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Green national accounting: why and how?

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ABSTRACT. The present paper gives an overview of the *theory* of green national accounting. Three purposes of green national accounting (measurement of welfare equivalent income, sustainable income, or net social profit) and two measures (Green NNP and wealth equivalent income) are considered. Under the assumption of no exogenous technological progress, Green NNP is shown to equal wealth equivalent income if there is a constant interest rate or if consumption is constant. It is established as a general result that sustainable income ⩽ wealth equivalent income ⩽ welfare equivalent income, while Green NNP ⩽ welfare equivalent income under no exogenous technological progress and a constant utility discount rate. Green NNP is shown to measure *gross* social profit rather than *net* social profit.

1. Introduction

During the last decades concern has been expressed for the long-term effects of natural resource depletion and environmental deterioration. This concern has spilled over into an interest in the question of whether national accounting can be 'greened' by taking into account the changes in the stocks of natural and environmental resources. In particular, would such an expanded concept of Net National Product ('Green NNP') serve as a welfare measure? Furthermore, would Green NNP be able to indicate whether the actual development is sustainable?

The present paper seeks to give an overview of the *theory* of green national accounting. In particular, I will pose the following two questions:

- What purposes should green national accounting serve (*Why* do green national accounting)?
- What measures are available for these purposes (*How* to do green national accounting)?

The immense practical problems associated with obtaining data to estimate the suggested measures are not discussed here. I abstract from such

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problems since any practical method of green national accounting needs to be based on theoretical results. The purpose of this paper is to present a survey of such theoretical results.

Section 2—which uses the elegant analysis of Dixit, Hammond and Hoel (1980) as its point of departure—gives an interpretation of the main theorems of welfare economics in the present intergenerational setting, thereby providing a welfare economic foundation for the subsequent analysis. Section 3 provides an overview of various purposes (measurement of welfare equivalent income, sustainable income, or net social profit) of green national accounting. Section 4 presents two measures that have been discussed in the literature—Green NNP (i.e. consumption plus the value of net investments) and wealth equivalent income—and reviews, by comparing these measures, Weitzman's (1976) fundamental result on national accounting as well as extending results on green national accounting and sustainability that I have presented in previous work (Asheim, 1994, 1997). Section 5 then investigates to what extent these measures can serve the purposes of green national accounting discussed in section 3. The objective is to give a comprehensive presentation of such results. Some of these results are novel, including the analysis of Green NNP and net social profit.

The present paper is a review of results, not a survey of literature. Hence, I have not attempted to include all relevant references; for an extensive bibliography as well as an interesting treatment of green national accounting, see Aronsson, Johansson and Löfgren (1997). As a consequence, some results may be presented without reference to the contributions where the results first appeared. In the mathematical analysis, I have put an emphasis on generality and elegance as well as economic interpretation at the expense of formal stringency. In particular, I have invoked an assumption of differentiability whenever needed. Where noted, proofs have been relegated to an appendix.

2. Setting

To concentrate on issues that are central to this paper (and to the debate on green national accounting), I will make the following simplifying assumptions:

- *Constant population*. I will assume that each generation lives for one instance; i.e., generations are not overlapping nor infinitely lived, implying that any intertemporal issue is of an intergenerational nature. Distributional issues within each generation will not be discussed.
- *Constant and inelastic supply of 'raw' labour*. This means that, by assumption, there is no trade-off between labour and leisure.
- *One consumption good*. This is an indicator of instantaneous well-being derived from the situation that people live in. This indicator depends not only on material goods, rather it is assumed to be increasing in the availability of environmental amenities, etc.

However, the analysis will allow for *multiple capital goods*. This is needed since the background for the interest in sustainability and green national accounting is that human economic activity leads to depletion of natural capital. It is evident that a question like: 'Is our accumulation of man-made capital sufficient to make up for the decreased availability of natural capital?', cannot be posed in a one-capital-good setting.

In the real world environmental externalities are not always internalised. This is one of many causes that prevents market economies from being fully efficient. Furthermore, for many capital stocks (e.g., stocks of natural and environmental resources or stocks of accumulated knowledge) it is hard to find market prices (or to calculate shadow prices) that can be used to estimate the value of such stocks. In the present setting, I will abstract from these problems by assuming the:

- *Existence of an intertemporal competitive equilibrium* that leads to efficiency and that provides market prices for all capital goods.

Such an unrealistic assumption would undermine the relevance of the analysis if I would show that, with this assumption, all problems of green national accounting would be solved. On the contrary, I will try to convince the reader that, even with the existence of an intertemporal competitive equilibrium, there are hard challenges that remain.

In my setting—where generations follow in sequence, each living only an instance—dynamic efficiency is equivalent to Pareto efficiency. This entails that there is an interesting relationships between techniques of dynamic optimisation and the main theorems of welfare economics. By pointing out this relationship through propositions 1 and 2 I provide a foundation for presenting this paper's analysis of intergenerational allocation as a special case of general equilibrium theory. Let me first indicate how the second welfare theorem applies.

To illustrate, consider the case with only two generations. Assume that the technology is such that the set of feasible utility paths is convex. Then, for any efficient utility path (u_1*, u_2*) there exist utility discount factors (λ_1, λ_2) such that (u_1*, u_2*) maximises $\lambda_1 u_1 + \lambda_2 u_2$ subject to (u_1, u_2) being feasible. This is illustrated by figure 1. Say that (λ_1, λ_2) *supports* (u_1*, u_2*).

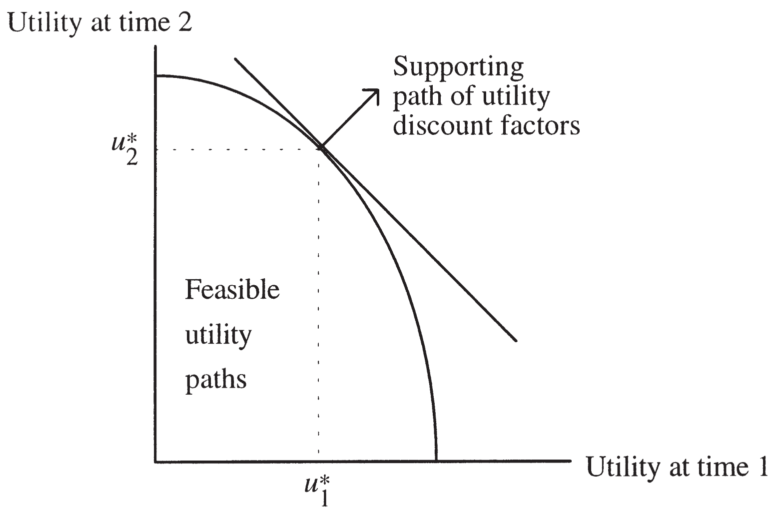

Figure 1.

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Turn now to the case with a continuum of generations and an infinite horizon. Assume that any efficient utility path is supported by a path of positive discount factors. In this context, the *second welfare theorem* can be restated as follows: any utility path that is supported by utility discount factors can be implemented as a *competitive path* (where at each point in time consumers maximise utility and producers maximise profit) provided that each generation is given an endowment that enables it to achieve the utility level at its point in time. Hence, the intergenerational distribution is taken as given, and it is shown that there exist prices to which agents maximise.

The precise statement of this result requires that the general model—to be used throughout this paper—is presented. Following Dixit, Hammond and Hoel (1980), I assume that consumption at time t, $c(t)$, the vector of capital stocks at time t, $\mathbf{k}(t)$, and the vector of investments at time t, $\dot{\mathbf{k}}(t)$, is feasible if $(c(t),\mathbf{k}(t),\dot{\mathbf{k}}(t))$ is in the set of feasible triples $F(t)$. Here, $c(t)$ is the indicator of well-being at time t, while $\mathbf{k}(t)$ comprises not only different kinds of man-made capital, but also stocks of natural capital and stocks of accumulated knowledge (thereby capturing *endogenous technological progress*). In contrast to Dixit, Hammond and Hoel (1980) I also allow for *exogenous technological progress* by permitting the set of feasible triples to be time dependent. I will assume that $F(t)$ is a closed and convex set that satisfies: (a) capital stocks are non-negative ($(c,\mathbf{k},\dot{\mathbf{k}}) \in F(t)$ implies $\mathbf{k} \geqslant 0$) and (b) free disposal of investment flows ($(c,\mathbf{k},\dot{\mathbf{k}}) \in F(t)$ implies $(c,\mathbf{k},\dot{\mathbf{k}}') \in F(t)$ if $\dot{\mathbf{k}}' \leqslant \dot{\mathbf{k}}$). The latter assumption means, e.g., that stocks of environmental resources are considered instead of stocks of pollutants. Lastly, consumption is non-negative and generates utility, $u(t) = u(c(t))$, where u is a time-invariant strictly increasing, concave and differentiable function.

Let $p(t)$ denote the present value price of consumption (i.e., the consumption discount factor) at time t, and let $\mathbf{q}(t)$ denote the vector of present value prices of the capital stocks at time t. The term 'present value' reflects that discounting is taken care of by the prices. If $p(t)$ is an exponentially decreasing function—i.e., $p(t) = p(0)e^{-rt}$—then there is one constant (consumption) interest rate: $r = -\dot{p}(t) / p(t) = p(t) / \int_t^\infty p(s)ds$. If not, there is a term structure of interest rates. The *instantaneous interest rate* is $r_0(t) = -\dot{p}(t) / p(t)$, while the *infinitely long-term interest rate* is $r_\infty(t) = p(t) / \int_t^\infty p(s)ds$. If there is a market for bonds with perpetual yield, then $r_\infty(t)$ is available as a market price at time t. To see this, observe that $1 / r_\infty(t) = \int_t^\infty p(s)ds / p(t)$ is the price in terms of current consumption of a bond that pays one unit of consumption in perpetuity (in other words, $1 / r_\infty(t)$ is the price of a consumption annuity).

The notion of a competitive path can now be defined.

DEFINITION 1. The path $(c^*(t),\mathbf{k}^*(t),\dot{\mathbf{k}}^*(t))_{t=0}^\infty$ is *competitive* at present value prices $(p(t), \mathbf{q}(t))_{t=0}^\infty$ and utility discount factors $(\lambda(t))_{t=0}^\infty$ if, at each t:

C1 instantaneous *utility* is maximised (i.e., $c^*(t)$ maximises $\lambda(t)u(c) - p(t)c$)

C2 instantaneous *profit* is maximised (i.e., $(c^*(t),\mathbf{k}^*(t)\dot{\mathbf{k}}^*(t))$ maximises $p(t)c + \mathbf{q}(t)\dot{\mathbf{k}} + \dot{\mathbf{q}}(t)\mathbf{k}$ subject to $(c,\mathbf{k},\dot{\mathbf{k}}) \in F(t)$).

Why is $p(t)c + \mathbf{q}(t)\dot{\mathbf{k}} + \dot{\mathbf{q}}(t)\mathbf{k}$ instantaneous *profit*? By writing $\mathbf{Q}(t) = \mathbf{q}(t) / p(t)$ for the capital prices in terms of current consumption, we have that

$$\dot{\mathbf{Q}}(t) = \frac{d}{dt}\left(\frac{\mathbf{q}(t)}{p(t)}\right) = \frac{\dot{\mathbf{q}}(t)}{p(t)} - \frac{\dot{p}(t)}{p(t)}\frac{\mathbf{q}(t)}{p(t)} = \frac{\dot{\mathbf{q}}(t)}{p(t)} + r_0(t)\mathbf{Q}(t).$$

Hence, $c + \mathbf{q}(t)\dot{\mathbf{k}} / p(t) + \dot{\mathbf{q}}(t)\mathbf{k} / p(t) = c + \mathbf{Q}(t)\dot{\mathbf{k}} - (r_0(\mathrm{t})\mathbf{Q}(t) - \dot{\mathbf{Q}}(t))\mathbf{k}$, where $c + \mathbf{Q}(t)\dot{\mathbf{k}}$ is the current value of production and $(r_0(\mathrm{t})\mathbf{Q}(t) - \dot{\mathbf{Q}}(t))\mathbf{k}$ is the current cost of holding capital.

PROPOSITION 1. (*Second welfare theorem*) *If* $(c^*(t),\mathbf{k}^*(t),\dot{\mathbf{k}}^*(t))_{t=0}^{\infty}$ *maximises* $\int_{t=0}^{\infty} \lambda(t)u(c(t))dt$ *subject to* $(c(t),\mathbf{k}(t),\dot{\mathbf{k}}(t)) \in F(t)$ for all t and $\mathbf{k}(0) = \mathbf{k}^0$, *then there exist present value prices* $(p(t),\mathbf{q}(t))_{t=0}^{\infty}$ *such that* $(c(t),\mathbf{k}^*(t),\dot{\mathbf{k}}^*(t))_{t=0}^{\infty}$ *is competitive at* $(p(t),\mathbf{q}(t))_{t=0}^{\infty}$ *and* $(\lambda(t))_{t=0}^{\infty}$.

The proof of this proposition (see appendix B) derives the vector of capital prices though optimal control theory, while the consumption price simply measurers the discounted value of marginal utility. The conclusion is that any utility path that is supported by utility discount factors is supported by present value prices of consumption and capital stocks. Hence, any Pareto-efficient path can be seen to be the outcome an intertemporal competitive equilibrium where the intergenerational distribution is given by the consumption path $(c^*(t))_{t=0}^{\infty}$.

Turn now to the *first welfare theorem*, which in the present context can be restated as follows: any competitive path (where consumers maximise utility and producers maximise profit) is efficient. This result is shown in appendix B by imposing the following regularity conditions.

DEFINITION 2. The competitive path $(c(t),\mathbf{k}^*(t),\dot{\mathbf{k}}^*(t))_{t=0}^{\infty}$ is *regular* at present value prices $(p(t),\mathbf{q}(t))_{t=0}^{\infty}$ and utility discount factors $(\lambda(t))_{t=0}^{\infty}$ if

R1 $\int_0^{\infty} \lambda(t)u(c^*(t))dt$ exists (and is finite)

R2 $\mathbf{q}(t)\mathbf{k}^*(t) \to 0$ as $t \to \infty$.

PROPOSITION 2. (*First welfare theorem*) *If* $(c(t),\mathbf{k}^*(t),\dot{\mathbf{k}}^*(t))_{t=0}^{\infty}$ *is regular at present value prices* $(p(t),\mathbf{q}(t))_{t=0}^{\infty}$ *and utility discount factors* $(\lambda(t))_{\mathrm{t}=0}^{\infty}$, *then* $(c(t),\mathbf{k}^*(t),\dot{\mathbf{k}}^*(t))_{t=0}^{\infty}$ *maximises* $\int_0^{\infty} \lambda(t)u(c^*(t))dt$ *subject to* $(c(t),\mathbf{k}(t),\dot{\mathbf{k}}(t)) \in F(t)$ *for all t and* $\mathbf{k}(0) = \mathbf{k}^0$.

Provided that the utility discount factors are positive, this means that any competitive path satisfying the regularity conditions R1 and R2 is efficient. Hence, with these qualifications, any intertemporal competitive equilibrium is Pareto efficient.

I end this section with the following useful lemma, which is proven in appendix B.

LEMMA 1. (*i*) *If* $(c^*(t),\mathbf{k}^*(t),\dot{\mathbf{k}}^*(t))_{t=0}^{\infty}$ *is a competitive path with* $c^*(t) > 0$, *then* $\lambda(t)u'(c^*(t)) = p(t)$. (*ii*) (*Dixit, Hammond and Hoel, 1980*) *If* $F(t)$ *is smooth and time invariant* (*i.e., no exogenous technological progress*) *and* $(c^*(t),$

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$\mathbf{k}^*(t),\dot{\mathbf{k}}^*(t))_{t=0}^{\infty}$ *is a competitive path, then* $p(t)\dot{c}^*(t) + d(\mathbf{q}(t)\dot{\mathbf{k}}^*(t)) / dt = 0$.

Hence, as pointed out by Aronsson *et al*. (1997, p. 105), if there is no exogenous technological progress and $(c^*(t),\mathbf{k}^*(t),\dot{\mathbf{k}}^*(t))_{t=0}^{\infty}$ is a competitive path satisfying that $\mathbf{q}(T)\dot{\mathbf{k}}^*(T) \to 0$ as $T \to \infty$, then $\mathbf{q}(t)\dot{\mathbf{k}}^*(t) = \int_t^{\infty} p(s)\dot{c}^*(s)ds$. Thus, the value of net investments at time t measures the present value of future changes in consumption. This in turn implies that the value of net investments is zero at time t if consumption is constant at all future times. The latter result is called the converse of Hartwick's rule (see Hartwick, 1977; Dixit, Hammond and Hoel, 1980 and Withagen and Asheim, 1998).

3. Purpose

What purpose should green national accounting serve? There are at least three different purposes that have been mentioned in the literature.

3.1 Welfare equivalent income

Assume that generation t seeks to maximise a social welfare functional, $\int_t^{\infty} \lambda(s)u(c(s))ds$, over all feasible paths given the capital stocks $\mathbf{k}$ that generation t has inherited. Refer to $\int_t^{\infty}(\lambda(s) / \lambda(t))\, u(c^*(s))ds$ as *welfare* at time t. Weitzman (1970) considers the level of utility $v(t)$ that if held constant will yield the same welfare as the welfare-maximising path $(c^*(s),\mathbf{k}^*(s),\dot{\mathbf{k}}^*(s))_{s=t}^{\infty}$. Such a *stationary equivalent* is defined by $\int_t^{\infty}\lambda(s)v(t)ds = \int_0^{\infty}\lambda(t)u(c^*(t))dt$, or

$$v(t) = \frac{\int_t^{\infty}\lambda(s)u(c^*(s))ds}{\int_t^{\infty}\lambda(s)ds}.$$

Since the present paper analyses notions and measures of income in terms of consumption, I will here consider the *consumption* index of welfare $w(t)$ uniquely defined by $w(t) = u^{-1}(v(t))$.

DEFINITION 3. The *welfare equivalent income* $w(t)$ at time t is the consumption that if held constant will yield the same welfare as the welfare maximising path $(c^*(t),\mathbf{k}^*(t),\dot{\mathbf{k}}^*(t))_{t=0}^{\infty}$, i.e., $\int_t^{\infty}\lambda(s)u(w(t))ds = \int_t^{\infty}\lambda(s)u(c^*(s))ds$, or

$$w(t) = u^{-1}\left(\frac{\int_t^{\infty}\lambda(s)u(c^*(s))ds}{\int_t^{\infty}\lambda(s)ds}\right).$$

The notation w refers to Weitzman (1970), who first suggested stationary welfare equivalence.

A measure of welfare expressed in terms of consumption can be of interest for different reasons. One reason is that one wants to pose the question: Which of two economies is better off at the same point in time? If $(\lambda(s))_{s=t}^{\infty}$ is proportional in the two different economies, a comparison of welfare equivalent income will answer this question. Another reason is that one wants to measure whether an economy grows. If $\lambda(s)$ is an exponentially decreasing function—i.e., $\lambda(s) = \lambda(0)e^{-\delta s}$, meaning that there is a constant utility discount rate $\delta = -\dot{\lambda}(s) / \lambda(s)$, and implying that the mapping from $(c^*(s))_{s=t}^{\infty}$ to $w(t)$ is time invariant—then an increase in welfare-equivalent income can be interpreted as growth. A foundation for the case where $\lambda(s)$ is an exponentially decreasing function has been provided by Koopmans (1960).

In my definition of $w(t)$ I have not imposed that $\lambda(s)$ be an exponentially decreasing function. The non-exponential case is of interest e.g. when—following Solow (1974)—the maximin principle is applied in the Dasgupta–Heal–Solow model where production is a Cobb–Douglas function of the stock of reproducible capital and the flow of exhaustible resource extraction (see Dasgupta and Heal, 1974 and Solow, 1974). Then the economy at time t behaves *as if* it maximises $\int_t^\infty \lambda(s)u(c(s))ds$ with the instantaneous discount rate $-\dot{\lambda}(s) / \lambda(s)$ decreasing over time. Since $\lambda(s)$ depends on absolute time s only—not on relative time $(s - t)$—this does not lead to a time-inconsistency problem of the kind considered by Strotz (1955–6).

3.2 Sustainable income

Definition 4. *Sustainable income $m(t)$ at time t is the maximum consumption that can be sustained from time t on, given the capital stocks* $\mathbf{k}$ *that generation t has inherited:*

$$m(t) = \sup(\inf_{s \geqslant t}(c(s))).$$

In the wake of the World Commission on Environment and Development—which popularised the term 'sustainable development' through its report (WCED, 1987)—the following question has gained attention: Is the present management of natural and environmental resources compatible with sustainable development? If $m(t)$ could be measured, it would in fact be possible to answer this question by comparing actual consumption to the sustainable income. The normative relevance of sustainable income was discussed by Solow (1974) in his intertemporal application of the Rawlsian maximin principle. More recently, Buchholz (1997) has provided a normative foundation for sustainability by imposing two seemingly weak axioms on any (quasi-)ordering of the set of feasible consumption paths; for details see appendix A.

A sustainability requirement to the effect that $c(t)$ should not exceed $m(t)$ is satisfied if $\dot{c}(s) \geqslant 0$ for all $s \geqslant t$ or equivalently (since there is only one consumption good) $\dot{u}(s) \geqslant 0$ for all $s \geqslant t$. This practise of associating sustainability with a requirement on the path of instantaneous utility has been supported in a number of references, see, e.g., Pezzey (1997). However, if the welfare at time t equals $\int_t^\infty (\lambda(s) / \lambda(t))\, u(c(s))ds$, then one can argue that a requirement of sustainability should instead be imposed on the path of welfare equivalent income; one such approach in the case when $\lambda(s) = \lambda(0)e^{-\delta s}$ is to define *welfare sustainability* by $\dot{w}(s) \geqslant 0$ for all $s \geqslant t$.

One argument against imposing sustainability on $(w(s))_{s=t}^\infty$ (and instead using $(c(s))_{s=t}^\infty$) is that it might be desirable to separate the definition of sustainability from the forces (e.g. altruism towards future generations) that can motivate our generation to act in accordance with the requirement of sustainability. This view is in principle supported by Rawls (1971, Paragraph 22). Another, more pragmatic, argument against imposing sustainability on welfare equivalent income is that national accounting based on current prices and quantities cannot yield any indication of whether development satisfies such sustainability; for more on this, see section 5.2.

Note that a comparison of sustainable income $m(t)$ will not necessarily give the correct answer when comparing which of two economies is better off at the same point in time. The following example illustrates this. Let two economies 1 and 2 have the same constant utility discount rate δ, and the same capital stock $k^1(t) = k^2(t) = 1$ at time t (capital is here assumed to be a scalar). Furthermore, assume that the stationary set of feasible triples $(c^i,k^i,\dot{k}^i)$ in economy i is given by $c^i + \dot{k}^i \leqslant (k^i)^{\alpha^i}$, where $\delta = \alpha^1$ and $\delta < \alpha^2 < 1$. Hence, both countries are endowed with a Ramsey technology, entailing that sustainable income is the level of consumption that results if capital is held constant (see section 5.2). Standard calculations yield that economy 1 will wish to keep its capital stock constant since $\alpha^1(k^1(t))^{\alpha^1-1} = \alpha^1 = \delta$, while economy 2 will wish to accumulate capital since $\alpha^2(k^2(t))^{\alpha^2-1} = \alpha^2 > \delta$. Hence, for economy 2, the welfare equivalent income is greater than the level of consumption that results if capital is held constant. Thus, $1 = m^1(t) = w^1(t) = m^2(t) < w^2(t)$. In this example it is correct to say that economy 2 is better off (since the optimal path of economy 1 is feasible but not optimal for economy 2). This conclusion will be obtained if welfare equivalent income is measured, but not if sustainable income is measured.

3.3 Net social profit

A third purpose of green national accounting is to develop a criterion function for social cost–benefit analysis. Such a criterion function is an index having the property that the acceptance of a 'small' policy change increases the index if and only the policy change leads to a welfare improvement. This approach is based on the theory of social cost–benefit analysis (see Dasgupta, Marglin and Sen, 1972, and Little and Mirrlees, 1974) and has been promoted in a series of paper by Dasgupta, Kriström and Mäler (1995, 1997). In the remainder of this paper, the value of such an index at time t will be denoted $\pi(t)$ and referred to as *net social profit*, Proposition 10 of section 5.3 establishes an explicit expression for $\pi(t)$ under a given interpretation of its purpose.

4. Measures

Green national accounting seeks to serve one or more of the purposes described in section 3 by calculating a measure based on current prices and quantities. The standard candidate for such a measure is *Green NNP*, which is defined in section 4.1. However, section 5 will reveal that this measure has serious limitations, particularly in the presence of exogenous technological progress. Hence, as an alternative measure, I will also consider *wealth equivalent income*, which will be defined in section 4.2 and given an expression in terms of current prices and quantities in section 4.4. Section 4.3 compares the two measures.

4.1 Green NNP

DEFINITION 5. Green NNP is the sum of consumption and the value of net investments:

$$g(t) = c^*(t) + \mathbf{Q}(t)\dot{\mathbf{k}}^*(t).$$

I assume that the vector of capital goods, $\mathbf{k}$, comprises all kinds of man-made capital (including stocks of accumulated knowledge) and all kinds of natural capital (including stocks of environmental resources). Since Green NNP is inclusive in this respect, one may follow the terminology of Pemberton and Ulph (1998) and refer to Green NNP as *inclusive income*. As in section 2, $\mathbf{Q}(t) = \mathbf{q}(t) / p(t)$ denotes the vector of capital prices in terms of current consumption.

4.2 Wealth equivalent income

If a regular path $(c^*(s),\mathbf{k}^*(s),\dot{\mathbf{k}}^*(s))_{s=t}^{\infty}$ is followed, then, by the proof of proposition 2, $\int_t^{\infty} p(s)c(s)ds$ is maximised over all feasible paths given the capital stocks $\mathbf{k}$ that generation t has inherited. Refer to $\int_t^{\infty} (p(s) / p(t))\ c^*(s)ds$ as *wealth* at time t. Then—in analogy with welfare equivalence—one can construct the consumption index of wealth $h(t)$ defined as follows.

DEFINITION 6. The *wealth equivalent income* $h(t)$ at time t is the consumption that if held constant will yield the same wealth as the wealth maximising path $(c^*(s),\mathbf{k}^*(s),\dot{\mathbf{k}}^*(s))_{s=t}^{\infty}$, i.e., $\int_t^{\infty} p(s)h(t)ds = \int_t^{\infty} p(s)c^*(s)ds$, or

$$h(t) = \frac{\int_t^{\infty} p(s)c^*(s)ds}{\int_t^{\infty} p(s)ds}.$$

The notation h refers to Hicks (1946), who suggested stationary wealth equivalence in Chapter 14 of *Value and Capital*. After considering the possibilities of changing interest rates and changing prices, Hicks associates income with both sustainable income and wealth equivalent income:

> Income No. 3 must be defined as the maximum amount of money which the individual can spend this week, and still expect to be able to spend the same amount in real terms in each ensuing week. Hicks (1946, p. 174) The standard stream corresponding to Income No. 3 is constant in real terms .... We ask ... how much he would be receiving if he were getting a standard stream of the same present value as his actual expected receipts. This amount is his income (Hicks, 1946, p. 184)

Whether income is associated with sustainable income (as on p. 174) or with wealth equivalent income (as on p. 184) does not make any difference at a personal level since a price taker can turn the actual consumption path into a constant consumption path with the same present value. At a national level, however, these notions need not coincide (see section 5.1 below).

Weitzman (1976) introduced the notion of wealth equivalent *national* income in a setting where wealth equivalent income coincides with welfare equivalent income since, by one of his interpretations (1976, p. 157), utility is assumed to be a linear function of consumption.

4.3 Comparison of Green NNP and wealth equivalent income

It follows from the expression for the infinitely long-term interest rate $r_\infty(t) = p(t) / \int_t^\infty p(s)ds$ that

$$h(t) = r_\infty(t) \int_t^\infty \frac{p(s)}{p(t)} c^*(s)ds.$$

Moreover, if there is no exogenous technological progress, then we have that $d(p(t)c^*(t)) / dt = p(t)\dot{c}^*(t) + \dot{p}(t)c^*(t) = -\, d(\mathbf{q}(t)\dot{\mathbf{k}}^*(\mathbf{t})) / dt + \dot{p}(t)c^*(t)$ by lemma 1(ii). Since $\lim_{T\to\infty} p(T)c^*(T) = 0$, and assuming that $\lim_{T\to\infty} \mathbf{q}(T)\dot{\mathbf{k}}^*(T) = 0$, it follows through integration that $-p(t)c^*(t) = \mathbf{q}(t)\dot{\mathbf{k}}^*(t) + \int_t^\infty \dot{p}(s)c^*(s)ds$. Hence, as shown by Sefton and Weale (1996) and Pemberton and Ulph (1998), Green NNP can be written as $g(t) = c^*(t) + \mathbf{Q}(t)\dot{\mathbf{k}}^*(t) = (p(t)c^*(\mathrm{t}) + \mathbf{q}(t)\dot{\mathbf{k}}^*(t)) / p(t) = -\int_t^\infty (\dot{p}(s) / p(t))c^*(s)ds$, or, using the expression for the instantaneous interest rate $r_0(s) = -\,\dot{p}(s) / p(s)$

$$g(t) = \int_t^\infty r_0(s) \frac{p(s)}{p(t)} c^*(s)ds.$$

This yields the following result.

PROPOSITION 3. *If there is no exogenous technological progress, then Green NNP is greater than wealth equivalent income if and only if* $\int_t^\infty r_0(s)\,(p(s) / p(t))\, c^*(s)ds$ *is greater than* $r_\infty(t) \int_t^\infty (p(s) / p(t))\, c^*(s)ds$.

Given the assumption of no exogenous technological progress, there are two special cases for which Green NNP and wealth equivalent income coincide (see also Asheim, 1997, p. 362):

(i) If there is a constant interest rate (i.e., $r_0(s) = r_\infty(t) = r$ for all s), then $g(t) = h(t) = r \int_t^\infty (p(s) / p(t))\, c^*(s)ds$. This is Weitzman's (1976) fundamental result on green national accounting in his seminal contribution: if there is a constant interest rate under no exogenous technological progress, then Green NNP equals wealth equivalent income. Hence

$$\int_t^\infty p(s)(c^*(t) + \frac{\mathbf{q}(t)}{p(t)}\dot{\mathbf{k}}^*(t))ds = \int_t^\infty p(s)c^*(s)ds.$$

Note that even if $\lambda(s) = \lambda(0)e^{-\delta s}$, so that there is a constant utility discount rate, there need not be a constant interest rate, e.g. in the Ramsey model, there is a constant interest rate only if consumption is constant. In certain resource models, like the Dasgupta–Heal–Solow model, not even a constant consumption path leads to a constant interest rate.

(ii) If consumption is constant (i.e., $c^*(s) = c^*$ for all s), then $g(t) = h(t) = c^*$, since it follows from the definitions of $r_0(s)$ and $r_\infty(t)$ that $\int_t^\infty r_0(s) (p(s) / p(t))\, ds = r_\infty(t) \int_t^\infty (p(s) / p(t))\, ds = 1$.

If these two special cases do not apply, but the assumption of no exogenous technological progress holds, then it follows from proposition 3 that wealth equivalent income exceeds Green NNP whenever consumption tends to increase (decrease) and interest rates tend to decrease (increase). This is indeed the case in the Ramsey model when there is a constant utility discount rate and the initial capital stock is smaller (larger) than the size for which the marginal productivity of capital equals the utility dis-

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count rate. However, Green NNP exceeds wealth equivalent income whenever both consumption and interest rates tend to decrease. This can occur in the Dasgupta–Heal–Solow model; see the example of Asheim (1994, section IV).

4.4 An expression for wealth equivalent income

The purpose of this sub-section is to express wealth equivalent income, $h(t) = r_\infty(t) \int_t^\infty (p(s) / p(t))\, c^*(s)ds$, in terms of current prices and quantities. Since $d(\int_t^\infty (p(s) / p(t))\, c^*(s)ds) / dt = -c^*(t) + r_0(t) \int_t^\infty (p(s) / p(t))\, c^*(s)ds$, it follows that $\int_t^\infty (p(s) / p(t))\, c^*(s)ds = (c^*(t) + d(\int_t^\infty (p(s) / p(t))\, c^*(s)ds) / dt) / r_0(t)$. Hence

$$h(t) = \frac{r_\infty(t)}{r_0(t)} \left(c^*(t) + \frac{d}{dt} \left(\int_t^\infty \frac{p(s)}{p(t)}\, c^*(s)ds \right) \right).$$

Assume now that the technology exhibits constant returns to scale (CRS). This assumption is in the spirit of Lindahl (1933, pp. 401–2) and implies that all factors of production, including labour, are dealt with as capital that is evaluated by the present value of future earnings. It amounts to assuming that all flows of future earnings can be treated as currently existing capital. CRS means that in the hypothetical case where all capital stocks were a given percentage larger, consumption and investments could be increased by the same percentage. This clearly allows for stocks in fixed supply—like 'raw labour' and land—that cannot actually be accumulated. Although being informationally demanding, the assumption of CRS is *not* technologically restrictive since the existence of an intertemporal competitive equilibrium entails that returns to scale are non-decreasing. The reason is that increasing returns to scale is incompatible with maximisation of instantaneous profit (see C2 of definition 1). Hence, CRS can be obtained by adding an additional fixed capital stock with which returns to scale become constant.

The following lemma, which is proven in appendix B, reveals the importance of assuming CRS by showing that the present value of future consumption is captured by current capital stocks.

LEMMA 2. *If, for each s, $F(s)$ is a convex cone (i.e., the technology exhibits* CRS*), and $(c^*(s), \mathbf{k}^*(s), \dot{\mathbf{k}}^*(s))_{s=t}^\infty$ is a regular path, then $\mathbf{q}(t)\mathbf{k}^*(t) = \int_t^\infty p(s)c^*(s)ds$.*

Hence, under CRS, wealth is equal to the value of current capital stocks: $\int_t^\infty (p(s) / p(t))c^*(s)ds = \mathbf{q}(t)\, \mathbf{k}^*(t) / p(t) = \mathbf{Q}(t)\mathbf{k}^*(t)$. Substituting $\mathbf{Q}(t)\mathbf{k}^*(t)$ for wealth in the above expression of $h(t)$ yields

$$h(t) = \frac{r_\infty(t)}{r_0(t)} \left(c^*(t) + \frac{d}{dt} (\mathbf{Q}(t)\mathbf{k}^*(t)) \right) = \frac{r_\infty(t)}{r_0(t)} (c^*(t) + \mathbf{Q}(t)\dot{\mathbf{k}}^*(t) + \dot{\mathbf{Q}}(t)\mathbf{k}^*(t)).$$

Thus, we have established the following result (which is a variant of Asheim (1997, eq. 10)).

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PROPOSITION 4. *If the technology exhibits constant returns to scale, then*

$$h(t) = \frac{r_\infty(t)}{r_0(t)} (g(t) + \dot{\mathbf{Q}}(t)\mathbf{k}^*(t)).$$

This result means that in order to arrive at wealth equivalent income ($h(t)$)

- *anticipated capital gains* ($\dot{\mathbf{Q}}(t)\mathbf{k}^*(t)$) must be added to Green NNP ($g(t)$),
- the sum $g(t) + \dot{\mathbf{Q}}(t)\mathbf{k}^*(t)$ must be adjusted for *interest rate effects* if there is not a constant interest rate, in which case $r_\infty(t) / r_0(t)$ need not equal 1.

*Un*anticipated capital gains (which fall outside the deterministic framework of the present paper) cannot be fully added to Green NNP. Only the interest on such windfall gains constitute current income (see, e.g. Hicks, 1946, p. 179).

Comparing proposition 4 with special case (i) of section 4.3 provides an alternative demonstration of the result shown in Asheim (1996, proposition 1), namely that in an economy with CRS and no technological progress, a constant interest rate implies that there are no anticipated capital gains. W.r.t. to special case (ii), where $h(t) = g(t)$ due to constant consumption, proposition 4 means that, in the Dasgupta–Heal–Solow model, positive anticipated capital gains ($\dot{\mathbf{Q}}(t)\mathbf{k}^*(t) > 0$) exactly offset the effect of decreasing interest rates ($r_\infty(t) / r_0(t) < 1$) along a constant consumption path.

One may discuss whether proposition 4 is a useful result for practical estimation given the informational burden of the CRS case. The result is, however, informative since it shows how, in principle, exogenous technological progress can be measured through its effect on capital gains. Such exogenous technological progress is relevant (i) for an economy where accumulated knowledge cannot be represented by augmented capital stocks (see, e.g., Aronsson and Löfgren (1995) Kemp and Long (1982), Weitzman (1997)), and (ii) for open economies whose 'technology' is changing exogenously when resource prices influence their terms of trade (see, e.g., Asheim, 1996; Sefton and Weale, 1996 and Vincent, Panayotou and Hartwick, 1977).

5. Ability to measure

The present section investigates to what extent Green NNP, $g(t)$, and wealth equivalent income, $h(t)$, can serve as measures of welfare equivalent income, $w(t)$, sustainable income, $m(t)$ or net social profit, $\pi(t)$.

5.1 Measuring welfare equivalent income

If there is a constant utility discount rate under no exogenous technological progress, then it follows from a generalisation of Weitzman's (1976) result that

$$\int_t^\infty \lambda(s)(u(c^*(t)) + \frac{\mathbf{q}(t)}{\lambda(t)} \dot{\mathbf{k}}^*(t))ds = \int_t^\infty \lambda(s)(u(c^*(s))ds$$

(see Weitzman, 1970 and Kemp and Long, 1982). This means that 'Green NNP in terms of utility', $u(c^*(t)) + \mathbf{q}(t)\dot{\mathbf{k}}^*(t) / \lambda(t)$, is equal to the utility derived from welfare equivalent income, $u(w(t))$

$$u(c^*(t)) + \frac{\mathbf{q}(t)}{\lambda(t)}\,\dot{\mathbf{k}}^*(t) = \frac{\int_t^\infty \lambda(s)u(c^*(s))ds}{\int_t^\infty \lambda(s)ds} = u(w(t)).$$

As a foundation for using Green NNP, $g(t)$, Hartwick (1990) observed that $u'(c^*(t))\cdot g(t)$ is a linear approximation of 'Green NNP in terms of utility' and thus of $u(w(t))$.

$$u'(c^*(t))\cdot g(t) = u'(c^*(t))\cdot c^*(t) + u'(c^*(t))\cdot \frac{\mathbf{q}(t)}{p(t)}\,\dot{\mathbf{k}}^*(t) = u'(c^*(t))\cdot c^*(t) + \frac{\mathbf{q}(t)}{\lambda(t)}\,\dot{\mathbf{k}}^*(t),$$

since $\lambda(t)u'(c^*(t)) = p(t)$ by Lemma 1(i). It turns out that this approximation is biased since the concavity of u implies that $u'(c^*(t))\cdot(w(t) - c^*(t)) \geqslant u(w(t)) - u(c^*(t)) = \mathbf{q}(t)\dot{\mathbf{k}}^*(t) / \lambda(t) = u'(c^*(t))\cdot(g(t) - c^*(t))$. Hence, $w(t) \geqslant g(t)$, thereby establishing the following (not previously noted) result.

PROPOSITION 5. *If there is no exogenous technological progress and the utility discount factor $\lambda(s)$ is an exponentially decreasing function, then Green NNP $g(t)$ is smaller than or equal to welfare equivalent income $w(t)$.*

Given the assumptions of no exogenous technological progress and a constant utility discount rate, there are two special cases for which Green NNP and welfare equivalent income coincide:

(i) If u is linear, so that $u'(c^*(t))\cdot(w(t) - c^*(t)) = u(w(t)) - u(c^*(t))$, then $g(t) = w(t)$.

(ii) If the value of net investments $\mathbf{Q}(t)\dot{\mathbf{k}}^*(t)$ is equal to zero, then $g(t) = c^*(t) = w(t)$. By lemma 1(ii) this latter case is equivalent to the present value of future changes in consumption $\int_t^\infty (p(s) / p(t))\ \dot{c}^*(s)ds$ being equal to zero. Hence, consumption being constant ($c^*(s) = c^*$ for all s) is sufficient, but not necessary for $\mathbf{Q}(t)\dot{\mathbf{k}}^*(t)$ being equal to zero.

Also *wealth equivalent income* tends to underestimate welfare equivalent income, as I have observed in Asheim (1997, p. 361). However, this is a perfectly general result which is valid also if there *is* exogenous technological progress and if there is *not* a constant utility discount rate.

PROPOSITION 6. *Wealth equivalent income $h(t)$ is smaller than or equal to welfare equivalent income $w(t)$.*

To establish this result, note that definition 6 implies that $\int_t^\infty \lambda(s)u'(c^*(s))(h(t) - c^*(s))ds = 0$, since $\lambda(s)u'(c^*(s)) = p(s)$ by lemma 1(i). Hence, since $\int_t^\infty \lambda(s)u(w(t))ds = \int_t^\infty \lambda(s)u(c^*(s))ds$ by the definition of $w(t)$, it follows from the concavity of u that

$$\begin{aligned}&\int_t^\infty \lambda(s)(u(w(t)) - u(h(t)))ds \\ &= \int_t^\infty \lambda(s)(u(c^*(s)) - u(h(t)) + u'(c^*(s))(h(t) - c^*(s)))ds \geqslant 0,\end{aligned}$$

showing that $h(t) \leqslant w(t)$ as claimed by proposition 6. Note that $h(t) = w(t)$ if consumption is constant.

If $u(c) = \ln c$, then $w(t)$ is a Cobb–Douglas functional of $(c^*(s))_{s=t}^\infty$

$$w(t) = \exp\left(\frac{\int_t^\infty \lambda(s)\ln c^*(s)ds}{\int_t^\infty \lambda(s)ds}\right).$$

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Likewise, if $u(c) = c^\rho / \rho$, $\rho \leqslant 1$, $\rho \neq 0$, then $w(t)$ is a CES functional of $(c^*(s))_{s=t}^{\infty}$

$$w(t) = \left(\frac{\int_t^\infty \lambda(s)(c^*(s))^\rho ds}{\int_t^\infty \lambda(s)ds} \right)^{1/\rho}.$$

Using the theory of indirect utility functions for Cobb–Douglas and CES functions, explicit expressions for welfare equivalent income can be found as functions of the path of consumption prices (i.e. consumption discount factors) and wealth. For the statements of these expressions, it is useful to define an interest rate $\tilde{r}_\infty(t)$ that relates welfare equivalent income to wealth:

$$\tilde{r}_\infty(t) = \begin{cases} \dfrac{p(t)}{(\int_t^\infty \lambda(s)ds) \cdot \exp\left(\dfrac{(\int_t^\infty \lambda(s)\ln(p(s)/\lambda(s))ds}{\int_t^\infty \lambda(s)ds)} \right)} & \text{if } u(c) = \ln c \\[2em] \dfrac{p(t)}{(\int_t^\infty \lambda(s)ds) \cdot \left(\dfrac{\int_t^\infty \lambda(s)(p(s)/\lambda(s))^{\rho/(\rho-1)}ds}{\int_t^\infty \lambda(s)ds} \right)^{(\rho-1)/\rho}} & \text{if } u(c) = \dfrac{c^\rho}{\rho} \end{cases}$$

The statements are provided by the following new proposition, which is proven in appendix B.

Proposition 7. *If* $u(c) = \ln c$ *or if* $u(c) = c^\rho / \rho$, $\rho < 1$, $\rho \neq 0$, *then*

$$w(t) = \tilde{r}_\infty(t) \int_t^\infty \frac{p(s)}{p(t)} c^*(s)ds = \frac{\tilde{r}_\infty(t)}{r_\infty(t)} h(t).$$

Furthermore, if the technology exhibits constant returns to scale, then

$$w(t) = \frac{\tilde{r}_\infty(t)}{r_0(t)} (g(t) + \dot{\mathbf{Q}}(t)\mathbf{k}^*(\mathbf{t})).$$

Since, by proposition 6, $h(t) \leqslant w(t)$, it follows that $r_\infty(t) \leqslant \tilde{r}_\infty(t)$. If consumption is constant, then Lemma 1(i) implies that $(p(s))_{s=t}^\infty$ and $(\lambda(s))_{s=t}^\infty$ are proportional, leading to $r_\infty(t)$ and $\tilde{r}_\infty(t)$ being equal. This is consistent with the observation subsequent to proposition 6, namely that $h(t) = c^* = w(t)$ in this case. The second part of proposition 7 supports the argument (made, for example, by Brekke, 1997, section 6.5) that anticipated capital gains must be taken into account when welfare comparisons over time and across economies are made.

5.2 Measuring sustainable income

At the level of a small open economy faced with given international prices, *wealth equivalent income* measures sustainable income. The reason is (as pointed out by, for example, Brekke, 1997 and Vincent, Panayotou and Hartwick, 1997) that the actual consumption path can be turned into a constant consumption path with the same present value. However, at the level

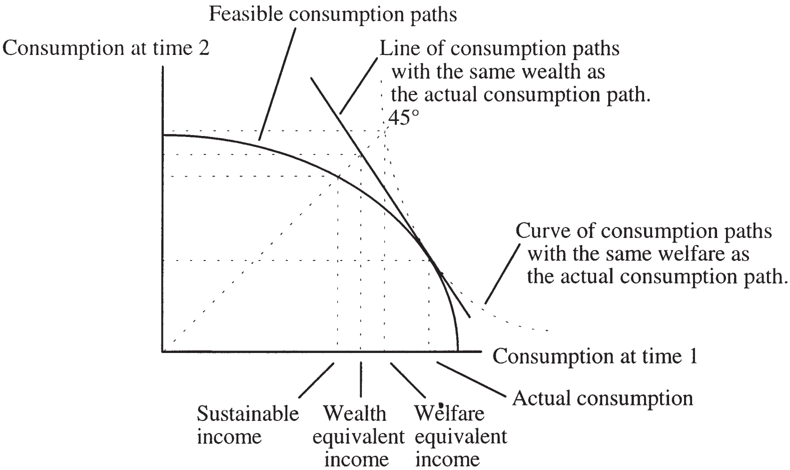

Figure 2.

of a closed economy with a non-linear technology, and at the level of a large open economy that influences international prices, wealth equivalent income overestimates sustainable income, since turning the actual consumption path into a constant consumption path leads to a loss of present value. Figure 2 illustrates this (as well as why welfare equivalent income exceeds wealth equivalent income; cf. proposition 6).

In appendix B it is formally shown how this argument yields the following proposition. Note that the result does *not* depend on an assumption of no exogenous technological progress.

PROPOSITION 8. *Wealth equivalent income $h(t)$ is greater than or equal to sustainable income $m(t)$, with equality if $c^*(s) = c^*$ for all s.*

In the Ramsey model—which one capital good technology is described by $c + \dot{k} \leqslant f(k)$, with f being an increasing and strictly concave function—*Green NNP* measures sustainable income. To see this, note that $g(t) = c^*(t) + \dot{k}^*(t)$, while $f(k^*(t))$ is the sustainable income given that generation t has inherited the capital stock $k^*(t) = k$. Hence, since efficiency implies that $c^*(t) + \dot{k}^*(t) = f(k^*(t))$, it follows that $g(t) = c^*(t) + \dot{k}^*(t) = f(k^*(t)) = m(t)$.

This result does not generalise. Since lemma 1(ii) implies that $\mathbf{q}(t)\dot{\mathbf{k}}^*(t) = 0$ if $c^*(s) = c^*$ for all s under the assumption of no exogenous technological progress, it follows that $g(t) = c^* = m(t)$ for a constant consumption path in a stationary technology. In the case of a constant interest rate under no exogenous technological progress, it follows from propositions 3 and 8 that Green NNP equals wealth equivalent income and thus overestimates sustainable income. No general result is, however, available on the relation between $g(t)$ and $m(t)$ when neither consumption nor the interest rate is constant, not even under the assumption of no exogenous technological progress. In Asheim (1994, section IV) I show by way of an example in the Dasgupta–Heal–Solow model that it is possible to construct situations in which $g(t) > c^*(t) > m(t)$. This can occur in their model under discounted utilitarianism since the scarcity of the non-renewable resource leads to an inverted-U-shaped consumption path and an decreasing interest rate. This means that a consumption path can be constructed (see Pezzey and Withagen (1995, figure 4; 1998)) where initial consumption $c^*(t)$ is slightly above $m(t)$, while $g(t) > h(t)$ (by proposition 3) and $h(t) > m(t)$ (by proposition 8). Hence, even when Green NNP exceeds consumption (implying that the value of net investments $\mathbf{Q}(t)\dot{\mathbf{k}}^*(t)$ is positive), consumption may be at an unsustainable level.

In spite of this negative result, it follows from lemma 1(ii) that $g(t)$ provides approximate information concerning the sustainability of consumption when there is no exogenous technological progress since the current difference between $g(t)$ and $c^*(t)$ measures future changes in consumption: $g(t) - c^*(t) = \mathbf{Q}(t)\dot{\mathbf{k}}^*(t) = \int_t^\infty (p(s) / p(t))\, \dot{c}^*(s)ds$. By assuming a constant utility discount rate, Pemberton and Ulph (1998) use this forward looking property of $g(t) - c^*(t)$ to establish the following result (for which I have included a proof in appendix B).

PROPOSITION 9. *If there is no exogenous technological progress and the utility discount factor $\lambda(s)$ is an exponentially decreasing function, then $g(t) \geqslant c^*(t)$ is equivalent to $\dot{w}(t) \geqslant 0$.*

Note, however, that $g(t) \geqslant c^*(t)$ does *not* indicate welfare sustainability ($\dot{w}(s) \geqslant 0$ for all $s \geqslant t$; see section 3b). In order to indicate welfare sustainability, one has to calculate $g(s) - c^*(s) = \mathbf{Q}(s)\dot{\mathbf{k}}^*(s)$ for all $s \geqslant t$; i.e., perform *today* national accounting for all future times based on future prices and quantities. It is probably fair to say that this is of little practical interest.

5.3 Measuring net social profit

Let the path $(c^*(s),\mathbf{k}^*(s),\dot{\mathbf{k}}^*(\mathbf{s}))_{s=0}^\infty$ be a regular path at present values prices $(p(s),\mathbf{q}(s))_{s=0}^\infty$ and utility discount factors $(\lambda(s))_{s=0}^\infty$ given the capital stocks $\mathbf{k}^0$ that generation 0 has inherited. Let $(c^*(s;\mathbf{k},t))_{s=t}^\infty$ be a consumption path maximising $\int_t^\infty (\lambda(s)u(c(s))ds$ subject to feasibility if generation t inherits $\mathbf{k}$. By Proposition 2, we can set $(c^*(s;\mathbf{k}^0,0))_{s=0}^\infty = (c^*(s))_{s=0}^\infty$.

Let a *policy change* at time s refer to the substitution of an alternative feasible set, $\tilde{F}(s)$, for $F(s)$ at time s. A policy change for the time interval $[0,t]$ is welfare improving if and only if there exists a path $(\tilde{c}(s),\tilde{\mathbf{k}}(s),\dot{\tilde{\mathbf{k}}}(s))_{s=0}^t$ satisfying $(\tilde{c}(s),\tilde{\mathbf{k}}(s),\dot{\tilde{\mathbf{k}}}(s)) \in \tilde{F}(s)$ and $\tilde{\mathbf{k}}(0) = \mathbf{k}^0$ such that

$$\int_0^t \lambda(s)(u(\tilde{c}(s)) - u(c^*(s)))ds + \int_t^\infty \lambda(s)(u(c^*(s;\tilde{\mathbf{k}}(t),t)) - u(c^*(s)))ds > 0.$$

Keeping in mind that $\lambda(s)u'(c^*(s)) = p(s)$ (see lemma 1(i)) and $q_i(t) = \partial V(\mathbf{k}^*(t),t) / \partial k_i$, where $V(\mathbf{k}, t) = \int_t^\infty \lambda(s)u(c^*(s;\mathbf{k},t))ds$ (see the proof of proposition 1), this is equivalent to

$$\int_0^t p(s)(\tilde{c}(s) - c^*(s))ds + \mathbf{q}(t)(\tilde{\mathbf{k}}(t) - \mathbf{k}^*(t)) > 0$$

if the policy change is *small* in the sense that the alternative path $(\tilde{c}(s),\tilde{\mathbf{k}}(s),\dot{\tilde{\mathbf{k}}}(s))_{s=0}^t$ satisfies $u'(\tilde{c}(s)) \approx u'(c^*(s))$ for all $s \in [0,t]$ and $\partial V(\tilde{\mathbf{k}}(t),t) / \partial k_i \approx \partial V(\mathbf{k}^*(t),t) / \partial k_i$. By rewriting the latter inequality as

$\int_0^t ((p(s)(\tilde{c}(s) + \mathbf{q}(s)\dot{\tilde{\mathbf{k}}}(s)) + \dot{\mathbf{q}}(s)\tilde{\mathbf{k}}(s)) - (p(s)c^*(s) + \mathbf{q}(s)\dot{\mathbf{k}}^*(s) + \dot{\mathbf{q}}(s)\mathbf{k}^*(s)))ds > 0,$

it follows that a small policy change can be evaluated by the discounted intertemporal sum of

$$\pi(s) = c^*(s) + \frac{\mathbf{q}(s)}{p(s)}\dot{\mathbf{k}}^*(s) + \frac{\dot{\mathbf{q}}(s)}{p(s)}\mathbf{k}^*(s),$$

where $\pi(s)$ is discounted by $p(s)$. Since

$$\dot{\mathbf{Q}}(s) = \frac{d}{ds}\left(\frac{\mathbf{q}(s)}{p(s)}\right) = \frac{\dot{\mathbf{q}}(s)}{p(s)} - \frac{\dot{p}(s)}{p(s)}\frac{\mathbf{q}(s)}{p(s)} = \frac{\dot{\mathbf{q}}(s)}{p(s)} + r_0(s)\mathbf{Q}(s)$$

it follows that

$$\pi(s) = c^*(s) + \mathbf{Q}(s)\dot{\mathbf{k}}^*(s) - (r_0(s)\mathbf{Q}(s) - \dot{\mathbf{Q}}(s))\mathbf{k}^*(s).$$

These arguments establish the following result, which appears to be novel.

PROPOSITION 10. *Net social profit $\pi(t)$ is an index for the evaluation of small policy changes if*

$$\pi(t) = g(t) - (r_0(t)\mathbf{Q}(t) - \dot{\mathbf{Q}}(t))\mathbf{k}^*(t).$$

Hence, net social profit is not equal to Green NNP, $g(t) = c^*(t) + \mathbf{Q}(t)\dot{\mathbf{k}}^*(t)$; rather, it is equal to Green NNP *minus* the cost of holding capital, $(r_0(t)\mathbf{Q}(t) - \dot{\mathbf{Q}}(t))\mathbf{k}^*(t)$.

The difference between $g(t)$ and $\pi(t)$ is that $g(t)$ measures *gross* social profit while $\pi(t)$ measures *net* social profit. In the Ramsey model it is possible—even if $\tilde{F}(s) = F(s)$ for all $s \in [0,t]$—to find an alternative feasible path $(\tilde{c}(s),\tilde{k}(s),\dot{\tilde{k}}(s))_{s=0}^t$ that increases the discounted intertemporal sum of $g(s) = c^*(s) + Q(s)\dot{k}^*(s) = (p(s)c^*(s) + q(s)\dot{k}^*(s)) / p(s)$ between 0 and t

$$\int_0^t p(s)g(s)ds = \int_0^t (p(s)c^*(s) + q(s)\dot{k}^*(s))ds.$$

This can be achieved by accumulating capital between 0 and $t/2$ and decumulating capital between $t/2$ and t, using a constant consumption path with its supporting prices as reference. The reason is that $\int_0^t p(s)(\tilde{c}(s) - c^*(s))ds$ is of second order, while $\int_0^t q(s)(\dot{\tilde{k}}(s) - \dot{k}^*(s))ds$ is of first order and positive since $q(s)$ is decreasing. Hence, *the discounted intertemporal sum of $g(s)$ is not a cost-benefit index for a small policy change that lasts for a non-trivial interval of time* unless the policy change does not influence the aggregate path of the vector of investments.

As observed by, for example Vellinga and Withagen (1996, section 5), it *does* hold that $g(t) = c^*(t)) + \mathbf{Q}(t)\dot{\mathbf{k}}^*(t) = (p(t)c^*(t) + q(t)\dot{\mathbf{k}}^*(t)) / p(t)$ is a cost–benefit index for a small policy change lasting only an instance. To see this in the present setting, note that, for a small policy change

$$\lambda(t)(u(\tilde{c}(t)) - u(c^*(t))) + \frac{d}{dt}\left(\int_t^\infty \lambda(s)(u(c^*(s;\tilde{\mathbf{k}}(t),t)) - u(c^*(s)))ds\right)$$
$$\approx p(t)(\tilde{c}(t) - c^*(t)) + \frac{d}{dt}(\mathbf{q}(t)(\tilde{\mathbf{k}}(t) - \mathbf{k}^*(t)))$$
$$= p(t)(\tilde{c}(t) - c^*(t)) + \mathbf{q}(t)(\dot{\tilde{\mathbf{k}}}(t) - \dot{\mathbf{k}}^*(t))$$

since $\tilde{\mathbf{k}}(t) = \mathbf{k}^*(t) = \mathbf{k}$. Hence, the change in the value of consumption measures the current change in utility, while the change in the value of investments measures the time derivative of the discounted intertemporal sum of future changes in utility. This result implies that Green NNP can be used to verify that no policy change should be implemented at any point in time, if it can be shown that any small policy change would contribute non-positively to Green NNP.

These results on the measurement of net social profit are general; i.e., they do *not* depend on there being a constant utility discount rate or there being no exogenous technological progress.

6. Conclusions

In the present paper I have given an overview of the theory of green national accounting by investigating three purposes that such accounting can be used for:

1 Measurement of *welfare equivalent income.*
2 Measurement of *sustainable income*.
3 Measurement of *net social profit*.

It has been shown that welfare equivalent income must be used for the purpose of welfare comparisons across economies; sustainable income will not give a correct result.

Furthermore, I have considered two measures that may potentially serve these purposes:

(i) *Green NNP* (being equal to consumption + value of net investments).
(ii) *Wealth equivalent income* (being the level of consumption with the same present value as the actual future consumption path).

It has been established as a general result that sustainable income $\leqslant$ wealth equivalent income $\leqslant$ welfare equivalent income. It has also been demonstrated how, in principle, wealth equivalent income can be expressed by current prices and quantities only. Practical estimation of such an expression is, however, likely to be informationally demanding.

To establish results concerning Green NNP it is necessary to assume no exogenous technological progress. This is restrictive since (i) it requires that accumulated knowledge is represented by augmented capital stocks, and (ii) it excludes open economies whose 'technology' is changing exogenously due to changing terms of trade. If, in addition, the utility discount rate is constant, it has been shown that Green NNP $\leqslant$ welfare equivalent income. No general result appears to be available concerning the relation between Green NNP and sustainable income except that Green NNP equals wealth equivalent income—and thus overestimates sustainable income—when the interest rate is constant under no exogenous technological progress.

It has been shown that it is *not* justified to associate Green NNP with net social profit. Rather, Green NNP measures *gross* social profit, from which it is necessary to subtract the cost of holding capital in order to arrive at *net* social profit. As a consequence, it is *not* correct to use the discounted intertemporal sum of Green NNP as a cost–benefit index for (even) a small policy change, provided that the policy change lasts for a non-trivial interval of time. Green NNP is, however, a cost–benefit index for a small policy change lasting only an instance.

Throughout I have assumed that all externalities are internalised, and that technological progress is captured by current investments or capital gains. If such assumptions cannot be made, green national accounting must include forward-looking terms of the kind discussed by Aronsson, Johansson and Löfgren (1997, chapter 4) or adjustments like the one suggested by Weitzman (1997).

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Appendix A: A normative foundation for sustainability

Following Buchholz (1997) this appendix provides a normative foundation for sustainability by imposing the following two axioms on any (quasi-)orderings of the set of feasible consumption paths (the statement of the axioms requires that time is discrete):

- Weak Anonymity (WA): Indifference between two paths if the one can be obtained from the other by changing the sequence of a *finite* number of elements.
- Strong Pareto (SP): One path is strictly preferred to another path if it has higher consumption at some date, without having lower consumption at any other date.

Some degree of equity is ensured by WA, which is a weak requirement of intergenerational neutrality. (In fact, van Liederkerke and Lauwers (1997)

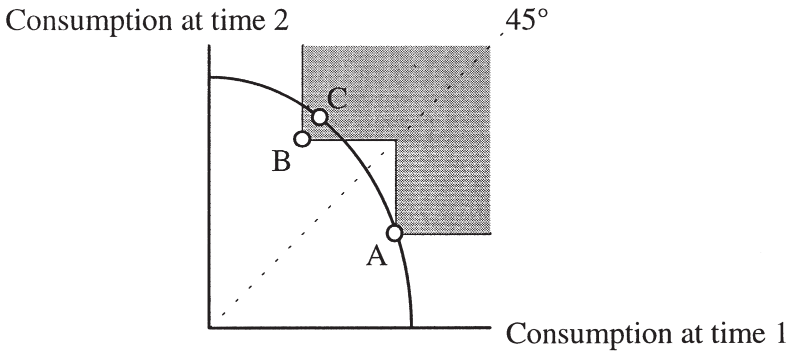

Figure 3.

argue that moving around only a *finite* number of elements is *not* sufficient to ensure equal treatment of an *infinite* number of generations.) Some sensitivity to the interest of any single generation is ensured by SP, which seems quite uncontroversial. The quasi-ordering induced by the axioms is called Suppes–Sen dominance (Suppes, 1966 and Sen, 1970). For a discussion of Suppes–Sen dominance in an intergenerational setting, see Svensson (1980) who defines the term *ethical preferences* by WA and SP.

Suppes–Sen dominance is illustrated by figure 3 in the two-generation case. Since A and B are symmetrical around the 45° line, WA deems the paths A and B as indifferent since one can be obtained from the other by permuting consumption at times 1 and 2. By SP and transitivity, a path like C in the shaded area is strictly preferred to A.

Consider now a *productive* economy, entailing that if $s < t$ and $c_s > c_t$, then a permutation of s and t is feasible, and moreover, there will be some consumption left over after such a permutation. In the two-generation case, an assumption of a productive economy means that the set of feasible consumption path looks like the set inside the curved line in figure 3. It follows that an efficient, but decreasing path like A cannot be Suppes-Sen maximal, since a path like C is feasible and strictly preferred to A. This argument carries over to the case with an infinite number of generations (see Asheim, Buchholz and Tungodden, 1999). A path is Suppes–Sen maximal in a productive economy if and only if it is efficient and non-decreasing, implying that consumption in excess of sustainable income is inconsistent with Suppes–Sen maximality. Thus, the Suppes–Sen dominance yields a normative foundation for sustainability (as often interpreted) in productive economies.

Appendix B: Proofs

Proof of proposition 1. Let $(c^*(t),\mathbf{k}^*(t),\dot{\mathbf{k}}^*(t))_{t=0}^{\infty}$ maximise $\int_0^\infty \lambda(t)u(c(t))dt$ subject to $(c(t),\mathbf{k}(t),\dot{\mathbf{k}}(t)) \in F(t)$ for all t and $\mathbf{k}(0) = \mathbf{k}^0$. Let $V(\mathbf{k}^0, 0) := \int_0^\infty \lambda(t)u(c^*(t))dt$ and define $V(\mathbf{k}, t)$ likewise. Then $V(\mathbf{k}^*(t), t) \geqslant \int_t^{t+\Delta t} \lambda(s)u(c(s))ds + V(\mathbf{k}(t+\Delta t), t+\Delta t)$, if $(c(s),\mathbf{k}(s),\dot{\mathbf{k}}(s))_{s=t}^{t+\Delta t}$ satisfies $(c(s),\mathbf{k}(s),\dot{\mathbf{k}}(s)) \in F(s)$ for all $s \in [t, t+\Delta t]$ and $\mathbf{k}(t) = \mathbf{k}^*(t)$, with equality for $(c^*(s),\mathbf{k}^*(s),\dot{\mathbf{k}}^*(s))_{s=t}^{t+\Delta t}$. Assuming differentiability, we obtain

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$$dV/dt = \frac{\sum \partial V(\mathbf{k}^*(t),t)}{\partial k_i} \dot{k}_i(t) + \frac{\partial V(\mathbf{k}^*(t),t)}{\partial t} \leqslant -\lambda(t)u(c(t))$$

$$dV/dt = \sum \frac{\partial V(\mathbf{k}^*(t),t)}{\partial k_i} \dot{k}^*{}_i(t) + \frac{\partial V(\mathbf{k}^*(t),t)}{\partial t} = -\lambda(t)u(c^*(t)).$$

Hence, for each t, $(c^*(t),\dot{\mathbf{k}}^*(t))$ maximises $\lambda(t)u(c(t)) + \sum (\partial V(\mathbf{k}^*(t),t) / \partial k_i)\, \dot{k}(t)$ subject to $(c(t),\mathbf{k}^*(t),\dot{\mathbf{k}}(t)) \in F(t)$. Let $q_i(t) := \partial V(\mathbf{k}^*(t),t) / \partial k_i$ denote the presence value *price* of capital good i. Let $H(t,c,\dot{\mathbf{k}},\mathbf{q}) := \lambda(t)u(c) + \mathbf{q}\dot{\mathbf{k}}$ denote the present value *Hamiltonian*. Given our assumption of differentiability, we have shown the *Maximum principle*: For each t, $(c^*(t),\dot{\mathbf{k}}^*(t))$ maximises $H(t,c,\dot{\mathbf{k}},\mathbf{q}(t))$ subject to $(c,\mathbf{k}^*(t),\dot{\mathbf{k}}) \in F(t)$. Let $H^*(t,\mathbf{k},\mathbf{q}) := \max_{(c,\mathbf{k},\dot{\mathbf{k}})\in F(t)} H(t,c,\dot{\mathbf{k}},\mathbf{q})$. Assuming differentiability, then for each capital good i, $\dot{\mathbf{q}}_i(t) = \partial(\partial V(\mathbf{k}^*(t),t) / \partial k_i) / \partial t = \partial(\partial V(\mathbf{k}^*(t),t) / \partial t) / \partial k_i = -\,\partial(\lambda(t)u(c^*(t)) + \mathbf{q}(t)\dot{\mathbf{k}}^*(t))/\partial k_i = -\partial(H^*(t,\mathbf{k}^*(t),\mathbf{q}(t)))/\partial k_i$.

By the convexity of $F(t)$, it follows that $H^*(t,\mathbf{k},\mathbf{q}(t))$ is a concave function of $\mathbf{k}$. Assume that $(c,\mathbf{k},\dot{\mathbf{k}}) \in F(t)$. Then $\lambda(t)u(c) + \mathbf{q}(t)\dot{\mathbf{k}} + \dot{\mathbf{q}}(t)\mathbf{k} \leqslant H^*(t,\mathbf{k},\mathbf{q}(t)) + \dot{\mathbf{q}}(t)\mathbf{k} \leqslant H^*(t,\mathbf{k}^*(t),\mathbf{q}(t)) + \sum(\partial(H^*(t,\mathbf{k}^*(t),\mathbf{q}(t)))/\partial k_i)(k_i - k_i^*(t)) + \dot{\mathbf{q}}(t)\mathbf{k} = \lambda(t)u(c^*(t)) + \mathbf{q}(t)\dot{\mathbf{k}}^*(t) + \dot{\mathbf{q}}(t)\mathbf{k}^*(t)$ since, for each capital good i, $\dot{q}_i(t) = -\partial(H^*(t,\mathbf{k}^*(t),\mathbf{q}(t)))/\partial k_i$. Hence, $(c^*(t),\mathbf{k}^*(t),\dot{\mathbf{k}}^*(t))$ maximises $\lambda(t)u(c) + \mathbf{q}(t)\dot{\mathbf{k}} + \dot{\mathbf{q}}(t)\mathbf{k}$ subject to $(c,\mathbf{k},\dot{\mathbf{k}}) \in F(t)$. By the convexity of $F(t)$ and the concavity of u, it follows that $c^*(t)$ maximises $\lambda(t)u(c) - p(t)c$, and $(c^*(t),\mathbf{k}^*(t),\dot{\mathbf{k}}^*(t))$ maximises $p(t)c + \mathbf{q}(t)\dot{\mathbf{k}} + \dot{\mathbf{q}}(t)\mathbf{k}$ subject to $(c,\mathbf{k},\dot{\mathbf{k}}) \in F(t)$, where $p(t) := \lambda(t)u'(c^*(t))$ denotes the present value *price* of the consumption good. □

Proof of proposition 2. Assume $(c(t),\mathbf{k}(t),\dot{\mathbf{k}}(t)) \in F(t)$ for all t and $\mathbf{k}(0) = \mathbf{k}^0$. Then

$$\begin{aligned}
&\textstyle\int_0^T \lambda(t)(u(c(t)) - u(c^*(t)))dt \leqslant \int_0^T p(t)(c(t) - c^*(t))dt \text{ (by C1)}\\
&\leqslant \textstyle\int_0^T [\mathbf{q}(t)(\dot{\mathbf{k}}^*(t) - \dot{\mathbf{k}}(t)) + \dot{\mathbf{q}}(t)(\mathbf{k}^*(t) - \mathbf{k}(t))]dt \text{ (by C2)}\\
&= \int_0^T \left[\frac{d}{dt}(\mathbf{q}(t)(\mathbf{k}^*(t) - \mathbf{k}(t)))\right]dt = \mathbf{q}(T)(\mathbf{k}^*(T) - \mathbf{k}(T)) - \mathbf{q}(0)(\mathbf{k}^*(0) - \mathbf{k}(0))\\
&\leqslant \mathbf{q}(T)(\mathbf{k}^*(T)
\end{aligned}$$

since $\mathbf{k}^*(0) = \mathbf{k}(0) = \mathbf{k}^0$, $\mathbf{q}(T) \geqslant 0$ (by free disposal of investment flows) and $\mathbf{k}(T) \geqslant 0$. By R1 and R2 the result follows. □

Proof of lemma 1. (i) follows directly from C1. (ii) Since $F(t)$ is time invariant, C2 implies that

$$\begin{aligned}
&p(t)c^*(t + \Delta t) + \mathbf{q}(t)\dot{\mathbf{k}}^*(t + \Delta t) + \dot{\mathbf{q}}(t)\mathbf{k}^*(t + \Delta t)\\
&\leqslant p(t)c^*(t) + \mathbf{q}(t)\dot{\mathbf{k}}^*(t) + \dot{\mathbf{q}}(t)\mathbf{k}^*(t).
\end{aligned}$$

Divide by Δt, and let Δt go to zero both from the right and from the left. This yields $0 = p(t)\dot{c}^*(t) + \mathbf{q}(t)\ddot{\mathbf{k}}^*(t) + \dot{\mathbf{q}}(t)\dot{\mathbf{k}}^*(t) = p(t)\dot{c}^*(t) + d(\mathbf{q}(t)\dot{\mathbf{k}}^*(t)) / dt$, where differentiability follows since $F(t)$ is smooth. □

Proof of lemma 2. If $F(s)$ is a convex cone, then $(c^*(s),\mathbf{k}^*(s),\dot{\mathbf{k}}^*(s))$ maximises $p(s)c + \mathbf{q}(s)\dot{\mathbf{k}} + \dot{\mathbf{q}}(s)\mathbf{k}$ subject to $(c,\mathbf{k},\dot{\mathbf{k}}) \in F(s)$ only if $p(s)c^*(s) + \mathbf{q}(s)\dot{\mathbf{k}}^*(s) + \dot{\mathbf{q}}(s)\mathbf{k}^*(s) = 0$. Hence, $p(s)c^*(s) + d(\mathbf{q}(s)\mathbf{k}^*(s)) / ds = 0$ for all s, such that, by R2, $\mathbf{q}(t)\mathbf{k}^*(t) = \int_t^\infty p(s)c^*(s)ds$.

Proof of proposition 7. Case (i): $u(c) = \ln c$. Since, by lemma 1(i), $\lambda(s) / c^*(s) = p(s)$ for all s,

$$\ln\left(\tilde{r}_\infty(t) \int_t^\infty \frac{p(s)}{p(t)} c^*(s)ds\right)$$

$$= -\ln\left(\int_t^\infty \lambda(s)ds\right) - \frac{\int_t^\infty \lambda(s)\ln(p(s)/\lambda(s))ds}{\int_t^\infty \lambda(s)ds} + \ln(\textstyle\int_t^\infty p(s)\, c^*(s)ds)$$

$$= -\ln(\textstyle\int_t^\infty \lambda(s)ds) + \frac{\int_t^\infty \lambda(s)\ln c^*(s)ds}{\int_t^\infty \lambda(s)ds} + \ln(\textstyle\int_t^\infty \lambda(s)ds) = \ln w(t).$$

Hence, $\tilde{r}_\infty(t) \int_t^\infty (p(s) / p(t))\, c^*(s)ds = w(t)$.

Case (ii): $u(c) = c^\rho / \rho$, $\rho < 1$, $\rho \neq 0$. Since, by lemma 1(i), $\lambda(s)\, \rho c^*(s)^{\rho-1} = p(s)$ for all s

$$\left(\tilde{r}_\infty(t) \int_t^\infty \frac{p(s)}{p(t)} c^*(s)ds\right)^\rho = \frac{(\int_t^\infty p(s)c^*(s)\mathrm{d}s)^\rho}{(\int_t^\infty \lambda(s)ds)^\rho \cdot \left(\frac{(\int_t^\infty \lambda(s)(p(s)/\lambda(s))^{\rho/(\rho-1)}ds}{\int_t^\infty \lambda(s)ds}\right)^{(\rho-1)}}$$

$$= \frac{\rho^\rho \cdot (\int_t^\infty \lambda(s)c^*(s)^\rho\, ds)^\rho}{(\int_t^\infty \lambda(s)ds)^\rho \cdot \rho^\rho \cdot \left(\frac{\int_t^\infty \lambda(s)c^*(s)^\rho\, ds}{\int_t^\infty \lambda(s)ds}\right)^{(\rho-1)}} = \frac{\int_t^\infty \lambda(s)c^*(s)^\rho\, ds}{\int_t^\infty \lambda(s)ds} = w(t)^\rho.$$

Hence, $\tilde{r}_\infty(t) \int_t^\infty (p(s) / p(t))c^*(s)ds = w(t)$.

In both cases definition 6 implies that $\tilde{r}_\infty(t) \int_t^\infty (p(s) / p(t))\, c^*(s)ds = (\tilde{r}_\infty(t) / r_\infty(t))\, h(t)$. The second part of the proposition follows since $\int_t^\infty (p(s) / p(t))\, c^*(s)ds = (g(t) + \dot{\mathbf{Q}}(t)\mathbf{k}^*(t)) / r_0(t)$ under the assumption of constant returns to scale (by proposition 4 and definition 6).

Proof of proposition 8. Suppose $m(t) > h(t)$. Then $(c(s))_{s=t}^\infty$ with $c(s) = m(t)$ for all s is feasible, and $\int_t^\infty p(s)c(s)ds > \int_t^\infty p(s)h(t)ds = \int_t^\infty p(s)c^*(s)ds$, where the last equality follows from definition 6. This yields a contradiction since the proof of proposition 2 implies that $(c^*(s))_{s=t}^\infty$ maximises $\int_t^\infty p(s)c(s)ds$ over all feasible consumption paths. Hence, $h(t) \geqslant m(t)$. Moreover, it follows directly from definition 6 that $h(t) = c^* = m(t)$ if $c^*(s) = c^*$ for all s.

Proof of proposition 9. Assume no exogenous technological progress and $\lambda(s) = \lambda(0)e^{-\delta s}$. Then

$$p(t)(g(t) - c^*(t)) = \mathbf{q}(t)\dot{\mathbf{k}}^*(t) = \textstyle\int_t^\infty p(s)\dot{c}^*(s)ds \text{ (by definition 5 and using lemma 1(ii))}$$

$$= \textstyle\int_t^\infty \lambda(s)u'(c^*(s))\dot{c}^*(s)ds \text{ (by lemma 1(i))}$$

$$= \lambda(t) \int_t^\infty \frac{\lambda(s)}{\lambda(t)} \frac{d}{ds} (u(c^*(s)))ds = \lambda(t) \frac{d}{dt} \left(\int_t^\infty \frac{\lambda(s)}{\lambda(t)} u(c^*(s))ds\right) \text{ (since } \lambda(s) = \lambda(0)e^{-\delta s})$$

$$= \lambda(t) \frac{d}{dt} \left(\int_t^\infty \frac{\lambda(s)}{\lambda(t)} u(w(t))ds\right) \text{ (by definition 3)}$$

$$= \lambda(t) \int_t^\infty \frac{\lambda(s)}{\lambda(t)}\, ds \cdot \frac{d}{dt}\,(u(w(t))) = \int_t^\infty \lambda(s) ds \cdot u'(w(t))\dot{w}(t)$$

$$(\text{since } \lambda(s) = \lambda(0)\, e^{-\delta s}).$$

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