  $n$  (each leaf carries its metric of sectional curvature  $-1$ ), such that the variation of the metric, as well as its derivatives, is continuous in the transverse directions. The unstable lamination in  $(T^1\mathcal{F}, \tilde{\mathcal{F}})$  will be called the unstable-horosphere lamination  $\mathcal{W}^u$ . The laminated geodesic flow is preserving the foliation  $\mathcal{W}^u$  (but not the horospheres themselves), being a uniform expansion on the horospheres. By a *horocycle measure* we mean a measure on  $T^1\mathcal{F}$  which is invariant under the geodesic flow and such that its disintegration with respect to  $\mathcal{W}^u$  is a constant multiple of the volume of the horospheres. In this case, they coincide with the Gibbs  $u$ -states, since in fact the unstable Jacobian equals  $n - 1$ , so  $\psi^u(x, y) \equiv 1$ .

**THEOREM 4.2.** [5, 11] *Let  $(M, \mathcal{F})$  be a compact lamination by hyperbolic manifolds.  $\pi_*$  induces a bijective correspondence between horocycle measures on  $T^1\mathcal{F}$  which are invariant under the laminated geodesic flow and harmonic measures on  $(M, \mathcal{F})$ .*

**COROLLARY 4.3.** *Every Gibbs  $u$ -state on  $T^1\mathcal{F}$  projects to a harmonic measure on  $M$ .*

### 5. Normal Lyapunov exponents and SRB measures

Let  $\mathcal{F}$  be a smooth foliation on a compact manifold  $M$  and  $X$  a uniformly  $C^r$  vector field tangent the leaves,  $r \geq 2$ . Let  $N_{\mathcal{F}}$  be the normal bundle of  $\mathcal{F}$ , that is, the quotient of the tangent bundle of  $M$  by the tangent bundle to the foliation. To each path  $\gamma$  in a leaf, one may associate the linear holonomy from  $N_{\mathcal{F}}(\gamma(0)) \rightarrow N_{\mathcal{F}}(\gamma(1))$ . In this way, one defines a flow  $\mathcal{X}$  on  $N_{\mathcal{F}}$  as follows:

- $\mathcal{X}$  is a linear cocycle over  $X$ , that is,  $\mathcal{X}$  leaves invariant the bundle structure over  $M$ , the action on the leaves is linear and  $\mathcal{X}$  projects on  $X$ ;
- $\mathcal{X}_t|_{N_{\mathcal{F}}(x)}: N_{\mathcal{F}}(x) \rightarrow N_{\mathcal{F}}(X_t(x))$  is the linear holonomy of  $\mathcal{F}$  over the path  $X_{[0,t]}(x)$ .

For time 1 we obtain a linear cocycle over a continuous dynamical system on a compact manifold, and we may apply Oseledets's theorem [16]. Hence for every  $X$ -invariant ergodic probability there is an associated measurable  $\mathcal{X}$ -invariant decomposition of the normal bundle in Lyapunov spaces on which the rate of exponential growth of the vectors is well defined and constant and called the *normal Lyapunov exponents* of the ergodic measure.

We will now give a proof of Proposition 2. The first statement follows from classical results due to Pesin and to Pugh and Shub, together with Oseledets's theorem.

*Proof of Proposition 2.*

(1) The flow  $X_t$  is a smooth flow on  $M$  whose Lyapunov exponents for the measure  $\mu$  are  $\lambda^c = 0$  (corresponding to the direction of the flow),  $\lambda_j^s < 0$ ,  $\lambda_j^u > 0$  (corresponding to the directions of the stable and unstable tangential directions to  $\mathcal{F}$ , respectively) and the normal exponents  $\lambda_j$  which are negative by hypothesis. Oseledets's theorem gives us a

set  $R$  (the *regular set*) that has full measure with respect to  $\mu$ , and over which there is an invariant measurable splitting

$$T_R M = E^s \oplus E^c \oplus E^u \oplus E^{\text{h}} \tag{5.1}$$

such that, for every  $x \in R$ ,

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \log \|D_x X_t(v)\| &< 0 && \text{if } v \in E^s \text{ or } v \in E^{\text{h}} \\ \lim_{t \rightarrow \infty} \frac{1}{t} \log \|D_x X_t(v)\| &= 0 && \text{if } v \in E^c \\ \lim_{t \rightarrow \infty} \frac{1}{t} \log \|D_x X_t(v)\| &> 0 && \text{if } v \in E^u. \end{aligned} \tag{5.2}$$

The Pesin stable manifold theorem (see [18] or [14]) tells us that, for  $\mu$ -almost every  $x \in R$ , the local stable manifold at  $x$ .

$$W^s(x) = \left\{ y \in M : \limsup_{t \rightarrow \infty} \frac{1}{t} \log d(X_t(x), X_t(y)) < 0 \right\}.$$

is an embedded smooth disk. Furthermore, Pesin proved that this foliation is absolutely continuous, see [18].

Under these circumstances, there is a well-known theorem due to Pugh and Shub (see [19, Theorem 3]) that guarantees that the measure  $\mu$  is an SRB measure for the flow  $X_t$ .

(2) The arguments in the proof of this proposition are contained in [8], although they are stated for and used in a different context – that of partial hyperbolicity with an additional assumption called ‘mostly contracting center direction’. The strong similarity of the ‘mostly contracting’ condition in the partially hyperbolic setting and negativity of normal Lyapunov exponents in the setting of foliated hyperbolicity allows us to follow [8] closely.

More precisely, the difficulty in proving the finiteness of ergodic attractors stems from the fact that the Pesin stable manifolds (tangent to the stable direction and the transverse direction) vary in a measurable way. In particular, we have no *a priori* lower bound on their size. This problem is handled in [8, Proposition 2.1], where the absolute continuity of the Pesin stable ‘foliation’ is also established. This in turn implies, as a consequence of [8, Corollary 2.2] and Hopf’s argument [9], that accessibility classes are open, which is [8, Lemma 2.7]. The finiteness of Gibbs  $u$ -states follows, as in [8, Lemma 2.9]. □

### 6. SRB measures for the foliated geodesic flow: visibility functions of the attractors

In this section we will consider a smooth foliation  $\mathcal{F}$  of the compact manifold  $M$  and a  $C^3$  Riemannian metric on  $M$  which induces negative curvature on all the leaves of  $\mathcal{F}$ . We denote by  $\tilde{\mathcal{F}}$  the foliation on the unit tangent bundle of the foliation whose leaves are the unit tangent bundles of the leaves. The geodesic flow  $X_t$  leaves the foliation  $\tilde{\mathcal{F}}$  invariant.

We assume here that all the transverse Lyapunov exponents of the foliated geodesic flow are negative for all ergodic Gibbs  $u$ -states. In other words, one assume the hypotheses of part 2 in Proposition 2. As a corollary of Propositions 1 and 2 one obtains the following result.

**COROLLARY 6.1.** *Given any point  $x \in M$ , for Lebesgue almost all unit vectors  $v \in T_x^1 \mathcal{F}$ , the corresponding orbit of the foliated geodesic flow belongs to the basin  $\mathcal{B}(\mu_i)$  of one of the SRB measures  $\mu_i$ .*

*Proof.* Consider a plaque  $\Delta$  of  $\mathcal{F}$  containing the point  $x$  and consider the unit tangent bundle  $T^1\Delta$ . According to Proposition 1, for almost every vector  $v$  in  $T^1\Delta$ , any accumulation point of the averages  $v_{T,v} = X_*(\cdot, v)|_{[0,T]}dt/T$  is a Gibbs  $u$ -state. Let  $\mu_v$  be one of these accumulation points. As stated in Proposition 3.8, its ergodic components are also Gibbs  $u$ -states. Proposition 2 tells us that there are only finitely many ergodic Gibbs  $u$ -states  $\mu_1, \dots, \mu_n$ , and therefore  $\mu_v$  is simply a finite convex linear combination  $\mu_v = \sum_{i=1}^n \alpha_i \mu_i$ . Let us pick an  $i$  for which  $\alpha_i \neq 0$ . We will prove that  $\mu_v = \mu_i = \lim_{T \rightarrow \infty} v_{T,v}$ .

The orbit of  $v$  must come arbitrarily close to the minimal set  $\mathcal{M}_i$  which is the support of  $\mu_i$ . Inside  $\mathcal{M}_i$ , the set of points for which the Pesin stable manifold exists is of full  $\mu_i$  measure. The size of the stable manifolds may not be uniformly big, but there exists a subset of  $\mathcal{M}_i$  of positive measure of points for which both the size of the Pesin stable manifold and the angle between  $E^{\text{rh}}$  and  $T\mathcal{F}$  are uniformly bounded below. The orbit of  $v$  must also accumulate on this positive measure set, and therefore there exist a time  $t$  and a point  $w$  which is regular for  $\mu_i$  such that  $X_t(v) \cap W^s(w) \neq \emptyset$ . This readily implies that  $\mu_i = \lim_{T \rightarrow \infty} v_{T,v}$ .

We have therefore established for almost every vector  $v$  in  $T^1\Delta$  that there is  $i$  so that the orbit of  $X_t$  through  $v$  belongs to the basin of  $\mu_i$ . Therefore, by the Fubini theorem, there is a point  $y \in \Delta$  so that the set  $T_y^1\Delta \cap \bigcup_i \mathcal{B}(\mu_i)$  has total Lebesgue measure. Now each  $\mathcal{B}_i$  is invariant under the  $\mathcal{F}$ -center-stable foliation. Thus the image of  $T_y^1\Delta \cap \bigcup_i \mathcal{B}(\mu_i)$  by the center-stable holonomy from  $T_y^1\Delta$  to  $T_x^1\Delta$  is contained in  $T_x^1\Delta \cap \bigcup_i \mathcal{B}(\mu_i)$ . As the center-stable foliation is absolutely continuous in the leaves of  $\mathcal{F}$ , one gets that  $T_x^1\Delta \cap \bigcup_i \mathcal{B}(\mu_i)$  has total Lebesgue measure, as claimed.  $\square$

This property makes it natural to make the following definition.

*Definition 6.2.* The *visibility function* of the measure  $\mu_i$ , denoted by  $f_i : M \rightarrow [0, 1]$ , is defined by

$$f_i(x) = \text{Leb}_x(T_x^1\mathcal{F} \cap \mathcal{B}(\mu_i)),$$

where  $\text{Leb}_x$  is the spherical measure on  $T_x^1\mathcal{F}$ .

Then Corollary 6.1 implies that  $\sum_i f_i(x) = 1$  for every  $x \in M$ .

*Remark 6.3.*  $\mathcal{B}(\mu_i)$  is invariant for the center-stable foliation inside the leaves of  $\tilde{\mathcal{F}}$ .

According to the above remark, the functions  $f_i$  vary smoothly along the leaves, and the set of points for which  $f_i(x) > 0$  is saturated for the foliation  $\mathcal{F}$ . More precisely, each basin corresponds to some subset of the boundary at infinity of the universal cover of the leaf, and  $f_i$  is the mass of the vectors corresponding to this set at infinity.

We are now in a position to prove Theorem 1.

*Proof of Theorem 1.* Every orbit in the basin of an SRB measure has well-defined transverse Lyapunov exponents for positive time, and these Lyapunov exponents are negative. This is because any orbit in this basin must, at some point, intersect the Pesin stable manifold of some point which is regular for the measure (as explained earlier in this section, in the proof of Corollary 6.1). Hence the orbits have a transverse Pesin stable

manifold, and this family of stable manifolds is absolutely continuous with respect to Lebesgue measure. One deduces that the functions  $f_i$  are lower semicontinuous. As the sum of the  $f_i$  is 1, the semicontinuity of the  $f_i$  implies the continuity.  $\square$

7. *Transversely conformal foliations with a  $C^2$   $\mathcal{F}$ -hyperbolic vector field*

7.1. *Negativity of the Lyapunov exponent.* Throughout this section, let  $(M, \mathcal{F})$  be a compact manifold with a smooth transversely conformal foliation and  $X$  a  $C^2$   $\mathcal{F}$ -hyperbolic vector field. In this case, the normal Lyapunov exponents are all equal.

We will make the additional assumption that the foliation is endowed with a Riemannian metric along the leaves that has a Hölder variation in the transverse direction.

Under the above hypothesis, Proposition 2 says that if  $\mu$  is an ergodic Gibbs  $u$ -state for  $X$  and its normal Lyapunov exponent is negative, then  $\mu$  is an SRB measure.

A special case of this is the following result.

**COROLLARY 7.1.** *Let  $(M, \mathcal{F})$  be a compact manifold with a smooth transversely conformal foliation, and suppose that  $M$  has a  $C^2$ -Riemannian metric that induces negative curvature on the leaves. Denote by  $X$  the generator of the foliated geodesic flow on  $T^1\mathcal{F}$  and let  $\mu$  be an ergodic Gibbs  $u$ -state for  $X$ . If the Lyapunov exponent of the normal bundle of  $\mathcal{F}$  along  $X$  is negative, then  $\mu$  is an SRB measure for the foliated geodesic flow.*

In the special case where the leaves have not only negative curvature but also constant curvature  $-1$ , this gives rise to Theorem 2, the proof of which is the object of the present section.

Let us fix a transverse metric compatible with the conformal structure on transversals;  $|\cdot|$  is its associated norm. If  $\gamma : [0, +\infty) \rightarrow M$  is a path lying on a single leaf, let  $h_{\gamma|_{[0,t]}}$  be the holonomy transformation corresponding to the path  $\gamma$  between times 0 and  $t$ . The Lyapunov exponent corresponding to  $\gamma$  is the limit

$$\lambda(\gamma) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log |Dh_{\gamma|_{[0,t]}}|, \tag{7.3}$$

if it exists. (When writing  $Dh_{\gamma|_{[0,t]}}$  we omit the point  $\gamma(0)$  where we are taking the differential.)

The Birkhoff ergodic theorem and the random ergodic theorem tell us that for  $\nu$ -almost every  $x \in M_0$  and almost every Brownian path  $\gamma$  starting at  $x$ , the above limit exists and is independent of both  $x$  and  $\gamma$ . It is called the Lyapunov exponent of the measure  $\nu$ ,  $\lambda(\nu)$ .

The following result can be found in [12].

**THEOREM 7.2.** (Deroin–Kleptsyn) *Let  $(M, \mathcal{F})$  be a compact minimal subset of a transversely conformal foliation having a Riemannian metric whose variation in the transverse direction is Hölder-continuous. Let  $\nu$  be an  $(M, \mathcal{F})$  harmonic measure. If  $(M, \mathcal{F})$  admits no holonomy invariant measures, then  $\lambda(\nu) < 0$ .*

The requirement that the transverse variation of the metric be Hölder continuous is necessary; see [13].

Let  $\mu^+$  be the ergodic horocycle measure on  $T^1\mathcal{F}$  which is invariant under the geodesic flow and projects onto  $\nu$ . It is an ergodic measure for the geodesic flow, and we can define

the Lyapunov exponent  $\lambda(\mu^+)$  as the Lyapunov exponent of the holonomy along a generic geodesic, namely

$$\lambda(\mu^+) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log |Dh_{\gamma|_{[0,t]}}|, \tag{7.4}$$

where  $\gamma(s) = g_s(v)$  and  $v \in T^1\mathcal{F}$  is generic for  $\mu^+$ .

LEMMA 7.3. *With the above notation,*

$$\lambda(v) = \lambda(\mu^+). \tag{7.5}$$

The remainder of this subsection will be devoted to the proof of this result. We will need expressions for the Lyapunov exponents  $\lambda(v)$  and  $\lambda(\mu^+)$  which are easier to deal with. We follow [12, §3.1].

Consider a holonomy-invariant local normal vector field  $u$  in  $T^1\mathcal{F}$ . Such a vector field can be defined in every foliated chart, and its norm  $|u|$  is globally defined up to a multiplicative constant on plaques. Therefore,  $\omega = d \log |u|$  is a globally defined foliated 1-form on  $M$ . (Here  $d$  is the exterior derivative in the leaf direction.) Together with the Riemannian metric on the leaves, it determines  $\tilde{Y} = \nabla \log |u|$ , which is a vector field along the leaves of  $T^1\mathcal{F}$ .

Applying the same argument to the foliation  $\mathcal{F}$ , we obtain the vector field  $Y = \nabla \log |u|$  on  $M$  which is tangent to the leaves of  $\mathcal{F}$ .

As in [12], we can write the Lyapunov exponent of the ergodic harmonic measure  $\nu$  as

$$\lambda(v) = \int_M \Delta \log |u| d\nu = \int_M \operatorname{div} Y d\nu, \tag{7.6}$$

where  $\operatorname{div}$  is the leafwise divergence.

Now consider a path  $\gamma : [0, T] \rightarrow T^1\mathcal{F}$  lying on a leaf, and let  $h_\gamma$  be the holonomy transformation it determines. The contraction or expansion of the holonomy along  $\gamma$  is

$$|Dh_\gamma| = e^{\int_\gamma \omega}.$$

Therefore, the Lyapunov exponent along a geodesic orbit is

$$\lambda(v) = \lim_{t \rightarrow +\infty} \frac{1}{t} \int_0^t \omega(g_s(v)) ds. \tag{7.7}$$

If  $X$  is the vector field that directs the foliated geodesic flow on  $T^1\mathcal{F}$ , the Lyapunov exponent of the measure  $\mu^+$  is

$$\lambda(\mu^+) = \int_{T^1\mathcal{F}} \lambda(v) d\mu^+(v) = \int_{T^1\mathcal{F}} \omega(X) d\mu^+. \tag{7.8}$$

Disintegration of the harmonic measure  $\nu$  on a flow box of  $(M, \mathcal{F})$  yields a measure on its transversal and conditional measures of the form  $h \cdot \operatorname{vol}$  on its plaques, where  $h$  is a positive function which is harmonic in the leaf direction and  $\operatorname{vol}$  is the volume on leaves. On two overlapping flow boxes, the corresponding harmonic densities  $h$  and  $h'$  coincide up to multiplication by a positive function which is constant on plaques. Therefore,  $\eta = d \log h$  is a well-defined foliated 1-form, which is called the *modular form* of  $\mathcal{F}$ .

The following result, which we will use to relate the exponents  $\lambda(v)$  and  $\lambda(\mu^+)$ , appears in [10, p. 210].

LEMMA 7.4. *The modular form satisfies*

$$\int_M \operatorname{div} Y \, d\nu = - \int_M \eta(Y) \, d\nu$$

for every vector field  $Y$  along the leaves with an integrable divergence.

Taking  $Y = \log |u|$  as above, the term on the left is the Lyapunov exponent of the measure  $\nu$ . Equation (7.5) translates into

$$- \int_M \eta(Y) \, d\nu = \int_{T^1\mathcal{F}} \langle \tilde{Y}, X \rangle \, d\mu^+. \tag{7.9}$$

(Here  $\langle \cdot, \cdot \rangle$  is the leafwise metric on  $\mathcal{F}$  in the term on the left and the leafwise metric on  $T^1\mathcal{F}$  in the term on the right.)

Consider a partition of unity  $\{\varphi_i\}$  subordinated to a foliated atlas on  $M$  and the corresponding partition of unity  $\{\varphi_i \circ \pi\}$  on  $T^1\mathcal{F}$ . The proof of equation (7.9) is reduced to a local computation. So let  $E \simeq D \times T$  be a foliated chart on  $M$ , where  $D$  is a ball in hyperbolic space, and  $\pi^{-1}(E) \simeq D \times T \times S^{n-1}$  its preimage on  $T^1\mathcal{F}$ . Notice that the transverse components of  $\nu$  and  $\mu^+$  are equal, and therefore equation (7.9) must actually be an equation on each plaque. Namely, we have to prove that

$$- \int_D \langle Z(x), Y(x) \rangle h(x) \, dx = \int_{D \times S^{n-1}} \langle X, \tilde{Y} \rangle \, d\mu_{D \times S^{n-1}}^+, \tag{7.10}$$

where  $h$  is the harmonic density induced by  $\nu$  in a plaque of  $E$ ,  $\mu_{D \times S^{n-1}}^+$  is the conditional measure induced by  $\mu^+$  on the corresponding plaque in  $\pi^{-1}(E)$ , and  $Z = \nabla \log h$ .

Each plaque in  $\pi^{-1}(E)$  is itself foliated by (local) center-unstable leaves, parametrized by  $\xi \in \partial D$ . We define a 1-form which is a foliated form for this center-unstable foliation by the expression

$$\tilde{\eta} = d \log k(\cdot, \xi)$$

on the leaf corresponding to  $\xi$ , where  $k$  is the Poisson kernel on the  $D$ . This is a globally defined 1-form on  $T^1\mathcal{F}$ .

LEMMA 7.5.

$$\int_{T^1\mathcal{F}} \tilde{\eta}(\tilde{Y}) \, d\mu^+ = \int_M \eta(Y) \, d\nu. \tag{7.11}$$

Equation (7.10) follows from this lemma, and the proof of the lemma is straightforward. Nevertheless, we present it in full detail.

*Proof.* It is enough to prove the lemma in a foliated chart  $E \simeq D \times T$  where  $\pi^{-1}(E) \simeq D \times S^{n-1} \times T$ . Since disintegration of the measures  $\nu$  in  $E$  and  $\mu^+$  in  $\pi^{-1}(E)$  yields the same measure on  $T$ , it is in fact enough to prove it at plaque level.

Notice that  $\tilde{Y} = \pi^* Y$ , and therefore it is constant on the fibers of  $\pi$ .

If  $h$  is a positive harmonic function on  $D$ , we call  $m_h$  the measure on  $S^{n-1} = \partial D$  which corresponds to  $h$  via Poisson’s representation formula. The symbol  $\nabla$  will always refer to

the gradient with respect to  $x \in D$  with the leafwise metric. We have

$$\begin{aligned}
 & \int_{S^{n-1}} \int_D \tilde{\eta}(\tilde{Y})k(x, \xi) dx dm_h(\xi) \\
 &= \int_{S^{n-1}} \int_D \langle \nabla \log k(x, \xi), \nabla \log |u|(x) \rangle k(x, \xi) dx dm_h(\xi) \\
 &= \int_D \int_{S^{n-1}} \left\langle \frac{\nabla k(x, \xi)}{k(x, \xi)}, \nabla \log |u|(x) \right\rangle k(x, \xi) dm_h(\xi) dx \\
 &= \int_D \left\langle \int_{S^{n-1}} \nabla k(x, \xi) dm_h(\xi), \nabla |u|(x) \right\rangle dx \\
 &= \int_D \left\langle \nabla \int_{S^{n-1}} k(x, \xi) dm_h(\xi), \nabla |u|(x) \right\rangle dx \\
 &= \int_D \langle \nabla h(x), \nabla \log |u|(x) \rangle dx \\
 &= \int_D \langle \nabla \log h(x), \nabla \log |u|(x) \rangle h(x) dx.
 \end{aligned}$$

Finally, we will complete the proof of Theorem 2. □

*Proof of Theorem 2.* Equation (7.10) follows directly from Lemma 7.5 when we observe that on the center-unstable leaf for the geodesic flow in  $D \times S^{n-1} \simeq T^1D$  corresponding to geodesics born at  $\xi \in S^{n-1}$ ,  $X = -\nabla \log k(\cdot, \xi)$ . □

**7.2. The regular orbits.** We have seen that almost every positive orbit of the foliated geodesic flow follows one of the SRB measures corresponding to the harmonic measures. That is, if  $\nu_1, \dots, \nu_k$  are the ergodic harmonic measures on  $(M, \mathcal{F})$ , they give rise to  $\mu_1^+, \dots, \mu_k^+$  which are SRB measures for the positive orbits of the foliated geodesic flow. We can follow the same line of reasoning for the negative orbits of the geodesic flow, thus obtaining measures  $\mu_1^-, \dots, \mu_k^-$  which also project onto the  $\nu_i$ . In general the measures  $\mu_i^+$  and  $\mu_i^-$  do not coincide. They do so if and only if the harmonic measure  $\nu_i$  is *totally invariant*, which means that in each foliated chart it can be expressed as a transverse holonomy-invariant measure times the leaf volume. This is the content of the following proposition.

**PROPOSITION 7.6.** *If  $\mu$  is a measure on  $T^1\mathcal{F}$  which is invariant by the geodesic flow and which has constant density along both stable and unstable horospheres, then it projects on  $M$  as a measure whose disintegration in the leaves is the volume in the leaves, implying that this measure is locally the product of the volume in the leaves by a transverse invariant measure.*

*Proof.* This is exactly the same proof as in [7], and can also be found in [11, Theorem 5.2]. □

Recall that, according to Birkhoff’s theorem, almost every orbit (for any invariant measure) is regular in the sense that the positive and negative averages exist and are equal.

PROPOSITION 7.7. *Assume that  $\mathcal{F}$  does not admit any transverse invariant measure. Then the set of regular points in  $T^1\mathcal{F}$  has Lebesgue measure equal to 0.*

*Proof.* For each  $i$ , let  $\mathcal{B}(\mu_i^+)$  be the basin of attraction of  $\mu_i^+$ , and  $\mathcal{B}(\mu_i^-)$  the set of points whose negative orbit is contained in the basin of attraction of  $\mu_i^-$ . Assume that  $E \subset \tilde{M}$  is a positive Lebesgue measure set of regular points of the foliated geodesic flow. Up to removing from  $E$  a zero Lebesgue measure set, one may assume that almost every  $x \in E$  belongs to some  $\mathcal{B}(\mu_i^+)$  and to some  $\mathcal{B}(\mu_j^-)$ . As there are finitely many positive and negative SRB measures, this implies that there are  $i, j$  and a positive Lebesgue measure set  $E_0 \subset E$  included in  $\mathcal{B}(\mu_i^+) \cap \mathcal{B}(\mu_j^-)$ . However, as the points in  $E$  (hence in  $E_0$ ) are assumed to be regular, the positive and negative averages coincide, that is, the integral of every continuous function for  $\mu_i^+$  or  $\mu_j^-$  are equal, meaning that  $\mu_i^+ = \mu_j^-$ . Since  $\mu_i^+$  and  $\mu_j^-$  project onto  $v_i$  and  $v_j$  respectively, this immediately means that  $i = j$ . Therefore  $\mu_i^+ = \mu_i^-$  is a measure which is invariant by the geodesic flow, and it has constant density along both stable and unstable horospheres. This, together with Proposition 7.6, gives a contradiction. □

COROLLARY 7.8. *If  $\mathcal{F}$  has no transverse holonomy-invariant measure, each of the measures  $\mu_i^+$  is singular with respect to Lebesgue measure on  $T^1\mathcal{F}$ .*

For  $x \in M$  and  $v \in T_x^1\mathcal{F}$ , let  $\ell_{(x,v)}(T)$  be the geodesic segment in the leaf  $\mathcal{L}_x$  centered at  $x$  and directed by  $v$ . Let  $m_{(x,v)}(T)$  the normalized length measure of  $\ell_{(x,v)}(T)$ , which is a probability measure on  $M$ .

PROPOSITION 7.9. *If  $\mathcal{F}$  admits only one harmonic measure  $\nu$  which is not totally invariant, then for Lebesgue almost every  $x \in M$  and Lebesgue almost every  $v \in T_x^1\mathcal{F}$ , the measures  $m_{(x,v)}(T)$  converge to  $\nu$  as  $T \rightarrow +\infty$ .*

*Proof.* If  $(M, \mathcal{F})$  admits only one harmonic measure  $\nu$  which is not totally invariant, Lebesgue almost every orbit of the geodesic flow in  $T^1\mathcal{F}$  distributes according to  $\mu^+$  in the future and  $\mu^-$  in the past. Such orbits project onto geodesics with the desired property. □

*Example 2.* (Riccati equations) Let  $S$  be a compact hyperbolic Riemann surface and  $\rho : \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{C})$  a representation of its fundamental group into the group of Möbius transformations of the Riemann sphere  $\mathbb{P}^1$ . Let  $p : M_\rho \rightarrow S$  be the (ruled) algebraic surface obtained by the suspension of  $\rho$  (i.e.  $\tilde{S} \times_{\pi_1(S)} \mathbb{P}^1$  with the holomorphic foliation  $\mathcal{F}_\rho$  transversal to the ruling obtained by taking the quotient of the trivial foliation  $\bigcup_{t \in \mathbb{P}^1} \tilde{S} \times \{t\}$ ). It is a Riccati equation [7, 17]. Let  $T^1S \rightarrow S$  be the unit tangent bundle to  $S$  with the induced morphism of fundamental groups  $p_* : \pi_1(T^1S) \rightarrow \pi_1(S)$ . The unit tangent bundle  $T^1\mathcal{F}$  is obtained by suspending the representation  $\rho \circ p_* : \pi_1(T^1S) \rightarrow \text{PSL}(2, \mathbb{C})$ . The geodesic flow and the foliated geodesic flow commute with the projection  $\tilde{p} : T^1\mathcal{F} \rightarrow T^1S$ . It is shown in [7] that there are measurable sections  $\sigma^\pm : T^1S \rightarrow T^1\mathcal{F}$  commuting with the geodesic flows, such that  $\mu^\pm := \sigma^\pm(dLiouv_{T^1S})$  are the attractors and repellers (the Gibbs  $u$ -states) of the foliated geodesic flow. If  $\rho$  is the representation of the universal cover of  $S$ , or another quasi-Fuchsian representation, the sections are

continuous, the limit sets of the discrete groups are circles or quasi-circles, and the support of the Gibbs  $u$ -states are the 3-manifolds  $\sigma^\pm(T^1S)$ . We have in this case a north to south pole dynamics. If the representation is non-discrete, so that the limit set of the representation is the Riemann sphere, the general orbit of  $\rho(\pi_1(S))$  will be dense in  $\mathbb{P}^1$ , the general leaf of  $\mathcal{F}$  will be dense in  $M_\rho$  and the support of the Gibbs  $u$ -state is all of  $T^1\mathcal{F}$ . In this last case one has a strange attractor/repeller dynamics.

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