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Compact metric spaces as minimal-limit sets in domains of bottomed sequences

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Every compact metric space X is homeomorphically embedded in an ω-algebraic domain D as the set of minimal limit (that is, non-finite) elements. Moreover, X is a retract of the set $L(D)$ of all limit elements of D. Such a domain D can be chosen so that it has property M and finite-branching, and the height of $L(D)$ is equal to the small inductive dimension of X. We also show that the small inductive dimension of $L(D)$ as a topological space is equal to the height of $L(D)$ for domains with property M. These results give a characterisation of the dimension of a space X as the minimal height of $L(D)$ in which X is embedded as the set of minimal elements. The domain in which we embed an n-dimensional compact metric space X ($n \leqslant \infty$) has a concrete structure in that it consists of finite/infinite sequences in $\{0, 1, \bot\}$ with at most n copies of $\bot$.

1. Introduction

When D is an ω-algebraic domain, we can consider the set $L(D)$ of limit (that is, non-finite) elements of D as a topological space with the subspace topology of the Scott topology of D, and the set $K(D)$ of finite elements of D as its approximation structure. That is, $K(D)$ forms a base of the topology of $L(D)$ through the identification of $d \in K(D)$ with the open set $\uparrow d \cap L(D)$. We can use this domain-theoretic viewpoint for a topological space X when X is embedded in $L(D)$. In this case, $K(D)$ also forms a base of X, and each element of X can be identified as the limit of an infinite strictly increasing sequence in $K(D)$.

This viewpoint is particularly effective when D is composed of infinite sequences in $\Sigma_\bot = \{0, 1, \bot\}$. In this case, each cell of a sequence can be considered as representing boolean information and an infinite strictly increasing sequence in $K(D)$ can be considered as an infinite (possibly uncomputable) process that incrementally outputs 0 or 1 to the cells based on the partial information about the point obtained so far. The order the cells are filled may not be unique, and is regulated by the structure of $K(D)$. Some of the cells may be left unfilled even after the infinite time of execution, and in that case, the corresponding cell has the value $\bot$.

Many of the computational notions over topological spaces that have been studied so far are related to this idea of representing a computation as an infinite process with incremental outputs based on partial information. A Type-2 machine (Weihrauch 2000) can implement this kind of output because we can encode, as an infinite sequence of characters, an infinite list of pairs composed of the index and the value of a cell. An IM2-machine

(Tsuiki 2002) can directly manipulate this kind of sequence with bottoms because it has the ability to skip some of the cells with multiple-heads and indeterministic rules. And RealPCF (Escardó 1996) realises computation over the continuous domain of closed intervals of $\mathbb{R}$ so that better and better approximations to an interval are obtained as the evaluation proceeds. Embeddings of topological spaces into domains have been studied by many authors (Weihrauch and Schreiber 1981; Blanck 2000; Edalat 1997; Edalat and Sünderhauf 1998) with the motivation being the use of effective structures of domains (Smyth 1977) for the study of computation over topological spaces, and, in particular, the embedding of $\mathbb{R}$ in an ω-algebraic domain is studied in Di Gianantonio (1999).

To ensure that this programme to embed a space X in $L(D)$ for the study of the topological and computational structure of X works very well, we assume that all the infinite increasing sequences in $K(D)$ are meaningful, and identify one point of X. That is, every process whose output at each finite time is valid and that continues to output infinitely should be considered as designating a unique point of X. We first show that, when D has property M (which is equivalent to Lawson-compactness because we only consider ω-algebraic domains), this condition is equivalent to requiring that X is a Hausdorff space densely embedded in D as the set of minimal elements of $L(D)$ (Section 4). Note that many of the domains studied in computer science such as $P_\omega = \{u \mid u \subseteq N\}$ and Plotkin's T^ω(Plotkin 1978) do not have minimal limit elements. We introduce a condition on $K(D)$ that guarantees the existence of enough minimal limit elements. That is, $K(D)$ is a finite-branching poset. A domain with this condition on $K(D)$ is called a finite-branching domain (fb-domain in short). In any fb-domain, the minimal limit elements form a compact space.

We show that for each compact metric space X, there is an fb-domain D that contains X as the set of minimal limit elements of D. Moreover, X is a retract of $L(D)$. We first present an fb-domain RD that has $\mathbb{I} = [0, 1]$ as the set of minimal limit elements (Section 5). RD is usually defined as the domain corresponding to the signed digit representation of real numbers, and this retract structure has already been investigated in Di Gianantonio (1999). In this paper, based on the Gray-code embedding (Tsuiki 2002), we present this domain as a subdomain of BD_1, which is the set of finite/infinite sequences in $\{0, 1, \perp\}$ with at most one copy of $\perp$. Then we define a new product (called the synchronous product) of fb-domains, and construct domains corresponding to the n-dimensional Euclidean cube $\mathbb{I}^n$ $(n = 0, 1, 2, \ldots)$ and the Hilbert cube $\mathbb{I}^\omega$ (Section 7). Finally, we prove the existence of such an fb-domain for a compact metric space in general, based on Nöbeling's universal n-dimensional space (Nöbeling 1931) for the finite dimensional case, and the universality of the Hilbert cube $\mathbb{I}^\omega$ for the infinite dimensional case (Section 8).

When X is n-dimensional $(n \leqslant \infty)$, we construct all the fb-domains mentioned above so that they are composed of finite/infinite sequences in $\{0, 1, \perp\}$ with at most n copies of $\perp$. In addition, we show that we need to use at least n copies of $\perp$ when X is n-dimensional and D has property M. For this purpose, the topological dimension of the set of limit elements of a domain is studied. It is proved that the small inductive dimension of $L(D)$ is equal to the maximal length of a chain in $L(D)$ when D has property M (Section 5). Thus, we have a characterisation of the dimension of a space X as the minimal height of $L(D)$ in which X is embedded as the set of minimal elements. This is a generalisation of the result in Tsuiki (2000), and the proof is greatly simplified by thinking about the dimension of $L(D)$ in general.

2. Preliminaries and notation

First note that in this paper we use the word *domain* to mean an ω-algebraic pointed dcpo.

Infinite sequences

In this paper, we fix the character set Σ as $\{0, 1\}$ unless we state otherwise. We write Σ^* for the set of finite sequences of Σ, and Σ^ω for the set of infinite sequences of Σ. Σ^* forms a tree (and thus a poset) with respect to the prefix ordering. We sometimes identify an infinite sequence with an infinite tape, and call each place to write a character a cell. A *bottomed sequence* is an infinite sequence of $\Sigma_\perp = \Sigma \cup \{\perp\}$, where $\perp$ means undefinedness. In other words, it is an infinite tape some of whose cells may not be filled by a character in Σ. We write $\Sigma_\perp^\omega$ for the set of bottomed sequences. When $\alpha \in \Sigma_\perp^\omega$, we write $\alpha[j]$ $(j = 0, 1, 2\ldots)$ for the j-th component of α. When $\alpha[j] = \perp$ for $j \geqslant n$, we say that α is a finite bottomed sequence.

Domain theory

Let $(P, \leqslant)$ and $(Q, \leqslant)$ be partially ordered sets (posets). When $d, e \in P$, we write $d < e$ for $d \leqslant e$ and $d \neq e$, $\uparrow d$ for the set $\{d' \in P \mid d' \geqslant d\}$, and $\downarrow d$ for the set $\{d' \in P \mid d' \leqslant d\}$. We also write $\uparrow A$ (or $\downarrow A$) for the set $\cup_{a \in A} \uparrow a$ (or $\cup_{a \in A} \downarrow a$) and say that a subset A is *upper-closed* (or *down-closed*) when $\uparrow A = A$ (or $\downarrow A = A$). We say that a pair of elements d and e are *bounded* if d and e have an upper bound, and write $d \uparrow_P e$, or $d \uparrow e$ when P is obvious.

A subset A of a poset P is *directed* if it is non-empty and each pair of elements of P has an upper bound in A. A *directed complete partial order* (*dcpo*) is a partial order $(D, \leqslant)$ where every directed subset A has a *least upper bound* (*lub*) $\sqcup A$, also called the *supremum* of A. A poset P is *pointed* if it has a least element. A *finite element* of a dcpo D is an element $d \in D$ such that for every directed subset A, if $d \leqslant \sqcup A$, then $d \leqslant a$ for some element $a \in A$. We write $K(D)$ for the set of finite elements of D. An element of D is called a *limit element* when it is not finite. We write $L(D)$ for the set of limit elements of D. We write K_x for $K(D) \cap \downarrow x$. A dcpo D is *algebraic* if K_x is directed and $\sqcup K_x = x$ for each $x \in D$; and it is ω-*algebraic* if D is algebraic and $K(D)$ is countable. In this paper, we use the word *domain* to mean an ω-algebraic pointed dcpo. See, for example, Abramsky and Jung (1994), Plotkin (1981) and Stoltenberg-Hansen *et al* (1994) for expositions of the theory of domains.

An *ideal* of D is a directed down-closed subset. When P is a countable poset with least element, $Idl(P)$, the set of ideals of P ordered by set inclusion, becomes a domain called the *ideal completion* of P, and satisfies $K(Idl(P)) \cong P$. On the other hand, when D is a domain, we have $Idl(K(D)) \cong D$. Therefore, $K(D)$, the set of finite elements of D, determines the structure of D. We say that an ideal of $K(D)$ is *principal* (or *non-principal*) if its supremum is in $K(D)$ (or $L(D)$). When D is a domain and $a_1 < a_2 < \dots$ is an infinite strict increasing sequence in $K(D)$, it determines a non-principal ideal $\{x \in K(D) \mid x \leqslant a_i \text{ for some } i\}$ of $K(D)$ and thus determines a point of $L(D)$. A domain D is *bounded complete* if every bounded pair has a supremum.

The *Scott topology* of a dcpo P is defined so that a subset O is open iff it is upper-closed and for each directed subset S of P with $\sqcup S \in O$, $s \in O$ for some $s \in S$. When D is an algebraic dcpo, the set $\{\uparrow d \mid d \in K(D)\}$ forms a base of the Scott topology on D.

When $(D, \leqslant)$ is a domain, we call $E \subseteq D$ a *subdomain* if $(E, \leqslant)$ is a domain, $K(E) \subseteq K(D)$, and the embedding of E in D preserves the least element and the supremums of directed sets. In this case, the Scott topology of $(E, \leqslant)$ is the subspace topology of that of $(D, \leqslant)$.

Two domains of bottomed sequences

The set $\Sigma^\infty = \Sigma^\omega \cup \Sigma^*$ is a domain with $K(\Sigma^\infty) = \Sigma^*$ and $L(\Sigma^\infty) = \Sigma^\omega$. Σ^ω is called *Cantor space* and the topology on Σ^ω induced as the subspace topology of the Scott topology of Σ^∞ is called the *Cantor topology*. The set of bottomed sequences $(\Sigma_\perp^\omega, \leqslant)$ also forms a domain with $x \leqslant y$ iff $x[k] \leqslant y[k]$ for all $k = 0, 1, \dots$. Here, the order on $\Sigma_\perp$ is defined as $\perp \leqslant a$ for $a \in \Sigma$. In $(\Sigma_\perp^\omega, \leqslant)$, d is a finite element iff d is a finite bottomed sequence. Domains that are subdomains of $\Sigma_\perp^\omega$ (and thus composed of bottomed sequences) will play an important role in this paper.

Topology

When O is a subset of a topological space X, we write $cl_X(O)$ and $int_X(O)$ for the closure and interior of O in X, respectively, and $B_X(O)$ for the *boundary* of O in X, that is, $cl_X(O) - int_X(O)$. We write $cl(O)$, $int(O)$, and $B(O)$ when these are unambiguous. A space X is said to be a *retract* of a space Y if there is a pair $s : X \to Y$, $r : Y \to X$ of continuous functions such that $r \circ s$ is the identity on X. When X is a subspace of Y, we say that X is a *retract* of Y if r and the embedding of X in Y form a retract. In this paper, we say that a topological space is *compact* when each open cover has a finite subcover and we do not assume the Hausdorff property. See, for example, Smyth (1992) and Engelking (1989) for topological notions.

Filter and filter-base

A *filter* in a topological space X is a non-empty family $\mathscr{F}$ of subsets of X that satisfies the following conditions:

1 If $A \in \mathscr{F}$ and $A \subseteq B$, then $B \in \mathscr{F}$.
2 If $A_1 \in \mathscr{F}$ and $A_2 \in \mathscr{F}$, then $A_1 \cap A_2 \in \mathscr{F}$.
3 $\emptyset \notin \mathscr{F}$.

A *filter-base* in X is a non-empty family $\mathscr{B}$ of subsets of X that satisfies:

1 If $A_1 \in \mathscr{B}$ and $A_2 \in \mathscr{B}$, then there exists an $A_3 \in \mathscr{B}$ such that $A_3 \subseteq A_1 \cap A_2$.
2 $\emptyset \notin \mathscr{B}$.

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When $\mathscr{B}$ is a filter-base, the family

$$\mathscr{F}_{\mathscr{B}} = \{A \subseteq X \mid \text{there exists a } B \in \mathscr{B} \text{ such that } B \subseteq A\}$$

is a filter. A point x is called a *limit* of a filter $\mathscr{F}$ if every neighbourhood of x belongs to $\mathscr{F}$, and x is called a *limit* of a filter-base $\mathscr{B}$ if x is a limit of $\mathscr{F}_{\mathscr{B}}$. When x is a limit of a filter (or a filter-base) $\mathscr{F}$, we say that $\mathscr{F}$ converges to x. A point x is called a *cluster point* of a filter $\mathscr{F}$ (or a filter base $\mathscr{B}$) if x belongs to the closure of every element of $\mathscr{F}$ (or $\mathscr{B}$). We say that a filter (or a filter-base) $\mathscr{F}_1$ *refines* $\mathscr{F}_2$ if $\mathscr{F}_1 \supseteq \mathscr{F}_2$. We say that a filter (or a filter-base) $\mathscr{F}$ is *infinite* when $\mathscr{F}$ is an infinite family. See, for example, Engelking (1989) for more about filters.

The real line

A *dyadic number* is a rational number of the form $m \times 2^{-n}$ for integers m and n. We write $\mathbb{I}$ for the unit closed interval $[0, 1]$.

3. Domains with property M

In this section, we give some fundamental properties of domains with property M.

We use domains to represent topological structures; we embed a topological space X in $L(D)$ and consider $K(D)$ as a topological base of X through the identification of $d \in K(D)$ with the subset $\uparrow d \cap X$ of X. Therefore, when $\uparrow d \cap L(D)$ is empty, d does not contribute to defining the topology of X. It is easy to show the following lemma.

Lemma 3.1. When D is a domain, the followings are equivalent.

(1) $\uparrow d \cap L(D) \neq \varnothing$ for all $d \in K(D)$.
(2) $L(D)$ is dense in D.
(3) D has no maximal finite element.

In the following, we will refer to this property as D *has no maximal finite element*. In this paper, we are particularly interested in domains without maximal finite elements. However, most of the theorems hold without this condition and thus we do not assume it in general. We will write $\hat{D}$ for the domain $D - \{d \in K(D) \mid d \neq \bot \text{ and } \uparrow d \cap L(D) = \varnothing\}$.

Lemma 3.2. When D is a domain and $L(D) \neq \varnothing$, we have $\hat{D}$ is a domain without maximal finite elements. When D is a domain, $L(D) = L(\hat{D})$.

The notion of the set of minimal elements appears frequently in this paper.

Definition 3.3. Let P be a poset.

(1) $x \in P$ is a *minimal element* if $y \leqslant x$ implies $y = x$ for all $y \in P$.
We write M_P for the set of all minimal elements of P.
(2) We say that P has *enough minimal elements* if, for all $y \in P$, there exists $x \in M_P$ such that $x \leqslant y$.

Many of the results of this paper are based on the following completeness condition, which is more general than bounded completeness.

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Definition 3.4.

(1) We say that a poset P is *mub-complete* if for every finite subset $X \subseteq P$, the set of upper bounds of X has enough minimal elements. That is, when y is an upper bound of X, there exists a minimal upper bound y' of X such that $y' \leqslant y$.

(2) We say that a domain (that is, ω-algebraic pointed dcpo) D has *property M* if $K(D)$ is mub-complete and each finite subset $X \subseteq K(D)$ has a finite set of minimal upper bounds.

Property M is equivalent to Lawson-compactness for ω-algebraic domains by the 2/3 SFP Theorem (Plotkin 1981), and domains with property M are studied in Jung (1989). Though only ω-algebraic domains are considered in this paper, the results of this section and Section 6 can be generalised to Lawson-compact continuous domains, as discussed in Section 9.

Lemma 3.5. When D has property M, $\hat{D}$ also has property M.

Proposition 3.6. Suppose that D is a domain with property M.

(1) $\alpha \uparrow \beta$ for $\alpha, \beta \in D$ iff $d \uparrow e$ for all $d \in K_\alpha$ and $e \in K_\beta$.

(2) $cl_D(\uparrow d) = \{\alpha \in D \mid d \uparrow \alpha\}$ $(= \downarrow\uparrow d)$ for $d \in K(D)$.

(3) If $L(D)$ is a T_1 space, then $L(D)$ is a Hausdorff space.

(4) Suppose also that $L(D)$ has enough minimal elements. If $M_{L(D)}$ is a retract of $L(D)$, then $M_{L(D)}$ is a Hausdorff space.

Proof.

(1) *If part*: Let γ be an upper bound of α and β. Then, γ is also an upper bound of e and f.
Only if part: First note that when $d \uparrow e$ for $d, e \in K(D)$, an upper bound of d and e exists in $K(D)$ because if $\gamma \in L(D)$ is an upper bound of d and e, then K_γ is directed. Let $\bot = d_0 < d_1 < \ldots$ and $\bot = e_0 < e_1 < \ldots$ be strictly increasing sequences in $K(D)$ with least upper bounds α and β, respectively. Choose an upper bound $f_i \in K(D)$ of d_i and e_i for every i. Now we will form an infinite increasing sequence $g_0 < g_1 < \ldots$ such that $d_i \leqslant g_i$, $e_i \leqslant g_i$ and the set $N_k = \{f_i \mid f_i > g_k, i > k\}$ is infinite for every k. First take $g_0 = \bot$. Suppose that $g_0, \ldots, g_k$ are defined. Consider the set $G_k = \{g_k, d_{k+1}, e_{k+1}\}$. Note that N_k is an infinite set of upper bounds of G_k. Since the set of minimal upper bounds of G_k is finite, we can choose a minimal upper bound g_{k+1} of G_k so that N_{k+1} is infinite. The least upper bound of such a sequence is greater than both α and β.

(2) We need to show $cl_D(\uparrow d) \ni \alpha$ iff $d \uparrow \alpha$. $cl_D(\uparrow d) \ni \alpha$ means that $\uparrow d \cap \uparrow e \neq \emptyset$ for all $e \in K_\alpha$. $\uparrow d \cap \uparrow e \neq \emptyset$ iff $d \uparrow e$, and it is equivalent to $f \uparrow e$ for all $f \in K_d$ because d is finite. Therefore, by applying (1), we have the result.

(3) First consider the case in which D has no maximal finite elements. If $L(D)$ is a T_1 space and $x, y \in L(D)$ are different elements, then x and y do not have an upper bound in $L(D)$, and therefore they do not have an upper bound in D, because if a finite element is an upper bound, there is also an upper bound that is a limit element.

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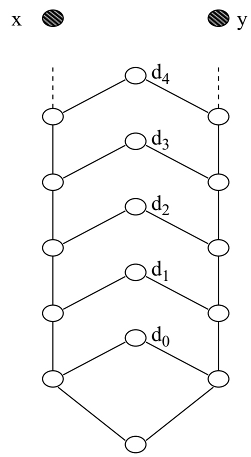

Fig. 1. An ω-algebraic domain that does not have property M.

Therefore, from (1), for some $d, e \in K(D)$ such that $d < x$ and $e < y$, we have d and e do not have an upper bound in D. This means that $\uparrow d$ and $\uparrow e$ do not intersect. For the case in which D has a maximal finite element and $L(D) \neq \emptyset$, x and y may have an upper bound in D even when $L(D)$ is a T_1 space, as Figure 2 shows. Therefore, we consider $\hat{D}$ instead of D. $\hat{D}$ comes to be a domain with property M and with no maximal finite element, and we have $L(D) = L(\hat{D})$ by Lemmas 3.5 and 3.2.

(4) As in (3), we only need to show this theorem for the case in which D has no maximal finite elements. Let r be the retract map from $L(D)$ to $M_{L(D)}$. Since r is monotonic and $L(D)$ has enough minimal elements, we have $r^{-1}(x) = \uparrow x$. This means that every pair of different elements of $M_{L(D)}$ do not have an upper bound in either $L(D)$ or D. Therefore, they are separated by open sets in D by (1), and thus they are separated by open sets in $M_{L(D)}$. □

Example 3.7. A counterexample to Proposition 3.6 (1) and (2) when D does not have property M is given in Figure 1. Note that there is no order relation between d_i $(i = 0, 1, \ldots)$.

Example 3.8. As a counterexample to Proposition 3.6 (3) and (4), one can add, to each d_i in Figure 1, a strictly increasing sequence $d_i = e_{i,0} < e_{i,1} < \ldots$ and its limit p_i $(i = 0, 1, \ldots)$. Then, $L(D) = \{x, y, p_0, p_1, \ldots\}$ is flat in that whenever $t, u \in L(D)$ and $t \leqslant u$, we have $t = u$. Thus $L(D)$ is a T_1 space. However, $L(D)$ is not Hausdorff because every open neighbourhood of x (and also y) contains all p_n $(n \geqslant k)$ for some k.

Note that Proposition 3.6 (1) can also be stated as x and y are separated by open sets iff $x \not\uparrow y$. In addition, when D has no maximal finite element, we can take an upper bound of x and y in $L(D)$. Therefore, in this case, we can restate (1) in the following form, which connects the order structures of $L(D)$ and $K(D)$.

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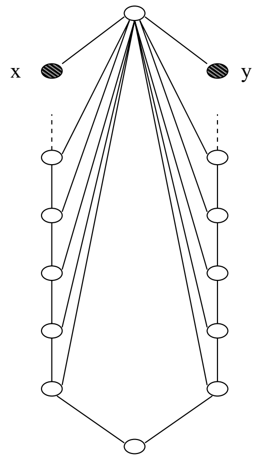

Fig. 2. An ω-algebraic domain that has property M and has a maximal finite element.

Proposition 3.9. Suppose that D is a domain with property M and with no maximal finite element. $x \uparrow_{L(D)} y$ iff $d \uparrow_{K(D)} e$ for all $d \in K_x$ and $e \in K_y$.

Figure 2 shows a counterexample when D has a maximal finite element.

4. Embeddings in minimal-limit sets of domains

When D is a domain, we can consider $L(D)$ as a topological space and $K(D)$ as its approximation structure. That is, through the identification of $d \in K(D)$ with the open set $\uparrow d \cap L(D)$, we have $K(D)$ forms a base of the topology of $L(D)$, which is the subspace topology of the Scott topology of D. Through this identification, each $y \in L(D)$, viewed as the ideal $K_y \subseteq K(D)$, defines a filter-base $\mathscr{F}(K_y) = \{\uparrow d \cap L(D) \mid d \in K_y\}$ of $L(D)$, which converges as follows.

Proposition 4.1. The set of limits of $\mathscr{F}(K_y)$ is $\downarrow y \cap L(D)$.

Proof. A point x is a limit of $\mathscr{F}(K_y)$ iff, for every $d \in K_x$, there exists $e \in K_y$ such that $\uparrow d \cap L(D) \supseteq \uparrow e \cap L(D)$, which is equivalent to $d \leqslant e$. Therefore, x is a limit of $\mathscr{F}(K_y)$ iff $x \leqslant y$. □

When X is a subspace of $L(D)$, $K(D)$ also forms a base of the topology of X, through the identification of $d \in K(D)$ with the open set $\uparrow d \cap X$. Through this identification, each $y \in L(D)$, viewed as the ideal $K_y \subseteq K(D)$, defines a family $\mathscr{F}_X(K_y) = \{\uparrow d \cap X \mid d \in K_y\}$ of subsets of $L(D)$. It is easy to show that $\mathscr{F}_X(K_y)$ becomes a filter-base for all $y \in L(D)$ iff X is dense in D; X is dense in D iff $\uparrow d \cap X$ is not empty for each $d \in K(D)$ and the first condition of the definition of a filter-base holds because K_y is an ideal.

Now, suppose that X is dense in D, and consider the condition that for each $y \in L(D)$, the filter-base $\mathscr{F}_X(K_y)$ converges to a unique point of X. When this holds, each infinite strictly increasing sequence in $K(D)$, which identifies an element of $L(D)$ and determines

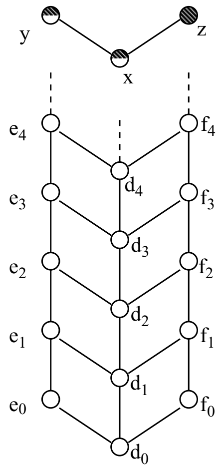

Fig. 3. An example showing that there is no limit of $\mathscr{F}_Y(K_x)$ for $Y = \{y, z\}$.

a non-principal ideal of $K(D)$, specifies an element of X as the limit of the corresponding filter-base.

The uniqueness of such a point, if it exists, is guaranteed when X is a Hausdorff space because each filter-base converges to at most one point in a Hausdorff space. The converse is also true when D has property M, as follows.

Proposition 4.2. Suppose that D is a domain with property M and X is a dense subset of $L(D)$. All the filter-bases of the form $\mathscr{F}_X(K_y)$ ($y \in L(D)$) have at most one limit point iff X is Hausdorff.

Proof. We only need to show this for domains with no maximal finite element because, in the domains D and $\hat{D}$, the sets of limit elements are the same and the filter-bases of the forms $\mathscr{F}_X(K_y)$ are the same.
If part: This is as described above.
Only if part: Suppose that X is not Hausdorff. Then there are two points x and y that are not separated by open sets. That is, for all pairs of finite elements $d < x$ and $e < y$, $\uparrow d$ and $\uparrow e$ intersect in X, and thus d and e have an upper bound in $K(D)$. Therefore, from Proposition 3.9, x and y have an upper bound z in $L(D)$. Then, $\mathscr{F}_X(K_z)$ converges to both x and y. □

For the existence of such a limit, if $\downarrow y \cap X \neq \emptyset$, then $\mathscr{F}_X(K_y)$ converges to every point of $\downarrow y \cap X$. However, when $\downarrow y \cap X$ is empty, there may be no limit of $\mathscr{F}_X(K_y)$.

Example 4.3. Consider the domain D in Figure 3. Let Y be the subset $\{y, z\}$ of $L(D)$. Then the filter-base $\mathscr{F}_Y(K_x)$ does not converge to a point.

When $\downarrow y \cap X$ is empty, there are also cases in which a limit of $\mathscr{F}_X(K_y)$ exists but is not a limit of $\mathscr{F}(K_y)$.

Example 4.4. In Example 4.3, consider the set $Y = \{y\}$ and the filter-base $\mathscr{F}_Y(K_x)$. This converges to y, while $\mathscr{F}(K_x)$ converges only to x.

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In order to exclude these cases, we consider the condition that $\mathscr{F}_X(K_y)$ ($y \in L(D)$) converges to a unique point that is among the limits of $\mathscr{F}(K_y)$. It is immediate that under this condition D has enough minimal limit elements and X is the minimal-limit set of D defined as follows.

Definition 4.5. Let D be a domain. $x \in L(D)$ is a *minimal limit element* of D if it is a minimal element in $L(D)$. We say that D *has enough minimal limit elements* if $L(D)$ has enough minimal elements (Definition 3.4). In this case, $M_{L(D)}$ is called the *minimal-limit set* of $L(D)$.

On the other hand, these conditions on D and X are sufficient for the above condition.

Proposition 4.6. Suppose that D is a domain which has enough minimal limit elements and that $X = M_{L(D)}$ is a Hausdorff dense subspace of D.

(1) X is a retract of $L(D)$.
Let $y \in L(D)$ in the remaining parts.
(2) The filter-base $\mathscr{F}_X(K_y)$ converges to a unique point $r(y)$ for r the retract map from $L(D)$ to X.
(3) $\cap\mathscr{F}_X(K_y) = \{y\}$ if $y \in X$.
(4) $\cap\mathscr{F}_X(K_y) = \emptyset$ if $y \notin X$.
(5) $\cap\{cl(s) \mid s \in \mathscr{F}_X(K_y)\} = \{r(y)\}$. That is, $r(y)$ is the unique cluster point of $\mathscr{F}_X(K_y)$.

Proof.

(1) From the minimality, for every $y \in L(D)$, there is an element x in X such that $x \leqslant y$. Suppose that there is another element $z \neq x$ in X such that $z \leqslant y$. Since X is Hausdorff, we have $c \in K_x$ and $d \in K_z$ such that $\uparrow c \cap \uparrow d \cap X = \emptyset$. Since $\uparrow c \cap \uparrow d$ includes y and thus is non-empty, from the density of X in D, we have $u \in X$, which is in this set, and thus contradicts the assumption. Therefore, there is only one element x in X such that $x \leqslant y$. We define this element as $r(y)$. r is a continuous function from $L(D)$ to X; $r^{-1}(x) = \uparrow x$ for each $x \in X$, and $r^{-1}(\uparrow d \cap X) = \uparrow d \cap L(D)$ for each $d \in K(D)$. Thus, X is a retract of $L(D)$.
(2) Since $r(y) \leqslant y$ and thus $d \in K_y$ for all $d \in K_{r(y)}$, every neighbourhood $\uparrow d \cap X$ of $r(y)$ is a member of $\mathscr{F}_X(K_y)$. Uniqueness of the limit is guaranteed by the Hausdorff property of X.
(3) $\cap\mathscr{F}_X(K_y)$ is a subset of the set of limits of $\mathscr{F}_X(K_y)$. Thus, we have $\cap\mathscr{F}_X(K_y) \subseteq \{r(y)\}$. Since each element of $\mathscr{F}_X(K_y)$ contains $y = r(y)$ when $y \in X$, we have $\cap\mathscr{F}_X(K_y) \supseteq \{y\}$.
(4) $y \notin X$ means that $y > r(y)$ and, therefore, there is an element $d \in K_y$ such that $\uparrow d$ does not contain $r(y)$.
(5) Let $d \in K_y$. For all $e \in K_{r(y)}$, we have y is an upper bound of d and e. Therefore, an upper bound $f \in K(D)$ of d and e exists, and since X is dense, $\uparrow f \cap X$ is not empty. Therefore, $(\uparrow e \cap X) \cap (\uparrow d \cap X) = \uparrow e \cap \uparrow d \cap X \supseteq \uparrow f \cap X$ is not empty. Therefore, $r(y) \in cl(\uparrow d \cap X)$. On the other hand, when $x \in X$ and $x \neq r(y)$, since X is Hausdorff, there exists $f \in K_x$ and $e \in K_{r(y)}$ such that $\uparrow f \cap \uparrow e \cap X$ is empty. Therefore, $x \notin cl(\uparrow e \cap X)$ for $e \in K_y$. □

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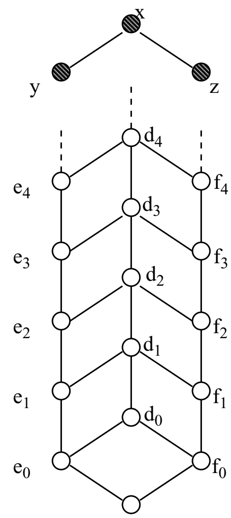

Fig. 4. A counterexample to Proposition 4.6 (1) for the non-dense case.

Example 4.7. A counterexample to Proposition 4.6 (1) for the non-dense case is given in Figure 4. In this example, $M_{L(D)}$ is a Hausdorff subspace of D, but is not a retract of D.

As we have noted, our idea is to consider an infinite increasing sequence in $K(D)$ as giving a code for a point of X. This proposition suggests two interpretations of such sequences when D and X satisfy the conditions of this proposition. One is to consider that $d \in K(D)$ has the information that the point is in $cl(\uparrow d \cap X)$, and consider a strictly increasing sequence $\mathscr{I} = d_0 < d_1 < \ldots$ as specifying the point $\cap_i cl(\uparrow d_i \cap X)$, which is actually the only limit of the filter-base $\mathscr{F}_X(K_y)$, and is equal to the unique cluster point of $\mathscr{F}_X(K_y)$, where K_y is the ideal corresponding to $\mathscr{I}$. In this case, all the infinite increasing sequences have meaning as a unique point of X. However, the representation is not unique in that when $x \in X$, all the ideals K_y with $y \in \uparrow x$ specify the same point x. This kind of interpretation is used in Di Gianantonio (1999) and many other calculi of real numbers. The other interpretation is to consider that $d \in K(D)$ has the information that the point is in $\uparrow d \cap X$, and that $\mathscr{I} = d_0 < d_1 < \ldots$ is specifying the point $\cap_i \uparrow d_i$. In this case, only those infinite increasing sequences with the limits in X have meanings. However, the representation becomes unique in that the ideal representing a point is unique. This kind of interpretation is used in Tsuiki (2002). In this paper, we do not care which interpretation is used, and find, for each Hausdorff space X, a domain D with enough minimal limit elements such that X is homeomorphically and densely embedded in D as the minimal-limit set.

Many of the domains studied in computer science do not have enough minimal limit elements. For instance, $P_\omega = \{u \mid u \subseteq N\}$ and $\Sigma_\perp^\omega$ do not have minimal limit elements. We consider a condition on a domain (Definition 4.11 below) that guarantees the existence of enough minimal limit elements.

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Definition 4.8. When P is a poset, we define the *level* of $d \in P$ as the maximal length of a chain $\bot_P = a_0 < a_1 < \ldots < a_n = d$, when it exists. A poset P is *stratified* if each $e \in P$ has a level. When P is a stratified poset, we write $K_n(P)$ for the set of level-n elements of P. A domain D is *stratified* if $K(D)$ is a stratified poset. We write $K_n(D)$ for the set $K_n(K(D))$ of level-n finite elements of D. We call $K_n(D) \cap K_x$ the set of *level-n approximations* of x.

Thus, when D is a stratified domain, $K(D)$ is stratified as $K(D) = K_0(D) \cup K_1(D) \cup \ldots$ and $K_0(D) = \{\bot_D\}$.

Example 4.9. All the domains P_ω, $\Sigma_\bot^\omega$, Σ^∞, and Figures 1, 3 and 4 are stratified domains, whereas Figure 2 is not.

Lemma 4.10. When D is a stratified domain:

(1) Every subset of $K(D)$ has enough minimal elements.

(2) No finite element is bigger than a limit element. In particular, it has no maximal finite element if $L(D) \neq \varnothing$.

In a poset P, when $d < d'$ and there is no element e such that $d < e < d'$, we say that d' is an *immediate successor* of d and call the pair (d, d') a *successor pair* or an *edge* from d to d'. We write $succ(d)$ for the set of immediate successors of d.

Definition 4.11. A stratified poset P is *finite-branching* if $succ(d) \subseteq K_{n+1}(P)$ and $succ(d)$ is finite for every $d \in K_n(P)$. A *finite-branching domain* (*fb-domain* in short) is a domain D such that $K(D)$ is a finite-branching poset.

Each element of $L(D)$ may have an infinite number of immediate successors for an fb-domain D. An example is the fb-domain RD^∞ in Proposition 7.7 corresponding to the Hilbert cube. When D is an fb-domain, $K_n(D)$ is a finite set for each n.

Proposition 4.12. When D is an fb-domain, $L(D)$ is compact.

Proof. Suppose that $\{\uparrow d \mid d \in S\}$ forms an open cover of $L(D)$ for $S \subseteq K(D)$. The set S has enough minimal elements by Lemma 4.10 (1), and we define $T = M_S$. Then, $\mathscr{T} = \{\uparrow d \cap L(D) \mid d \in T\}$ is an open subcovering of $L(D)$. Suppose that T is an infinite set. Let $J = \{j \in K(D) \mid \uparrow j \cap T \text{ is infinite}\}$. We have $\bot \in J$, and when $j \in J$, at least one member of $succ(j)$ is also in J. Therefore, we have an infinite strictly increasing sequence $\bot = j_0 < j_1 < \ldots$ in J. Let $x \in L(D)$ be the limit of this sequence. Since J is down-closed by definition, we have $K_x \subseteq J$. Since $\mathscr{T}$ is a covering, we have $x > d$ for some $d \in T$. Then $d \in K_x$, and $d \notin J$ because $\uparrow d \cap T = \{d\}$ by the minimality of T, which is a contradiction. □

From the compactness of a space Y, we can show the existence of enough minimal elements of Y with respect to the specialisation order (Neumann-lara and Wilson 1998; Kopperman and Wilson (to appear)). We will show the proof for the case $Y = L(D)$.

Proposition 4.13. When $L(D)$ is compact:

(1) $L(D)$ has enough minimal limit elements.

(2) $M_{L(D)}$ is compact.

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Proof.

(1) Let $y \in L(D)$. By Zorn's lemma, we have a maximal co-directed set $A \subseteq L(D)$ containing y. Then $\{\downarrow a \cap L(D) \mid a \in A\}$ is a family of closed sets in $L(D)$ with the finite intersection property. Since $L(D)$ is compact, this family has non-empty intersection, and we let x be in this intersection. Since A is maximal, $x \in A$. Therefore, x is the least element of A, which is minimal in $L(D)$ because A is maximal. Thus we have a minimal limit element less than or equal to y.

(2) Since $L(D)$ is compact and every open covering of $M_{L(D)}$ also covers $L(D)$, this result is immediate. □

From Propositions 4.12 and 4.13, we have a condition for the existence of enough minimal limit elements.

Theorem 4.14.

(1) An fb-domain D has enough minimal limit elements.

(2) $M_{L(D)}$ is compact.

Thus finite-branchingness is a sufficient condition for the existence of enough minimal limit elements. In addition, in this case, the set $M_{L(D)}$ is a compact set. Therefore, in the rest of this paper we restrict our attention to the case X is compact, and we find, for each compact metric space X, a finite-branching domain D such that $X = M_{L(D)}$ and $M_{L(D)}$ is dense in D.

Note that $M_{L(D)}$ may not be dense in D even when D is finite-branching. For example, the fb-domain in Example 4.3 has $\{x\}$ as the minimal-limit set, which is not dense in D. However, we can have a subdomain that contains $M_{L(D)}$ as a dense subset by simply taking the closure of $M_{L(D)}$. Therefore, we will consider the construction of an fb-domain D such that $M_{L(D)}$ contains X in the following sections, and then obtain the desired fb-domain by taking the closure of X (Theorems 8.5 and 8.8).

5. fb-domains composed of bottomed sequences

In this section we give some examples of fb-domains composed of bottomed sequences.

Definition 5.1. A domain D is a *domain of bottomed sequences* if it is a subdomain of $\Sigma_\perp^\omega$ and the embedding of D in $\Sigma_\perp^\omega$ preserves the level.

In this case, each element of $K_n(D)$ has n filled cells and an edge corresponds to filling one unfilled cell with a character in Σ.

When D is a domain of bottomed sequences, we introduce a labelling of edges of D by the character set $\Gamma = \{a^{(i)} \mid a \in \{0,1\}, i \in \{0,1,\ldots\}\}$ so that the label $a^{(i)}$ is assigned to an edge filling the i-th (counting from 0) unfilled cell with a. For example, the edge from $\perp^\omega$ to $\perp 1 \perp^\omega$ is labelled with $1^{(1)}$, and the edges from $\perp 1 \perp^\omega$ to $\perp 10 \perp^\omega$ and from $\perp 10 \perp^\omega$ to $010 \perp^\omega$ are labelled with $0^{(1)}$ and $0^{(0)}$, respectively. Let $\Gamma^{(n)}$ be the finite set $\{a^{(i)} \mid a \in \{0,1\}, i \in \{0,1,\ldots,n\}\}$. When D is an fb-domain of bottomed sequences, $K_n(D)$ is a finite set for all $n = 0,1,2,\ldots,$ Therefore, for each n, there is a number l such that all the edges from level-n finite elements are labelled with $\Gamma^{(l)}$ $(n = 0,1,\ldots)$.

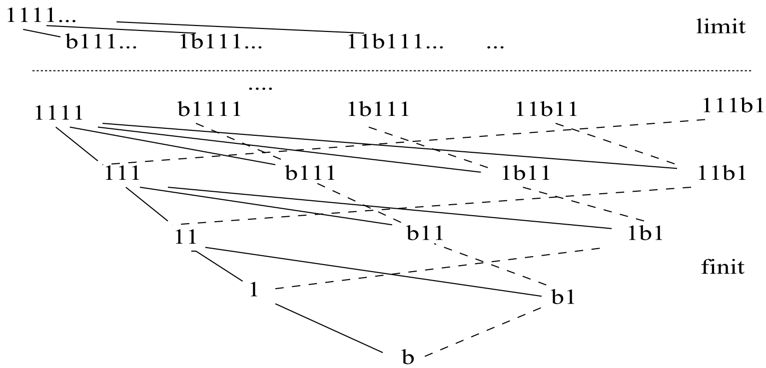

Fig. 5. The structure of BD_1 for the case $\Sigma = \{1\}$ ($\perp$ is shown as 'b').

We write $\Sigma^{\omega}_{\perp,n}$ for the set of infinite bottomed sequences in which at most n undefined cells are allowed to exist. Therefore, for example, $\Sigma^{\omega}_{\perp,0} = \Sigma^{\omega}$ and $\Sigma^{\omega}_{\perp,1} = \Sigma^{\omega} \cup \Sigma^{*}\perp\Sigma^{\omega}$. We write $\Sigma^{*}_{\perp,n}$ for the sets of finite bottomed sequences in which at most n undefined cells are allowed to exist. More precisely, $\Sigma^{*}_{\perp,n}$ is a subset of $\Sigma^{\omega}_{\perp}$ such that all the cells are $\perp$ after the $(n+1)$-th $\perp$ cell.

Definition 5.2.

(1) Let P be a poset and $d \in P$. The *co-level* of d is the maximal length n of a chain (that is, strictly increasing sequence) $d = a_0 < a_1 < \ldots < a_n$ in P. If there is an arbitrary long chain starting with d, we define the co-level of d to be ∞.

(2) The *upper-n* subset of P is the set of elements whose co-level is not greater than n.

The upper-n subset of $\Sigma^{\omega}_{\perp}$ is $\Sigma^{\omega}_{\perp,n}$. Now we define $BD_n = \Sigma^{*}_{\perp,n} \cup \Sigma^{\omega}_{\perp,n}$ $(n = 0, 1, \ldots)$. This is obviously a subdomain of $\Sigma^{\omega}_{\perp}$ with $K(BD_n) = \Sigma^{*}_{\perp,n}$ and $L(BD_n) = \Sigma^{\omega}_{\perp,n}$. BD_n are obviously bounded-complete fb-domains of bottomed sequences. As a special case, BD_0 is the domain Σ^{∞}.

We will now study the structures of BD_n more carefully. In BD_1, the least element of $\Sigma^{*}_{\perp,1}$, which is the empty string, has 4 successors:'0', '1', '$\perp$0' and '$\perp$1'. This is also the case for other elements; every finite element has 4 outgoing edges labelled with $0^{(0)}$, $1^{(0)}$, $0^{(1)}$, and $1^{(1)}$. Therefore, BD_1 is the subdomain of $\Sigma^{\omega}_{\perp}$ in which the edges are restricted to $\Gamma^{(1)}$. In the same way, each finite element of BD_n has $2n$ successors. Figure 5 shows the order structure of BD_1 for the case $\Sigma = \{1\}$. Note that the open sets $\uparrow d$ $(d \in K(D))$ are all isomorphic to each other.

Definition 5.3. An fb-domain D is *homogeneous* if $\uparrow d$ is isomorphic to D for each $d \in K(D)$.

Proposition 5.4. BD_n is homogeneous.

Proof. Let $d \in K(D)$, $K = \{k \mid d[k] \in \Sigma\}$ and $e \in \uparrow d$. Since all the bottom cells of e have an index not in K, the number of bottoms in e does not change if we omit the cells with index in K. Therefore, by deleting K from the index set ω and re-indexing, we can make an isomorphism between $\uparrow d$ and BD_n. □

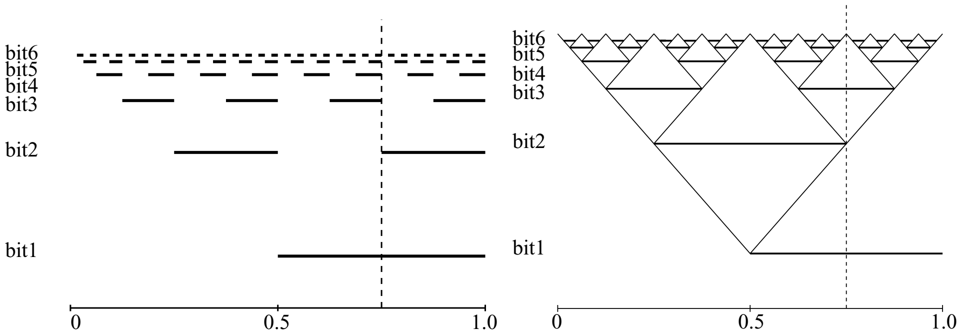

Fig. 6. Binary Expansion and Gray code Expansion of real numbers.

As for the limit elements, $\Sigma^{\omega}_{\perp,1}$ has 2 levels of elements. The upper level is isomorphic to Σ^{ω}, and the lower level, which is the minimal-limit set of $\Sigma^{\omega}_{\perp,1}$, consists of infinite sequences with one bottom. Each lower level element is smaller than two upper level elements obtained by specifying the value of the bottom cell as 0 or 1, and each upper level element is bigger than countably many lower level elements obtained by substituting the value of each cell with $\perp$. Similarly, $\Sigma^{\omega}_{\perp,n}$ has (n+1)-level structures ($n = 0, 1, \ldots$).

Proposition 5.5. $M_{L(BD_n)}$ is not Hausdorff when $n \geqslant 1$.

Proof. If it is Hausdorff, then it is a retract of $L(BD_n)$ by Proposition 4.6 (1). This means that for each maximal element $x \in L(BD_n)$, there is only one $y \in M_{L(BD_n)}$ such that $y \leqslant x$. This contradicts the structure of $\Sigma^{\omega}_{\perp,n}$ mentioned above. □

Next we consider a more important example of a domain of bottomed sequences whose minimal-limit set is Hausdorff and homeomorphic to $\mathbb{I} = [0, 1]$. The Gray code embedding G (see Tsuiki (2002) and Definition 5.6 below) is an embedding of $\mathbb{I} = [0, 1]$ in the set $\Sigma^{\omega}_{\perp,1}$. It is based on the Gray code expansion, which is another expansion of real numbers. Figure 6 shows the usual binary expansion and the Gray-code expansion of $\mathbb{I}$. Here, a horizontal line means that the corresponding bit has value 1 on the line and value 0 otherwise. In the usual binary expansion of x, the head h of the expansion indicates whether x is in [0, 1/2] or [1/2, 1], and the tail is the expansion of $f(x, h)$ for f the following function:

$$f(x, h) = \begin{cases} 2x & (h = 0) \\ 2x - 1 & (h = 1). \end{cases}$$

Note that the value of f depends not only on x but also on the choice of h when $x = 1/2$.

On the other hand, the head of the Gray-code expansion is the same as that of the binary expansion, whereas the tail is the expansion of $t(x)$ for t the so-called tent function. Note that t is continuous at 1/2.

$$t(x) = \begin{cases} 2x & (0 \leqslant x \leqslant 1/2) \\ 2(1 - x) & (1/2 < x \leqslant 1). \end{cases}$$

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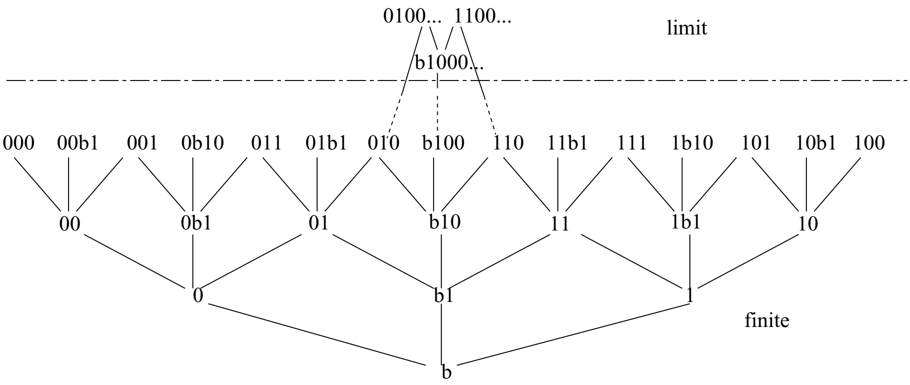

Fig. 7. The structure of RD. Here, b means $\perp$.

As is the case for the usual binary expansion, we have two expansions for dyadic numbers. For example, we have two Gray code expansions $111000\ldots$ and $101000\ldots$ for $3/4$, corresponding to the two binary expansions $11000\ldots$ and $10111\ldots$. However, the two expansions differ only at one digit (in this case the second digit). This means that the second digit does not contribute to the fact that this number is $3/4$. Therefore, it is natural not to select a $\{0,1\}$ value for such a digit, but instead to leave it unspecified as $\perp$. In this way, we define the Gray code embedding G as follows.

Definition 5.6. Let $P : \mathbb{I} \to \Sigma_\perp$ be the map

$$P(x) = \begin{cases} 0 & (x < 1/2) \\ \perp & (x = 1/2) \\ 1 & (x > 1/2) \end{cases}$$

and $t : \mathbb{I} \to \mathbb{I}$ be the tent function defined above. The Gray code embedding G is a function from $\mathbb{I} = [0,1]$ to $\Sigma^\omega_{\perp,1}$ defined as $G(x)[n] = P(t^n(x))$.

Note that G is an injective function from $\mathbb{I}$ to $\Sigma^\omega_{\perp,1}$ with the image $im(G) = \Sigma^\omega \cup \Sigma^*\perp 10^\omega - \Sigma^*\Sigma 10^\omega$. Next we consider an fb-domain RD of bottomed sequences that corresponds to $im(G)$. Let $L(RD)$ be the set $\Sigma^\omega \cup \Sigma^*\perp 10^\omega$ and $K(RD)$ be the set $\Sigma^* \cup \Sigma^*\perp 10^*$. Then $RD = L(RD) \cup K(RD)$ is a bounded complete fb-domain with $K(RD)$ and $L(RD)$ the sets of finite and limit elements, respectively. The structure of RD is represented in Figure 7. RD was introduced in Di Gianantonio (1999) as a domain corresponding to the signed digit representation of $\mathbb{I}$, and as an fb-domain of bottomed sequences in Tsuiki (2002). This domain corresponds to the way an IM2-machine manipulates Gray-code; $K(RD)$ represents finite-time states of the input/output tapes of an IM2-machine (Tsuiki 2002). Comparing $im(G)$ with $L(RD)$, one can see that G is a bijection to $M_{L(RD)}$.

We consider how the function G interacts with the topological structure. For each finite element d of RD, $G^{-1}(\uparrow d)$ has the form $(m/2^k, (m+1)/2^k)$ or $((2m-1)/2^k, (2m+1)/2^k)$, depending on whether $d \in \Sigma^*$ or $d \in \Sigma^*\perp 10^*$, with the exception that $G^{-1}(\uparrow\epsilon) = [0,1]$, $G^{-1}(\uparrow 0^k) = [0, 1/2^k)$, and $G^{-1}(\uparrow 10^{k-1}) = ((2^k-1)/2^k, 1]$ (k $= 1, 2, \ldots$). Since these intervals form a base of $\mathbb{I}$, G gives a correspondence between bases of $\mathbb{I}$ and RD. Therefore, G is a

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topological embedding of $\mathbb{I}$ in RD. Since the set of real numbers $\mathbb{R}$ is homeomorphic to $(0, 1)$, $\mathbb{R}$ can also be embedded in $M_{L(RD)}$.

Proposition 5.7.

(1) $\mathbb{I}$ is homeomorphic to $M_{L(RD)}$, and $M_{L(RD)}$ is dense in $L(RD)$.
(2) $\mathbb{I}$ is a retract of $L(RD)$.
(3) $\mathbb{R}$ can be embedded in $M_{L(RD)}$.

As we said in Section 4, we have two interpretations of finite elements of RD as information about a point in $\mathbb{I}$. One is to interpret $d \in K(RD)$ as $G^{-1}(\uparrow d)$ and $x \in M_{L(RD)}$ as expressing $G^{-1}(x)$. Thus, for example, $0, 1, \perp 1$ express the open intervals $[0, 1/2)$, $(1/2, 1]$, and $(1/4, 3/4)$, and only $\perp 1000\ldots$ represents $1/2$. The alternative is to interpret d as $cl(G^{-1}(\uparrow d))$, and $x \in L(RD)$ as expressing $G^{-1}(r(x))$. Thus, $0, 1, \perp 1$ express the closed intervals $[0, 1/2]$, $[1/2, 1]$, and $[1/4, 3/4]$, and the three sequences $\perp 1000\ldots$, $11000\ldots$, $01000\ldots$ represent $1/2$.

Since the embeddings of RD in BD_1 and BD_1 in $\Sigma_\perp^\omega$ are topological ones, G can be considered as topological embeddings also in BD_1 and in $\Sigma_\perp^\omega$.

6. The dimension of $L(D)$

Dimension is one of the most important invariants of topological spaces, which is useful in proving, for instance, the non-existence of an embedding of a space into another space. In this section, we calculate the dimension of $L(D)$ for the case in which D has property M, and induce a requirement for the existence of an embedding of X in $L(D)$.

There are three major definitions of the dimension of a topological space X, the small (or weak) inductive dimension ind X, the large (or strong) inductive dimension Ind X, and the covering dimension dim X. The three dimension functions coincide and have good properties for the class of separable metric spaces. However, they diverge in T_0 spaces in general. Actually, ind $\Sigma_{\perp,1}^\omega = 1$, as we will show, whereas one can calculate dim $\Sigma_{\perp,1}^\omega = \infty$. In this paper, we will consider the small inductive dimension, since it has good properties even for such a general class of spaces.

Definition 6.1. The *small inductive dimension* ind X of a topological space X is defined to be:

(i) ind $X = -1$ if $X = \varnothing$.
(ii) ind $X \leqslant n$ if for every neighbourhood U of a point $p \in X$ there exists an open set V such that $x \in V \subseteq U$ and ind $B(V) \leqslant n - 1$, where $B(V)$ is the boundary of V, see Section 2.

If ind $X \leqslant n$ and ind $X \nleqslant n - 1$, we define ind $X = n$. If ind $X \nleqslant n$ for every n, then ind $X = \infty$.

The following proposition is straightforward, and we will use it in calculating the dimension.

Proposition 6.2. If X has a base $\mathcal{O}$ such that every element $U \in \mathcal{O}$ satisfies ind $B(U) \leqslant n-1$, then ind $X \leqslant n$.

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Proposition 6.3 (heredity property of ind**).**

(1) If ind $X \leqslant n$ and Y is a subspace of X, then ind $Y \leqslant n$.
(2) When ind $X <$ ind Y, Y has no topological embedding in X.

Proof.

(1) The proof is by induction on n. The case $n = -1$ is trivial. Assume the result for $n-1$. Since ind $X \leqslant n$, for all $x \in Y$ and $O \ni x$, there exists $x \in O' \subseteq O$ such that ind $B(O') \leqslant n-1$. Since $B_Y(O' \cap Y) \subseteq B(O')$, by the induction hypothesis, we have ind $B_Y(O' \cap Y) \leqslant n-1$.
(2) This is immediate from (1). □

This heredity property does not hold for T_0 spaces in general when we consider the covering dimension or the large inductive dimension. See the Appendix of Hurewicz and Wallman (1948) for details. Below, when we use the word *dimension* we will mean small inductive dimension.

Definition 6.4. The *height* of a poset P (denoted by height P) is the maximal length of a chain in P. If P is empty, we define height $P = -1$.

Proposition 6.5.

(1) height $\{a_0 < a_1 < \ldots < a_n\} = n$.
(2) height $\Sigma^\omega = 0$.
(3) height $\Sigma^\omega_{\perp,n} = n$.
(4) height $\Sigma^\omega_\perp = \infty$.

Proposition 6.6. For a poset P, the height of P and the dimension of P with the Alexandroff topology coincide. Here, the Alexandroff topology of P has as open sets the upper-closed subsets of P.

However, when P is a subspace of a domain, with the subspace topology of the Scott topology, the height of P and the dimension of P do not coincide. For example, the image of the Gray code embedding $im(G) \subseteq \Sigma^\omega_{\perp,1}$ has height 0 because there is no order relation among elements of $im(G)$, whereas it has dimension 1 because it is homeomorphic to $\mathbb{I}$.

Proposition 6.7. When P is a subspace of a domain D with the subspace topology of the Scott topology of D, ind $P \geqslant$ height P.

Proof. Let $n =$ height P. A chain of length n has dimension n by Proposition 6.5 (1), and is embedded in P as a subspace. The result then follows by heredity (Proposition 6.3). □

Lemma 6.8.

(1) If D is a domain and A is a closed subset of D, then A is also a domain such that $K(A) = A \cap K(D)$.
(2) In addition, when D has property M, A also has property M.

Proposition 6.9. Let D be a domain with property M and with no maximal finite element. Let $d \in K(D)$.

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(1) $cl_{L(D)}(\uparrow d \cap L(D)) = cl_D(\uparrow d) \cap L(D)$.
(2) $B_D(\uparrow d) = \{\alpha \in D \mid d \uparrow \alpha \text{ and } d \not\leqslant \alpha\}$, and $B_D(\uparrow d)$ is a domain with property M such that $L(B_D(\uparrow d)) = B_{L(D)}(\uparrow d \cap L(D))$.
(3) When $L(D)$ is not empty, height $L(B_D(\uparrow d)) \leqslant$ height $L(D) - 1$.

Proof.

(1) Since the right-hand side is a closed subset of $L(D)$ including $\uparrow d \cap L(D)$, we have the $\subseteq$ direction. Let $x \in cl_D(\uparrow d) \cap L(D)$. Since D has no maximal finite element, d and x have an upper bound $z \in L(D)$ by Propositions 3.6 and 3.9. Therefore, $z \in cl_{L(D)}(\uparrow d \cap L(D))$ and, since this set is down-closed, x must also belong to this set.
(2) By Lemma 6.8, $B_D(\uparrow d)$ is a domain with property M such that $L(B_D(\uparrow d)) = B_D(\uparrow d) \cap L(D)$. On the other hand, from (1)

$$\begin{aligned} B_{L(D)}(\uparrow d \cap L(D)) &= cl_{L(D)}(\uparrow d \cap L(D)) - \uparrow d \cap L(D) \\ &= cl_D(\uparrow d) \cap L(D) - \uparrow d \cap L(D) \\ &= (cl_D(\uparrow d) - \uparrow d) \cap L(D) \\ &= B_D(\uparrow d) \cap L(D). \end{aligned}$$

(3) Let $a_0 < a_1 < \ldots < a_n$ be a chain in $L(B_D(\uparrow d))$. Then, by (2), $a_n \uparrow d$. So there exists z with $a_n \leqslant z \geqslant d$. As D has no maximal finite element, we can assume $z \in L(D)$. Also, by (2), $z \notin B_D(\uparrow d)$. So $a_n < z$. Therefore, $a_0 < a_1 < \ldots < a_n < z$ is a chain in $L(D)$. □

Proposition 6.10. When D is a domain with property M, ind $L(D) \leqslant$ height $L(D)$.

Proof. The proof is by induction on height $L(D)$. The proposition is obvious when $L(D)$ is empty. Suppose that height $L(D) = n \geqslant 0$. Consider the domain $\hat{D}$. Since $L(\hat{D}) = L(D)$, $L(\hat{D})$ also has height n. From Proposition 6.9 (3), we have height $L(B_{\hat{D}}(\uparrow d)) \leqslant n-1$ for any finite element d of $\hat{D}$. We apply the induction hypothesis to the domain $B_{\hat{D}}(\uparrow d)$ to have ind $L(B_{\hat{D}}(\uparrow d)) \leqslant n-1$. Therefore, by Proposition 6.2, we have ind $L(\hat{D}) \leqslant n$. Therefore, ind $L(D) \leqslant n$. □

Propositions 6.7 and 6.10 now give us our result.

Theorem 6.11. When D is a domain with property M, ind $L(D) =$ height $L(D)$.

Since we view a domain D as the space $L(D)$ with the approximation structure given by $K(D)$, we also refer to the dimension of $L(D)$ as the *dimension* of the domain D, and write ind D for it.

This theorem, with the heredity property, derives the main result of Tsuiki (2000) as follows.

Corollary 6.12.

(1) ind $\Sigma^{\omega}_{\perp,n} = n$.
(2) There are no embeddings of n-dimensional topological spaces in $\Sigma^{\omega}_{\perp,m}$ when $n > m$. In particular, there are no embeddings of $\mathbb{I}^n$ in $\Sigma^{\omega}_{\perp,n-1}$ for any character set Σ of countable cardinality.
(3) There are no embeddings of infinite-dimensional topological spaces in $\Sigma^{\omega}_{\perp,n}$ for any n.

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Proof.

(1) This follows from Proposition 6.5 (3).

(2) ind $\mathbb{I}^n = n$. See Engelking (1978). □

The domain in Example 3.8, which does not have property M, satisfies ind $L(D) = 1$ whereas height $L(D) = 0$, so it gives a counterexample to Theorem 6.11 when D does not have property M.

7. The synchronous product of stratified domains

We have shown that $\mathbb{I}$ is homeomorphic to $M_{L(RD)}$ and the real line $\mathbb{R}$ can be embedded in $M_{L(RD)}$. To consider corresponding results for higher dimensional spaces such as the n-dimensional Euclidean cube $\mathbb{I}^n$ $(n = 0, 1, 2, \ldots)$ and the Hilbert cube $\mathbb{I}^\omega$, we define a new product of stratified domains. When we use the usual product, we have ind $D_1 \times D_2 = \infty$ if ind $D_1 \geqslant 0$ and ind $D_2 \geqslant 0$, because any pair of a finite element and an infinite element is an infinite element of $D_1 \times D_2$. Therefore, we use the following definition.

Definition 7.1. Let D_1 and D_2 be stratified domains. The *synchronous product* $D_1 \times^s D_2$ of D_1 and D_2 is the stratified domain defined by the following set of finite elements

$$K_n(D_1 \times^s D_2) = \{\langle a, b\rangle \mid a \in K_n(D_1), b \in K_n(D_2)\} \ (n = 0, 1, \ldots),$$

with the pointwise order.

Proposition 7.2. Let D_1 and D_2 be stratified domains.

(1) $D_1 \times^s D_2$ is a subdomain of $D_1 \times D_2$ such that $L(D_1 \times^s D_2)$ is homeomorphic to $L(D_1) \times L(D_2)$.

(2) When D_1 and D_2 are finite-branching, $D_1 \times^s D_2$ is finite-branching also.

(3) $M_{L(D_1 \times^s D_2)}$ is homeomorphic to $M_{L(D_1)} \times M_{L(D_2)}$.

(4) When D_1 and D_2 have property M, $D_1 \times^s D_2$ also has property M and ind $D_1 \times^s D_2 =$ ind $D_1 +$ ind D_2.

Proof.

(1) $D_1 \times^s D_2$ is obviously a subdomain of $D_1 \times D_2$; the embedding maps a finite element of $D_1 \times^s D_2$ to a finite element of $D_1 \times D_2$.
Let p_1 and p_2 be projection functions from $D_1 \times^s D_2$ to D_1 and D_2, respectively, and defined in the obvious way for finite elements and continuously extended to limit elements. Let I be an ideal of $D_1 \times^s D_2$. Then $p_1 I$ and $p_2 I$ are obviously ideals of D_1 and D_2. Let x_1 and x_2 be the limits of $p_1 I$ and $p_2 I$, respectively. Let $a_1 \in K_{x_1}$ and $a_2 \in K_{x_2}$ such that a_1 and a_2 have the same level. Then, for some b_1 and b_2, we have $\langle a_1, b_2\rangle \in I$ and $\langle b_1, a_2\rangle \in I$. Since I is directed, we have $\langle c_1, c_2\rangle$ such that $c_i \geqslant a_i$ and $c_i \geqslant b_i$ for $i = 1, 2$. Since I is lower closed, we have $\langle a_1, a_2\rangle \in I$. Therefore, each non-principal ideal I has the following form for some $x \in L(D_1)$ and $y \in L(D_2)$:

$$I = \{\langle a, b\rangle \mid a \in K_x, b \in K_y, \ a \text{ and } b \text{ have the same level}\}.$$

Thus, there is a one-to-one correspondence between $L(D_1) \times L(D_2)$ and the set of non-principal ideals of $D_1 \times^s D_2$. It is obviously a homeomorphism.

(2) We have $succ(\langle a,b\rangle) = succ(a) \times succ(b)$.
(3) This is immediate from (1), because $M_{L(D_1)} \times M_{L(D_2)}$ is the set of minimal elements of $L(D_1) \times L(D_2)$.
(4) Let $N = \{(a_1,b_1),\ldots,(a_l,b_l)\}$ be a finite subset of $K(D_1 \times^s D_2)$ and S_1 and S_2 be the sets of minimal upper bounds of $\{a_1,\ldots,a_l\}$ and $\{b_1,\ldots,b_l\}$, respectively. Let n be the maximal level of the elements of $S_1 \cup S_2$. Define $T_i = \uparrow S_i \cap K_n(D_i)$ for $i = 1,2$. $T_1 \times T_2$ is a finite subset of $K(D_1 \times^s D_2)$, which is the set of minimal elements of $\{d \in K(D_1 \times^s D_2) \mid level(d) \geqslant n, d \text{ is an upper bound of } N\}$. Therefore, take $T = T_1 \times T_2 \cup \{d \in K(D_1 \times^s D_2) \mid level(d) < n, d \text{ is an upper bound of } N\}$. Since T is a finite set, the set of minimal elements of T is also finite and is the set of minimal upper bounds of N. Thus, $D_1 \times^s D_2$ has property M.
The height of $L(D_1 \times^s D_2)$ is equal to the height of $L(D_1) \times L(D_2)$ by (1), and is equal to the sum of the heights of $L(D_1)$ and $L(D_2)$. □

The domain $D_1 \times^s D_2$ can be extended to a domain of bottomed sequences when D_1 and D_2 are themselves domains of bottomed sequences. Let $in : \Sigma_\perp^\omega \times \Sigma_\perp^\omega \to \Sigma_\perp^\omega$ be the interleaving function defined as

$$\begin{aligned} in(a,b)[2n] &= a[n], \\ in(a,b)[2n+1] &= b[n]. \end{aligned}$$

Through in, $\Sigma_\perp^\omega \times \Sigma_\perp^\omega$ and $\Sigma_\perp^\omega$ become order-isomorphic. Thus, $D_1 \times^s D_2$ becomes a subdomain of $\Sigma_\perp^\omega$ by Proposition 7.2(1). Since this embedding is not level-preserving, we add to the set of finite elements of $D_1 \times^s D_2$ the sets

$$K_n'(D_1 \times^s D_2) = \{\langle a,b\rangle \mid a \in K_{n+1}(D_1), b \in K_n(D_2)\} \ (n = 0,1,\ldots)$$

so that $K_n(D_1 \times^s D_2)$ and $K_n'(D_1 \times^s D_2)$ become the set of $2n$-level and $(2n+1)$-level finite elements, respectively. We write $D_1 \times_\perp^s D_2$ for the domain thus constructed embedded in $\Sigma_\perp^\omega$ by in. This insertion of intermediate finite elements does not change the structure of the space of limit elements.

Proposition 7.3. When D_1 and D_2 are domains of bottomed sequences, $D_1 \times_\perp^s D_2$ is a domain of bottomed sequences such that $L(D_1 \times_\perp^s D_2)$ is homeomorphic to $L(D_1 \times^s D_2)$.

Proof. When I is an ideal of $K(D_1 \times_\perp^s D_2)$, we have that $I \cap K(D_1 \times^s D_2)$ is also a directed set of $K(D_1 \times_\perp^s D_2)$ with the same limit. To see this, it is enough to show that when $e < f$ in $K(D_1 \times_\perp^s D_2)$, there is $g \in K(D_1 \times^s D_2)$ such that $e \leqslant g \leqslant f$. □

We can also define the synchronous product $\times_\perp^s$ of arity n by adding $n-1$ levels of intermediate finite elements between K_n and K_{n+1}, and using the interleaving function of arity n. We write D^n for the n-arity synchronous product $D \times_\perp^s D \times_\perp^s \ldots \times_\perp^s D$ of n copies of D.

Corollary 7.4.

(1) RD^n is an n-dimensional finite-branching domain of bottomed sequences with property M.
(2) $L(RD^n)$ is an upper-closed subset of $\Sigma_{\perp,n}^\omega$.

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(3) $\mathbb{I}^n$ is homeomorphic to $M_{L(RD^n)}$.
(4) $\mathbb{I}^n$ is a retract of $L(RD^n)$.
(5) $\mathbb{R}^n$ can be embedded in $M_{L(RD^n)}$.

We write G^n for the homeomorphism from $\mathbb{I}^n$ to $M_{L(RD^n)}$.

Next, we consider infinite products.

Definition 7.5. Let D_i $(i = 1, 2, \ldots)$ be stratified domains. We can define a stratified domain $\prod_{i=1}^{\infty}{}^{s} D_i$ as the ideal completion of the following stratified poset:

$$K_n\left(\prod_{i=1}^{\infty}{}^{s} D_i\right) = \{\langle a_1, a_2, \ldots, a_n\rangle \mid a_k \in K_{n-k+1}(D_k) \ \ (k = 1, \ldots, n)\},$$

with the order $\langle a_1, a_2, \ldots, a_n\rangle \leqslant \langle b_1, b_2, \ldots, b_m\rangle$ if $n \leqslant m$ and $a_k \leqslant b_k$ in $K(D_k)$ $(k = 1, 2, \ldots, n)$.

Proposition 7.6. Let D_i $(i = 1, 2, \ldots)$ be stratified domains.

(1) $\prod_{i=1}^{\infty}{}^{s} D_i$ is a subdomain of $\prod_{i=1}^{\infty} D_i$ such that $L(\prod_{i=1}^{\infty}{}^{s} D_i)$ is homeomorphic to $\prod_{i=1}^{\infty} L(D_i)$.
(2) When D_i $(i = 1, 2, \ldots)$ are finite-branching, $\prod_{i=1}^{\infty}{}^{s} D_i$ is finite-branching also.
(3) $M_{L(\prod_{i=1}^{\infty}{}^{s} D_i)}$ is homeomorphic to $\prod_{i=1}^{\infty} M_{L(D_i)}$.
(4) When D_i $(i = 1, 2, \ldots)$ have property M, $\prod_{i=1}^{\infty}{}^{s} D_i$ also has property M and

$$\text{ind}\left(\prod_{i=1}^{\infty}{}^{s} D_i\right) = \sum_{i=1}^{\infty} \text{ind}\,(D_i).$$

Proof. The proof is similar to the proof of Proposition 7.2. □

$\prod_{i=1}^{\infty}{}^{s} D_i$ can also be extended to a domain of bottomed sequences when D_i $(i = 1, 2, \ldots)$ are. Let $in^{\infty} : \prod_{i=1}^{\infty} \Sigma_{\perp}^{\omega} \to \Sigma_{\perp}^{\omega}$ be the isomorphism defined as

$$in^{\infty}(\langle a_1, a_2, \ldots\rangle)[\langle n, k\rangle] = a_k[n] \tag{1}$$

for $\langle n, k\rangle = (n + k - 1)(n + k)/2 + k - 1$ with $n = 0, 1, \ldots$ and $k = 1, 2, \ldots$. Through in^{∞}, we have $\prod_{i=1}^{\infty}{}^{s} D_i$ becomes a subdomain of $\Sigma_{\perp}^{\omega}$ by Proposition 7.6(1). Since this embedding is not level-preserving, we add n levels of finite elements between K_n and K_{n+1}:

$$K_n^t\left(\prod_{i=1}^{\infty}{}^{s} D_i\right) = \left\{\langle a_1, a_2, \ldots, a_n\rangle \ \middle|\ \begin{array}{l} a_k \in K_{n-k+2}(D_k)(1 \leqslant k \leqslant t) \\ a_k \in K_{n-k+1}(D_k)(t < k \leqslant n) \end{array}\right\} \quad (t = 1, 2, \ldots, n).$$

We define the domain $\prod_{i=1}^{\infty}{}^{s}_{\perp} D_i$ of bottomed sequences as the ideal completion of this domain embedded in $\Sigma_{\perp}^{\omega}$ by in^{∞}, and we use D^{∞} to denote the synchronous product $\prod_{i=1}^{\infty}{}^{s}_{\perp} D$.

Corollary 7.7.

(1) RD^{∞} is an ∞-dimensional finite-branching domain of bottomed sequences with property M.
(2) $L(RD^{\infty})$ is an upper-closed subset of $\Sigma_{\perp}^{\omega}$.
(3) The Hilbert cube $\mathbb{I}^{\omega}$ is homeomorphic to $M_{L(RD^{\infty})}$.

(4) $\mathbb{I}^{\omega}$ is a retract of $L(RD^{\infty})$.
(5) $\mathbb{R}^{\infty}$ can be embedded in $M_{L(RD^{\infty})}$.

8. Embeddings of compact metric spaces

Now we consider embeddings of separable metric spaces. For finite-dimensional cases, our construction is based on the universality of Nöbeling's universal n-dimensional space.

Definition 8.1. We define a subspace N_k^n of I^n as follows:

$$N_k^n = \{(x_1, \ldots, x_n) \in I^n \mid \text{at most } k \text{ of } x_1, \ldots, x_n \text{ are dyadic}\}.$$

It is known that N_k^n has dimension k (Engelking 1978). The space N_n^{2n+1} is essentially the same as Nöbeling's universal n-dimensional space, and it has the following universality.

Proposition 8.2. For any n-dimensional separable metric space X, there is a topological embedding of X in N_n^{2n+1}.

Proof. See Engelking (1978), for example. □

Consider the embedding G^m of $\mathbb{I}^m$ in $M_{L(RD^m)} \subseteq \Sigma_{\perp}^{\omega}$. Since it is an interleaving of the Gray code, the number of $\perp$ that appear in $G^m(x)$ is equal to the number of dyadic coordinates that $x \in \mathbb{I}^m$ has. Therefore,

$$G^m\left(N_n^m\right) \subseteq \Sigma_{\perp,n}^{\omega} \cap M_{L(RD^m)}.$$

Theorem 8.3. Let n be a finite number. When X is an n-dimensional separable metric space, X has an embedding in $M_{L(RD^{2n+1})}$. The image is in the upper-n subspace of RD^m.

Proof. The proof follows from Proposition 8.2. □

Next, we consider the case when X is compact.

Lemma 8.4. When D is an fb-domain with property M and Y is a closed subset of $M_{L(D)}$, $cl_D(Y)$ is an fb-domain with property M such that $M_{L(cl_D(Y))} = Y$.

Proof. Being a closed subset, $Y = E \cap M_{L(D)}$ for some closed subset E of D. Since $cl_D(Y) \subseteq E$, we have $Y = cl_D(Y) \cap M_{L(D)}$. From Lemma 6.8, $cl_D(Y)$ is a domain with property M, which is also finite-branching because $cl_D(Y)$ is down-closed. □

Theorem 8.5. Let n be a finite number. For each n-dimensional compact metric space X, there is an n-dimensional fb-domain D of bottomed sequences with property M such that:

(1) $M_{L(D)}$ is homeomorphic to X and $M_{L(D)}$ is dense in D.
(2) X is a retract of $L(D)$.
(3) D is a subdomain of BD_n.

Proof.

(1) It is known that a compact metric space is separable. Therefore, X has an embedding in N_n^{2n+1}, and thus in $M_{L(RD^{2n+1})}$ by Theorem 8.3. Let Y be the image of the embedding. $Y \subseteq \Sigma_{\perp,n}^{\omega} \cap M_{L(RD^{2n+1})}$. Since $M_{L(RD^{2n+1})}$ is Hausdorff, Y is a closed subset of $M_{L(RD^{2n+1})}$.

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Therefore, by Lemma 8.4, $cl_{RD^{2n+1}}(Y)$ is an fb-domain with property M, which we denote by D. Since $M_{L(D)} = Y$ and $Y \subseteq \Sigma^{\omega}_{\perp,n}$, we have $L(D) \subseteq \Sigma^{\omega}_{\perp,n}$, and thus $L(D)$ is n-dimensional. Since D is the closure of $M_{L(D)}$, we have $M_{L(D)}$ is dense in D.

(2) This part follows from (1) and Proposition 4.6.

(3) This part is obvious from the construction. □

For the infinite-dimensional case, we can use the universality of the Hilbert cube.

Proposition 8.6. Every separable metric space X can be embedded in the Hilbert cube $\mathbb{I}^{\omega}$.

Proof. See Engelking (1978), for example. □

Theorem 8.7. Every separable metric space X can be embedded in $M_{L(RD^{\infty})}$.

Proof. The proof follows from Proposition 8.6 and Corollary 7.7. □

Also, from Lemma 8.4 and Theorem 8.7, we have the following theorem.

Theorem 8.8. Theorem 8.5 ((1) and (2)) holds also for the case of $n = \infty$.

As a corollary to Theorems 6.11, 8.5 and 8.8, we have the following theorem.

Theorem 8.9. The dimension of a compact metric space X is equal to the minimal height of $L(D)$ such that D is a domain with property M and X is homeomorphic to $M_{L(D)}$.

9. Concluding remarks

In Theorem 6.11, we have shown that the dimension of $L(D)$ is equal to the height of $L(D)$ when D is an ω-algebraic domain with property M. It is not hard to show that this theorem is also true for Lawson-compact continuous domains in general. Proposition 3.6(2) can be proved for Lawson-compact continuous domains, and from this the algebraic-domain case of Theorem 6.11 is derived. Since the height of $L(D)$ is always ∞ for non-algebraic continuous domains, this theorem holds trivially for the non-algebraic case.

We have shown that every n-dimensional compact metric space can be realised as the minimal-limit set of an n-dimensional fb-domain of bottomed sequences. This means that we can view every compact metric space as a kind of space of infinite sequences. The minimality of the subspace elements means that, through this embedding, each strictly increasing sequence in $K(D)$, which can be realised as a process of filling a tape infinitely, can be interpreted as a point of X. In addition, because D is finite-branching, we have only a finite number of candidates to fill at every finite stage. When X is n-dimensional, D can be constructed as a subdomain of BD_n, and thus the candidates for the next cell are the first $n+1$ unfilled cells.

In Tsuiki (2002), the author presented the notion of an IM2-machine, which has, on each input/output tape, $n+1$ heads that move so that they are always located at the first $n+1$ undefined cells, and thus can input/output sequences in BD_n. Therefore, an IM2-machine can be used to input/output representations of n-dimensional spaces. As a special case, when the Gray-code embedding is used to represent $\mathbb{I}$ in RD, some algorithms

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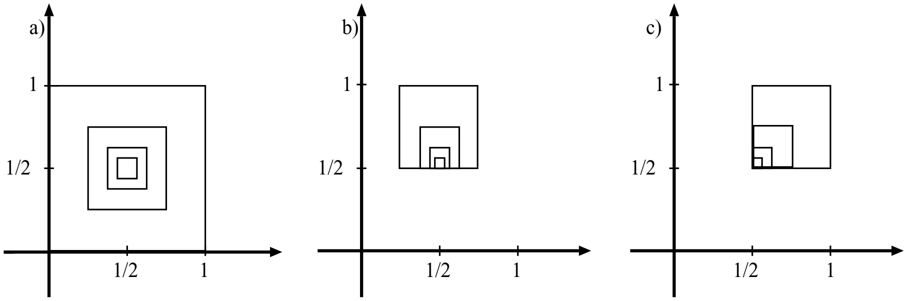

Fig. 8. The structure of infinite filters-bases converging to (1/2,1/2) on $\mathbb{I}^2$. (a) is the lower level element (unique), (a) + (b) is a middle level element (4 of this kind exist), (a) + 2 copies of type (b) + (c) is the upper level element (4 of this kind exist). Only the first one contains the point (1/2,1/2).

such as addition can be expressed using an IM2-machine (Tsuiki 2002), and it can be shown that the rules of an IM2-machine can easily be translated into a parallel logic programming language GHC, and executed on many platforms(Tsuiki 2001).

In this paper, we have proved the existence of a domain D that represents an n-dimensional separable metric space X via a classical theorem in dimension theory. In order to apply IM2-machines to give algorithms on a space X, we need to select a concrete structure for D and an embedding of X in D, as we did for $\mathbb{I}$. The question of how to define such a concrete structure when some effective structure of X is given is an interesting open problem.

When X is embedded in the space $L(D)$ of limit elements of a domain D, $K(D)$ gives a base of the topology of X. To conclude this paper, we will express some properties of this base in topological terms. When B is a base of X, we use $\mathbf{F}(B)$ to denote the set of infinite filter-bases that are composed of elements of B. We can consider $\mathbf{F}(B)$ as a poset by defining $\mathscr{F}_1 \leqslant \mathscr{F}_2$ iff $\mathscr{F}_2$ refines $\mathscr{F}_1$. When we combine Theorems 8.5 and 8.8 and Proposition 4.6, we have the following theorem.

Theorem 9.1. When X is a compact metric space of dimension n ($n \leqslant \infty$), there is a base B of X such that:

(1) The poset $(B, \supseteq)$ is finite-branching.
(2) Every infinite filter-base $\mathscr{F} \in \mathbf{F}(B)$ converges to a unique point of X (denoted by $lim\ \mathscr{F}$).
(3) $lim\ \mathscr{F}$ is the unique cluster point of $\mathscr{F}$.
(4) $\mathbf{F}(B)$ is a poset of height n.
(5) If $\mathscr{F}$ is a minimal element of $\mathbf{F}(B)$, then $\cap\mathscr{F} = \{lim\ \mathscr{F}\}$.
(6) If $\mathscr{F}$ is not a minimal element of $\mathbf{F}(B)$, then $\cap\mathscr{F} = \emptyset$.

Such a base is given by the Gray-code expansion for the case of $\mathbb{I}$, by the synchronised product of the base of $\mathbb{I}$ for $\mathbb{I}^n$ ($n = 2, 3, \ldots, \infty$), and as a subspace of $\mathbb{I}^{2n+1}$ (or $\mathbb{I}^\infty$ when $n = \infty$) for general cases. Figure 8 depicts the structure of the filter-bases in $\mathbf{F}(B)$ that are converging to $(1/2, 1/2)$, for the case of $\mathbb{I}^2$.

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