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THE MINIMAL MODULAR FORM ON QUATERNIONIC E_8

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Abstract Suppose that G is a simple reductive group over $\mathbf{Q}$, with an exceptional Dynkin type and with $G(\mathbf{R})$ quaternionic (in the sense of Gross–Wallach). In a previous paper, we gave an explicit form of the Fourier expansion of modular forms on G along the unipotent radical of the Heisenberg parabolic. In this paper, we give the Fourier expansion of the minimal modular form θ_{Gan} on quaternionic E_8 and some applications. The $Sym^8(V_2)$-valued automorphic function θ_{Gan} is a weight 4, level one modular form on E_8, which has been studied by Gan. The applications we give are the construction of special modular forms on quaternionic E_7, E_6 and G_2. We also discuss a family of degenerate Heisenberg Eisenstein series on the groups G, which may be thought of as an analogue to the quaternionic exceptional groups of the holomorphic Siegel Eisenstein series on the groups GSp_{2n}.

Keywords: minimal representation; modular forms; exceptional groups; arithmetic invariant theory; Eisenstein series

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Contents

1. Introduction

This paper is a sequel to paper [18]. In [18], we studied 'modular forms' on the quaternionic exceptional groups, following the beautiful work of Gan–Gross–Savin [6] and Wallach [21]. We proved that these modular forms possess a refined Fourier expansion,

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similar to the Siegel modular forms on the symplectic groups GSp_{2n}. This is, in a sense, a purely Archimedean result: The representation theory at the infinite place on the quaternionic exceptional groups forces the modular forms to have a robust theory of the Fourier expansion.

Suppose that $G_{/\mathbf{Q}}$ is a quaternionic exceptional group of adjoint type. The maximal compact subgroup $K_\infty \subseteq G(\mathbf{R})$ is $(\mathrm{SU}(2) \times L)/\mu_2$ for a certain group L. Denote by $\mathbb{V}_n = Sym^{2n}(V_2) \boxtimes \mathbf{1}$ the representation of $K_\infty = (\mathrm{SU}_2 \times L)/\mu_2$ that is the $(2n)$th symmetric power of the defining representation V_2 of $\mathrm{SU}(2)$ and the trivial representation of L. Recall from [18] that if $n \geqslant 1$ is an integer, a modular form on G of weight n is a smooth, moderate growth function $F : G(\mathbf{Q})\backslash G(\mathbf{A}) \to \mathbb{V}_n^\vee$ that satisfies

(1) $F(gk) = k^{-1} \cdot F(g)$ for all $k \in K_\infty \subseteq G(\mathbf{R})$ and
(2) $\mathcal{D}_n F = 0$.

Here, $\mathcal{D}_n$ is a certain first-order differential operator, closely related to the so-called Schmid operator for the quaternionic discrete series representations on $G(\mathbf{R})$. It is that F is annihilated by $\mathcal{D}_n$ that is the crucial piece of the definition of modular forms.

There is a weight 4, level one modular form on quaternionic E_8 that is associated with the automorphic minimal representation, studied by Gan [2, 3], which we denote by θ_{Gan}. The automorphic minimal representation is spherical at every finite place, but is not spherical at infinity; at the Archimedean place, it has minimal K_∞-type $\mathbb{V}_4$. If $v \in \mathbb{V}_4$, then pairing θ_{Gan} with v gives the vector in the minimal representation that is v at the Archimedean place and spherical at all the finite places. Our main result is the complete and explicit Fourier expansion of this modular form. See Theorem 1.0.1. Using θ_{Gan}, we construct special modular forms on E_7, E_6 and G_2; see Corollaries 1.0.2 and 1.0.3. Moreover, we study a family of absolutely convergent Eisenstein series on the quaternionic exceptional groups and prove that all their nontrivial Fourier coefficients are Euler products; see Theorem 3.2.5.

To set up the statements of these results, let us recall from [18] the shape of the Fourier expansion of modular forms on the quaternionic exceptional groups. Thus suppose $G = G_J$ is a quaternionic exceptional group of adjoint type, associated with a cubic norm structure J over $\mathbf{Q}$ with positive definite trace form. Then G has a rational Heisenberg parabolic $P_J = H_J N_J$, with Levi subgroup H_J and unipotent radical N_J. The group $N_J \supseteq N_0$ is a two-step unipotent group, with center $N_0 = [N, N]$ and abelianization $N/N_0 \simeq W_J$. Here $W_J = \mathbf{Q} \oplus J \oplus J^\vee \oplus \mathbf{Q}$ is Freudenthal's defining representation of the group H_J (see [17, 18]).

Suppose F is a modular form of weight n for G. Denote by F_0 the constant term of F along N_0, i.e.,

$$F_0(g) = \int_{N_0(\mathbf{Q})\backslash N_0(\mathbf{A})} F(ng)\, dn.$$

A simple argument using the left invariance of F under $G(\mathbf{Q})$ proves that F_0 determines F for the groups studied in [18]. The Fourier expansion of F_0 is then given as follows: For $x \in (N/N_0)(\mathbf{R}) \simeq W_J(\mathbf{R})$ and $g \in H_J(\mathbf{R})$,

$$F_0(xg) = F_{00}(g) + \sum_{\omega \in W_J(\mathbf{Q}), \omega \geqslant 0} a_F(\omega) e^{2\pi i \langle \omega, x\rangle} \mathcal{W}_{2\pi\omega}(g),$$

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where the notation is as follows:

- F_{00} denotes the constant term of F along N;
- $a_F(\omega)$ is the Fourier coefficient associated with ω;
- $\langle\cdot,\cdot\rangle$ is Freudenthal's symplectic form on W_J;
- $\mathcal{W}_\omega : H_J(\mathbf{R}) \to \mathbb{V}_n^\vee$ is a special function on $H_J(\mathbf{R})$ defined in terms of the K-Bessel functions $K_v(\cdot)$ for $v \in \{-n, -n+1, \dots, n-1, n\}$ and the element $\omega \in W_J(\mathbf{R})$.

Denote by x, y the fixed basis of V_2 from [18] so that $\{x^{2n}, x^{2n-1}y, \dots, xy^{2n-1}, y^{2n}\}$ is a basis of $\mathbb{V}_n$. The basis elements $x^{n+v}y^{n-v}$ of $\mathbb{V}_n$ are essentially characterized by the fact that $k \cdot x^{n+v}y^{n-v} = j(k,i)^v x^{n+v}y^{n-v}$ for $k \in K_H^1$ a certain compact subgroup of $H_J(\mathbf{R})$ and j the factor of automorphy on H_J specified in *loc. cit.*; see [18, § 9]. Then

$$\mathcal{W}_\omega(g) = \sum_{-n\leqslant v\leqslant n} \mathcal{W}_\omega^v(g)\frac{x^{n+v}y^{n-v}}{(n+v)!(n-v)!} \tag{1}$$

with

$$\mathcal{W}_\omega^v(g) = \nu(g)^n|\nu(g)|\left(\frac{|\langle\omega, gr_0(i)\rangle|}{\langle\omega, gr_0(i)\rangle}\right)^v K_v(|\langle\omega, gr_0(i)\rangle|) \tag{2}$$

and $r_0(i) = (1, -i, -1, i) \in W_J \otimes \mathbf{C}$. Here, $\nu : H_J \to \mathrm{GL}_1$ is the similitude character of H_J. Moreover, the constant term

$$F_{00}(g) = \nu(g)^n|\nu(g)|\left(\Phi(g)\frac{x^{2n}}{(2n)!} + \beta\frac{x^n y^n}{n!n!} + \Phi'(g)\frac{y^{2n}}{(2n)!}\right)$$

for some holomorphic modular form Φ of weight n on H_J and $\Phi'(g) = \Phi(gw_0)$ for a specific element $w_0 \in H_J$ that exchanges the upper and lower half-spaces $\mathcal{H}_J^\pm$.

With this result recalled, let us now state the Fourier expansion of θ_{Gan}. Denote by Θ_0 Coxeter's integral octonions [1, (5.1)] and $J_0 = H_3(\Theta_0)$ the associated integral lattice in the exceptional cubic norm structure J. The Freudenthal space W_J has a natural integral lattice $W_J(\mathbf{Z}) = \mathbf{Z} \oplus J_0 \oplus J_0^\vee \oplus \mathbf{Z}$. For $\omega \in W_J(\mathbf{Z})$, define $\Delta(\omega)$ to be the largest positive integer so that $\omega \in \Delta(\omega)W_J(\mathbf{Z})$. For $T \in J_0$, define $\Delta(T)$ analogously.

Recall Kim's modular form $H_{Kim}(Z)$ [13] on the exceptional tube domain, which has Fourier expansion

$$H_{Kim}(Z) = \frac{1}{240} + \sum_{T\in J_0, T\geqslant 0 \text{ rank one}} \sigma_3(\Delta(T))q^T.$$

Denote by Φ_{Kim} the automorphic form on $H_J = GE_7$ so that $j(g,i)^4\Phi_{Kim}(g)$ descends to $\mathcal{H}_J^\pm$, is holomorphic on $\mathcal{H}_J^+$, antiholomorphic on $\mathcal{H}_J^-$, and on $\mathcal{H}_J^+$ one has $H_{Kim}(Z) = j(g,i)^4\Phi_{Kim}(g)$ if $Z = g \cdot i$.

Theorem 1.0.1. *Let the notations be as above. Then*

$$\theta_{Gan,0}(xg) = \theta_{Gan,00}(g) + \sum_{\omega\in W_J(\mathbf{Z}) \textit{ rank one}} \sigma_4(\Delta(\omega))e^{2\pi i\langle\omega,x\rangle}\mathcal{W}_{2\pi\omega}(g)$$

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with

$$\theta_{Gan,00}(g) = |\nu(g)|^5 \left(\frac{12\zeta(5)}{(2\pi)^4} \frac{x^4 y^4}{4!4!} + 8 \left(\Phi_{Kim} \frac{x^8}{8!} + \Phi'_{Kim} \frac{y^8}{8!} \right) \right).$$

There is a degenerate Heisenberg Eisenstein series on each of the quaternionic exceptional groups, which we write as $E(g, s; n)$. This is a function $E(g, s; n) = E^G(g, s; n) : G(\mathbf{Q})\backslash G(\mathbf{A}) \to \mathbb{V}_n^\vee$ (depending on s) satisfying $E(gk, s; n) = k^{-1}E(g, s; n)$ for all $k \in K_\infty$. When G is quaternionic E_8 and $n = 4$, it turns out that the Eisenstein series $E(g, s; 4)$ is regular at $s = 5$, and θ_{Gan} is defined [2] (up to a nonzero scalar multiple) as the value of this Eisenstein series at this point.

Combining the Archimedean results of [18] with the work of Gan [2–4], Kim [13], Gross–Wallach [10], Kazhdan–Polishchuk [12] and Gan–Savin [7], most of Theorem 1.0.1 was known. See the discussion in § 2.2. What is left is to pin down a couple of constants. We do this by analyzing the Fourier expansion $E^{G_2}(g, s = 5; 4)$ of the weight 4 Eisenstein series on G_2 and applying the Siegel–Weil theorem of [3], which relates θ_{Gan} to this Eisenstein series on G_2.

The Eisenstein series $E^{G_2}(g, s = 5; 4)$ is easier to compute with in relation to $E^{E_8}(g, s = 5; 4)$ (which defines θ_{Gan}) because $s = 5$ is in the range of absolute convergence of $E^{G_2}(g, s; 4)$ but not of $E^{E_8}(g, s; 4)$. In fact, we study absolutely convergent degenerate Heisenberg Eisenstein series $E^G(g, s; n)$ on the quaternionic exceptional groups G in general, which is our second main result. More precisely, if n is even and the special point $s = n + 1$ is in the range of absolute convergence, the Eisenstein series $E(g, s = n + 1; n)$ is a modular form on G of weight n. We prove that at such an n, all of the nontrivial Fourier coefficients of $E(g, s = n + 1; n)$ are Euler products. This is the analogue to the exceptional groups of the corresponding classical fact about holomorphic Siegel Eisenstein series on symplectic groups [19]; we defer a precise statement of this result to § 3. The proof is an easy consequence of a weak form of the main result of [18]: The Fourier expansion of $E(g, s; n)$ has many terms, some of which are Euler products and some of which are not. However, applying [18], one can deduce that all of the terms that are not Euler products vanish at $s = n + 1$ for purely Archimedean reasons.

We also give a few applications of Theorem 1.0.1 to modular forms on E_7, E_6 and G_2. Namely, one can pull back the minimal modular form θ_{Gan} on E_8 to the simply connected quaternionic E_7 and E_6. Denote these pull-backs by $\theta^{(2)}_{E_7}$ and $\theta^{(4)}_{E_6}$, respectively. These pull-backs give interesting singular and distinguished modular forms on E_7^{sc} and E_6^{sc}. The modular form $\theta^{(2)}_{E_7}$ is singular in that it has no rank three or rank four Fourier coefficients, but it does have nonzero rank two Fourier coefficients. The modular form $\theta^{(4)}_{E_6}$ is not singular – it has nonzero rank four Fourier coefficients. However, it is distinguished in that it has only one orbit of nonzero rank four Fourier coefficients.

Corollary 1.0.2. *The automorphic functions $\theta^{(2)}_{E_7}$ and $\theta^{(4)}_{E_6}$ define nonzero modular forms on E_7^{sc} and E_6^{sc} of weight* 4*. Moreover, we have the following:*

(1) *The modular form $\theta^{(2)}_{E_7}$ has nonzero rank two Fourier coefficients, but all of its rank three and rank four Fourier coefficients are* 0*.*

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(2) *The modular form $\theta^{(4)}_{E_6}$ is distinguished: it has only one orbit of nonzero rank four Fourier coefficients.*

The distinguished nature of these Fourier expansions is a more or less immediate consequence of the results of [17, §§ 7 and 8].

Note that Theorem 1.0.1 says that (a scalar multiple of) θ_{Gan} just fails to have integral Fourier coefficients. All the rank one Fourier coefficients are integers, and all the Fourier coefficients of Φ_{Kim} are integers. Thus, it is reasonable to ask for a nonzero modular form on an exceptional group for which *all* of its Fourier coefficients are integers. Our final application of Theorem 1.0.1 is to produce such a modular form on G_2.

Following [1, 5, 6], there is a unique (up to scaling) automorphic function ϵ on a certain anisotropic form of F_4, which is right invariant under $F_4(\widehat{\mathbf{Z}})F_4(\mathbf{R})$ and orthogonal to the constant functions. Denote by F_Δ the theta lift of ϵ to G_2 via θ_{Gan}. The modular form F_Δ is discussed in [6] and [5]. The following is an essentially immediate corollary of Theorem 1.0.1 and results of *loc. cit.*

Corollary 1.0.3. *The modular form F_Δ has rational Fourier coefficients with bounded denominators. Its constant term is proportional to Ramanujan's function Δ.*

1.1. Notation

Throughout the paper, the notation is as in [18]. In particular, F denotes a field of characteristic 0, J denotes a cubic norm structure over F, and $\mathfrak{g}_J$ or $\mathfrak{g}(J)$ the Lie algebra associated with J in [18, § 4]. The field F will frequently be $\mathbf{Q}$ or $\mathbf{R}$. We will assume that J is either F or $H_3(C)$ with C a composition algebra over F. Thus $\mathfrak{g}(J)$ is of type G_2, F_4, E_6, E_7 or E_8.

We write $\mathfrak{h}(J)$ for the Lie algebra of the Freudenthal group H_J and $\mathfrak{m}(J)$ for the Lie algebra of the group that preserves the cubic norm on J up to similitude. Then (see [18, § 4] for our normalizations)

$$\begin{aligned}\mathfrak{g}(J) &= \mathfrak{sl}_2 \oplus \mathfrak{h}(J)^0 \oplus V_2 \otimes W_J \\ &\simeq \mathfrak{sl}_3 \oplus \mathfrak{m}(J)^0 \oplus V_3 \otimes J \oplus (V_3 \otimes J)^\vee.\end{aligned} \tag{3}$$

The first displayed line above is a $\mathbf{Z}/2$-grading on the Lie algebra $\mathfrak{g}(J)$, with $\mathfrak{sl}_2 \oplus \mathfrak{h}(J)^0$ in degree 0 and $V_2 \otimes W_J$ in degree 1; we refer to this as the $\mathbf{Z}/2$-model of $\mathfrak{g}(J)$. Similarly, the second line gives a $\mathbf{Z}/3$-grading of $\mathfrak{g}(J)$, with $\mathfrak{sl}_3 \oplus \mathfrak{m}(J)^0$ in degree 0, $V_3 \otimes J$ in degree 1 and $(V_3 \otimes J)^\vee$ in degree 2; we refer to this as the $\mathbf{Z}/3$-model of $\mathfrak{g}(J)$. We will sometimes use the $\mathbf{Z}/2$-model to specify elements of the Lie algebra $\mathfrak{g}(J)$, and other times the $\mathbf{Z}/3$-model. See [18, § 4] for more on these Lie algebras and, in particular, refer to paragraph 4.2.4 of *loc. cit.*, where an explicit isomorphism is given between the $\mathbf{Z}/2$-model and the $\mathbf{Z}/3$-model of $\mathfrak{g}(J)$.

When working in the $\mathbf{Z}/2$-model of $\mathfrak{g}(J)$, we use the letters e, f for a fixed symplectic basis of the V_2 appearing in (3). If $w \in W_J$, then we consider $e \otimes w$ and $f \otimes w$ as elements of $\mathfrak{g}(J)$.

When working in the $\mathbf{Z}/3$-model of $\mathfrak{g}(J)$, if $i \neq j$, we write E_{ij} for the element of $\mathfrak{sl}_3 \subseteq \mathfrak{g}(J)$ with a 1 in the (i, j) coordinate and 0's elsewhere. One defines $P_J = H_J N_J$

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(or $P = HN$, if J is fixed) to be the Heisenberg parabolic of G_J, which by definition is the stabilizer of the line FE_{13} in $\mathfrak{g}(J)$. We write $N_0 = [N, N]$, which is also the center of N. The letter ν denotes the similitude character of P; one has $\nu : P \to \mathrm{GL}_1$ given by $p \cdot E_{13} = \nu(p)E_{13}$.

Recall that H_J preserves a symplectic form on W_J so that there is an induced map $\mathfrak{h}(J)^0 \to \mathfrak{sp}(W_J) \simeq Sym^2(W_J)$. From the nondegeneracy of the Killing form, one obtains an H_J-equivariant map $Sym^2(W_J) \to \mathfrak{h}(J)^0$, unique up to a scalar multiple. A specific choice of such a map, together with an explicit formula is given in [18, § 3.4.2]: for $w_1, w_2 \in W_J$, we denote by $\Phi_{w_1,w_2} = \Phi_{w_2,w_1}$ this element of $\mathfrak{h}(J)^0$.

The letter K or K_∞ denotes the maximal compact subgroup of $G_J(\mathbf{R})$ defined in *loc. cit.*, where $G = G_J$ is the adjoint group associated with the Lie algebra $\mathfrak{g}(J)$. We write $\mathbb{V}_n$ for the representation of $K = (\mathrm{SU}(2) \times L)/\mu_2$ on $Sym^{2n}(V_2) \boxtimes \mathbf{1}$. *Modular forms* on G_J are by definition certain functions $F : G_J(\mathbf{Q})\backslash G_J(\mathbf{A}) \to \mathbb{V}_n^\vee$ satisfying $F(gk) = k^{-1} \cdot F(g)$ for all $g \in G_J(\mathbf{A})$ and $k \in K$, which are annihilated by a first-order differential operator $\mathcal{D}_n$.

Finally, if $z \in \mathbf{C}$ and $j \geqslant 0$ an integer, $(z)_j = z(z+1)(z+2)\cdots(z+j-1) = \frac{\Gamma(z+j)}{\Gamma(z)}$ is the Pochhammer symbol.

2. Statement of results, and applications

In this section, we state our main results more precisely and give the proofs of Corollaries 1.0.2 and 1.0.3. We begin by defining the degenerate Heisenberg Eisenstein series on the quaternionic groups G_J, as these Eisenstein series are central to everything that follows. We then review what was known about the automorphic form θ_{Gan}. Finally, we restate Corollaries 1.0.2 and 1.0.3 and give the proofs of these results.

2.1. The degenerate Heisenberg Eisenstein series

In this subsection, we define the degenerate Heisenberg Eisenstein series $E_J(g, s; n)$ on the quaternionic exceptional groups G_J. By definition, such an Eisenstein series is associated with a section $f(g, s) \in Ind_{P(\mathbf{A})}^{G_J(\mathbf{A})}(|\nu|^s)$. More precisely, we use the final parameter n in $E(g, s; n)$ to indicate that $E(g, s; n)$ is $\mathbb{V}_n^\vee$-valued and satisfies $E(gk, s; n) = k^{-1} \cdot E(g, s; n)$. Throughout, *we will assume that* $n \geqslant 0$ *is even.*

We now construct such an Eisenstein series explicitly; this makes it easier to do computations. Suppose Φ_f is a Schwartz–Bruhat function on $\mathfrak{g}_J(\mathbf{A}_f)$. We will define a $\mathbb{V}_n$-valued Schwartz function $\Phi_{\infty,n}$ on $\mathfrak{g}_J(\mathbf{R})$ satisfying $\Phi_{\infty,n}(kv) = k \cdot \Phi_{\infty,n}(v)$ momentarily. With this definition, we set $\Phi = \Phi_f \otimes \Phi_{\infty,n}$ and then

$$f(g, \Phi, s) = \int_{\mathrm{GL}_1(\mathbf{A})} |t|^s \Phi(tg^{-1}E_{13})\, dt,$$

absolutely convergent for $Re(s) > 1$. It is clear that $f(g, \Phi, s)$ is a section in the induced representation $Ind_{P_J}^{G_J}(|\nu|^s)$ (although not a flat section), and we set

$$E(g, \Phi, s) = \sum_{\gamma \in P_J(\mathbf{Q})\backslash G_J(\mathbf{Q})} f(\gamma g, \Phi, s).$$

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We will be interested in this Eisenstein series at the special value $s = n + 1$. When $n > \dim(W_J)/2 = \dim J + 1$, the Eisenstein series converges absolutely at $s = n + 1$ and defines a modular form there; see the remarks after Corollary 1.2.4 in [18].

The special Archimedean function $\Phi_{\infty,n}$ is defined as follows. Denote by $\mathfrak{k}_2$ the $\mathfrak{su}_2$ part of $\mathfrak{k}$, the Lie algebra of the maximal compact K, and denote by $pr : \mathfrak{g}(J) \to \mathfrak{k}_2$ the K-equivariant projection. This $\mathfrak{su}_2 \otimes \mathbf{C}$ is identified with $\mathbb{V}_1 = Sym^2(V_2)$ as follows. In [18, § 5.1], an $\mathfrak{sl}_2$-triple (e_ℓ, h_ℓ, f_ℓ) of $\mathfrak{su}_2 \otimes \mathbf{C}$ is specified, and under our identification $\mathfrak{su}_2 \otimes \mathbf{C} \simeq \mathbb{V}_1$, $e_\ell \mapsto x^2$, $h_\ell \mapsto -2xy$ and $f_\ell \mapsto -y^2$. For $n \geqslant 0$, define $\Phi_{\infty,n}(v) = pr(v)^n e^{-\pi \|v\|^2}$. Here $pr(v)^n$ is considered as an element of $\mathbb{V}_n$ and $\|v\|^2 = B_{\mathfrak{g}}(v, -\Theta(v))$, with $B_{\mathfrak{g}}$ and Θ defined in [17, § 4]. It is clear that $\Phi_{\infty,n}(kv) = k \cdot \Phi_{\infty,n}(v)$.

2.2. The minimal automorphic forms on quaternionic E_8

In this subsection, we briefly discuss the automorphic minimal representation on quaternionic E_8. The reader should see [2] and [7] and the references contained therein for more details.

For this subsection, let $J = H_3(\Theta)$ with Θ the octonion algebra over $\mathbf{Q}$ whose trace pairing is positive definite. Then G_J is the quaternionic E_8. Suppose that $f_s \in Ind_{P_J(\mathbf{A})}^{G_J(\mathbf{A})}(|\nu|^s)$ is a flat section, and $E_J(g, f_s)$ the associated Eisenstein series. It is proved in [2] that for appropriate f_s, $E_J(g, f_s)$ has a simple pole at $s = 24$. Moreover, this pole can be achieved when f_s is spherical at every finite place. The automorphic minimal representation Π is defined [2] to be the space of residues of the $E_J(g, f_s)$ at $s = 24$. By, e.g., [16] and also [7], the space of such automorphic forms only has rank 1 and rank 0 Fourier coefficients along N_J; for instance, this follows by the analogous local fact for one finite place.

Denote by $E_J(g, s)$ the Eisenstein series associated with the flat section $f_J(g, s; n)$, which has the following properties:

(1) $f_J(g, s; n)$ is valued in $\mathbb{V}_n^\vee \simeq \mathbb{V}_n$, and satisfies $f_J(gk, s; n) = k^{-1} f_J(g, s; n)$ for all $g \in G_J(\mathbf{A})$ and $k \in K \subseteq G_J(\mathbf{R})$.

(2) f_J is spherical at every finite place.

(3) $f_J(1, s; n) = \frac{x^n y^n}{n!n!} \in \mathbb{V}_n$.

One defines θ_{Gan} to be a certain nonzero multiple of $Res_{s=24}(E_J(g, s; 4))$. It is proved in [7] by a somewhat indirect method that θ_{Gan} is nonzero, i.e., that $E_J(g, s; 4)$ does have a nontrivial pole at $s = 24$.

In more detail, in [2] it is proved that the degenerate Heisenberg Eisenstein series $E_J(g, f_s)$ on G_J has at most a simple pole at $s = 24$, and that this simple pole is attained for a flat section that is spherical at all finite places. However, from [2] alone, one does not know that the K-type $\mathbb{V}_4$ survives in the residue, i.e., that $Res_{s=24}(E_J(g, s; 4))$ is nonzero. That this residue is nonzero follows from [7, Corollary 12.12] (see also the paragraph after Theorem 12.9 of *loc. cit.*). Corollary 12.12 of [7] is proved somewhat indirectly, by using minimality of the residue at a finite place and appealing to a rigidity property of these representations proved by those authors.

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For the convenience of the reader, we give a direct proof of the fact that $Res_{s=24}(E_J(g,s;4)) \neq 0$, by computing an appropriate Archimedean intertwining operator. This computation is done in § 4. More precisely, we shall use (and prove) the following fact.

Proposition 2.2.1. *The Eisenstein series $E_J(g,s;4)$ is regular at $s=5$, and defines a modular form on G_J of weight 4. Up to nonzero scalar multiples in each of the following equalities,*

$$E_J(g,s=5;4) = Res_{s=24}E_J(g,s;4) = \theta_{Gan}.$$

As mentioned, this proposition is essentially contained in the union of [2, 3, 7, 10]. For the reader reviewing this literature, the following remark may be helpful: Corollary 5.13 of [2] implies directly that $E_J(g,s;4)$ is regular at $s=5$. However, the remarks given in [2] do not prove that corollary, and it may be false. Nevertheless, as it is stated, [2, Corollary 5.13] is not used in [2, 3, 7], and this possible error does not impact the other results of those papers.

Now, because θ_{Gan} is a modular form on G_J, the results of [18] imply that its Fourier expansion takes the following shape. Denote by Θ_0 Coxeter's integral subring [1, (5.1)] of Θ, $J_0 = H_3(\Theta_0)$, and $W_J(\mathbf{Z}) = \mathbf{Z} \oplus J_0 \oplus J_0^\vee \oplus \mathbf{Z} \subseteq W_J(\mathbf{Q})$. Then for $x \in N_J(\mathbf{R})$ and $m \in H_J(\mathbf{R})$, one has

$$\theta_{Gan,0}(n(x)m) = \theta_{00}(m) + \sum_{\omega \in W_J(\mathbf{Z})} a_\theta(\omega) e^{2\pi i \langle x,\omega\rangle} \mathcal{W}_{2\pi\omega}(m)$$

with the constant term $\theta_{00}(m)$ given by

$$\theta_{00}(m) = \beta_1 \Phi(m) x^8 + \beta_0 x^4 y^4 + \beta_1 \Phi'(m) y^8$$

for a holomorphic weight 4 modular form Φ on $H_J = \mathrm{GE}_7$. Because θ_{Gan} is minimal, $a_\theta(\omega)$ is nonzero only for ω rank one. Denote by $\Delta(\omega)$ the largest positive integer so that $\omega \in \Delta(\omega) W_J(\mathbf{Z})$. By [4, Theorem 2.3], which follows from [12, Theorem 1.1.3], we may scale θ_{Gan} so that

$$a_\theta(\omega) = \begin{cases} \sigma_4(\Delta(\omega)) & \text{if } \omega \text{ is rank one} \\ 0 & \text{if } \omega \text{ is rank two, three, or four.} \end{cases}$$

Moreover, from [3], Φ is proportional to Kim's [13] level one, weight 4 modular form on $H_J = \mathrm{GE}_7$. Thus, applying the results of [2, 3, 7, 10, 12, 13, 18], what is left is to pin down the constants β_0 and β_1. This is precisely what Theorem 1.0.1 does. We restate the result now.

Theorem 2.2.2. *The Eisenstein series $E_J(g,s;4)$ is regular at $s=5$ and defines a modular form on $G_J = E_{8,4}$ of weight 4 at this point. The Fourier coefficient corresponding to the rank one element $(0,0,0,1) \in W_J$ is nonzero. Denote by θ_{Gan} the scalar multiple of $E_J(g,s;4)$ for which this Fourier coefficient is equal to 1. Moreover, denote by Φ_{Kim} the spherical automorphic form on $H_J = GE_7$ so that $H_{Kim} = j(g,i)^4 \Phi_{Kim}$ descends to $\mathcal{H}_J^{\pm}$,*

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is holomorphic on $\mathcal{H}_J^+$, antiholomorphic on $\mathcal{H}_{\mathcal{J}}{}^-$, and on $\mathcal{H}_J^+$ has the Fourier expansion

$$H_{Kim} = \frac{1}{240} + \sum_{T \in J_0, T \geqslant 0 \text{ rank one}} \sigma_3(\Delta(T))q^T.$$

Then one has

$$\theta_{Gan,0} = |\nu(g)|^5 \left(\frac{12\zeta(5)}{(2\pi)^4} \frac{x^4 y^4}{4!4!} + 8 \left(\Phi_{Kim} \frac{x^8}{8!} + \Phi'_{Kim} \frac{y^8}{8!} \right) \right) + \sum_{\omega \in W_J(\mathbf{Z}),\ \text{rank one}} \sigma_4(\Delta(\omega)) \mathcal{W}_{2\pi\omega}(g).$$

We will prove this theorem in § 4 after understanding the Fourier expansion of degenerate absolutely convergent Heisenberg Eisenstein series in § 3. We now detail and prove the corollaries of Theorem 2.2.2 that were mentioned in § 1.

2.3. The singular modular form

In this subsection, we consider the singular modular form $\theta_{E_7}^{(2)}$ on the simply connected quaternionic E_7. This modular form is defined as follows. First, fix a quaternion algebra B over $\mathbf{Q}$, which is ramified at the Archimedean place. Recall that the quaternionic Lie algebra $\mathfrak{e}_7$ is $\mathfrak{g}(H_3(B))$, in the notation of [18]. For ease of notation, we write $J_B = H_3(B)$ and $J_\Theta = H_3(\Theta)$.

Now, fix $\gamma \in \mathbf{Q}^\times$ not representing the identity coset in $\mathbf{Q}^\times/N(B^\times)$; in other words, $\gamma < 0$. By the Cayley–Dickson construction, one can form an octonion algebra Θ out of B and γ. See [17, § 8]. With such a γ, Θ is ramified at infinity. The Cayley–Dickson construction induces an identification $J_B \oplus B^3 \simeq J_\Theta$, an embedding $\mathfrak{h}(J_B) \to \mathfrak{h}(J_\Theta)$, and then consequently an embedding $\mathfrak{g}(J_B) \hookrightarrow \mathfrak{g}(J_\Theta)$.

More precisely, denote by W_6 the defining representation of GSp_6, and define an identification $W_{J_B} \oplus W_6 \otimes B \simeq W_{J_\Theta}$ as in [17, § 8.1.2]. From [17, Proposition 8.1.5], one gets a group H'_B (this is the group $G(\gamma, C)$ in the notation of that proposition) together with maps $H'_B \to H_{J_B}$ and $H'_B \to H_{J_\Theta}$, where the first map induces an isomorphism of Lie algebras. Consequently, one obtains a map $\mathfrak{h}^0(J_B) \to \mathfrak{h}^0(J_\Theta)$. As $\mathfrak{g}(J) = \mathfrak{sl}_2 \oplus \mathfrak{h}^0(J) \oplus V_2 \otimes W_J$, one obtains a specific embedding $\mathfrak{g}(J_B) \to \mathfrak{g}(J_\Theta)$.

Denote by A_B the connected component of the identity of the subgroup of G_{J_Θ} that preserves $\mathfrak{g}(J_B)$. Write $B^{n=1}$ for the norm 1 elements of B. We define a map $B^{n=1} \to A_B$ as follows. First, define $B^{n=1} \to H_{J_\Theta}$ via its action on $W_{J_\Theta} \simeq W_{J_B} \oplus B^6$ as $s \cdot (w, v) = (w, vs^{-1})$ for $s \in B^{n=1}$, $w \in W_{H_3(B)}$ and $v \in B^6$. Because the quadratic norm on Θ is $n_\Theta(x, y) = n_B(x) - \gamma n_B(y)$ for $x, y \in B$, it is easy to see directly that this action preserves the symplectic and quartic form on $W_{H_3(\Theta)}$. Now because $H_{J_\Theta} \to G_{J_\Theta}$, this defines $B^{n=1} \to G_{J_\Theta}$. Finally, it is clear by construction that this $B^{n=1}$ preserves $\mathfrak{g}(J_B)$, and thus we obtain $B^{n=1} \to A_B$, as claimed. At the Archimedean place, this $B^{n=1}$ is a compact SU(2). In terms of the maximal compact subgroup $K = (\mathrm{SU}(2) \times L)/\mu_2$ of $G_J(\mathbf{R})$, this SU(2) sits inside the image of L.

Denote by E_B the connected component of the identity of the centralizer of $B^{n=1}$ in A_B.

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Lemma 2.3.1. *The group E_B is the simply connected quaternionic E_7.*

Proof. Indeed, one verifies without difficulty that $\mathfrak{g}(J_\Theta)^{B^1} = \mathfrak{g}(J_B)$, and thus E_B has the correct Lie algebra. Moreover, the $\mu_2 \subseteq B^1$ centralizes B^1 in A_B, and this μ_2 sits in H'_B; see [17, Proposition 8.1.5]. Because H'_B is connected and centralizes B^1, this proves that μ_2 is at the center of E_B. Thus E_B is connected, has Lie algebra equal to $\mathfrak{g}(J_B)$, and contains μ_2 at its center, so $E_B \simeq E_{7,4}^{sc}$. □

At the Archimedean place, the subgroup $\mathrm{SU}(2) \times E_B(\mathbf{R}) = \mathrm{SU}(2) \times E_{7,4}$ of $E_{8,4}$ just constructed was considered in [9, § 6]; see also [14, 15].

Denote by $\theta^{(2)}$ the automorphic function that is the pull-back of θ_{Gan} to E_B via the embedding $E_7^{sc} \simeq E_B \to G_{J_\Theta}$. We have the following result, which is a restatement of Corollary 1.0.2 (1).

Proposition 2.3.2. *The automorphic function $\theta^{(2)}$ is a modular form on E_7^{sc} of weight 4. It has nonzero rank two Fourier coefficients, but all of its rank three and rank four Fourier coefficients are 0.*

The definitions and results of [18] were made for adjoint groups, not simply connected ones. However, it is easy to see that they carry over immediately for the simply connected E_7. Indeed, because the μ_2 that is the center of E_7^{sc} acts trivially on $\mathbb{V}_n$, and because the map of real groups $E_7^{sc}(\mathbf{R}) \to E_7^{ad}(\mathbf{R})$ is surjective [20, Table II], the Archimedean theory is identical for modular forms on the adjoint E_7 and modular forms on the simply connected E_7.

Proof of Proposition 2.3.2. To see that $\theta^{(2)}$ is a modular form, one must only check the condition $\mathcal{D}_4\theta^{(2)} = 0$. One way to do this is simply observe that since θ_{Gan} satisfies the equations of [18, Theorem 7.3.1 or Theorem 7.5.1], so too does $\theta^{(2)}$. One can also reason directly with the definition of $\mathcal{D}_4$ in terms of a basis of $\mathfrak{p}(J_B)$ and $\mathfrak{p}(J_\Theta)$, or apply results of [9, 14, 15].

For the analysis of the Fourier coefficients, this is a direct consequence of [17, Theorem 8.1.4]. Namely, if $x \in W_{J_B}$ is nonzero, and $a(x)$ denotes the x-Fourier coefficient of the modular form $\theta^{(2)}$, then

$$a(x) = \sum_{u \in W_6(B)} a_{\theta_{Gan}}(x+u).$$

(The sum has only finitely many nonzero terms.) Because all the numbers $a_{\theta_{Gan}}(\omega)$ are nonnegative, it is clear that $\theta^{(2)}$ is nonzero. Finally, because $a_{\theta_{Gan}}(\omega)$ is only nonzero for ω rank one, all the rest of the claims of the proposition follow immediately from [17, Theorem 8.1.4]. This completes the proof. □

2.4. The distinguished modular form

Pulling back θ_{Gan} to the semisimple simply connected quaternionic E_6, we obtain a modular form $\theta^{(4)}$. In this subsection, we discuss the automorphic form $\theta^{(4)}$ and explain why it is distinguished.

Fix a quadratic imaginary extension K of $\mathbf{Q}$. Recall that the Lie algebra $\mathfrak{g}(H_3(K))$ is the quaternionic Lie algebra of type E_6. Using $H_3(K)$ and some additional data, the

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so-called second construction of Tits produces an exceptional cubic norm structure J. We will use this construction to define the map $E_6^{sc} \to G_J \simeq E_{8,4}$ and analyze the Fourier coefficients of $\theta^{(4)}$.

Thus, suppose $\lambda \in K^\times$, $S \in H_3(K)$ and that $\lambda\lambda^* = N(S)$. In this subsection, we let J_K denote $H_3(K)$ and B denote $M_3(K)$. Set $J = H_3(K) \oplus M_3(K) = J_K \oplus B$. Then one can make J into a cubic norm structure using λ and S; see, e.g., [17, § 7.1]. If S is positive definite, then so is the trace pairing on J, and thus the Lie algebra $\mathfrak{g}(J)$ is of type E_8 and quaternionic at infinity. We will choose $\lambda = 1$ and $S = 1_3$, so that $J \simeq H_3(\Theta)$ over $\mathbf{Q}$, but other choices of λ, S should yield interesting results.[1]

Now, via this construction of Tits, we obtain $J_K \to J$ and then $\mathfrak{h}(J_K) \to \mathfrak{h}(J)$ and then finally $\mathfrak{g}(J_K) \to \mathfrak{g}(J)$. More precisely, from [17, § 7.2], there is an identification $W_{J_K} \oplus B^2 \simeq W_J$. From [17, Proposition 7.2.2], there is a group H'_K (denoted by G in that proposition) that comes with maps $H'_K \to H_{J_K}$ and $H'_K \to H_J$, and it is easy to see that the first map induces an isomorphism of Lie algebras. Consequently, as above, these constructions define an embedding $\mathfrak{g}(J_K) \to \mathfrak{g}(J)$.

Denote by A_K the connected component of the identity of the subgroup of G_J that preserves $\mathfrak{g}(J_K)$. We will construct explicitly the simply connected quaternionic E_6 inside A_K, just as we did for E_7 in the previous subsection. More precisely, consider the subgroup SU_3 of $B^\times$ defined as the $g \in B$ with $\det(g) = 1$ and $gSg^* = S$. Let this SU_3 act on $J \simeq J_K \oplus B$ as $g \cdot (X, \alpha) = (X, \alpha g^{-1})$ for $X \in J_K$, $\alpha \in B$, and $g \in \mathrm{SU}_3$. It is clear from the formulas defining the second construction of Tits [17, § 7.1] that this action preserves the norm and pairing on J. It acts on $W_J = W_{J_K} \oplus B^2$ as $g \cdot (w, \eta) = (w, \eta g^{-1})$, for $w \in W_{J_K}$ and $\eta \in B^2$. Consequently, one obtains maps $\mathrm{SU}_3 \to H_J \to G_J$, and this SU_3 lands in A_K because its action on $\mathfrak{g}(J)$ fixes $\mathfrak{g}(J_K)$. At the Archimedean place, this SU_3 is compact and sits inside the image of the L-piece of the maximal compact $(\mathrm{SU}(2) \times L)/\mu_2$ of $G_J(\mathbf{R}) = E_{8,4}$.

Denote by E_K the connected component of the identity of the centralizer of this SU_3 in A_K.

Lemma 2.4.1. *The group E_K is the simply connected quaternionic E_6.*

Proof. Indeed, one verifies quickly that $\mathfrak{g}(J)^{\mathrm{SU}_3} = \mathfrak{g}(J_K)$, and thus the Lie algebra of E_K is $\mathfrak{g}(J_K)$. Moreover, the μ_3 that is the center of SU_3 is identified with the diagonal μ_3 in H'_K. Because H'_K is connected and in E_K, this μ_3 is in E_K. Hence E_K is connected, has Lie algebra $\mathfrak{g}(J_K)$ and contains μ_3 at its center, which proves that $E_K \simeq E_6^{sc}$. □

Denote by $\theta^{(4)}$ the automorphic function that is the pull-back to E_K via the embedding $E_K \to G_J$. The following result proves Corollary 1.0.2 (2).

Proposition 2.4.2. *The automorphic function $\theta^{(4)}$ is a modular form on E_6 of weight 4. It has nonzero rank four Fourier coefficients. However, $\theta^{(4)}$ is distinguished in the following sense: if $a(\omega)$ denotes the Fourier coefficient associated with $\omega \in W_{J_K}$ and ω is rank four, then $a(\omega) \neq 0$ implies that $q(\omega) = \kappa^2$ for some $\kappa \in K^\times$ with $\kappa^* = -\kappa$.*

[1] At this point, we have only understood the automorphic form θ_{Gan} when $J = H_3(\Theta)$ because in the computations above and below, we used that θ_{Gan} was spherical at every finite place.

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Note that the converse to Proposition 2.4.2 is false: if $q(\omega) = \kappa^2$ for some $\kappa \in K^\times$ with $\kappa^* = -\kappa$, then $a(\omega)$ might still vanish. For example, this occurs if ω is not sufficiently integral.

Because the quaternionic adjoint group is connected [20, Table II], there is literally no difference between the Archimedean theory for the simply connected quaternionic E_6 and the adjoint form. Thus, the definitions and results of [18] – which were proved in the adjoint case – apply immediately to the group E_K.

Proof of Proposition 2.4.2. To see that $\theta^{(4)}$ is a modular form of weight 4, again it suffices to check that $\mathcal{D}_4\theta^{(4)} = 0$, which may be done by, e.g., applying the results of [18, § 7]. Alternatively, one can apply results of [15]. The Fourier coefficients of $\theta^{(4)}$ are controlled by [17, Theorem 7.3.1], which gives the result.

More precisely, suppose $\omega \in W_{J_K}$, and denote by $a(\omega)$ the Fourier coefficient of $\theta^{(4)}$ associated with ω. As explained in [17, § 7], if $\omega \in W_{J_K}$ and $\eta \in B^2$, then $\omega + \eta$ can be regarded as an element of W_J, by applying the second Tits construction. Then for $\omega \neq 0$,

$$a(\omega) = \sum_{\eta \in B^2} a_{\theta_{Gan}}(\omega + \eta).$$

Again, the sum has only finitely many nonzero terms. It follows immediately from [17, Theorem 7.3.1 part (1)] that for $a(\omega)$ to be nonzero and ω rank four, we need $q(\omega) = \kappa^2$ for some $\kappa \in K^\times$ with $\kappa^* = -\kappa$.

To see that $\theta^{(4)}$ has nonzero rank four Fourier coefficients for our particular choice $\lambda = 1$, $S = 1$, one may proceed as follows. Suppose $K = \mathbf{Q}(\kappa)$ with $\kappa^* = -\kappa$. Then set $\eta = \langle 1, -\frac{\kappa}{2}\rangle$ and $\omega = \left(0, 1, 0, \frac{\kappa^2}{4}\right)$. Then

$$\omega + \eta = \left(0, (1, 1), \left(0, \frac{\kappa}{2}\right), \frac{\kappa^2}{4}\right)$$

is rank one in W_J. As ω is rank 4, this completes the proof. □

2.5. The integral modular form on G_2

Recall from above that ϵ denotes the automorphic function on F_4^{an} that takes on just two values and is orthogonal to the constant function [5, 6]. More precisely, as mentioned above and following [1, 5, 6], one has

$$\#F_4^{an}(\mathbf{Q})\backslash F_4^{an}(\mathbf{A})/F_4^{an}(\widehat{\mathbf{Z}})F_4^{an}(\mathbf{R}) = 2,$$

where F_4^{an} is the stabilizer of the element $I = 1_3 \in H_3(\Theta_0)$. The two double cosets are denoted by U_I and U_E. Here

$$E = \begin{pmatrix} 2 & \beta & \beta^* \\ \beta^* & 2 & \beta \\ \beta & \beta^* & 2 \end{pmatrix}$$

with $\beta = \frac{1}{2}(-1 + e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7) \in \Theta_0$. See, e.g., [1]. The set U_I has measure $\frac{91}{691}$ and the set U_E has measure $\frac{600}{691}$ [5]. The function ϵ takes the value $\frac{691}{91}$ on U_I and value $-\frac{691}{600}$ on U_E.

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Denote by Δ Ramanujan's elliptic modular cusp form of weight 12, and recall that we set

$$F_\Delta(g) = \int_{F_4^{an}(\mathbf{Q})\backslash F_4^{an}(\mathbf{A})} \theta_{Gan}((g,h))\epsilon(h)\,dh,$$

the θ-lift of ϵ to G_2. The rank four Fourier coefficients of F_Δ are discussed in [6], and it is explained there that F_Δ is a level one, weight 4 modular form on G_2. The constant term is essentially in [1, 5]. Thus much of the following result is contained in [6] and [5].

To state the Fourier expansion of F_Δ, we make some notations. Suppose $\omega_0 \in W_F = Sym^3(V_2)$. As in [3], define

$$\Omega_I(\omega_0) = \left\{(a,b,c,d) \in W_J^{rk=1} = (F \oplus J \oplus J^\vee \oplus F)^{rk=1} : \left(a, \frac{(b, I^\#)}{3}, \frac{(c, I)}{3}, d\right) = \omega_0\right\}$$

and similarly define

$$\Omega_E(\omega_0) = \left\{(a,b,c,d) \in W_J^{rk=1} = (F \oplus J \oplus J^\vee \oplus F)^{rk=1} : \left(a, \frac{(b, E^\#)}{3}, \frac{(c, E)}{3}, d\right) = \omega_0\right\}.$$

(In our normalization, ω_0 is integral if it has coefficients in $\mathbf{Z} \oplus \frac{\mathbf{Z}}{3} \oplus \frac{\mathbf{Z}}{3} \oplus \mathbf{Z}$.)

The following is an immediate corollary of Theorem 1.0.1 and what has been said above.

Corollary 2.5.1. *The weight 4, level one modular form F_Δ on G_2 is nonzero and has rational Fourier coefficients with bounded denominators. If θ_{Gan} is normalized to have $a_\theta((1,0,0,0)) = 1$, then the constant term of F_Δ is*

$$F_{\Delta,00}(g) = 24|\nu(g)|^5 \left(\Phi_\Delta(g)\frac{x^8}{8!} + \Phi'_\Delta(g)\frac{y^8}{8!}\right).$$

Here Φ_Δ is the automorphic form on GL_2 with[2] *$j(g,i)^4\Phi_\Delta(g)$ equal to Ramanujan's Δ function. Moreover, for ω_0 as above,*

$$a_{F_\Delta}(\omega_0) = \left(\sum_{\omega \in \Omega_I(\omega_0)} \sigma_4(\Delta(\omega))\right) - \left(\sum_{\omega \in \Omega_E(\omega_0)} \sigma_4(\Delta(\omega))\right).$$

Proof. The key point is that the $\zeta(5)$ term drops out because ϵ is orthogonal to the constant functions. Everything else follows immediately from what has been said and the fact [1] that

$$\int_{F_4^{an}(\mathbf{Q})\backslash F_4^{an}(\mathbf{A})} \Phi_{Kim}((g,h))\epsilon(h)\,dh = 3\Phi_\Delta(g).$$

□

[2]Here, we are using the definition of the automorphy factor $j(g,i)$ from [18]. This automorphy factor is essentially the cube of the usual automorphy factor on GL_2.

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In [3, Proposition 12.3], the Fourier coefficients $a_{F_\Delta}(\omega_0)$ for ω_0 with $\Delta(\omega_0) = 1$ were expressed in terms of the number of embeddings of a cubic ring corresponding to ω_0 into integral cubic norm structures. See *loc. cit.* Proposition 6.9 for the relation between $\#\Omega_I(\omega_0)$, $\#\Omega_E(\omega_0)$ and the number of these embeddings.

The automorphic function F_Δ and the Eisenstein series $E^{G_2}(g, s = 5; 4)$ are modular forms of weight four. With appropriated normalizations, their Fourier coefficients are integers that are congruent modulo 691. That this congruence holds for the nondegenerate coefficients is immediate from, e.g., [6, Proposition 10.1] and the Siegel–Weil theorem of [3]. That the congruence continues to hold for the degenerate coefficients is now a triviality, again using [3].

Corollary 2.5.2. *Denote by $E_4(g)$ the scalar multiple of the Eisenstein series $E^{G_2}(g, s = 5; 4)$ with Fourier coefficient $a_{E_4}((1, 0, 0, 0)) = 691$. Then $91F_\Delta \equiv E_4$ modulo 691, in the sense that all the nontrivial Fourier coefficients of F_Δ and E_4 are integers, and $91a_{F_\Delta}(\omega_0) \equiv a_{E_4}(\omega_0) \pmod{691}$ for every $\omega_0 \in \mathbf{Z} \oplus \frac{\mathbf{Z}}{3} \oplus \frac{\mathbf{Z}}{3} \oplus \mathbf{Z}$. Moreover, this congruence is not trivial in the sense that not all the Fourier coefficients of F_Δ (or E_4) are divisible by 691.*

Proof. Let $\mathbf{1}_{U_I}$ denote the characteristic function of U_I in $F_4(\mathbf{A})$ and similarly let $\mathbf{1}_{U_E}$ denote the characteristic function of U_E. Define

$$F_I(g) = \int_{F_4^{an}(\mathbf{Q})\backslash F_4^{an}(\mathbf{A})} \theta_{Gan}((g, h))\mathbf{1}_{U_I}(h)\, dh$$

and similarly define $F_U(g)$. Then F_I and F_U are modular forms of weight 4 on G_2. To compare with [6], set $\theta_I = \frac{691}{91}F_I$ and $\theta_E = \frac{691}{600}F_E$, and these modular forms have integer Fourier coefficients, with $\theta_I((1, 0, 0, 0)) = \theta_E((1, 0, 0, 0)) = 1$. This last fact follows from Lemma 4.2.2 at the end of the paper.

The modular form $F_\Delta = \theta_I - \theta_E$. By Gan's Siegel–Weil theorem [3], $E_4(g) = 91\theta_I + 600\theta_E$. Hence $91F_\Delta$ has all of its Fourier coefficients integers, and congruent modulo 691 to those of the Eisenstein series $E_4(g)$.

To see that the congruence is not trivial, i.e., that the Fourier coefficients of F_Δ are not all divisible by 691, we compute that the Fourier coefficient $a_{F_\Delta}((0, 1, -1, 0)) = 6$.

To see this, first note that one has

$$\Omega_I((0, 1, -1, 0)) = \{(0, X, Y, 0) \in W_J^{rk=1} : (X, I^\#) = 1, (Y, I) = 1\}$$

and similarly for $\Omega_E((0, 1, -1, 0))$. Now, for $X \in J$ and $Y \in J^\vee$, one has that $(0, X, Y, 0)$ is rank at most one if and only if $X^\# = 0$, $Y^\# = 0$, $(X, Y) = 0$, and $\Phi_{Y,X} = 0$, where $\Phi_{Y,X} : J \to J$ is the linear map

$$\Phi_{Y,X}(z) = -Y \times (X \times z) + (Y, z)X + (Y, X)z.$$

If $X \neq 0$, then these conditions are equivalent to $X^\# = 0$, $Y \in X \times J$ and $Y^\# = 0$.

By [1, Proposition 5.5], it follows that $\Omega_E((0, 1, -1, 0))$ is empty. Recall the element $e_{11} = \mathrm{diag}(1, 0, 0) \in J$, and set $e_{22} = \mathrm{diag}(0, 1, 0)$, $e_{33} = \mathrm{diag}(0, 0, 1)$. Again, by [1, Proposition 5.5], one now obtains that $\Omega_I((0, 1, -1, 0))$ consists of the six elements $(0, e_{ii}, -e_{jj}, 0)$ with $i \neq j$. This completes the proof. □

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3. Fourier expansion of the Heisenberg Eisenstein series

In this section, we prove results about the degenerate Heisenberg Eisenstein series on G_J. In the first subsection, we give an abstract discussion of its Fourier expansion, not utilizing any of the Archimedean results of [18]. In the second subsection, we analyze the Fourier expansions of the special values of $E_J(g, s, \Phi; n)$ at $s = n+1$ when this point is in the range of absolute convergence for the Eisenstein series. It is proved that the Fourier coefficients are Euler products. Finally, in the third and fourth subsections, we analyze the constant term and rank one Fourier coefficients of these Eisenstein series directly.

3.1. Abstract Fourier expansion

In this subsection, we give the 'abstract' Fourier expansion of a Heisenberg Eisenstein series $E(g, s)$. Thus, we assume that $E(g, s) = \sum_{\gamma \in P(F)\backslash G(F)} f(\gamma g, s)$ for $f(g, s)$ a section in $Ind_P^G(|\nu|^s)$, but we do not assume anything special about this section.

Lemma 3.1.1. *Suppose that $J \neq F$, so that $G = G_J$ is not G_2. Then $P(F)\backslash G(F)/P(F)$ has five elements, represented by $\{w_0 = 1, w_1, w_2, w_3, w_4\}$ with $w_1(E_{13}) \in e \otimes W_J$, $w_2(E_{13}) \in \mathfrak{h}(J)^0$, $w_3(E_{13}) \in f \otimes W_J$ and $w_4(E_{13}) = E_{31}$. The Lie algebra elements $w_1(E_{13})$ and $w_3(E_{13})$ are rank one when considered in W_J, and $w_2(E_{13})$ in the minimal orbit in $\mathfrak{h}(J)^0$. If $J = F$ so that $G_J = G_2$, then the double coset $P(F)\backslash G(F)/P(F)$ has four elements, and is represented by $\{w_0 = 1, w_1, w_3, w_4\}$, with these w_i as above.*

Proof. Recall from [18, § 4.1.2] the nondegenerate G_J-invariant symmetric form $B_{\mathfrak{g}}$ on the Lie algebra $\mathfrak{g}(J)$, proportional to the Killing form. Say an element X – or the line spanned by X – in the Lie algebra $\mathfrak{g}(J)$ is *minimal* if $[X, [X, y]] + 2B_{\mathfrak{g}}(X, y)X = 0$ for all $y \in \mathfrak{g}(J)$. The orbit $G(F)E_{13}$ consists of the nonzero elements $X \in \mathfrak{g}_J(F)$ spanning a minimal line; see, e.g., [18, § 4.3.2]. The double coset $P(F)\backslash G(F)/P(F)$ is thus identified with the $P(F)$-orbits on the minimal lines in $\mathfrak{g}_J$.

Denote by h the element $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{sl}_2 \subseteq \mathfrak{g}(J)$. By the Bruhat decomposition, the coset representatives w_i for $P(F)\backslash G(F)/P(F)$ can be chosen to be normalizers of a maximal torus; thus we may assume that the vectors $w_i(E_{13})$ are eigenvectors for the one-parameter subgroup with Lie algebra spanned by h. Consequently, we may assume that the elements $w_i(E_{13})$ only have one nonzero component of the 5-grading on $\mathfrak{g}(J)$ determined by the Levi subgroup H_J of P; see [18, § 4.3.1]. In particular, representatives can taken to be in FE_{13}, $e \otimes W_J$, $Fh \oplus \mathfrak{h}(J)^0$, $f \otimes W_J$, and FE_{31}. Here recall from § 1.1 that e, f are a symplectic basis of the V_2 in (3).

It is easy to see that there exist w_1, w_3, and w_4 as in the statement of the lemma, and that furthermore that there is one $M(F)$-orbit of such w_i's. Thus, there are at least four $P(F)$-orbits on the minimal lines in $\mathfrak{g}_J(F)$ in all cases. When $J = F$ so that $G_J = G_2$, these are the only orbits. One can verify this by hand – it is because there are no rank two elements in $W_{J=F}$ – or see, e.g., [11, Equation (9)].

Now suppose $J = H_3(C)$, so that G is not G_2. Suppose $X \in Fh + \mathfrak{h}(J)^0$ spans a minimal line. We claim that $X \in \mathfrak{h}(J)^0$. Indeed, write $X = \mu h + \phi \in Fh \oplus \mathfrak{h}(J)^0$. By taking $y = E_{13}$ in $[X, [X, y]] + 2B_g(X, y)X = 0$, one sees that $\mu = 0$. Thus $X \in \mathfrak{h}(J)^0$ as claimed. The elements so obtained in $\mathfrak{h}(J)^0$ are the minimal elements of this Lie algebra, i.e., are in

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the orbit of the highest root. In particular, the group $H_J(F)$ acts transitively on them. This completes the proof of the lemma. □

Via the lemma, we have $E(g,s) = \sum_{i=0}^{4} E_i(g,s)$, with $E_i(g,s) = \sum_{\gamma \in P(F)\backslash P(F)w_i P(F)} f(\gamma g, s)$. Thus, $E_0(g,s) = f(g)$. For $G = G_2$, we understand $E_2(g,s) = 0$. We now write out more explicit expressions for the other E_i.

Recall that if $\ell \subseteq W_J$ is a rank one line, there is associated with it a flag

$$W_J \supseteq (\ell)^{\perp} \supseteq W(\ell) \supseteq \ell \supseteq 0$$

with $W(\ell)$ a certain maximal isotropic subspace. Precisely,

$$W(\ell) = \{x \in W_J : \langle x, \mu \rangle = 0 \text{ and } \Phi_{x,\mu} = 0 \text{ for all } \mu \in \ell\}.$$

Here, recall $\Phi_{\bullet,\bullet} : Sym^2(W_J) \to \mathfrak{h}(J)^0$ is the $H(J)$-equivariant map described in § 1.1.

Lemma 3.1.2. *Assume that $Re(s) \gg 0$ so that the sum defining $E(g,s)$ converges absolutely. Then one has the following expressions for the $E_i(g,s)$.*

(1) *For each rank one line ℓ in $e \otimes W_J$, select $\gamma(\ell) \in G(F)$ with $\gamma(\ell)E_{13} \in \ell$. Then*

$$E_1(g,s) = \sum_{\ell \subseteq W_J^{rk=1}} \sum_{\mu \in (\ell)^{\perp} N_0(F)\backslash N(F)} f(\gamma(\ell)^{-1}\mu g, s).$$

(2) *For each minimal line $F\phi \subseteq \mathfrak{h}(J)^0$, select $\gamma(\phi) \in G(F)$ with $\gamma(\phi)E_{13} \in F\phi$. Then*

$$E_2(g,s) = \sum_{F\phi \subseteq \mathfrak{h}(J)^0 \ minimal} \sum_{\mu \in (\ker(\phi) N_0(F))\backslash N(F)} f(\gamma(\phi)^{-1}\mu g, s).$$

(3) *For each minimal line $F\ell \in f \otimes W_J$, select $\gamma(\ell) \in G(F)$ with $\gamma(\ell)E_{13} \in \ell$. Then*

$$E_3(g,s) = \sum_{\ell \subseteq W_J^{rk=1}} \sum_{\mu \in W(\ell)\backslash N(F)} f(\gamma(\ell)^{-1}\mu g, s).$$

(4) *One has*

$$E_4(g,s) = \sum_{\mu \in N(F)} f(w_4^{-1}\mu g, s).$$

Proof. The expression for $E_4(g,s)$ is clear. The rest follows easily from what has already been said. The only thing that must still be computed is the stabilizers in $N(F)$ of the minimal lines in $e \otimes W_J$, $\mathfrak{h}(J)^0$, and $f \otimes W_J$. And for this, it suffices to work on the level of Lie algebras. We write $n = e \otimes x + cE_{13}$ for a typical element of the Lie algebra of $N(F)$.

We separate into cases. For $E_1(g,s)$, write a typical rank one element of $e \otimes W_J$ as $e \otimes v$. Then $[n, e \otimes v] = \langle x, v \rangle E_{13}$, thus verifying the expression for $E_1(g,s)$. For $E_2(g,s)$, suppose that $\phi \in \mathfrak{h}(J)^0$ spans a minimal line. Then $[n, \phi] = [e \otimes x, \phi] = e \otimes \phi(x)$. This gives the stated expression for $E_2(g,s)$. Finally, suppose $f \otimes v \in f \otimes W_J$ is a rank one element. Then $[n, f \otimes v] = ce \otimes v + \left(\langle x, v \rangle \frac{ef}{2} + \frac{1}{2}\Phi_{x,v}\right)$. Thus $[n, f \otimes v] = 0$ if and only if $c = 0$ and $x \in W(\ell)$, where $\ell = Fv$. This completes the proof of the lemma. □

We now consider the Fourier expansions of the $E_i(g, s)$ along $N(F)$; because the $E_i(g, s)$ are $P(F)$-invariant, this makes sense. Fix an additive character $\psi : F\backslash\mathbf{A} \to \mathbf{C}^\times$. For $v \in W_J$, define $\chi_v : N(F)\backslash N(\mathbf{A}) \to \mathbf{C}^\times$ as $\chi_v(n) = \psi(\langle v, \overline{n}\rangle)$, where $\overline{n}$ denotes the image of $n \in N/[N, N] \simeq W_J$. We set

$$E_i^v(g, s) = \int_{N(F)\backslash N(\mathbf{A})} \chi_v^{-1}(n) E_i(ng, s)\, dn.$$

Our measure is normalized so that if U is a closed algebraic subgroup of N, then $[U] := U(F)\backslash U(\mathbf{A})$ has volume 1.

Recall that elements of W_J have a *rank*, which is 0, 1, 2, 3 or 4.

Lemma 3.1.3. *If* $\mathrm{rank}(v) > i$, *then* $E_i^v(g, s) = 0$.

Proof. Let us write

$$E_i(g, s) = \sum_{\ell} \sum_{\mu \in N_\ell(F)\backslash N(F)} f(\gamma(\ell)^{-1}\mu g, s) \tag{4}$$

for the expression given in Lemma 3.1.2. Then

$$\begin{aligned} E_i^v(g, s) &= \sum_{\ell} \int_{[N]} \chi_v(n)^{-1} \left(\sum_{\mu \in N_\ell(F)\backslash N(F)} f(\gamma(\ell)^{-1}\mu n g, s) \right) dn \\ &= \sum_{\ell} \int_{N(\ell)(F)\backslash N(\mathbf{A})} \chi_v^{-1}(n) f(\gamma(\ell)^{-1} n g, s)\, dn \\ &= \sum_{\ell} \int_{N(\ell)(\mathbf{A})\backslash N(\mathbf{A})} \left(\int_{[N(\ell)]} \chi_v^{-1}(r)\, dr \right) \chi_v^{-1}(n) f(\gamma(\ell)^{-1} n g, s)\, dn \\ &= \sum_{\ell, \chi_v|_{N(\ell)} = 1} \int_{N(\ell)(\mathbf{A})\backslash N(\mathbf{A})} \chi_v^{-1}(n) f(\gamma(\ell)^{-1} n g, s)\, dn. \end{aligned}$$

It follows that $E_i^v(g, s)$ vanishes if χ_v restricted to $N(\ell)$ is nontrivial for all ℓ appearing in sum (4). If $\mathrm{rank}(v) > i$, then it is not hard to see that this indeed happens. We consider the different cases:

(1) First, assume $i = 1$. Then $N(\ell) = (\ell)^\perp N_0 \subseteq N$ for ℓ a rank one line in W_J. Thus $\chi_v|_{N(\ell)} = 1$ implies $v \in \ell$, so $\mathrm{rank}(v) = 0$ or 1.

(2) Suppose next that $i = 2$. Then $N(\ell) = \ker(\phi) N_0 \subseteq N$ for some $\phi \in \mathfrak{h}(J)^0$ in the orbit of the highest root. Then $\chi_v|_{N(\ell)} = 1$ implies $v \in I(\phi) := \ker(\phi)^\perp \subseteq W_J$. Denote by $e_{11} \in J$ the element $e_{11} = \begin{pmatrix} 1 & & \\ & 0 & \\ & & 0 \end{pmatrix}$. Without loss of generality, we can assume $\phi = n_L(e_{11})$, in which case $I(\phi)$ is easily computed to be $(0, Fe_{11}, e_{11} \times J, F)$. If $v \in (0, Fe_{11}, e_{11} \times J, F)$, then indeed $\mathrm{rank}(v) = 0, 1, 2$, as desired.

(3) Finally, assume $i = 3$. Then $N(\ell) = W(\ell) \subseteq N$ for some rank one line $\ell \subseteq W_J$. One has $\chi_v|_{N(\ell)} = 1$ implies $v \in W(\ell)^\perp = W(\ell)$. Without loss of generality, one can assume $\ell = (0, 0, 0, F) \subseteq W_J$, in which case $W(\ell) = (0, 0, J^\vee, F)$ and $v \in W(\ell)$ implies $\mathrm{rank}(v) \leqslant 3$, as desired. $\square$

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Lemma 3.1.4. *If $i = \mathrm{rank}(v)$, then $E_i^v(g,s)$ is Eulerian. More precisely, we have the following:*

(1) *Suppose v is rank one. Define $N_v^1 = (Fv)^{\perp} N_0 \subseteq N$. Then*

$$E_1^v(g,s) = \int_{N_v^1(\mathbf{A})\backslash N(\mathbf{A})} \chi_v^{-1}(n) f(\gamma(\ell_v)^{-1} ng, s)\, dn.$$

(2) *Suppose v is rank two. Recall the element $\Phi_{v,v} \in \mathfrak{h}(J)^0$, and write $\ker(\Phi_{v,v})$ for its kernel when considered as an endomorphism of $e \otimes W_J$. Define $N_v^2 = \ker(\Phi_{v,v}) N_0 \subseteq N$, and take $\gamma(\Phi_{v,v}) \in G_J(F)$ with $\gamma(\Phi_{v,v}) E_{13} = \Phi_{v,v} \in \mathfrak{h}(J)^0$. Then*

$$\int_{(N_v^2)(\mathbf{A})\backslash N(\mathbf{A})} f(\gamma(\phi)^{-1} ng, s) \chi_v^{-1}(n)\, dn.$$

(3) *Suppose v is rank three. Recall the element $v^\flat = t(v,v,v)$ (e.g., [18, § 3.4.2]), which is rank one because v is rank three. Define $N_v^3 = W(Fv^\flat) \subseteq N$, and take $\gamma(v^\flat)$ with $\gamma(v^\flat) E_{13} = f \otimes v^\flat$. Then*

$$E_3^v(g,s) = \int_{N_v^3(\mathbf{A})\backslash N(\mathbf{A})} \chi_v^{-1}(n) f(\gamma(v^\flat)^{-1} ng, s)\, dn.$$

(4) *Suppose v is rank four. Then*

$$E_4^v(g,s) = \int_{N(\mathbf{A})} \chi_v^{-1}(n) f(w_4^{-1} ng, s)\, dn.$$

Proof. The cases of v rank one and v rank four follow immediately from Lemma 3.1.3.

Consider first the case of v rank two. For $\phi \in \mathfrak{h}(J)^0$ spanning a minimal line, define $I(\phi) = \ker \phi^{\perp}$. Then χ_v is 1 on $\ker \phi(\mathbf{A})$ if and only if $v \in I(\phi)$. Thus we have

$$E_2^v(g,s) = \sum_{F\phi \text{ with } v \in I(\phi)} \int_{(\ker(\phi) N_0)(\mathbf{A})\backslash N(\mathbf{A})} f(\gamma(\phi)^{-1} ng, s) \chi_v^{-1}(n)\, dn.$$

If v is rank two, the only minimal line $F\phi$ with $v \in I(\phi)$ is $F\Phi_{v,v}$, so in this case $E_2^v(g,s)$ is Eulerian, as in the statement of the lemma.

Now consider the case of v rank three. One has that χ_v is 1 on $W(\ell)(\mathbf{A}) \subseteq N(\mathbf{A})$ if and only if $v \in W(\ell)^{\perp} = W(\ell)$. Thus

$$E_3^v(g,s) = \sum_{\ell \text{ rank one with } v \in W(\ell)} \int_{W(\ell)(\mathbf{A})\backslash N(\mathbf{A})} \chi_v^{-1}(n) f(\gamma(\ell)^{-1} ng, s)\, dn.$$

Thus, if v is rank three, then $F\ell = Fv^\flat$, so the line ℓ is determined by v and $E_3^v(g,s)$ is Eulerian as specified above. This completes the proof of the lemma. □

For the constant term $E_1^0(g,s)$, we record now that

$$E_1^0(g,s) = \sum_{\ell \subseteq W_J^{rk=1}} \int_{N_\ell^1(\mathbf{A})\backslash N(\mathbf{A})} f(\gamma(\ell)^{-1} ng, s)\, dn.$$

This function is as an Eisenstein series for the Levi subgroup H_J attached to the parabolic that stabilizes a rank one line in W_J.

3.2. Euler product

The purpose of this subsection is to prove that the special (modular form) values of the Heisenberg Eisenstein series have nonconstant Fourier coefficients that are Euler products. More precisely, for $v \in W_J$ set

$$E^v(g,s;n) = \int_{N(F)\backslash N(\mathbf{A})} \chi_v^{-1}(u) E(ug,s;n)\, du.$$

We know from the previous section that if $v \in W_J$ is nonzero, then $E^v(g,s;n) = \sum_{i \geqslant rk(v)} E_i^v(g,s;n)$, and that the term $E^v_{rk(v)}(g,s;n)$ is an Euler product. In this subsection, we prove that if the special value $s = n+1$ is in the range of absolute convergence for the Eisenstein series, then the terms $E_i^v(g,s,n)$ vanish at this point when $i > \mathrm{rank}(v)$.

Proposition 3.2.1. *Suppose n is even, and $n > \dim(J)+1$ so that the sum defining $E(g,s;n)$ converges absolutely at $s = n+1$ and defines a modular form of weight n at this point. Then if $v \neq 0$ and $i > \mathrm{rank}(v)$, $E_i^v(g,s;n)$ vanishes at $s = n+1$.*

We will prove Proposition 3.2.1 by applying the following corollary of the main result of [18].

Lemma 3.2.2. *Suppose $\omega \in W_J$ is nonzero, $m \in H_J(\mathbf{R})$, and F is a modular form on G_J of weight n. Denote by F_ω the ω Fourier coefficient of F, and set $\tilde{m} = \nu(m)m^{-1}$. Then if $\tilde{m}\omega = \omega$, $F_\omega(mg) = \nu(m)^n|\nu(m)|F_\omega(g)$. For the constant term F_{00}, one has the following: F_{00} is the sum of two terms f_1 and f_2, which are distinguished by the following properties. For $z \in Z_H \simeq \mathrm{GL}_1(\mathbf{R})$, one has $f_1(zg) = z^{2n+2}f_1(g)$ and $f_2(zg) = z^{n+2}f_2(g)$.*

Remark 3.2.3. We have derived Lemma 3.2.2 as a consequence of the complete formula for the function $\mathcal{W}_\omega(g)$, which is [18, Theorem 1.2.1] or see equations (1) and (2). However, the lemma is useful by itself. Can it be proved directly, by a 'softer' method?

The idea of the proof of Proposition 3.2.1 is to show that $E_i^v(g,s)$ is an absolutely convergent sum of terms, each of which vanishes at $s = n+1$ because it does not satisfy the required equivariance property enforced by Corollary 3.2.2. We now make some preparatory remarks that we will use to carry out this strategy.

For $\omega \in W_J$, recall that we denote $\chi_\omega(n) = \psi(\langle \omega, n\rangle)$. Suppose we have an absolutely convergent integral

$$f^\omega(g,s) = \int_{Y(\mathbf{R})} f(w^{-1}yg,s)\chi_\omega(y)^{-1}\, dy,$$

and $m \in H_J$ with $\widetilde{m}\omega = \omega$, and m preserves Y. Here $Y \subseteq N$. Furthermore, assume $f(pg,s) = |\nu(p)|^s f(g,s)$ for $p \in P$ the Heisenberg parabolic, and that $w^{-1}mw \in P$. Then we have

$$f^\omega(mg,s) = J(m,Y)|\nu(w^{-1}mw)|^s f^\omega(g,s),$$

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where

$$J(m, Y) = \left| \frac{d(mym^{-1})}{dy} \right|_Y$$

is the measure term that comes from a change of variable in the integral.

For a rank one line $\ell \subseteq \mathfrak{g}(J)$, and $g \in G_J$ stabilizing ℓ, define $\mu_\ell(g) \in \mathrm{GL}_1$ to be the element with $g\ell = \mu_\ell(g)\ell$. Then $\nu(w^{-1}mw) = \mu_{wE_{13}}(m)$. Thus, with assumptions and notations as above, we have

$$f^\omega(mg, s) = J(m, Y)|\mu_{wE_{13}}(m)|^s f^\omega(g, s).$$

We will now analyze term $C(m, Y, s) := J(m, Y)|\mu_{wE_{13}}(m)|^s$ for the integrals that appear in the previous subsection. We first consider the constant term, i.e., $\omega = 0$.

(1) The case $i = 0$. We have $f_0(zg, s) = |\nu(z)|^s f_0(g, s) = |z|^{2s} f_0(g, s)$. At $s = n + 1$, this character is z^{2n+2}.

(2) The case $i = 1$. Fix $\ell \in e \otimes W_J$ rank one, and set $Y_1 = N_\ell^1 \backslash N$. We have $J(z, Y_1) = |z|$. Because $\gamma(\ell)E_{13} \in e \otimes W_J$, $\mu_{\gamma(\ell)E_{13}}(z) = z$. Thus, $C(z, Y_1, s) = |z|^{s+1}$, which gives z^{n+2} at $s = n + 1$.

(3) The case $i = 2$. Fix $\phi \in h(J)^0$ spanning a minimal line, and set $Y_2 = \ker(\phi)N_0 \backslash N$. Then $J(z, Y_2) = |z|^{4+dim(C)}$, where recall $J = H_3(C)$. In this case, $\gamma(\phi)E_{13} \in \mathfrak{h}(J)^0$, so $\mu_w(z) = 1$. Thus $C(z, Y_2^0, s) = |z|^{4+\dim(C)}$, independent of s.

(4) The case $i = 3$. Fix $\ell \subseteq f \otimes W_J$ a rank one line, and set $Y_3 = W(\ell)\backslash N$. In this case, $J(z, Y_3) = |z|^{\dim(J)+3}$. (Note that z scales the measure on N_0 by $|z|^2$.) Furthermore, $\gamma(\ell)E_{13} \in f \otimes W_J$, and thus $\mu_{wE13}(z) = z^{-1}$. Hence $C(z, Y_3, s) = |z|^{\dim(J)+3-s}$, which specializes to $|z|^{\dim(J)+2-n}$ at $s = n + 1$.

(5) The case $i = 4$. Set $Y_4 = N$. Then $J(z, Y_4) = |z|^{\dim W_J + 2} = |z|^{2\dim(J)+4}$. We have $\mu_{w_4 E_{13}}(z) = z^{-2}$; thus $C(z, Y_4, s) = |z|^{2\dim(J)+4-2s}$, which at $s = n + 1$ is $|z|^{2\dim(J)+2-2n}$.

We obtain the following lemma. Recall our running assumption that $n > 0$ is even.

Lemma 3.2.4. *Suppose that the meromorphic continuation of $E(g, s, n)$ to $s = n + 1$ is regular and defines a modular form of weight n at this point. Suppose moreover that n and J are such that the numbers $2n + 2, n + 2, 4 + \dim(C), \dim(J) + 2 - n$, and $2\dim(J) + 2 - 2n$ are pairwise distinct. If $G = G_2$, then exclude the term $4 + \dim(C)$. Then only $E_0(g, s)$ and $E_1(g, s)$ contribute to the constant term of $E(g, s, n)$ at $s = n + 1$.*

Note that in Lemma 3.2.4, we do not need to assume that the Eisenstein series $E(g, s, n)$ converges absolutely at $s = n + 1$.

Proof of Lemma 3.2.4. If $E(g, s, n)$ is regular at $s = n + 1$, then so is its constant term. Each $E_i(g, s)$ contributes to the constant term. Now, under the assumptions of the lemma, the constant terms of the $E_i(g, s)$ transform by different characters under left translation $g \mapsto zg$ for $z \in Z_H(\mathbf{R})$. It follows the constant terms of the $E_i(g, s)$ are each individually regular at $s = n + 1$. Because they are regular at $s = n + 1$, the values of these constant terms at $s = n + 1$ must transform by the characters $\{2n + 2, n + 2, 4 +$

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$\dim(C), \dim(J)+2-n, 2\dim(J)+2-2n\}$. However, the constant term of a modular form can only transform by z^{2n+2} and z^{n+2}. Again, because these numbers are assumed all distinct, only $E_0^0(g,s)$ and $E_1^0(g,s)$ contribute; the other $E_i^0(g,s)$ for $i=2,3,4$ must vanish at $s=n+1$. □

We consider next the nonconstant Fourier coefficients. For the nonconstant terms, we will need to assume that the sum defining the Eisenstein series converges absolutely. Let the Y_i be as above.

We will use the action of a maximal $\mathbf{Q}$-split torus of the groups $H_J \subseteq G_J$. For convenience, we specify a particular choice of torus T when $J = H_3(C)$. We parametrize the elements of T as ordered four-tuples $t = (\lambda, t_1, t_2, t_3)$. In these coordinates, set $\delta(t) := t_1 t_2 t_3$. Then T acts on W_J as

$$t(a,b,c,d) = (\lambda^{-1}\delta^{-1}(t)a, \delta^{-1}(t)(t\cdot b), \lambda\delta(t)(t^{-1}\cdot c), \lambda^2\delta(t)d),$$

where $t \cdot X = \operatorname{diag}(t) X \operatorname{diag}(t)$ for $X \in H_3(C)$. Moreover, one has $\nu(t) = \lambda$ and $Ad(t)n_L(X) = \lambda n_L(t\cdot X)$.

We set $T^1 \subseteq T$ the subtorus consisting of elements with $\lambda = 1$. Additionally, define $T' \subseteq T$ the subtorus consisting of those elements where $t_1 = t_2 = t_3$; T' is a maximal $\mathbf{Q}$-split torus of G_2. Then when $G = G_2$, $T' \subseteq P \subseteq G_2$ and T' acts on W_J by the same formula for T just given. We are now ready to prove Proposition 3.2.1.

Proof of Proposition 3.2.1. Because of the assumption that the Eisenstein series converges absolutely, we may analyze the terms $E_i^{\omega}(g,s)$ separately for each i. Assume $\omega \neq 0$.

The case $i=4$: Consider first the case $i=4$. Suppose $m \in H_J$. Then $J(m, Y_4) = |\nu(m)|^{\dim J+2}$, and $\mu_{w_4 E_{13}}(m) = \nu(m)^{-1}$. Thus $C(m,Y,s) = |\nu(m)|^{\dim J+2-s}$, which specializes to $\dim(J)+1-n$ at $s=n+1$. Note that, if ω rank four and $\widetilde{m}\omega = \omega$, then necessarily $|\nu(m)| = 1$. By contrast, if ω is rank 1, 2 or 3, there exists $m \in M$ with $\widetilde{m}\omega = \omega$ but $|\nu(m)| \neq 1$. Indeed, if ω is rank two or three, then ω has an $H_J(F)$-translate ω' of the form $(0,*,0,0)$, and if ω is rank one, then ω has a translate ω' of the form $(1,0,0,0)$. In the first case, ω' is fixed by $t = (\lambda,(1,1,1))$, while in the second, ω' is fixed by $(\lambda^3,(\lambda^{-1},\lambda^{-1},\lambda^{-1}))$. Thus, so long as $\dim J+1-n \neq n+1$, the term $E_4^{\omega}(g,s)$ will vanish for ω not rank 4. However, the condition $\dim(J) = 2n$ is never satisfied for an even n, as the allowable $\dim(J)$'s are $\{1,6,9,15,27\}$.

The case $i=3$: Now consider the case $i=3$. Recall that we are analyzing

$$E_3^{\omega}(g,s) = \sum_{\ell:\omega\in W(\ell)} \int_{W(\ell)(\mathbf{A})\backslash N(\mathbf{A})} \chi_{\omega}^{-1}(n) f(\gamma(\ell)^{-1}ng,s)\,dn,$$

where $\gamma(\ell)E_{13}$ spans $\ell \subseteq f \otimes W_J$. We are interested in seeing if this vanishes at $s=n+1$ for ω of rank one or two.

Because of the absolute convergence, we may consider each term separately, and then without loss of generality we may assume that ℓ is spanned by $f \otimes (0,0,0,1) = E_{21}$, so that $W(\ell) = (0,0,*,*)$. Recall that the condition on ω is $\omega \in W(\ell)$. If $G = G_2$, then necessarily ω is rank one and in $(0,0,0,*)$. If G is not G_2, then without loss of generality we may assume that $\omega = (0,0,\omega_2,0)$, with $\omega_2 \in e_{11} \times J \subseteq J^{\vee}$, i.e., ω_2 has (i,j) coefficient equal to 0 if either i or j equals 1. We set $Y_3 = W(\ell)\backslash N$ as above.

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Suppose first that $G = G_2$, and $t' = (\lambda, (t, t, t)) \in T'$. Then with notation as above, $J(t', Y_2) = |t|^{-4}$. Furthermore, one has $\mu_{E_{21}}(t') = \lambda t^3$. Thus, $C(t', Y_3, s) = |\lambda t^3|^s |t|^{-4} = |\lambda t^3|^{n+1}|t|^{-4}$ at $s = n+1$. Now, $\widetilde{t'}(0,0,0,1) = \lambda^{-1}t^{-3}(0,0,0,1)$, and thus the condition $\widetilde{t'}\omega = \omega$ is $\lambda t^3 = 1$. Thus for such t', $C(t', Y_3, s = n+1) = |\nu(t')|^{n+1}$ if and only if $|t|^{-4} = |\lambda|^{n+1} = |t|^{-3n-3}$. But $4 \neq 3n+3$, and we can find t' with $\widetilde{t'}\omega = \omega$ and $|t| \neq 1$. Thus, in the case $G = G_2$, this term vanishes.

Now suppose that $G \neq G_2$. Consider the action of the subgroup of T^1 with $t_1 = 1$ and $t_2 = t_3$. Then this group fixes ω. One computes that $J(t, Y_3) = |\delta|^{-1-\dim J/3}$. The term $\mu_\ell(t) = \delta$; thus $C(t, Y_3, s) = |\delta|^{s-1-\dim J/3}$. Because $\nu(t) = 1$ but the elements $(1, t, t) \in T$ have $\delta \neq 1$, this only transforms the right way at $s = n+1$ if $3n = \dim(J)$. But $n > 2$ is even, so this cannot occur for $\dim(J) \in \{6, 9, 15, 27\}$. This completes the proof of the vanishing for $i = 3$.

The case $i = 2$: We are now left with the case of $i = 2$. This case cannot occur when $G = G_2$, so we assume $J = H_3(C)$ so that $G \neq G_2$. Recall that we are interested in

$$E_2^{\omega}(g, s) = \sum_{\phi:\omega \in I(\phi)} \int_{(\ker(\phi)N_0)(\mathbf{A})\backslash N(\mathbf{A})} \chi_\omega^{-1}(n) f(\gamma(\phi)^{-1}ng, s)\, dn.$$

We would like to see that this vanishes at $s = n+1$ if ω is rank one. If ϕ is fixed, set $Y_2 = \ker(\phi)N_0\backslash N$.

By absolute convergence, we may consider the individual terms separately. As explained in the proof of Lemma 3.1.3, we may assume that $\phi = n_L(e_{11})$ and the condition on ω is that $\omega \in I(\phi) = (0, Fe_{11}, e_{11} \times J, F)$. Without loss of generality, we may furthermore assume that $\omega = (0, 0, 0, 1)$. Note that the subgroup of T^1 with $\delta = t_1t_2t_3 = 1$ fixes ω.

Now, for $t \in T^1$, we have $\mu_{F\phi}(t) = t_1^2$. Moreover, one computes $J(t, Y_2) = |t_1|^{-(\dim(C)+4)}$. Thus, $C(t, Y_2, s) = |t_1|^{2s-\dim(C)-4}$, which is $|t_1|^{2n-\dim(C)-2}$ at $s = n+1$. However, $\nu(t) = 1$, so if $|t_1| \neq 1$, this transforms the correct way only if $2n = \dim(C)+2$. But then $n < \dim(J)$, so this cannot occur. This completes the argument in the case $i = 2$. □

Summarizing, we have proved the following result.

Theorem 3.2.5. *Suppose $n > \dim(J)+1$ is even, so that $E(g, s; n)$ converges absolutely, and defines a modular form at $s = n+1$. Then, the Fourier expansion of $E(g, s; n)$ at $s = n+1$ is given as follows:*

(1) *If ω is rank one, then*

$$E^{\omega}(g, s = n+1; n) = \int_{((F\omega)^{\perp}N_0)(\mathbf{A})\backslash N(\mathbf{A})} \chi_\omega^{-1}(u) f(\gamma(\omega)^{-1}ug, s = n+1)\, du,$$

where $\gamma(\omega)E_{13} = e \otimes \omega$.

(2) *If ω is rank two (which cannot occur in the case $G = G_2$), then*

$$E^{\omega}(g, s = n+1; n) = \int_{(\ker(\Phi_{\omega,\omega})N_0)(\mathbf{A})\backslash N(\mathbf{A})} \chi_\omega^{-1}(u) f(\gamma(\Phi_{\omega,\omega})^{-1}ug, s = n+1)\, du,$$

where $\gamma(\Phi_{\omega,\omega})E_{13} = \Phi_{\omega,\omega}$.

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(3) *If ω is rank three, then*

$$E^{\omega}(g, s = n+1; n) = \int_{(W(\omega^{\flat})(\mathbf{A})\backslash N(\mathbf{A})} \chi_{\omega}^{-1}(u) f(\gamma(\omega^{\flat})^{-1} ug, s = n+1)\, du,$$

where $\gamma(\omega^{\flat})E_{13} = f \otimes \omega^{\flat}$.

(4) *If ω is rank four, then*

$$E^{\omega}(g, s = n+1; n) = \int_{N(\mathbf{A})} \chi_{\omega}^{-1}(u) f(w_4^{-1} ug, s = n+1)\, du.$$

Finally, the constant term of $E(g, s = n+1; n)$ is $f_0(g, s = n+1) + E_1^0(g, s = n+1; n)$, where

$$E_1^0(g, s = n+1; n) = \sum_{\ell \subseteq W_J^{rk1}} \int_{(\ell)^{\perp} N_0(\mathbf{A})\backslash N(\mathbf{A})} f(\gamma(\ell)^{-1} ug, s = n+1)\, du$$

is an absolutely convergent Siegel Eisenstein series on H_J and $\gamma(\ell)E_{13}$ spans $e \otimes \ell$.

3.3. Computation of constant term

We now compute the constant terms more explicitly.

3.3.1. The $i = 0$-term. The first thing that we do is compute the simplest term, $f_0(g, s)$.

Lemma 3.3.1. *Suppose $g \in P_{Heis}(\mathbf{A})$. Then*

$$f_0(g, s) = |\nu(g)|^s \zeta(s) \frac{(-1)^{n/2}}{2^n} \Gamma_{\mathbf{R}}(s+n) x^n y^n,$$

which at $s = n+1$ becomes

$$f_0(g, s = n+1) = |\nu(g)|^{n+1} \zeta(n+1) \frac{(-1)^{n/2}}{2^n} \pi^{-n} (1/2)_n x^n y^n.$$

Proof. Assume $g \in P \subseteq G_J$ the Heisenberg parabolic. Then we have

$$\begin{aligned} f_0(g, s) &= \int_{\mathrm{GL}_1(\mathbf{A})} |t|^s \Phi(tg^{-1}E_{13})\, dt \\ &= \int_{\mathrm{GL}_1(\mathbf{A})} |t|^s \Phi(t\nu(g)^{-1}E_{13})\, dt \\ &= |\nu(g)|^s \int_{\mathrm{GL}_1(\mathbf{A})} |t|^s \Phi(tE_{13})\, dt. \end{aligned}$$

Thus, the contribution to $f_0(g, s)$ from the finite places is $|\nu(g)|^s \zeta(s)$.

Let us now analyze the Archimedean section $f(g, \Phi_n, s)$. Observe that for $v \in \mathfrak{g}(J)$,

$$pr(v) = B(v, f_{\ell})e_{\ell} + \frac{1}{2} B(v, h_{\ell})h_{\ell} + B(v, f_{\ell})e_{\ell}.$$

Also, recall [18, § 5.1]

- $e_\ell = \frac{1}{4}(ie+f)\otimes r_0(i)$,
- $f_\ell = \frac{1}{4}(ie-f)\otimes r_0(-i)$, and
- $h_\ell = \frac{i}{2}\left(\left(\begin{smallmatrix} & 1 \\ -1 & \end{smallmatrix}\right) + n_L(-1) + n_L^\vee(1)\right)$,

and under the map $pr : \mathfrak{k} \to Sym^2(V_2)$, $e_\ell \mapsto x^2$, $f_\ell \mapsto -y^2$, and $h_\ell \mapsto -2xy$ [18, § 9]. Consequently,

$$pr(E_{13}) = -\frac{i}{4}h_\ell \mapsto \frac{i}{2}xy.$$

Thus if n is even,

$$f_\infty(1, \Phi_{\infty,n}, s) = pr(E_{13})^n \int_{\mathrm{GL}_1(\mathbf{R})} |t|^{s+n} e^{-\pi t^2}\, dt = \frac{(-1)^{n/2}}{2^n}\Gamma_{\mathbf{R}}(s+n)x^n y^n. \tag{5}$$

Combining the finite places and the Archimedean place, we obtain the lemma. □

3.3.2. The $i = 1$-term. Denote by ℓ_0 the line spanned by $E_{23} = e \otimes (0, 0, 0, 1)$, and suppose $\gamma_0 E_{13} = E_{23}$. Define

$$f_1^0(g, s) = \int_{((\ell_0)^\perp N_0)(\mathbf{A})\backslash N(\mathbf{A})} f(\gamma_0^{-1} ng, s)\, dn.$$

Then for $Re(s)$ large,

$$E_1^0(g, s) = \sum_{\gamma \in P_{Sieg}(\mathbf{Q})\backslash H_J(\mathbf{Q})} f_1^0(\gamma g, s),$$

where P_{Sieg} denotes the parabolic subgroup of H_J that stabilizes the line $\mathbf{Q}(0, 0, 0, 1)$. Note that this sum defines a Siegel Eisenstein series on H_J.

Proposition 3.3.2. *Suppose that Φ_v restricted to $\mathbf{Q}_p E_{13} \oplus \mathbf{Q}_p E_{23}$ is the characteristic function of $\mathbf{Z}_v E_{13} \oplus \mathbf{Z}_v E_{23}$ for every $v < \infty$. Then for $p \in P_{Sieg}(\mathbf{A})$,*

$$f_1^0(p, s = n+1) = \frac{\zeta(n)\Gamma(n)}{(4\pi)^n}|\nu(p)\|\lambda(p)|^n(x^{2n} + y^{2n})$$

with $\lambda(p)$ defined by $p(0, 0, 0, 1) = \lambda(p)(0, 0, 0, 1)$.

See also [3, § 13] and especially [3, Lemma 13.14], where the case of G_2 is discussed.

Proof. We have

$$\begin{aligned} f_1^0(g, s) &= \int_{((\ell_0)^\perp N_0)(\mathbf{A})\backslash N(\mathbf{A})} f(\gamma_0^{-1} ng, s)\, dn \\ &= \int_{\mathrm{GL}_1(\mathbf{A})} \int_{\mathbf{A}} |t|^s \Phi(tg^{-1}n^{-1}E_{23})\, dn\, dt \\ &= \int_{\mathrm{GL}_1(\mathbf{A})} \int_{\mathbf{A}} |t|^s \Phi(tg^{-1}(E_{23} - xE_{13}))\, dx\, dt. \end{aligned}$$

Here we have used that if $n = e \otimes (x, 0, 0, 0)$, then $n^{-1}E_{23} = E_{23} - xE_{13}$.

Now suppose $g = p \in P_{Sieg} \subseteq H_J$, the Siegel parabolic subgroup of the Levi H_J. Denote by λ the character of P_{Sieg} that defines its Siegel Eisenstein series, i.e., $pE_{23} = \lambda(p)E_{23}$.

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Then $p^{-1}(E_{23}-xE_{13})=\lambda(p)^{-1}E_{23}-x\nu(p)^{-1}E_{13}$. So, we must evaluate

$$\begin{aligned}f_1^0(g,s)&=\int_{\mathrm{GL}_1(\mathbf{A})}\int_{\mathbf{A}}|t|^s\Phi(t\lambda(p)^{-1}E_{23}-xt\nu(p)^{-1}E_{13})\,dx\,dt\\&=|\nu(p)||\lambda(p)|^{s-1}\int_{\mathrm{GL}_1(\mathbf{A})}\int_{\mathbf{A}}|t|^s\Phi(tE_{23}-xtE_{13})\,dx\,dt,\end{aligned}$$

where we have changed variables $x\mapsto\nu(p)\lambda(p)^{-1}x$ and $t\mapsto\lambda(p)t$.

For a place v of $\mathbf{Q}$, set

$$f_{1,v}^0(g,s)=|\nu(p)||\lambda(p)|^{s-1}\int_{\mathbf{Q}_v^\times}\int_{\mathbf{Q}_v}|t|^s\Phi_v(tE_{23}-xtE_{13})\,dx\,dt.$$

Now, assuming that Φ_v is as in the statement of the proposition, the integral over t and x gives $\zeta_v(s-1)$. Thus, at the finite place v with Φ_v as above and $p\in P_{Sieg}\subseteq H_J$, we get

$$f_{1,v}^0(p,s)=\zeta_v(s-1)|\nu(p)||\lambda(p)|^{s-1}.$$

We now must calculate $f_{1,v}^0(g,s)$ when $v=\infty$ is the Archimedean place. We require an explicit expression for $pr(u)$ for $u=e\otimes v+\mu E_{13}$. We have $B(u,e_\ell)=\frac{1}{4}\langle v,r_0(i)\rangle$, $B(u,f_\ell)=-\frac{1}{4}\langle v,r_0(-i)\rangle$, and $B(u,h_\ell)=-\frac{i}{2}\mu$. Thus

$$\begin{aligned}-4pr(e\otimes v+\mu E_{13})&=\langle v,r_0(-i)\rangle e_\ell+i\mu h_\ell-\langle v,r_0(i)\rangle f_\ell\\&\mapsto\langle v,r_0(-i)\rangle x^2-2i\mu xy+\langle v,r_0(i)\rangle y^2.\end{aligned}$$

Therefore

$$pr(E_{23}-\beta E_{13})=-\frac{1}{4}(-x^2+2i\beta xy-y^2)=\frac{1}{4}(x^2-2i\beta xy+y^2).$$

Thus

$$\begin{aligned}f_{1,\infty}^0(g,s)&=\int_{\mathbf{R}^\times}\int_{\mathbf{R}}|t|^s\Phi_{\infty,n}(t(E_{23}-\beta E_{13}))\,dx\,dt\\&=\int_{\mathbf{R}^\times}\int_{\mathbf{R}}|t|^{s+n}pr(E_{23}-\beta E_{13})^ne^{-\pi t^2(1+\beta^2)}\,d\beta\,dt\\&=\frac{1}{4^n}\int_{\mathbf{R}^\times}\int_{\mathbf{R}}|t|^{s+n}(x^2-2i\beta xy+y^2)^ne^{-\pi t^2(1+\beta^2)}\,d\beta\,dt.\end{aligned}$$

The final integral is

$$\frac{1}{4^n}\sum_{0\leqslant k\leqslant n,\text{ even}}\binom{n}{k}2^k(-1)^{k/2}(x^2+y^2)^{n-k}(xy)^k\int_{\mathrm{GL}_1}|t|^{s+n}e^{-\pi t^2}\left(\int_{\mathbf{R}}\beta^ke^{-\pi t^2\beta^2}\,d\beta\right)dt.$$

The inner β-integral above gives

$$\begin{aligned}\int_{\mathbf{R}}\beta^ke^{-\pi t^2\beta^2}\,d\beta&=|t|^{-k-1}\int_{\mathbf{R}}\beta^ke^{-\pi\beta^2}\,d\beta\\&=\pi^{-k/2}|t|^{-k-1}\left(\frac{1}{2}\right)_{k/2}.\end{aligned}$$

Therefore, we obtain

$$|\nu(p)|^{-1}|\lambda(p)|^{1-s} f_1^0(p,s) = \frac{1}{4^n}\sum_{0\leqslant k\leqslant n,\ \text{even}} \binom{n}{k} 2^k(-1)^{k/2}(x^2+y^2)^{n-k}(xy)^k\pi^{-k/2}\left(\frac{1}{2}\right)_{k/2} \times \Gamma_{\mathbf{R}}(s+n-k-1)$$

$$\overset{s=n+1}{=} \frac{\pi^{-n}}{4^n}\sum_{0\leqslant k\leqslant n,\ \text{even}} \binom{n}{k} 2^k(-1)^{k/2}(x^2+y^2)^{n-k}(xy)^k\left(\frac{1}{2}\right)_{k/2} \times \Gamma(n-k/2).$$

Lemma 3.3.3. *One has*

$$\sum_{0\leqslant k\leqslant n,\ even} \binom{n}{k} 2^k(-1)^{k/2}(x^2+y^2)^{n-k}(xy)^k\left(\frac{1}{2}\right)_{k/2}\Gamma(n-k/2) = \Gamma(n)(x^{2n}+y^{2n}).$$

Proof. The proof is by generating series. First, write $u=x^2$, $v=y^2$, $n=2m$ and $k=2j$. Then we must evaluate

$$\sum_{0\leqslant j\leqslant m}\binom{2m}{2j}2^{2j}(-1)^j(u+v)^{2m-2j}(uv)^j\left(\frac{1}{2}\right)_j\Gamma(2m-j).$$

Because $\left(\frac{1}{2}\right)_j = \frac{(2j)!}{2^{2j}(j!)}$, this is

$$n!\sum_{0\leqslant j\leqslant m}(-1)^j\binom{2m-j}{j}\frac{1}{2m-j}(u+v)^{2m-2j}(uv)^j.$$

To evaluate this, ignore the $n!$ and sum over all n (not just even n) to obtain

$$\sum_{n\geqslant 1}\sum_{0\leqslant j\leqslant n-j}(-1)^j\binom{n-j}{j}\frac{1}{n-j}(u+v)^{(n-j)-j}(uv)^j$$

$$=\sum_{p\geqslant 1}\sum_{0\leqslant j\leqslant p}(-1)^j\binom{p}{j}\frac{1}{p}(u+v)^{p-j}(uv)^j$$

$$=\sum_{p\geqslant 1}\frac{((u+v)-uv)^p}{p}.$$

But this last sum is

$$-\log(1-u-v+uv) = -\log(1-u)-\log(1-v) = \sum_{n\geqslant 1}\frac{u^n+v^n}{n}.$$

This proves the lemma. □

We have thus proved at $s=n+1$ the Archimedean section gives

$$f_1^0(p,s=n+1) = |\nu(p)\|\lambda(p)|^n(4\pi)^{-n}\Gamma(n)(x^{2n}+y^{2n}).$$

This completes the proof of the proposition. □

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3.4. The rank one Fourier coefficients

In [11], the authors computed the non-Archimedean part of the rank four Fourier coefficients of the degenerate Heisenberg Eisenstein series on G_2. The work of *loc. cit.* is an involved computation, and in general, the rank two, three and four coefficients of $E(g, \Phi, s = n+1)$ appear to be somewhat difficult to evaluate. However, the Fourier coefficients $E_1^{\omega}(g, s = n+1)$ for ω rank one can be computed directly. In this subsection, we make this computation.

For $\omega \in W_J$ rank one, define

$$f_1^{\omega}(g,s) = \int_{(F\omega)^{\perp}N_0\backslash N} \chi_{\omega}^{-1}(n) f(\gamma(\omega)^{-1} n g, s)\, dn,$$

where $\gamma(\omega)E_{13} = e \otimes \omega$. We will assume that $\omega = ae \otimes (0,0,0,1) = ae \otimes \omega_0$ with $a \in F$. So, $\omega_0 = (0,0,0,1)$ in this subsection.

3.4.1. Evaluation at spherical finite places. First, we evaluate $f_1^{\omega}(g,s)$ at spherical finite places. The Fourier coefficient will vanish unless $a \in \mathcal{O}$, so we can assume $a \in \mathcal{O} = \mathbf{Z}_p$. We get, as above,

$$\begin{aligned} f_1^{\omega}(g,s) &= \int_{(\ell)^{\perp}N_0\backslash N} \chi_{\omega}^{-1}(n) f(\gamma_0^{-1} n g, s)\, dn \\ &= \int_{\mathbf{Q}_p^{\times}} \int_{\mathbf{Q}_p} \psi^{-1}(ax)|t|^s \Phi(t g^{-1}(E_{23} + x E_{13}))\, dx\, dt. \end{aligned}$$

(We have changed variables $x \mapsto -x$; note that $\chi_{\omega}(n) = \psi(\langle \omega, n\rangle) = \psi(-ax)$.) Now, because we are interested in the Fourier expansion of the spherical vector, we will take $g = 1$ at the finite places. Thus, at the finite places, we obtain

$$\begin{aligned} f_1^{\omega}(1,s) &= \int_{\mathbf{Q}_p^{\times}} \int_{\mathbf{Q}_p} |t|^s \psi^{-1}(ax)\Phi(tE_{23} + txE_{13})\, dx\, dt \\ &= \int_{\mathbf{Q}_p^{\times}} \int_{\mathbf{Q}_p} |t|^{s-1} \psi^{-1}(ax/t)\Phi(tE_{23} + xE_{13})\, dx\, dt \\ &= \sum_{t|a} |t|^{s-1}. \end{aligned}$$

3.4.2. Evaluation at the Archimedean place. We now evaluate $f_1^{\omega}(g,s)$ at the Archimedean place. Suppose $\omega_0 \in W_J$ is rank one, $z \in W_J$, $n = n(z) = \exp(e \otimes z)$, and $g \in H_J$. Then

$$\begin{aligned} g^{-1}n^{-1}e \otimes \omega_0 &= g^{-1}(e \otimes \omega_0 + \langle \omega_0, z\rangle E_{13}) \\ &= \nu(g)^{-1}(e \otimes \widetilde{g}\omega_0 + \langle \omega_0, z\rangle E_{13}). \end{aligned}$$

Here, recall that $\widetilde{g} = \nu(g) g^{-1}$. Thus we have

$$\|g^{-1}n^{-1}e \otimes \omega_0\|^2 = |\nu(g)|^{-2}\big(|\langle \widetilde{g}\omega_0, r_0(i)\rangle|^2 + |\langle \omega_0, z\rangle|^2\big)$$

using that ω_0 is rank one. Additionally, we have

$$\begin{aligned} pr(g^{-1}n^{-1}e \otimes \omega_0) &= -\tfrac{1}{4}\nu(g)^{-1}\big(\langle \widetilde{g}\omega_0, r_0(-i)\rangle x^2 - 2i\langle \omega_0, z\rangle xy + \langle \widetilde{g}\omega_0, r_0(i)\rangle y^2\big) \\ &= -\tfrac{1}{4}\nu(g)^{-1}\big(\alpha^* x^2 - 2i\beta xy + \alpha y^2\big), \end{aligned}$$

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where $\alpha = \langle \omega_0, gr_0(i)\rangle$ and $\beta = \langle \omega_0, z\rangle$. Thus we obtain

$$f_1^{\omega}(g,s) = \frac{|\nu(g)|^s}{4^n}\int_{\mathbf{R}^\times}\int_{\mathbf{R}} \psi^{-1}(\langle \omega, z\rangle)|t|^{s+n}\left(\alpha^* x^2 - 2i\beta xy + \alpha y^2\right)^n e^{-\pi t^2(|\alpha|^2+\beta^2)}\,dz\,dt$$
$$= \frac{|\nu(g)|^s}{4^n}\int_{\mathbf{R}^\times}\int_{\mathbf{R}} \psi^{-1}(\lambda\beta)|t|^{s+n}\left(\alpha^* x^2 - 2i\beta xy + \alpha y^2\right)^n e^{-\pi t^2(|\alpha|^2+\beta^2)}\,d\beta\,dt,$$

where $\lambda = a$ is the constant satisfying $\langle \omega, z\rangle = \lambda\langle \omega_0, z\rangle = \lambda\beta$.

Hence

$$f_1^{\omega}(g,s) = \frac{|\nu(g)|^s}{4^n}\sum_k \binom{n}{k}(xy)^k(\alpha^* x^2 + \alpha y^2)^{n-k}(-2i)^k\int_{\mathrm{GL}_1}|t|^{s+n}e^{-\pi t^2|\alpha|^2}$$
$$\times\left(\int_{\mathbf{R}}\beta^k e^{-\pi t^2\beta^2}e^{-2\pi i\lambda\beta}\,d\beta\right)dt.$$

Because

$$t^2\beta^2 + 2i\lambda\beta = (|t|\beta + i\lambda/|t|)^2 + \lambda^2/|t|^2,$$

this inner integral is

$$e^{-\pi\lambda^2/|t|^2}\int_{\mathbf{R}}\beta^k e^{-\pi(|t|\beta+i\lambda/|t|)^2}\,d\beta.$$

The $k=0$ term then gives

$$k=0 \text{ term} = \frac{|\nu(g)|^s}{4^n}(\alpha^* x^2+\alpha y^2)^n\int_{\mathbf{R}^\times}|t|^{s+n-1}e^{-\pi(t^2|\alpha|^2+\lambda^2/t^2)}\,dt$$
$$= 2\frac{|\nu(g)|^s}{4^n}(\alpha^* x^2+\alpha y^2)^n\left(\frac{|\lambda|}{|\alpha|}\right)^{(s+n-1)/2}K_{(s+n-1)/2}(2\pi|\lambda||\alpha|).$$

Note that we are using the normalization of the K-Bessel function

$$K_s(y) = \frac{1}{2}\int_0^\infty t^s e^{-y(t+t^{-1})/2}\,\frac{dt}{t}$$

for $y > 0$.

Thus, at $s = n+1$, the coefficient of x^{2n} is

$$2\frac{|\nu(g)|^{n+1}}{4^n}|\lambda|^n\left(\frac{\alpha^*}{|\alpha|}\right)^n K_n(2\pi|\lambda||\alpha|)$$
$$= 2\frac{|\nu(g)|^{n+1}}{4^n}|\lambda|^n\left(\frac{|\langle 2\pi\omega, gr_0(i)\rangle|}{\langle 2\pi\omega, gr_0(i)\rangle}\right)^n K_n(|\langle 2\pi\omega, gr_0(i)\rangle|)$$
$$= 2\frac{|\lambda|^n}{4^n}\mathcal{W}^n_{2\pi\omega}(g).$$

Here, recall that $|\lambda||\alpha| = |\langle \omega, gr_0(i)\rangle|$, and we have used that n is even so that we do not have to worry about the sign of λ.

Remark 3.4.1. By the multiplicity one result [18, Theorem 1.2.1], to compute $f_1^{\omega}(g, s = n+1)$ it is enough to compute the coefficient of x^{2n}, which is what we have done. The coefficient of $x^{n+v}y^{n-v}$ with $v \neq n$ comes from various terms with $k > 0$, and gives a complicated mess of K-Bessel functions. However, by this multiplicity one result, these K-Bessel functions must ultimately combine and simplify, using the various identities among Bessel functions, to a single $K_v(\bullet)$.

3.4.3. Combining finite and Archimedean. Combining the above computations of $f_1^{\omega}(g,s)$ at the finite and Archimedean places, we have proved the following.

Proposition 3.4.2. *Suppose $g \in H_J(\mathbf{R})$, $\omega = ae \otimes (0,0,0,1)$, and Φ_p restricted to $\mathbf{Q}_p E_{13} + \mathbf{Q}_p E_{23}$ is the characteristic function of $\mathcal{O}E_{13} + \mathcal{O}E_{23} = \mathbf{Z}_p E_{13} \oplus \mathbf{Z}_p E_{23}$ for every finite prime p. Then*

$$f_1^{\omega}(g, s = n+1) = \frac{2(2n)!}{4^n}\sigma_n(|a|)\mathcal{W}_{2\pi\omega}(g).$$

3.5. The degenerate Heisenberg Eisenstein series

Putting everything together, we have proved the following result.

Corollary 3.5.1. *Suppose that $n > 0$ is even such that $E(g,\Phi,s;n)$ is a modular form of weight n at $s = n+1$. Assume moreover that for all $p < \infty$, Φ_p is such that when restricted to $\mathbf{Q}_p E_{13} + \mathbf{Q}_p E_{23}$ it is the characteristic function of $\mathbf{Z}_p E_{13} \oplus \mathbf{Z}_p E_{23}$. Denote by $E_{hol}(g,s,n)$ the Siegel Eisenstein series on H_J defined as*

$$E_{hol}(g,s,n) = \sum_{\gamma \in P_{Sieg}(\mathbf{Q})\backslash H_J(\mathbf{Q})} f(\gamma g, n)$$

with $f(p,s,n) = |\nu(p)\|\lambda(p)|^s$ for $p \in P_{Sieg}(\mathbf{A})$ and $f(gk,n) = j(k,i)^{-n} f(g,n)$ for $k \in K_H^1$. Then for $g \in H_J(\mathbf{R})$,

$$\begin{aligned} E(n(x)g,\Phi,s=n+1)_0 = {} & \frac{\zeta(n)\Gamma(n)}{(4\pi)^n}\left(E_{hol}(g,n)x^{2n} + E'_{hol}(g,n)y^{2n}\right) \\ & + |\nu(g)|^{n+1}\frac{\zeta(n+1)(-1)^{n/2}\left(\frac{1}{2}\right)_n}{(2\pi)^n}x^n y^n \\ & + \sum_{\omega \in W_J(\mathbf{Q})} a(\omega)e^{2\pi i\langle\omega,x\rangle}\mathcal{W}_{2\pi\omega}(g) \end{aligned}$$

for some coefficients $a(\omega)$. If $n > \dim(J)+1$ and $\omega = a(0,0,0,1)$ with $a \in \mathbf{Z}$, then $a(\omega) = \frac{2(2n)!}{4^n}\sigma_n(|a|)$.

4. The minimal modular form

In this section, we prove Theorem 2.2.2. In the first subsection, we prove Proposition 2.2.1. In the second subsection, we put together all the pieces to complete the proof of Theorem 2.2.2.

4.1. The value of the Eisenstein series at its special point

As mentioned, the purpose of this section is to spell out a proof of Proposition 2.2.1, which is essentially contained in [2, 3, 7, 10]. Crucial to our arguments is the following proposition.

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Proposition 4.1.1. *Denote by w_0 the element of the Weyl group that takes N to its opposite, and by $M(w_0, s)$ the intertwiner, which for $Re(s) \gg 0$ is given by the absolutely convergent integral*

$$M(w_0, s) f(g, s; 4) = \int_{N(\mathbf{R})} f(w_0^{-1} n g, s; 4)\, dn.$$

Here we are working on $G(\mathbf{R}) = E_{8,4}(\mathbf{R})$ and f is our special inducing section defined above. Then $M(w_0, s) f(g, s; 4) = h(s) f(g, 29 - s; 4)$, where $h(s)$ is a meromorphic function of s, which is regular at $s = 5$ and vanishes there.

We give the proof of the proposition below. It is via factorization of the intertwining operator and reduction to rational rank one. Denote by P_0 the minimal standard parabolic on $G = E_8$, so that P_0 defines a root system of type F_4. We first prove the following corollary of Proposition 4.1.1.

Corollary 4.1.2. *The Eisenstein series $E(g, s; 4)$ is regular at $s = 5$ and defines a modular form of weight 4 at this point. Moreover, its constant term along P_0 consists of just two terms: $f(g, s = 5; 4)$ and $f_1^0(g, s = 5; 4)$.*

Proof. First, from the functional equation of Eisenstein series, one obtains

$$E(g, s; 4) = c_f(w_0, s) h(s) E(g, 29 - s; 4). \tag{6}$$

The Eisenstein series $E(g, s; 4)$ is spherical at *every* finite place (because the octonions are split at every finite place) and thus the finite part $c_f(w_0; s)$ of the c-function can be computed. In fact, one gets (e.g. [2, 8])

$$c_f(w_0; s) = \frac{\zeta(2s - 29)\zeta(s - 28)\zeta(s - 23)\zeta(s - 19)}{\zeta(2s - 28)\zeta(s)\zeta(s - 5)\zeta(s - 9)}.$$

It is clear that the left-hand side of (6) is nonzero at $s = 5$, by examining the contribution of $f(g, s; 4)$ to its constant term. Because $h(5) = 0$, it follows that $E(g, s; 4)$ has a pole at $s = 24$. Because, as proved in [2], the order of the pole is at most one at $s = 24$, one concludes that $E(g, s; 4)$ is regular at $s = 5$. More precisely, $E(g, 5; 4)$ is a nonzero constant times $Res_{s=24} E(g, s; 4)$.

Now, as explained in [2], at most two terms contribute to the constant term along P_0 of the residue $Res_{s=24} E(g, s; 4)$. It follows that all but two terms in the constant term of $E(g, s; 4)$ along P_0 vanish at $s = 5$. Since we have already seen two of these terms above, $f(g, s; n)$ and $f_1^0(g, s; n)$, these are the only terms that contribute to the constant term of $E(g, s; 4)$ at $s = 5$.

It follows immediately from our calculation of $f(g, s = n + 1; n)$ and $f_1^0(g, s = n + 1; n)$ above and [18, § 11] that $\mathcal{D}_4$ annihilates both $f(g, s = 5; 4)$ and $f_1^0(g, s = 5; 4)$. Thus $\mathcal{D}_4$ annihilates the constant term of the $E(g, s = 5; 4)$. Because applying $\mathcal{D}_4$ commutes with taking the constant term, $\mathcal{D}_4 E(g, s; 4)$ has constant term 0. But an easy application of [18, Theorem 7.3.1] yields $\mathcal{D}_4 E(g, s; 4) = (s - 5) E(g, f_s'; 4)$ for another $G(\widehat{\mathbf{Z}})$-spherical, K-finite flat section f_s'. Thus the constant term of $E(g, f_s'; 4)$ is regular at $s = 5$, and so $E(g, f_s'; 4)$ is regular, and thus $\mathcal{D}_4 E(g, s = 5; 4) = 0$. That is, $E(g, s = 5; 4)$ is a modular form on G of weight 4. This completes the proof of the corollary. □

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4.1.1. Archimedean intertwiner. It remains to explain the proof of Proposition 4.1.1.

Denote by α_i the simple roots of our F_4-root system, so that they are labeled (in order) 1, 2, 3, 4 when the root diagram is

$$\circ - - - \circ ==>== \circ - - - \circ.$$

That is, 1, 2 label the long simple roots and 3, 4 label the short simple roots. Denote by w_i the simple reflection in the root α_i and $[i_1, i_2, \ldots, i_k]$ for the composition $w_{i_1} w_{i_2} \cdots w_{i_k}$. The long intertwiner [2] in this notation is $w_0 = [1, 2, 3, 2, 1, 4, 3, 2, 1, 3, 2, 4, 3, 2, 1]$.

To compute $M(w_0, s)$ at the Archimedean place, we factorize into the intertwiners for the simple roots α_i and use the cocycle property. The short roots, fortunately, give spherical intertwiners on groups isogenous to $\mathrm{SO}(9, 1)$. Normalize the inner product $(\cdot, \cdot)$ on the F_4 root spaces so that the long roots have norm squared equal to 2. With this normalization, if μ is a character of P_0, the result is that the c-function is

$$c(w_k, \mu) = (\text{nonzero constant}) \frac{\Gamma((\mu, \alpha_k))}{\Gamma((\mu, \alpha_k) + 4)}$$

for $k = 3, 4$ corresponding to the short roots.

For the long roots, the intertwiners are no longer spherical, but we must only make a GL_2-computation. To do this, first denote by V_+ the three-dimensional subspace of $\mathbb{V}_4$ spanned by

$$b_2^2 := x^8 + y^8, b_2^1 =:= x^2 y^2 (x^4 + y^4), b_2^0 := x^4 y^4.$$

As in [3, § 13], define $f_1 = \frac{x+y}{2}$, $f_2 = \frac{x-y}{2}$. Then V_+ is also the span of

$$b_1^2 := f_1^8 + f_2^8, b_1^1 := f_1^2 f_2^2 (f_1^4 + f_2^4), b_1^0 := f_1^4 f_2^4.$$

When we compute $M(w_2, \mu)$, it is convenient to use the first basis b_2^i, and when we compute $M(w_1, \mu)$ it is convenient to use the basis b_1^j.

More precisely, denote by $[\mu, b_j^k]$ the K-equivariant inducing section for $Ind_{P_0}^G(\delta_{P_0}^{1/2}\mu)$ whose value at $g = 1$ is b_j^k. Then for $i \in \{1, 2\}$, $j \in \{1, 2\}$ and $k \in \{0, 1, 2\}$, one has

$$M(w_i, \mu)[\mu, b_j^k] = \frac{\zeta_{\mathbf{R}}(s)}{\zeta_{\mathbf{R}}(s+1)} \frac{\left(\frac{1-s}{2}\right)_k}{\left(\frac{1+s}{2}\right)_k} [w_i(\mu), b_j^k],$$

where $s = \langle \mu, \alpha_i^\vee \rangle$. Here $\zeta_{\mathbf{R}}(s) = \Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $(s)_k = s(s+1)\cdots(s+k-1)$.

To carry out the computation of $M(w_0, s)$, one then just puts together the above information. The change of basis matrix between the b_2^i's and the b_1^i's is

$$A = \begin{pmatrix} 2 & 2 & 1 \\ 56 & 8 & -4 \\ 140 & -20 & 6 \end{pmatrix}$$

as

$$x^8+y^8 = 2(f_1^8+f_2^8)+2(f_1^6f_2^2+f_1^2f_2^6)+f_1^4f_2^4$$
$$x^6y^2+x^2y^6 = 56(f_1^8+f_2^8)+8(f_1^6f_2^2+f_1^2f_2^6)-4f_1^4f_2^4$$
$$x^4y^4 = 140(f_1^8+f_2^8)-20(f_1^6f_2^2+f_1^2f_2^6)+6f_1^4f_2^4.$$

Putting together the pieces, one gets the following proposition, which immediately implies Proposition 4.1.1.

Proposition 4.1.3. *Denote by* $\lambda_s = \delta_{P_0}^{-1/2}|\nu(s)|^s$ *the normalized character defining* $f(g,s;n)$. *Then up to exponential factors, the intertwiner*

$$M(w_0)[\lambda_s, b_2^0] = Z(s)A(s)[\lambda_{29-s}, b_2^0]$$

with

$$Z(s) = \left(\frac{\Gamma_{\mathbf{R}}(2s-29)\Gamma_{\mathbf{R}}(s-28)\Gamma_{\mathbf{R}}(s-19)\Gamma_{\mathbf{R}}(s-11)\Gamma_{\mathbf{R}}(s-2)}{\Gamma_{\mathbf{R}}(2s-28)\Gamma_{\mathbf{R}}(s-26)\Gamma_{\mathbf{R}}(s-17)\Gamma_{\mathbf{R}}(s-9)\Gamma_{\mathbf{R}}(s)}\right)\left(\frac{\Gamma_{\mathbf{C}}(s-23)\Gamma_{\mathbf{C}}(s-14)}{\Gamma_{\mathbf{C}}(s-11)\Gamma_{\mathbf{C}}(s-2)}\right)$$

and

$$A(s) = \frac{(s-31)(s-29)(s-22)(s-20)(s-14)(s-12)(s-5)(s-3)}{(s-26)(s-24)(s-17)(s-15)(s-9)(s-7)s(s+2)}.$$

4.2. Proof of Theorem 1.0.1

At this point, we have shown that $E_J(g,s;4)$ is regular at $s=5$ and defines a modular form there. By the fact that the local representation $\pi_p \subseteq Ind_{P_J(\mathbf{Q}_p)}^{G_J(\mathbf{Q}_p)}(|\nu|^5)$ generated by the spherical vector is minimal [7, 16], $E_J(g, s=5;4)$ only has a constant term and rank one Fourier coefficients; all of its rank two, three and four Fourier coefficients are 0. Moreover, our computations above show the following.

Denote by P_{Sieg} the Siegel parabolic subgroup of H_J, which by definition is the stabilizer of the line spanned by $(0,0,0,1)$ in W_J. Let $E_{hol}(g,s;n)$ be the Siegel Eisenstein series on H_J defined as

$$E_{hol}(g,s;n) = \sum_{\gamma\in P_{Sieg}(\mathbf{Q})\backslash H_J(\mathbf{Q})} f(\gamma g, n)$$

with $f(p,s;n) = |\nu(p)||\lambda(p)|^s$ for $p \in P_{Sieg}(\mathbf{A})$ and $f(gk,s;n) = j(k,i)^{-n}f(g,s;n)$ for $k \in K_H^1$. The Eisenstein series $E_{hol}(g,s;4)$ is regular at $s=4$. The value $E_{hol}(g,s=4;4) = 240|\nu(g)|^5\Phi_{Kim}(g)$, i.e., it corresponds to the holomorphic modular form that is the multiple of H_{Kim} with constant term 1. Indeed, this is the result of [13].

Thus for $g \in H_J(\mathbf{R})$ and $x \in (N/N_0)(\mathbf{R}) \simeq W_J(\mathbf{R})$,

$$E(xg, s=5,\Phi;4)_0 = \frac{\zeta(4)\Gamma(4)}{(4\pi)^4}\left(E_{hol}(g,s=4;4)x^8 + E'_{hol}(g,s=4;4)y^8\right)$$
$$+|\nu(g)|^5\frac{\zeta(5)\left(\frac{1}{2}\right)_4}{(2\pi)^4}x^4y^4 + \sum_{\omega\in W_J(\mathbf{Q})} a(\omega)e^{2\pi i\langle\omega,x\rangle}\mathcal{W}_{2\pi\omega}(g)$$

for some coefficients $a(\omega)$.

4.2.1. The nonconstant terms. To finish the proof, we must analyze the nonconstant terms. We do this by applying Gan's Siegel–Weil theorem [3] for $G_2 \times F_4^{an} \subseteq G_J = E_{8,4}$. Here recall that F_4^{an} is the anisotropic F_4 defined to be the fixator of 1_J in the exceptional cubic norm structure $J = H_3(\Theta)$. We only require the following much weaker form of it.

Theorem 4.2.1 (Gan). *Denote by $\theta(\mathbf{1})(g)$ the theta lift of the constant function* 1 *on F_4^{an} to G_2, i.e.,*

$$\theta(\mathbf{1})(g) = \int_{[F_4^{an}]} E((g, h), 5; 4)\, dh.$$

Then the difference $E^{G_2}(g, 5; 4) - \theta(\mathbf{1})(g)$ is a weight 4*, level one* cuspidal *modular form on G_2.*

Gan's Siegel–Weil theorem proves that the above difference is 0, and moreover that it is 0 for a large family of inducing sections. However, we only require that the difference is cuspidal, and only for this one particular section. These simplifications make the necessary result easier to prove, which is why we state it in this weaker form.

Now, to finish the proof of Theorem 2.2.2, we must evaluate the rank one Fourier coefficients of $E(g, s = 5; 4)$. Because this modular form is spherical, it suffices to evaluate $a_\theta(a(0, 0, 0, 1))$ for positive integers a. To do this, we will use Theorem 4.2.1, together with the following lemma.

Recall the definitions of $\Omega_I(\omega_0)$ and $\Omega_E(\omega_0)$ from § 2.5.

Lemma 4.2.2. *Suppose $\omega_0 = (\alpha, \beta, \gamma, \delta) \in W_F$ is rank one. Then $\Omega_I(\omega_0)$ and $\Omega_E(\omega_0)$ are each singletons, consisting of the elements $(\alpha, \beta I, \gamma I^\#, \delta)$ and $(\alpha, \beta E, \gamma E^\#, \delta)$, respectively.*

Proof. By equivariance and scaling, we may suppose that $\omega_0 = (1, 0, 0, 0)$. But then,

$$\Omega_E((1, 0, 0, 0)) = \{(1, X, X^\#, N(X)) : (X, E^\#) = (X^\#, E) = N(X) = 0\}.$$

However, the only such X is 0, and similarly with I in place of E. This proves the lemma. □

Because modular forms that are cusp forms only have rank four coefficients, it follows from Lemma 4.2.2 that the rank one Fourier coefficients of $E_J(g, s = 5; 4)$ are equal to the rank one Fourier coefficients of the similar Eisenstein series on G_2. But these coefficients were computed in Corollary 3.5.1. This completes the proof of Theorem 2.2.2.

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