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Abstract

The principal aim of this article is to attach and study p-adic L-functions to cohomological cuspidal automorphic representations Π of GL_{2n} over a totally real field F admitting a Shalika model. We use a modular symbol approach, along the global lines of the work of Ash and Ginzburg, but our results are more definitive because we draw heavily upon the methods used in the recent and separate works of all three authors. By construction, our p-adic L-functions are distributions on the Galois group of the maximal abelian extension of F unramified outside $p\infty$. Moreover, we work under a weaker Panchishkine-type condition on Π_p rather than the full ordinariness condition. Finally, we prove the so-called Manin relations between the p-adic L-functions at *all* critical points. This has the striking consequence that, given a unitary Π whose standard L-function admits at least two critical points, and given a prime p such that Π_p is ordinary, the central critical value $L(\frac{1}{2}, \Pi \otimes \chi)$ is non-zero for all except finitely many Dirichlet characters χ of p-power conductor.

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Introduction

A crucial result in Shimura's work on the special values of L-functions of modular forms concerns the existence of a twisting character to ensure that a twisted L-value is non-zero at the center of symmetry (see [Shi77, Theorem 2]). Since then, it has been a very important problem in the analytic theory of automorphic L-functions to find characters to render a twisted L-value non-zero. Rohrlich [Roh89] proved such a non-vanishing result in the context of cuspidal automorphic representations of GL_2 over any number field. This was then generalized to GL_N over any number

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field by Barthel and Ramakrishnan [BR94] and further refined by Luo [Luo05]. However, neither [BR94] nor [Luo05] can prove this at the center of symmetry if $N \geqslant 4$ (for us the functional equation will be normalized so that $s = \frac{1}{2}$ is the center of symmetry). There have been other types of analytic machinery that have been brought to bear on this problem, for example, see [CFH05]. Even for simple situations involving L-functions of higher degree this problem is open. For example, suppose π is the unitary cuspidal automorphic representation associated to a primitive holomorphic cusp form for $\mathrm{GL}_2/\mathbb{Q}$, then it has been an open problem to find a Dirichlet character χ so that the twisted symmetric cube L-function $L(\frac{1}{2}, (\mathrm{Sym}^3 \pi) \otimes \chi)$ is non-zero at the center.

In this article, we prove the following result.

Theorem A. *Let F be a totally real field and Σ_∞ the set of all its real places. Let Π be a unitary cuspidal automorphic representation of GL_{2n}/F admitting a Shalika model and such that Π_∞ is cohomological with respect to a pure dominant integral weight μ such that*

$$\mu_{\sigma,n} > \mu_{\sigma,n+1}, \quad \text{for all } \sigma \in \Sigma_\infty. \tag{1}$$

Assume that for all primes $\mathfrak{p}$ above a given prime number p, $\Pi_\mathfrak{p}$ is unramified and Q-ordinary, where Q is the parabolic of type (n,n) of GL_{2n}/F (see (65)).

Then, for all but finitely many Dirichlet characters χ of p-power conductor we have

$$L\big(\tfrac{1}{2}, \Pi \otimes (\chi \circ \mathrm{N}_{F/\mathbb{Q}})\big) \neq 0.$$

For notions and notation that are not defined in the introduction, the reader should consult the main body of the paper. A more general statement is proven in Theorem 4.8. Furthermore, we can prove a stronger non-vanishing result covering the nearly ordinary case (see Corollary 4.9) as well as a simultaneous non-vanishing result (see Corollary 4.10). For example, with a classical normalization of L-functions, it follows from our results that there are infinitely many Dirichlet characters χ such that

$$L(6, \Delta \otimes \chi) \cdot L(17, \mathrm{Sym}^3(\Delta) \otimes \chi) \neq 0,$$

for the Ramanujan Δ-function. Our methods are purely arithmetic and involve studying p-adic distributions on $\mathscr{C}\ell_F^+(p^\infty)$, the Galois group of the maximal abelian extension of F unramified outside $p\infty$, that are attached to suitable eigenclasses in the cohomology of GL_{2n}.

Let us now describe our methods and results in greater detail. We begin with a purely cohomological situation, without any reference to automorphic forms or L-functions. Let $\mathcal{O}_F$ be the ring of integers of F and $\mathfrak{d}$ its different. Take a pure dominant integral weight μ for $G = \mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}} \mathrm{GL}_{2n}$, and let V_E^μ be the algebraic irreducible representation of $G(E)$, for some 'large enough' p-adic field E. If $\mathcal{O}$ is the ring of integers of E, then we also consider an $\mathcal{O}$-lattice $V_\mathcal{O}^\mu$ stabilized by $G(\mathcal{O})$. For any open compact subgroup K of GL_{2n} over the finite adeles of F, let $\mathcal{V}_\mathcal{O}^\mu$ be the associated sheaf on the locally symmetric space S_K^G of G with level structure K and let us consider the compactly supported cohomology $\mathrm{H}_c^q(S_K^G, \mathcal{V}_\mathcal{O}^\mu)$ endowed with the usual Hecke action. Assume that for $\mathfrak{p}$ dividing p, $K_\mathfrak{p}$ is the parahoric subgroup corresponding to the parabolic Q and consider an eigenclass $\phi \in \mathrm{H}_c^t(S_K^G, \mathcal{V}_\mathcal{O}^\mu)$ having a non-zero eigenvalue $\alpha_\mathfrak{p}$ for a particular Hecke operator $U_\mathfrak{p}$. Here and throughout the paper $t = |\Sigma_\infty|(n^2 + n - 1)$ denotes the top degree supporting cuspidal cohomology. The weight μ determines a contiguous string of integers $\mathrm{Crit}(\mu)$, which would correspond to the set of critical points for an L-function. For each $j \in \mathrm{Crit}(\mu)$ we attach an E-valued distribution $\boldsymbol{\mu}_\phi^j$ on $\mathscr{C}\ell_F^+(p^\infty)$ and show that it is $\mathcal{O}$-valued when ϕ is Q-ordinary, that is, it is a measure (see diagram (36) for a quick overview of the sheaf-theoretic maps that are involved in the construction). Most importantly we prove in Theorem 2.3 a Manin-type relation, namely for all $j, j' \in \mathrm{Crit}(\mu)$ we have

$$\varepsilon_{\mathrm{cyc}}^{j'-j}(\boldsymbol{\mu}_\phi^j) = \boldsymbol{\mu}_\phi^{j'},$$

where $\varepsilon : \mathscr{C}\ell_F^+(p^\infty) \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character and $\varepsilon_{\mathrm{cyc}}$ is the automorphism of $\mathcal{O}[[\mathscr{C}\ell_F^+(p^\infty)]]$ sending $[x]$ to $\varepsilon([x])[x]$, allowing us to define a measure $\boldsymbol{\mu}_\phi = \varepsilon_{\mathrm{cyc}}^{-j}(\boldsymbol{\mu}_\phi^j)$ that is independent of j.

Next we apply these considerations to the situation when ϕ is related to a cuspidal automorphic representation Π of GL_{2n}/F such that Π_∞ is cohomological with respect to the weight μ (see § 4.1.3). Friedberg and Jacquet related the period integral of cusp forms in Π over the Levi subgroup H of Q to the standard L-function $L(s,\Pi)$, and for the unfolding of this integral to see the Eulerian property the representation is assumed to have a Shalika model (see § 4.1). Such a cohomological interpretation was used in [GR14] to deduce algebraicity results for the critical values of $L(s,\Pi\otimes\chi)$. The following result further investigates their p-adic integrality properties. A more general p-adic interpolation statement is proven in Theorem 4.7 under the assumption that $\Pi_\mathfrak{p}$ admits a Q-regular refinement $\tilde{\Pi}_\mathfrak{p}$ for $\mathfrak{p} \mid p$ (see Definition 3.5), which is shown to be always fulfilled when $\Pi_\mathfrak{p}$ is Q-ordinary (see Lemma 4.4).

THEOREM B. *Let Π be a cuspidal automorphic representation of GL_{2n}/F admitting a (ψ,η)-Shalika model and such that Π_∞ is cohomological of weight μ. Assume that for all primes $\mathfrak{p}$ above a given prime number p, $\Pi_\mathfrak{p}$ is spherical and Q-ordinary, and let $\alpha_\mathfrak{p}$ denote the corresponding $U_\mathfrak{p}$-eigenvalue. Given any isomorphism $i_p : \mathbb{C} \xrightarrow{\sim} \bar{\mathbb{Q}}_p$, there exists a bounded p-adic distribution $\boldsymbol{\mu}_{\tilde{\Pi}}$ on $\mathscr{C}\ell_F^+(p^\infty)$ such that for any $j \in \mathrm{Crit}(\mu)$ and for any finite order character χ of $\mathscr{C}\ell_F^+(p^\infty)$ of conductor $\beta_\mathfrak{p} \geqslant 1$ at all $\mathfrak{p} \mid p$ one has*

$$\begin{aligned} i_p^{-1}&\left(\int_{\mathscr{C}\ell_F^+(p^\infty)} \varepsilon^j(x)\chi(x)\,d\boldsymbol{\mu}_{\tilde{\Pi}}(x)\right) \\ &= \gamma\cdot \mathrm{N}_{F/\mathbb{Q}}^{jn}(\mathfrak{d})\cdot\prod_{\mathfrak{p}\mid p}\left(\alpha_\mathfrak{p}^{-1}q_\mathfrak{p}^{n(j+1)}\right)^{\beta_\mathfrak{p}}\cdot\mathcal{G}(\chi_f)^n\cdot L\left(j+\frac{1}{2},\Pi_f\otimes\chi_f\right)\zeta_\infty\left(j+\frac{1}{2};W_{\Pi_\infty,j}^{(\varepsilon^j\chi\eta)_\infty}\right),\end{aligned}$$

where $\mathcal{G}(\chi_f)$ is the Gauss sum, the zeta factor is non-zero by (76), and $\gamma \in \mathbb{Q}^\times$ is as in (77).

Let us hint on how we deduce Theorem A. Theorem B, the formulation of which implicitly uses the earlier established Manin relations, gives congruence relations between successive critical values, whereas (1) translates into $\frac{3}{2} \in \mathrm{Crit}(\mu)$. As the complex L-function of the unitary cuspidal automorphic representation Π does not vanish for $\Re(s) \geqslant 1$ we deduce that $L(\frac{3}{2},\Pi\otimes\chi)$ never vanishes, which, in turn, implies the non-vanishing of $L(\frac{1}{2},\Pi\otimes(\chi\circ\mathrm{N}_{F/\mathbb{Q}}))$ for all but finitely many Dirichlet characters χ (see the proof of Theorem 4.8).

Let us mention some relevant studies in the literature. First, Ash and Ginzburg [AG94] started the study of p-adic L-functions for GL_{2n} over a totally real field by considering the analytic theory developed by Friedberg and Jacquet [FJ93]. However, to quote the authors of [AG94], their results are definitive only for GL_4 over $\mathbb{Q}$ and for cohomology with constant coefficients. Furthermore, they constructed their distributions on local units while only suggesting that one should really work, as we do in this paper, on $\mathscr{C}\ell_F^+(p^\infty)$. This article uses the more recent techniques developed in independent papers by all three of the current authors; namely, [Dim13],

[GR14], and [Jan15]. Finally, we mention Gehrmann's thesis [Geh18], which also constructs p-adic L-functions in essentially a similar context, but his methods are entirely different from ours.

To conclude the introduction, our emphasis is on the purely sheaf-theoretic nature of the construction of the distributions attached to eigenclasses in cohomology, which leads to a purely algebraic proof of Manin relations in a very general context. When specialized to a cohomology class related to a representation Π of GL_{2n}, we obtain p-adic interpolation of the critical values of the standard L-function $L(s,\Pi)$, and Manin relations give non-vanishing of twists $L(s,\Pi\otimes\chi)$ at the center of symmetry. A non-vanishing theorem in the realms of analytic number theory admitting a decidedly algebraic proof is philosophically piquant.

1. Automorphic cohomology

Recall that F is a totally real number field with ring of integers $\mathcal{O}_F$ and set of infinite places Σ_∞. For a set of places Σ, we denote by $\mathbb{A}^{(\Sigma)}$ the topological ring of adeles of $\mathbb{Q}$ outside Σ. Let $\mathbb{A}_F=\mathbb{A}\otimes_{\mathbb{Q}}F$ (respectively, $\mathbb{A}_{F,f}$) be the group of adeles (respectively, finite adeles) of F.

We consider $G=\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}(\mathrm{GL}_{2n})$ as a reductive group scheme over $\mathbb{Z}$, quasi-split over $\mathbb{Q}$ and let $Z=\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}(\mathrm{GL}_1)$ be the center of G. The standard Borel subgroup $B\subseteq G$ is defined as the restriction of scalars of the standard Borel subgroup of all upper triangular matrices in $\mathrm{GL}_{2n}/\mathcal{O}_F$. We have $B=TN$, where N is the unipotent radical of B and T is the standard torus of all diagonal matrices. Let $H=\mathrm{Res}_{\mathcal{O}_F/\mathbb{Z}}(\mathrm{GL}_n\times\mathrm{GL}_n)$, and $\iota: H\hookrightarrow G$ be the map that sends (h_1,h_2) to $\left(\begin{smallmatrix} h_1 & 0\\ 0 & h_2\end{smallmatrix}\right)$. Let $Q=HU$ be the standard parabolic subgroup of type (n,n) whose Levi subgroup is H and unipotent radical is U. Finally, the Shalika subgroup S of G is defined as $S=\left\{\left(\begin{smallmatrix} h & 0\\ 0 & h\end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & X\\ 0 & 1\end{smallmatrix}\right)\mid h\in\mathrm{GL}_n, X\in\mathrm{M}_n\right\}$.

For any commutative ring A, we let $\mathfrak{g}_A$, $\mathfrak{b}_A$, $\mathfrak{q}_A$, $\mathfrak{t}_A$, $\mathfrak{h}_A$, $\mathfrak{n}_A$, and $\mathfrak{u}_A$ denote the Lie algebras of G, B, Q, T, H, N, and U over A, respectively. For $\mathfrak{a}_A$ any amongst these, we let $\mathcal{U}(\mathfrak{a}_A)$ denote the enveloping algebra over A. In the particular case $A=\mathbb{R}$, let $\mathfrak{g}_\infty=\mathfrak{g}_{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}$ denote the complexification and likewise for the other groups. The reader is referred to [Jan03] as a general reference for integral Lie algebras and their enveloping algebras.

For any real reductive Lie group $\mathcal{G}$, we let $\mathcal{G}^\circ$ denote the connected component of the identity. Let $G_\infty=G(\mathbb{R})$, and similarly $Z_\infty=Z(\mathbb{R})$.

1.1 Pure weights

We identify integral weights μ of T with tuples of weights $\mu=(\mu_\sigma)_{\sigma\in\Sigma_\infty}$, $\mu_\sigma=(\mu_{\sigma,1},\dots,\mu_{\sigma,2n})\in\mathbb{Z}^{2n}$. A weight μ is B-dominant if

$$\mu_{\sigma,1}\geqslant\cdots\geqslant\mu_{\sigma,2n},\quad \text{for all } \sigma\in\Sigma_\infty. \tag{2}$$

Let $X_+^*(T)$ be the set of all such dominant integral weights. For $\mu\in X_+^*(T)$ denote by V^μ the unique algebraic irreducible rational representation of G of highest weight μ. For any field E over which μ is defined, we denote by V_E^μ its E-valued points. Denote by $\mu^\vee$ the highest weight of the contragredient $(V^\mu)^\vee$ of V^μ which we consider as a rational character of B.

We call μ *pure* if there exists $\mathsf{w}\in\mathbb{Z}$, called the *purity weight* of μ, such that

$$V^\mu=V^{\mu^\vee}\otimes(\mathrm{N}_{F/\mathbb{Q}}\circ\det)^{\mathsf{w}},$$

where $\mathrm{N}_{F/\mathbb{Q}} : \mathrm{Res}_{F/\mathbb{Q}}(\mathrm{GL}_1) \to \mathrm{GL}_1$ denotes the norm homomorphism. If μ is pure, then

$$\mu_{\sigma,i} + \mu_{\sigma,2n-i+1} = \mathsf{w}, \quad \text{for all } \sigma \in \Sigma_\infty \text{ and for all } 1 \leqslant i \leqslant n. \tag{3}$$

In particular, $\sum_{i=1}^{2n} \mu_{\sigma,i} = \mathsf{w}n$ is independent of σ. We let $X_0^*(T) \subset X_+^*(T)$ denote the pure dominant integral weights of T. Given any $\mu \in X_0^*(T)$, define the set

$$\mathrm{Crit}(\mu) = \{j \in \mathbb{Z} \mid \mu_{\sigma,n} \geqslant j \geqslant \mu_{\sigma,n+1}, \forall\, \sigma \in \Sigma_\infty\}. \tag{4}$$

It is well known that only pure weights support cuspidal cohomology, and the motivation for this definition comes from the fact proved in [GR14, Proposition 6.1] that if Π is a cuspidal automorphic representation of $G(\mathbb{A})$ that is cohomological with respect to μ (see § 4.1.3), then $\frac{1}{2} + j$ with $j \in \mathbb{Z}$ is critical for the standard L-function $L(s, \Pi \otimes \chi)$ for any finite order character χ if and only if $j \in \mathrm{Crit}(\mu)$. Note that the central point $(\mathsf{w}+1)/2$ of $L(s, \Pi \otimes \chi)$ is critical (i.e. $\mathsf{w}/2 \in \mathrm{Crit}(\mu)$) if and only if w is even.

1.2 Integral lattices

Let E be a finite extension of $\mathbb{Q}_p$ and let $\mathcal{O}$ be its ring of integers. Given $\mu \in X_+^*(T)$, we consider V_E^μ as a representation of $G(E)$.

Let $v_0 \in V_E^\mu$ be a non-zero lowest-weight vector. Then the unipotent radical $N^-(E)$ of the Borel subgroup $B^-(E)$ of lower triangular matrices fixes v_0, whereas $T(E)$ acts on v_0 via the character $-\mu^\vee = w_{2n}(\mu)$ where w_{2n} is the Weyl group element of longest length.

Observe that

$$V_{\mathcal{O}}^\mu = \mathcal{U}(\mathfrak{n}_{\mathcal{O}})v_0, \tag{5}$$

is an $\mathcal{O}$-lattice V_E^μ endowed with a natural action of $G(\mathcal{O})$.

We fix once and for all uniformizers $\varpi_{\mathfrak{p}} \in F_{\mathfrak{p}}$ and put $t_{\mathfrak{p}} = \iota(\varpi_{\mathfrak{p}} \cdot \mathbf{1}_n, \mathbf{1}_n) \in \mathrm{GL}_{2n}(F_{\mathfrak{p}})$. Define for any integral multi-exponent $\beta = (\beta_{\mathfrak{p}})_{\mathfrak{p}|p}$ the element

$$t_p^\beta = \prod_{\mathfrak{p}|p} t_{\mathfrak{p}}^{\beta_{\mathfrak{p}}} \in T(\mathbb{Q}_p), \tag{6}$$

and consider the semi-group

$$\Delta_p^+ = \{t_p^\beta \mid \beta_{\mathfrak{p}} \in \mathbb{Z}_{\geqslant 0},\ \forall\, \mathfrak{p} \mid p\}. \tag{7}$$

Then by our choice of dominance condition, we have for any $t \in \Delta_p^+$:

$$\mathrm{Ad}(t)Q(\mathcal{O}) = tQ(\mathcal{O})t^{-1} \subseteq Q(\mathcal{O}) \quad \text{and} \quad \mathrm{Ad}(t^{-1})U^-(\mathcal{O}) = t^{-1}U^-(\mathcal{O})t \subseteq U^-(\mathcal{O}). \tag{8}$$

Consider the standard maximal parahoric subgroup $J_p = \prod_{\mathfrak{p}|p} J_{\mathfrak{p}} \subset G(\mathbb{Z}_p)$, where

$$J_{\mathfrak{p}} = t_{\mathfrak{p}}^{-1}\,\mathrm{GL}_{2n}(\mathcal{O}_{F,\mathfrak{p}})t_{\mathfrak{p}} \cap \mathrm{GL}_{2n}(\mathcal{O}_{F,\mathfrak{p}}). \tag{9}$$

As $J_p \supset Q(\mathbb{Z}_p)$, the parahoric decomposition is given by

$$J_p = (J_p \cap U^-(\mathbb{Z}_p))Q(\mathbb{Z}_p) = Q(\mathbb{Z}_p)(J_p \cap U^-(\mathbb{Z}_p)). \tag{10}$$

Using (8) and (10) one sees that

$$\Lambda_p = J_p\Delta_p^+J_p = Q(\mathbb{Z}_p)\Delta_p^+(J_p \cap U^-(\mathbb{Z}_p)), \tag{11}$$

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is a semi-group. Moreover, because $U^-(\mathbb{Z}_p) \subset N^-(\mathbb{Z}_p)$ acts trivially on v_0, the J_p-action on $V_{\mathcal{O}}^{\mu}$ extends uniquely to an action $\bullet$ of the semi-group Λ_p by letting Δ_p^+ act trivially on the lowest weight vector v_0. Then for all $t \in \Delta_p^+$ and $v \in V_{\mathcal{O}}^{\mu}$ one has

$$t \bullet v = \mu^\vee(t)(t \cdot v). \tag{12}$$

In fact, by (5), one can write $v = m \cdot v_0$ for some $m \in \mathcal{U}(\mathfrak{n}_{\mathcal{O}})$ and using (8) one finds

$$t \bullet v = t \bullet (m \bullet v_0) = \mathrm{Ad}(t)(m) \bullet (t \bullet v_0) = \mathrm{Ad}(t)(m) \cdot v_0 = \mu^\vee(t)\mathrm{Ad}(t)(m)(t \cdot v_0) = \mu^\vee(t)(t \cdot v).$$

1.3 Local systems on locally symmetric spaces for $\mathbf{GL}_{2n}$

The standard maximal compact subgroup of G_∞ is denoted by $C_\infty = \prod_{\sigma \in \Sigma_\infty} C_\sigma$, where $C_\sigma \simeq \mathrm{O}_{2n}(\mathbb{R})$. The determinant identifies the group of connected components $C_\infty / C_\infty^\circ$ with $F_\infty^\times / F_\infty^{\times\circ} \cong \{\pm 1\}^{\Sigma_\infty}$. Let $K_\infty = C_\infty Z_\infty$ and for any open compact subgroup K of $G(\mathbb{A}_f)$ consider the locally symmetric space:

$$S_K^G = G(\mathbb{Q}) \backslash G(\mathbb{A}) / K K_\infty^\circ = G(\mathbb{Q}) \backslash \big((G_\infty / K_\infty^\circ) \times G(\mathbb{A}_f)/K \big). \tag{13}$$

Note that $K_\infty^\circ = C_\infty^\circ Z_\infty^\circ = C_\infty^\circ Z_\infty$ because $2n$ is even. In general, S_K^G is only a real orbifold. In the sequel, we assume that K is *sufficiently small* in the sense that for all $g \in G(\mathbb{A})$,

$$G(\mathbb{Q}) \cap g K K_\infty^\circ g^{-1} = Z(\mathbb{Q}) \cap K K_\infty^\circ, \tag{14}$$

which implies in particular that S_K^G is a real manifold.

Given a left $G(\mathbb{Q})$-module V, one can define $\mathcal{V}_K$ as the sheaf of locally constant sections of the local system:

$$G(\mathbb{Q}) \backslash (G(\mathbb{A}) \times V) / K K_\infty^\circ \to S_K^G,$$

where $\gamma(g, v)k = (\gamma g k, \gamma \cdot v)$ for all $\gamma \in G(\mathbb{Q})$, $k \in K K_\infty^\circ$. Consider the canonical fibration $\pi : (G(\mathbb{R})/K_\infty^\circ) \times G(\mathbb{A}_f)/K \to S_K^G$ given by going modulo the left action of $G(\mathbb{Q})$. Then, for any open $\mathcal{U} \subset S_K^G$, one has the sections $\mathcal{V}_K(\mathcal{U})$ over $\mathcal{U}$ to be the set of all locally constant $s : \pi^{-1}(\mathcal{U}) \to V$ such that $s(\gamma \cdot x) = \gamma \cdot s(x)$ for all $\gamma \in G(\mathbb{Q}), x \in \pi^{-1}(\mathcal{U})$. We denote by $\mathcal{V}_{K,E}^{\mu}$ the sheaf associated to V_E^{μ}. The sheaf $\mathcal{V}_{K,E}^{\mu}$ is non-trivial if and only if

$$\mu(Z(\mathbb{Q}) \cap K K_\infty^\circ) = \{1\}. \tag{15}$$

Condition (15) is always satisfied if μ is pure, because $\det(F^\times \cap K K_\infty^\circ) \subset \mathcal{O}_F^\times \cap F_\infty^{\times\circ}$.

In order to attach a sheaf to $V_{\mathcal{O}}^{\mu}$ we need a slightly different construction. Given a left K-module V satisfying (15) define $\mathcal{V}_K$ instead as the sheaf of locally constant sections of

$$G(\mathbb{Q}) \backslash (G(\mathbb{A}) \times V) / K K_\infty^\circ \to Y_K,$$

with left $G(\mathbb{Q})$-action and right $K K_\infty^\circ$-action given by $\gamma(g, v)k = (\gamma g k, k^{-1} \cdot v)$. As K acts on $V_{\mathcal{O}}^{\mu}$ through its p-component $K_p \subset G(\mathbb{Z}_p) \subset G(\mathcal{O})$ we obtain a sheaf $\mathcal{V}_{\mathcal{O}}^{\mu}$ on S_K^G.

When the actions of $G(\mathbb{Q})$ and K on V extend compatibly into a left action of $G(\mathbb{A})$, the two resulting local systems are isomorphic by $(g, v) \mapsto (g, g^{-1} \cdot v)$, justifying the abuse of notation.

1.4 Hecke operators

For any open compact subgroups $K' \subseteq K$ of $G(\mathbb{A}_f)$ the natural map $p_{K',K} : S_{K'}^G \to S_K^G$ induces an isomorphism of sheaves $p_{K',K}^* \mathcal{V}_K \xrightarrow{\sim} \mathcal{V}_{K'}$.

When the K-action on V extends to an action of a semi-group containing K and γ, then one can define a Hecke operator $[K\gamma K]$ as a composition of three maps:

$$[K\gamma K] = \mathrm{Tr}(p_{\gamma K\gamma^{-1}\cap K,K}) \circ [\gamma] \circ p^*_{K\cap\gamma^{-1}K\gamma,K} : \quad \mathrm{H}^q_c(S^G_K, \mathcal{V}_K) \to \mathrm{H}^q_c(S^G_K, \mathcal{V}_K),$$

where $p^*_{K\cap\gamma^{-1}K\gamma,K}$ is the pull-back, $\mathrm{Tr}(p_{\gamma K\gamma^{-1}\cap K,K})$ is the finite flat trace and

$$[\gamma] : \mathrm{H}^q_c(S^G_{K\cap\gamma^{-1}K\gamma}, \mathcal{V}_{K\cap\gamma^{-1}K\gamma}) \to \mathrm{H}^q_c(S^G_{\gamma K\gamma^{-1}\cap K}, \mathcal{V}_{\gamma K\gamma^{-1}\cap K}),$$

is induced by the morphism of local systems given by $(g,v) \mapsto (g\gamma^{-1}, \gamma\cdot v)$ in the case of a right K-action.

When $K_p \subset J_p$, the above construction applies to $V^\mu_{\mathcal{O}}$ on which the semi-group Λ_p acts by the $\bullet$-action (see (11)) yielding for each $t \in \Delta^+_p$ a Hecke operator $[KtK]$ on $\mathrm{H}^q_c(S^G_K, \mathcal{V}^\mu_{K,\mathcal{O}})$. Note that although the natural inclusion $V^\mu_{\mathcal{O}} \subseteq V^\mu_E$ is K_p-equivariant, it is not Λ_p-equivariant (see (12)). As a consequence, the natural map $\mathrm{H}^q_c(S^G_K, \mathcal{V}^\mu_{K,\mathcal{O}}) \to \mathrm{H}^q_c(S^G_K, \mathcal{V}^\mu_{K,E})$ is equivariant for the $\bullet$-action of $[KtK]$ on the source and the action of the optimally integral Hecke operator $[KtK]^\circ = \mu^\vee(t)[KtK]$ on the target. To ensure compatibility with extension of scalars, we also denote by $[KtK]^\circ$ the Hecke operator $[KtK]$ acting (via the $\bullet$-action) on $\mathrm{H}^q_c(S^G_K, \mathcal{V}^\mu_{K,\mathcal{O}})$.

For any prime $\mathfrak{p} \mid p$ of F, the following Hecke operators play an important role:

$$U_{\mathfrak{p}} = [Kt_{\mathfrak{p}}K] \quad \text{and} \quad U^\circ_{\mathfrak{p}} = \mu^\vee(t_{\mathfrak{p}})U_{\mathfrak{p}}. \tag{16}$$

For $\beta = (\beta_{\mathfrak{p}})_{\mathfrak{p}\mid p}$ with $\beta_{\mathfrak{p}} \in \mathbb{Z}_{\geqslant 0}$ we let $U_{p^\beta} = [Kt_{p^\beta}K]$ and $U^\circ_{p^\beta} = \mu^\vee(t_{p^\beta})U_{p^\beta}$.

As the image of $\mathrm{H}^q_c(S^G_K, \mathcal{V}^\mu_{\mathcal{O}})$ in $\mathrm{H}^q_c(S^G_K, \mathcal{V}^\mu_E)$ is a finitely generated $\mathcal{O}$-module, we may assume that E is large enough so that all $U^\circ_{\mathfrak{p}}$-eigenvalues belong to $\mathcal{O}$.

2. Distributions attached to cohomology classes for $\mathbf{GL}_{2n}$

Let $F_p = F \otimes_{\mathbb{Q}} \mathbb{Q}_p = \prod_{\mathfrak{p}\mid p} F_{\mathfrak{p}}$. For a prime $\mathfrak{p} \mid p$ of F we denote by $I_{\mathfrak{p}}$ (respectively, $J_{\mathfrak{p}}$) the standard Iwahori (respectively, parahoric) subgroup of $K^\circ_{\mathfrak{p}} = \mathrm{GL}_{2n}(\mathcal{O}_{F,\mathfrak{p}})$ consisting of elements whose reduction modulo the $\mathfrak{p}$ belongs to $B(\mathcal{O}_F/\mathfrak{p})$ (respectively, to $Q(\mathcal{O}_F/\mathfrak{p})$).

We let $K = K^{(p)} \times \prod_{\mathfrak{p}\mid p} K_{\mathfrak{p}}$ be an open compact subgroup of $G(\mathbb{A}_f)$ such that:

(K1) $K^{(p)}$ is the principal congruence subgroup of modulus $\mathfrak{m}$, an ideal of $\mathcal{O}_F$ which is relatively prime to p, and $K^{(p)}G(\mathbb{Z}_p)$ satisfies (14);

(K2) $\left(\begin{smallmatrix} T_n(\mathcal{O}_{F,\mathfrak{p}}) & M_n(\mathcal{O}_{F,\mathfrak{p}}) \\ \mathbf{0}_n & T_n(\mathcal{O}_{F,\mathfrak{p}}) \end{smallmatrix}\right) \subseteq K_{\mathfrak{p}} \subseteq J_{\mathfrak{p}}$ for all $\mathfrak{p} \mid p$.

An important role will be played by the matrix $\xi \in \mathrm{GL}_{2n}(\mathbb{A}_F)$, where

$$\xi_{\mathfrak{p}} = \begin{pmatrix} \mathbf{1}_n & w_n \\ \mathbf{0}_n & w_n \end{pmatrix} \in \mathrm{GL}_{2n}(\mathcal{O}_{F,\mathfrak{p}}), \quad \text{for all } \mathfrak{p} \mid p, \quad \text{and} \quad \xi_v = \mathbf{1}_{2n}, \quad \text{for all } v \nmid p. \tag{17}$$

Here $\mathbf{1}_n$ and $\mathbf{0}_n$ are the $n\times n$ identity and zero matrices, respectively, and w_n is the longest length element in the Weyl group of GL_n, whose (i,j)-entry is $\delta_{i,n-j+1}$.

We have $\xi^{-1}_{\mathfrak{p}} = \left(\begin{smallmatrix} \mathbf{1}_n & -\mathbf{1}_n \\ \mathbf{0}_n & w_n \end{smallmatrix}\right)$. Once and for all we record the identities

$$\xi^{-1}_{\mathfrak{p}} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \xi_{\mathfrak{p}} = \begin{pmatrix} A - C & (A - D + B - C)w_n \\ w_nC & w_n(C+D)w_n \end{pmatrix} \tag{18}$$

and

$$\xi_{\mathfrak{p}} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \xi_{\mathfrak{p}}^{-1} = \begin{pmatrix} A + w_n C & w_n D w_n - A + B w_n - w_n C \\ w_n C & w_n D w_n - w_n C \end{pmatrix}. \tag{19}$$

2.1 Automorphic cycles

For any open-compact subgroup $L \subset H(\mathbb{A}_f)$ we consider the locally symmetric space:

$$\tilde{S}_L^H = H(\mathbb{Q}) \backslash H(\mathbb{A}) / L L_\infty^\circ, \quad \text{where } L_\infty = H_\infty \cap K_\infty. \tag{20}$$

Note that for each $\sigma \in \Sigma_\infty$ one has $L_\sigma^\circ \simeq \left(\begin{smallmatrix} \mathrm{SO}_n(\mathbb{R}) & 0 \\ 0 & \mathrm{SO}_n(\mathbb{R}) \end{smallmatrix}\right)\mathbb{R}^{\times\circ}$. As in (14), $\tilde{S}_L^H$ is a real manifold when L is *sufficiently small* in the sense that for all $h \in H(\mathbb{A})$,

$$H(\mathbb{Q}) \cap h L L_\infty^\circ h^{-1} = Z(\mathbb{Q}) \cap L L_\infty^\circ. \tag{21}$$

Recall the notation $t_{\mathfrak{p}} = \iota(\varpi_{\mathfrak{p}} \cdot \mathbf{1}_n, \mathbf{1}_n)$ where $\varpi_{\mathfrak{p}}$ is an uniformizer at $\mathfrak{p} \mid p$. Recall also that for $\beta = (\beta_{\mathfrak{p}})_{\mathfrak{p}|p}$ with $\beta_{\mathfrak{p}} \in \mathbb{Z}_{\geqslant 0}$ we let $p^\beta = \prod_{\mathfrak{p}|p} \varpi_{\mathfrak{p}}^{\beta_{\mathfrak{p}}}$ and $t_p^\beta = \prod_{\mathfrak{p}|p} t_{\mathfrak{p}}^{\beta_{\mathfrak{p}}} \in G(\mathbb{Q}_p)$.

For any ideal $\mathfrak{m}$ of $\mathcal{O}_F$, we denote by $I(\mathfrak{m})$ the open-compact subgroup of $\mathbb{A}_{F,f}^\times$ of modulus $\mathfrak{m}$, and we consider the strict idele class group:

$$\mathscr{C}\ell_F^+(\mathfrak{m}) = F^\times \backslash \mathbb{A}_F^\times / I(\mathfrak{m}) F_\infty^{\times\circ}.$$

We let $L_\beta = L^{(p)} \prod_{\mathfrak{p}|p} L_{\mathfrak{p}}^{\beta_{\mathfrak{p}}}$ be an open compact subgroup of $H(\mathbb{A}_f)$ such that:

(L1) $L^{(p)} = K^{(p)} \cap H$ is the principal congruence subgroup of modulus $\mathfrak{m}$; and
(L2) $L_{\mathfrak{p}}^{\beta_{\mathfrak{p}}} = H(F_{\mathfrak{p}}) \cap K_{\mathfrak{p}} \cap \xi t_{\mathfrak{p}}^{\beta_{\mathfrak{p}}} K_{\mathfrak{p}} t_{\mathfrak{p}}^{-\beta_{\mathfrak{p}}} \xi^{-1}$ for all $\mathfrak{p} \mid p$.

Note that conditions (K1) and (L1) imply (21), in particular that $\tilde{S}_{L_\beta}^H$ is a real manifold.

Lemma 2.1. *$L_{\mathfrak{p}}^{\beta_{\mathfrak{p}}}$ consists of elements $(h_1, h_2) \in \mathrm{GL}_n(\mathcal{O}_{F,\mathfrak{p}}) \times \mathrm{GL}_n(\mathcal{O}_{F,\mathfrak{p}})$ such that*

$$\iota(h_1, h_2) \in K_{\mathfrak{p}} \cap \begin{pmatrix} \mathbf{1}_n & \\ & w_n \end{pmatrix} K_{\mathfrak{p}} \begin{pmatrix} \mathbf{1}_n & \\ & w_n \end{pmatrix}, \quad \textit{and} \quad h_1 h_2^{-1} \in 1 + \varpi_{\mathfrak{p}}^{\beta_{\mathfrak{p}}} M_n(\mathcal{O}_{F,\mathfrak{p}}).$$

Proof. By (18) for all $(h_1, h_2) \in H(F_{\mathfrak{p}}) \cap K_{\mathfrak{p}} = \mathrm{GL}_n(\mathcal{O}_{F,\mathfrak{p}}) \times \mathrm{GL}_n(\mathcal{O}_{F,\mathfrak{p}})$ one has

$$t_{\mathfrak{p}}^{-\beta_{\mathfrak{p}}} \xi^{-1} \begin{pmatrix} h_1 & \\ & h_2 \end{pmatrix} \xi t_{\mathfrak{p}}^{\beta_{\mathfrak{p}}} = \begin{pmatrix} h_1 & \varpi^{-\beta_{\mathfrak{p}}}(h_1 - h_2) w_n \\ & w_n h_2 w_n \end{pmatrix}.$$

Hence, $h_1 - h_2 \in \varpi_{\mathfrak{p}}^{\beta_{\mathfrak{p}}} M_n(\mathcal{O}_{F,\mathfrak{p}})$, and as $\left(\begin{smallmatrix} \mathbf{1}_n & M_n(\mathcal{O}_{F,\mathfrak{p}}) \\ \mathbf{0}_n & \mathbf{1}_n \end{smallmatrix}\right) \subseteq K_{\mathfrak{p}}$, we obtain $(h_1, w_n h_2 w_n) \in K_{\mathfrak{p}}$. □

Lemma 2.1 implies that the map $(1 + \varpi_{\mathfrak{p}}^{\beta_{\mathfrak{p}}} M_n(\mathcal{O}_{F,\mathfrak{p}})) \times \mathcal{O}_{F,\mathfrak{p}}^\times \to \det(L_{\mathfrak{p}}^{\beta_{\mathfrak{p}}})$ sending (x, y) to (xy, y) is an isomorphism. By the strong approximation theorem for $\mathrm{SL}_n(\mathbb{A}_F)$ the map

$$(h_1, h_2) \mapsto \left(\frac{\det(h_1)}{\det(h_2)}, \det(h_2) \right),$$

identifies the set of connected components of $\tilde{S}_{L_\beta}^H$ with a product of two idele class groups:

$$\pi_0(\tilde{S}_{L_\beta}^H) \xrightarrow{\sim} \mathscr{C}\ell_F^+(p^\beta \mathfrak{m}) \times \mathscr{C}\ell_F^+(\mathfrak{m}). \tag{22}$$

It is easy to see that the fiber $\tilde{S}^H_{L_\beta}[\delta]$ of $[\delta] \in \pi_0(\tilde{S}^H_{L_\beta})$ is connected of dimension

$$t = |\Sigma_\infty|(n^2 + n - 1). \tag{23}$$

If we consider a cohomology class on S^G_K in degree t, and pull it back to $\tilde{S}^H_{L_\beta}[\delta]$, then we end up with a top-degree class. The degree t happens to be the top-most degree with non-vanishing cuspidal cohomology of S^G_K. This *magical numerology* is at the heart of what ultimately permits us to give a cohomological interpretation to an integral representing an L-value (see [GR14]) and allows us to study its p-adic properties.

2.2 Evaluation maps

Fix $\mu \in X^*_0(T)$.

2.2.1 *Automorphic symbols.* By condition (L2), the map

$$\iota_\beta : \tilde{S}^H_{L_\beta} \to S^G_K, \quad [h] \mapsto [\iota(h)\xi t_p^\beta], \tag{24}$$

is well-defined. As ι_β is proper by a well-known result of Borel and Prasad (see, for example, [Ash80, Lemma 2.7]) one can consider the pull-back:

$$\iota^*_\beta : \mathrm{H}^q_c(S^G_K, \mathcal{V}^\mu_\mathcal{O}) \longrightarrow \mathrm{H}^q_c(\tilde{S}^H_{L_\beta}, \iota^*_\beta \mathcal{V}^\mu_\mathcal{O}). \tag{25}$$

2.2.2 *Twisting.* By condition (L1), the map $\iota : \tilde{S}^H_{L_\beta} \to S^G_K$, $[h] \mapsto [\iota(h)]$ is well-defined and proper. As $\xi t_p^\beta \in \Lambda_p$, using the $\bullet$-action from (12) one can consider the map

$$H(\mathbb{A}) \times V^\mu_\mathcal{O} \to H(\mathbb{A}) \times V^\mu_\mathcal{O}, \quad (h, v) \mapsto (h, (\xi t_p^\beta) \bullet v),$$

inducing a homomorphism of sheaves $\tau^\circ_\beta : \iota^*_\beta \mathcal{V}^\mu_\mathcal{O} \longrightarrow \iota^* \mathcal{V}^\mu_\mathcal{O}$, hence a map in cohomology

$$\tau^\circ_\beta : \mathrm{H}^q_c(\tilde{S}^H_{L_\beta}, \iota^*_\beta \mathcal{V}^\mu_\mathcal{O}) \longrightarrow \mathrm{H}^q_c(\tilde{S}^H_{L_\beta}, \iota^* \mathcal{V}^\mu_\mathcal{O}). \tag{26}$$

Similarly, using the natural action of $G(E)$ on V^μ_E instead of the $\bullet$-action one defines a map

$$\tau_\beta : \mathrm{H}^q_c(\tilde{S}^H_{L_\beta}, \iota^*_\beta \mathcal{V}^\mu_E) \longrightarrow \mathrm{H}^q_c(\tilde{S}^H_{L_\beta}, \iota^* \mathcal{V}^\mu_E), \tag{27}$$

and $\tau_\beta = \mu^\vee(t_p^{-\beta})\tau^\circ_\beta$, because by (12) one has $(\xi t_p^\beta) \bullet v = \mu^\vee(t_p^\beta)(\xi t_p^\beta) \cdot v$ for all $v \in V^\mu_E$.

2.2.3 *Critical maps.* For $j_1, j_2 \in \mathbb{Z}$ let $V^{(j_1,j_2)}$ be the 1-dimensional H-representation

$$(h_1, h_2) \mapsto \mathrm{N}_{F/\mathbb{Q}}(\det(h_1)^{j_1} \det(h_2)^{j_2}).$$

Let $V^{(j_1,j_2)}_\mathcal{O}$ be a free rank one $\mathcal{O}$-module on which the previously defined natural $H(\mathbb{Z}_p)$-action is extended to a $H(\mathbb{Q}_p)$-action by letting $p \in \mathbb{Q}_p^\times$ act trivially. Note that this action is similar to the Λ_p-action on $V^\mu_\mathcal{O}$ defined in § 1.2.

It follows from [GR14, Proposition 6.3] that $j \in \mathrm{Crit}(\mu)$ (see (4)) if and only if

$$\dim(\mathrm{Hom}_H(V^\mu, V^{(j,\mathsf{w}-j)})) = 1. \tag{28}$$

Fix a non-zero $\kappa_j \in \mathrm{Hom}_H(V^\mu, V^{(j,\mathsf{w}-j)})$ normalized so as to obtain an integral map:

$$\kappa_j : V^\mu_\mathcal{O} \to V^{(j,\mathsf{w}-j)}_\mathcal{O}.$$

Denoting by $\mathcal{V}_{\mathcal{O}}^{(j,\mathsf{w}-j)}$ the sheaf on $\tilde{S}^H_{L_\beta}$ attached to $V_{\mathcal{O}}^{(j,\mathsf{w}-j)}$ by the construction described in § 1.3, one obtains a homomorphism:

$$\kappa_j : \mathrm{H}^q_c(\tilde{S}^H_{L_\beta}, \iota^*\mathcal{V}^\mu_{\mathcal{O}}) \longrightarrow \mathrm{H}^q_c(\tilde{S}^H_{L_\beta}, \mathcal{V}_{\mathcal{O}}^{(j,\mathsf{w}-j)}). \tag{29}$$

Putting (25), (26) and (29) together, for each $j \in \mathrm{Crit}(\mu)$, we obtain a map:

$$\kappa_j \circ \tau_\beta^\circ \circ \iota_\beta^* : \mathrm{H}^q_c(S^G_K, \mathcal{V}^\mu_{\mathcal{O}}) \longrightarrow \mathrm{H}^q_c(\tilde{S}^H_{L_\beta}, \mathcal{V}_{\mathcal{O}}^{(j,\mathsf{w}-j)}). \tag{30}$$

2.2.4 *Trivializations.* Given any $\delta \in H(\mathbb{A}_f)$ the map

$$\mathrm{triv}_\delta : H(\mathbb{Q})\delta L_\beta H^\circ_\infty \times V_{\mathcal{O}}^{(j,\mathsf{w}-j)} \to H(\mathbb{Q})\delta L_\beta H^\circ_\infty \times V_{\mathcal{O}}^{(j,\mathsf{w}-j)}, \, (\gamma\delta l h_\infty, v) \mapsto (\gamma\delta l h_\infty, l_p^{-1}\cdot v),$$

is well-defined because $H(\mathbb{Q}) \cap L_\beta H^\circ_\infty \subset \ker(\mathrm{N}_{F/\mathbb{Q}} \circ \det)$ acts trivially on $V_{\mathcal{O}}^{(j,\mathsf{w}-j)}$. An easy check shows that triv_δ induces a homomorphism of local systems

$$\tilde{S}^H_{L_\beta}[\delta] \times V_{\mathcal{O}}^{(j,\mathsf{w}-j)} \to \big(\mathcal{V}_{\mathcal{O}}^{(j,\mathsf{w}-j)}\big)|_{\tilde{S}^H_{L_\beta}[\delta]},$$

where $[\delta]$ denotes the image of δ in $\pi_0(\tilde{S}^H_{L_\beta})$, hence yields a homomorphism:

$$\mathrm{triv}^*_\delta : \mathrm{H}^q_c(\tilde{S}^H_{L_\beta}[\delta], \mathcal{V}_{\mathcal{O}}^{(j,\mathsf{w}-j)}) \to \mathrm{H}^q_c(\tilde{S}^H_{L_\beta}[\delta], \mathbb{Z}) \otimes V_{\mathcal{O}}^{(j,\mathsf{w}-j)}.$$

We now render the trivializations independent of the choice of $\delta \in [\delta] \in \pi_0(\tilde{S}^H_{L_\beta})$. By definition, for any $\delta' \in H(\mathbb{Q})\delta l H_\infty$ one has

$$\mathrm{triv}^*_{\delta'} = (\mathrm{id} \otimes l_p^{-1}) \cdot \mathrm{triv}^*_\delta = \mathrm{N}^{-1}_{F_p/\mathbb{Q}_p}\big(\det(l_{1,p})^j \det(l_{2,p})^{\mathsf{w}-j}\big)\,\mathrm{triv}^*_\delta\,. \tag{31}$$

The p-adic cyclotomic character ε seen as idele class character $F^\times\backslash\mathbb{A}_F^\times \to \mathbb{Z}_p^\times$ sends y to $\mathrm{N}_{F_p/\mathbb{Q}_p}(y_p)|y_f|_F \prod_{\sigma\in\Sigma_\infty} \mathrm{sgn}(y_\sigma)$, is trivial on $F_\infty^{\times\circ}$ and given by $\mathrm{N}_{F_p/\mathbb{Q}_p}$ on $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$. Hence,

$$\mathrm{triv}^*_{[\delta]} = \varepsilon\big(\det(\delta_1^j\delta_2^{\mathsf{w}-j})\big)\,\mathrm{triv}^*_\delta : \mathrm{H}^q_c(\tilde{S}^H_{L_\beta}[\delta], \mathcal{V}_{\mathcal{O}}^{(j,\mathsf{w}-j)}) \to \mathrm{H}^q_c(\tilde{S}^H_{L_\beta}[\delta], \mathbb{Z}) \otimes V_{\mathcal{O}}^{(j,\mathsf{w}-j)}, \tag{32}$$

is independent of the particular choice of $\delta \in [\delta] \in \pi_0(\tilde{S}^H_{L_\beta})$.

2.2.5 *Connected components and fundamental classes.* Recall that for each $[\delta] \in \pi_0(\tilde{S}^H_{L_\beta})$, $\tilde{S}^H_{L_\beta}[\delta]$ is a t-dimensional connected orientable real manifold and that choosing an orientation amounts to choosing a fundamental class, that is, a basis $\theta_{[\delta]}$ of its Borel–Moore homology $H^{\mathrm{BM}}_t(\tilde{S}^H_{L_\beta}[\delta]) \simeq \mathbb{Z}$. We choose such orientations in a consistent manner when β and $[\delta]$ vary as follows. First, we fix, once and for all, an ordered basis on the tangent space of the symmetric space $H^\circ_\infty/L^\circ_\infty$ yielding fundamental classes θ_β of the connected components of identity $\tilde{S}^H_{L_\beta}[1]$, when β varies. Then for each $[\delta] \in \pi_0(\tilde{S}^H_{L_\beta})$, we consider the isomorphism $\tilde{S}^H_{L_\beta}[1] \xrightarrow[\sim]{\cdot\delta} \tilde{S}^H_{L_\beta}[\delta]$ and define $\theta_{[\delta]} = \delta_*\theta_\beta$, which is clearly independent of the particular choice of $\delta \in [\delta]$. Capping with $\theta_{[\delta]}$ and fixing a basis of $V_{\mathcal{O}}^{(j,\mathsf{w}-j)}$ (later in (40) we fix a particular basis in order to compare evaluations at different j) yields an isomorphism:

$$\mathrm{H}^t_c(\tilde{S}^H_{L_\beta}[\delta], \mathbb{Z}) \otimes V_{\mathcal{O}}^{(j,\mathsf{w}-j)} \xrightarrow{\sim} V_{\mathcal{O}}^{(j,\mathsf{w}-j)} \xrightarrow{\sim} \mathcal{O}.$$

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Combining this with (30) and (32) gives homomorphisms:

$$\begin{aligned}\mathcal{E}^{j,\mathsf{w}}_{\beta,\delta} &= (-\cap\theta_{[\delta]})\circ \mathrm{triv}^*_{\delta}\circ\kappa_j\circ\tau^{\circ}_{\beta}\circ\iota^*_{\beta} : \mathrm{H}^t_c(S^G_K,\mathcal{V}^{\mu}_{\mathcal{O}})\to\mathcal{O},\\ \mathcal{E}^{j,\mathsf{w}}_{\beta,[\delta]} &= \varepsilon\big(\det(\delta_1^j\delta_2^{\mathsf{w}-j})\big)\cdot\mathcal{E}^{j,\mathsf{w}}_{\beta,\delta} = (-\cap\theta_{[\delta]})\circ \mathrm{triv}^*_{[\delta]}\circ\kappa_j\circ\tau^{\circ}_{\beta}\circ\iota^*_{\beta} : \mathrm{H}^t_c(S^G_K,\mathcal{V}^{\mu}_{\mathcal{O}})\to\mathcal{O}.\end{aligned} \tag{33}$$

2.2.6 *Summing over the second component.* Consider a finite order $\mathcal{O}$-valued idele class character η_0 of F that is trivial on $I(\mathfrak{m})$, in particular unramified at all places above p. The character $\eta=\eta_0|\cdot|_F^{-\mathsf{w}}$ later plays a role when we discuss Shalika models for automorphic representations of G. The following map provides a section of (22):

$$\delta(x,y) = (\mathrm{diag}(xy,1,\ldots,1),\mathrm{diag}(y,1,\ldots,1))\in H. \tag{34}$$

When $(x,y)\in(\mathbb{A}_F^{\times})^2$ runs over a set of representatives of $\mathscr{C}\ell_F^+(p^{\beta}\mathfrak{m})\times\mathscr{C}\ell_F^+(\mathfrak{m})$, $\tilde{S}^H_{L_\beta}[\delta(x,y)]$ runs over the set of connected components of $\tilde{S}^H_{L_\beta}$. Define the level β evaluation:

$$\mathcal{E}^{j,\eta}_{\beta} = \sum_{[\bar{x}]\in\mathscr{C}\ell_F^+(p^{\beta})}\mathcal{E}^{j,\eta}_{\beta,[\bar{x}]}[\bar{x}],\quad\text{where } \mathcal{E}^{j,\eta}_{\beta,[\bar{x}]} = \sum_{[y]\in\mathscr{C}\ell_F^+(\mathfrak{m})}\sum_{[x]}\eta_0([y])\mathcal{E}^{j,\mathsf{w}}_{\beta,[\delta(x,y)]}, \tag{35}$$

where the last sum runs over all $[x]\in\mathscr{C}\ell_F^+(p^{\beta}\mathfrak{m})$ mapping to $[\bar{x}]$ under the natural projection.

The following diagram recapitulates the steps in the construction of $\mathcal{E}^{j,\eta}_{\beta}$.

$$\begin{array}{ccccc} \mathrm{H}^t_c(S^G_K,\mathcal{V}^{\mu}_{\mathcal{O}}) & & \xrightarrow{\kappa_j\circ\tau^{\circ}_{\beta}\circ\iota^*_{\beta}} & & \mathrm{H}^t_c(\tilde{S}^H_{L_\beta},\mathcal{V}^{(j,\mathsf{w}-j)}_{\mathcal{O}}) \\ \Big\downarrow{\scriptstyle \mathcal{E}^{j,\eta}_{\beta}} & & & & \Big\downarrow{\scriptstyle \sum_{[\delta]\in\pi_0(\tilde{S}^H_{L_\beta})}(-\cap\theta_{[\delta]})\circ\mathrm{triv}^*_{[\delta]}} \\ \mathcal{O}[\mathscr{C}\ell_F^+(p^{\beta})] & \xleftarrow{[(x,y)]\mapsto\eta_0([y])[\bar{x}]} & \mathcal{O}[\mathscr{C}\ell_F^+(p^{\beta}\mathfrak{m})\times\mathscr{C}\ell_F^+(\mathfrak{m})] & = & \mathcal{O}[\pi_0(\tilde{S}^H_{L_\beta})] \end{array} \tag{36}$$

2.3 Distributions on $\mathscr{C}\ell_F^+(p^{\infty})$

The object of this section is to relate when β varies the evaluation maps $\mathcal{E}^{j,\eta}_{\beta}$ whose definition is summarized in (36).

2.3.1 *The distributive property.* Fix a $\beta=(\beta_{\mathfrak{p}})_{\mathfrak{p}|p}$ with $\beta_{\mathfrak{p}}\in\mathbb{Z}_{>0}$ for all $\mathfrak{p}\mid p$.

THEOREM 2.2. *Given a prime $\mathfrak{p}\mid p$ we let $p^{\beta'}=p^{\beta}\mathfrak{p}$ and consider the canonical projection $\mathrm{pr}_{\beta',\beta}:\mathscr{C}\ell_F^+(p^{\beta'})\to\mathscr{C}\ell_F^+(p^{\beta})$. For all $[x]\in\mathscr{C}\ell_F^+(p^{\beta})$ we have $\mathcal{E}^{j,\eta}_{\beta}\circ U^{\circ}_{\mathfrak{p}}=\mathrm{pr}_{\beta',\beta}\circ\mathcal{E}^{j,\eta}_{\beta'}$, that is,*

$$\mathcal{E}^{j,\eta}_{\beta,[x]}\circ U^{\circ}_{\mathfrak{p}} = \sum_{[x']\in\mathrm{pr}^{-1}_{\beta',\beta}([x])}\mathcal{E}^{j,\eta}_{\beta',[x']}.$$

Proof. Using (33), (34), and (35) one has to show that for all $[x]\in\mathscr{C}\ell_F^+(p^{\beta}\mathfrak{m})$, $[y]\in\mathscr{C}\ell_F^+(\mathfrak{m})$:

$$\mathcal{E}^{j,\mathsf{w}}_{\beta,\delta(x,y)}\circ U^{\circ}_{\mathfrak{p}} = \sum_{[x']\in\mathrm{pr}^{-1}_{\beta',\beta}([x])}\mathrm{N}^j_{F_p/\mathbb{Q}_p}(u_{x'})\cdot\mathcal{E}^{j,\mathsf{w}}_{\beta',\delta(x',y)},$$

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where $u_{x'} \in I(p^\beta)$ is such that $x' \in F^\times x u_{x'} F_\infty^{\times\circ}$. We proceed as in the proof of [BDJ, Proposition 3.4]. Pulling back the definition of the Hecke operator $U_\mathfrak{p}^\circ$ (see § 1.4) by the automorphic symbols (see § 2.2.1) and the twisting operators (see § 2.2.2) yields the following commutative diagram (we use implicitly that $p_{K^0(\mathfrak{p}),K}$ and $\mathrm{pr}_{\beta',\beta}$ have the same degree as $L_\beta/L_{\beta'} \simeq M_n(\mathcal{O}/\mathfrak{p})$)

$$\begin{array}{ccccccc}
\mathrm{H}_c^t(S_K^G, \mathcal{V}_K^\mu) & \xrightarrow{p^*_{K_0(\mathfrak{p}),K}} & \mathrm{H}_c^t(S_{K_0(\mathfrak{p})}^G, \mathcal{V}_{K_0(\mathfrak{p})}^\mu) & \xrightarrow{[t_\mathfrak{p}]} & \mathrm{H}_c^t(S_{K^0(\mathfrak{p})}^G, \mathcal{V}_{K^0(\mathfrak{p})}^\mu) & \xrightarrow{\mathrm{Tr}(p_{K^0(\mathfrak{p}),K})} & \mathrm{H}_c^t(S_K^G, \mathcal{V}_K^\mu) \\
\downarrow{\scriptstyle \iota^*_{\beta'}} & & \downarrow{\scriptstyle \iota^*_{\beta'}} & & \downarrow{\scriptstyle \iota^*_{\beta}} & & \downarrow{\scriptstyle \iota^*_{\beta}} \\
\mathrm{H}_c^t(\tilde{S}_{L_{\beta'}}^H, \iota^*_{\beta'}\mathcal{V}_K^\mu) & = & \mathrm{H}_c^t(\tilde{S}_{L_{\beta'}}^H, \iota^*_{\beta'}\mathcal{V}_{K_0(\mathfrak{p})}^\mu) & \xrightarrow{[t_\mathfrak{p}]} & \mathrm{H}_c^t(\tilde{S}_{L_{\beta'}}^H, \iota^*_{\beta}\mathcal{V}_{K^0(\mathfrak{p})}^\mu) & \xrightarrow{\mathrm{Tr}\left(\mathrm{pr}_{\beta',\beta}\right)} & \mathrm{H}_c^t(\tilde{S}_{L_\beta}^H, \iota^*_{\beta}\mathcal{V}_K^\mu) \\
\downarrow{\scriptstyle \tau^\circ_{\beta'}} & & \downarrow{\scriptstyle \tau^\circ_{\beta'}} & & \downarrow{\scriptstyle \tau^\circ_{\beta}} & & \downarrow{\scriptstyle \tau^\circ_{\beta}} \\
\mathrm{H}_c^t(\tilde{S}_{L_{\beta'}}^H, \iota^*\mathcal{V}_K^\mu) & = & \mathrm{H}_c^t(\tilde{S}_{L_{\beta'}}^H, \iota^*\mathcal{V}_{K_0(\mathfrak{p})}^\mu) & = & \mathrm{H}_c^t(\tilde{S}_{L_{\beta'}}^H, \iota^*\mathcal{V}_{K^0(\mathfrak{p})}^\mu) & \xrightarrow{\mathrm{Tr}(\mathrm{pr}_{\beta',\beta})} & \mathrm{H}_c^t(\tilde{S}_{L_\beta}^H, \iota^*\mathcal{V}_K^\mu)
\end{array}$$

where the upper $[t_\mathfrak{p}]$ is induced by the morphism $(g, v) \mapsto (g \cdot t_\mathfrak{p}^{-1}, t_\mathfrak{p} \bullet v)$ of local systems, whereas the lower $[t_\mathfrak{p}]$ is induced by the morphism $(h, v) \mapsto (h, t_\mathfrak{p} \bullet v)$. Then

$$\begin{array}{ccc}
\mathrm{H}_c^t(\tilde{S}_{L_{\beta'}}^H[\delta(x', y)], \mathcal{V}_\mathcal{O}^{(j,\mathsf{w}-j)}) & \xrightarrow{\mathrm{Tr}\left(\mathrm{pr}_{\beta',\beta}\right)} & \mathrm{H}_c^t(\tilde{S}_{L_\beta}^H[\delta(x, y)], \mathcal{V}_\mathcal{O}^{(j,\mathsf{w}-j)}) \\
\downarrow{\scriptstyle \mathrm{triv}^*_{\delta(x',y)}} & & \downarrow{\scriptstyle \mathrm{triv}^*_{\delta(x,y)}} \\
\mathrm{H}_c^t(\tilde{S}_{L_\beta}^H[\delta(x', y)], \mathbb{Z}) \otimes V_\mathcal{O}^{(j,\mathsf{w}-j)} & \xrightarrow{\cdot \mathrm{N}^j_{F_p/\mathbb{Q}_p}(u_{x'})} & \mathrm{H}_c^t(\tilde{S}_{L_\beta}^H[\delta(x, y)], \mathbb{Z}) \otimes V_\mathcal{O}^{(j,\mathsf{w}-j)}
\end{array}$$

is another commutative diagram by (31), hence the claim. □

2.3.2 *Distributions for finite slope eigenvectors.* Let $\phi \in \mathrm{H}_c^t(S_K^G, \mathcal{V}_\mathcal{O}^\mu)$ be an eigenvector for $U_\mathfrak{p}^\circ$ with eigenvalue $\alpha_\mathfrak{p}^\circ$ for all $\mathfrak{p} \mid p$. Then, for all $\beta = (\beta_\mathfrak{p})_{\mathfrak{p}|p}$, it is an eigenvector for $U_{p^\beta}^\circ$ with eigenvalue $\alpha_{p^\beta}^\circ = \prod_{\mathfrak{p}|p} (\alpha_\mathfrak{p}^\circ)^{\beta_\mathfrak{p}}$. We say that ϕ is of *finite slope* if $\alpha_p^\circ \neq 0$ and in which case we define its slope as $v_p(\alpha_p^\circ)$. A eigenvector ϕ of slope 0 is called Q-ordinary. Being Q-ordinary is equivalent to saying that the $U_\mathfrak{p}$-eigenvalue $\alpha_\mathfrak{p}$ satisfies $|\alpha_\mathfrak{p}|_p = |\mu^\vee(t_\mathfrak{p})|_p^{-1}$ for all $\mathfrak{p} \mid p$ (see § 4.2 for more details).

Given any U_p-eigenvector ϕ of finite slope and any $j \in \mathrm{Crit}(\mu)$ by Theorem 2.2 one has a well-defined element

$$\boldsymbol{\mu}_\phi^{j,\eta} = ((\alpha_{p^\beta}^\circ)^{-1} \mathcal{E}_\beta^{j,\eta}(\phi))_\beta, \tag{37}$$

which is thought of as an E-valued distribution on $\mathcal{C}\ell_F^+(p^\infty)$.

We write $\mathrm{H}_c^t(S_K^G, \mathcal{V}_\mathcal{O}^\mu)^{Q-\mathrm{ord}}$ for the maximal $\mathcal{O}$-submodule of $\mathrm{H}_c^t(S_K^G, \mathcal{V}_\mathcal{O}^\mu)$ on which the operators $U_\mathfrak{p}^\circ$ are invertible for all $\mathfrak{p} \mid p$ (it is a direct $\mathcal{O}$-factor). Given any (not necessarily U_p°-eigen) non-torsion element $\phi \in \mathrm{H}_c^t(S_K^G, \mathcal{V}_\mathcal{O}^\mu)^{Q-\mathrm{ord}}$ one defines

$$\boldsymbol{\mu}_\phi^{j,\eta} = (\mathcal{E}_\beta^{j,\eta}((U_{p^\beta}^\circ)^{-1}(\phi)))_\beta \in \mathcal{O}[[\mathcal{C}\ell_F^+(p^\infty)]] = \varprojlim_\beta \mathcal{O}[\mathcal{C}\ell_F^+(p^\beta)], \tag{38}$$

which can be reinterpreted as a measure (i.e. a bounded distribution) on $\mathcal{C}\ell_F^+(p^\infty)$.

2.4 Manin relations

Consider the p-adic cyclotomic character $\varepsilon : \mathscr{C}\ell_F^+(p^\infty) \to \mathbb{Z}_p^\times$ defined by composing the norm $\mathrm{N}_{F/\mathbb{Q}} : \mathscr{C}\ell_F^+(p^\infty) \to \mathcal{C}\ell_{\mathbb{Q}}^+(p^\infty)$ with the p-adic cyclotomic character over $\mathbb{Q}$. In this section, we prove the following result.

THEOREM 2.3. *Let $\mu \in X_0^*(T)$ and suppose that j and $j+1$ both belong to* $\mathrm{Crit}(\mu)$. *For $\phi \in \mathrm{H}_c^t(S_K^G, \mathcal{V}_{\mathcal{O}}^\mu)^{Q-\mathrm{ord}}$ the following equality holds in $\mathcal{O}[[\mathscr{C}\ell_F^+(p^\infty)]]$:*

$$\varepsilon_{\mathrm{cyc}}(\boldsymbol{\mu}_\phi^{j,\eta}) = \boldsymbol{\mu}_\phi^{j+1,\eta},$$

where $\varepsilon_{\mathrm{cyc}}$ denotes the automorphism of $\mathcal{O}[[\mathscr{C}\ell_F^+(p^\infty)]]$ sending $[x]$ to $\varepsilon([x])[x]$. Hence,

$$\boldsymbol{\mu}_\phi^{\eta} = \varepsilon_{\mathrm{cyc}}^{-j}(\boldsymbol{\mu}_\phi^{j,\eta}) \in \mathcal{O}[[\mathscr{C}\ell_F^+(p^\infty)]], \tag{39}$$

is independent of $j \in$ $\mathrm{Crit}(\mu)$.

The overdetermination of $\boldsymbol{\mu}_\phi^\eta$ in the Q-ordinary case, when there are at least two critical values, plays a pivotal role in the proof of main theorem. Before embarking on the proof of this theorem, we begin with some technical preparation (see [Jan18, § 3]).

2.4.1 *Lie theoretic considerations.* By the distributive property (see Theorem 2.2) we may reduce to strict p^β-power level with integral exponents $\beta \in \mathbb{Z}_{>0}$, ignoring the finer components $\mathfrak{p} \mid p$ for simplicity of notation. Recall that $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ and $\mathfrak{q} = \mathfrak{h} \oplus \mathfrak{u}$. With the notation $t_p = \iota(p\mathbf{1}_n, \mathbf{1}_n)$, we observe for any $\beta \geqslant 0$ the relations

$$t_p^\beta \mathfrak{n}_{\mathcal{O}} t_p^{-\beta} \subseteq \mathfrak{n}_{\mathcal{O}}, \quad t_p^\beta \mathfrak{u}_{\mathcal{O}} t_p^{-\beta} = p^\beta \mathfrak{u}_{\mathcal{O}}.$$

Recall the matrix $\xi = \left(\begin{smallmatrix} \mathbf{1}_n & w_n \\ \mathbf{0}_n & w_n \end{smallmatrix}\right)$. A superscript ${}^{\xi}(-)$ denotes left conjugation action by ξ.

PROPOSITION 2.4. *We have the relations:*

(i) $\mathfrak{g}_{\mathcal{O}} = \mathfrak{h}_{\mathcal{O}} + {}^{\xi}\,\mathfrak{b}_{\mathcal{O}}^-$*; and*
(ii) ${}^{\xi}\big(\mathfrak{n}_{\mathcal{O}} \cap \mathfrak{h}_{\mathcal{O}}\big) \subseteq [\mathfrak{h}, \mathfrak{h}]_{\mathcal{O}} + {}^{\xi}\,\mathfrak{n}_{\mathcal{O}}^-$.

Proof. (i) As $\xi \in G(\mathcal{O})$, it suffices to verify it over E, where it amounts to show that $\dim_E(\mathfrak{h}_E \cap {}^{\xi}\,\mathfrak{b}_E^-) = n$. To this end, let l_1, l_2 be lower triangular matrices in $M_n(E)$ and $u \in M_n(E)$. Then

$$\xi \cdot \begin{pmatrix} l_1 & \\ u & l_2 \end{pmatrix} \cdot \xi^{-1} = \begin{pmatrix} l_1 + w_n u & w_n l_2 w_n - l_1 - w_n u \\ w_n u & w_n l_2 w_n - w_n u \end{pmatrix},$$

lies in $\mathfrak{h}_E$ if and only if $u = 0$ and $l_1 = w_n l_2 w_n$. Therefore, l_1 and l_2 are diagonal matrices determining each other uniquely.

(ii) Conjugation by ξ^{-1} reduces the claim to the problem of solving

$$\begin{pmatrix} n_1 & \\ & n_2 \end{pmatrix} = \begin{pmatrix} h_1 & (h_1 - h_2) w_n \\ & w_n h_2 w_n \end{pmatrix} + \begin{pmatrix} \bar{n}_1 & \\ & \bar{n}_2 \end{pmatrix},$$

for given $\iota(n_1, n_2) \in \mathfrak{h} \cap \mathfrak{n}_{\mathcal{O}}$ and unknowns $\iota(h_1, h_2) \in [\mathfrak{h}_{\mathcal{O}}, \mathfrak{h}_{\mathcal{O}}]$ and $\iota(\bar{n}_1, \bar{n}_2) \in \mathfrak{n}^-$. The choice

$$h_1 = h_2 = n_1 + w_n n_2 w_n, \quad \bar{n}_1 = -w_n n_2 w_n, \quad \bar{n}_2 = -w_n n_1 w_n,$$

is a solution with the desired properties. □

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Corollary 2.5. *For any $\beta \geqslant 0$, the following relations hold inside $\mathcal{U}(\mathfrak{g}_{\mathcal{O}})$:*

(i) $\mathcal{U}(\mathfrak{g}_{\mathcal{O}}) = \mathcal{U}(\mathfrak{h}_{\mathcal{O}}) \cdot \mathcal{U}({}^{\xi}\mathfrak{b}^{-}_{\mathcal{O}})$*; and*
(ii) $\mathcal{U}({}^{\xi t_p^{\beta}}\mathfrak{n}_{\mathcal{O}}) \subseteq \mathcal{U}([\mathfrak{h},\mathfrak{h}]_{\mathcal{O}} + p^{\beta}\mathfrak{h}_{\mathcal{O}}) \cdot \mathcal{U}({}^{\xi}\mathfrak{n}^{-}_{\mathcal{O}} + p^{\beta}\,{}^{\xi}\mathfrak{b}^{-}_{\mathcal{O}})$.

Proof. (i) This is a consequence of Proposition 2.4(i) and the Poincaré–Birkhoff–Witt theorem.

(ii) The decomposition $\mathfrak{n}_{\mathcal{O}} = (\mathfrak{h}_{\mathcal{O}} \cap \mathfrak{n}_{\mathcal{O}}) \oplus \mathfrak{u}_{\mathcal{O}}$, gives $t_p^{\beta}\mathfrak{n}_{\mathcal{O}}t_p^{-\beta} = (\mathfrak{h}_{\mathcal{O}} \cap \mathfrak{n}_{\mathcal{O}}) \oplus p^{\beta}\mathfrak{u}_{\mathcal{O}}$. Conjugating by ξ, we obtain

$$\xi t_p^{\beta}\mathfrak{n}_{\mathcal{O}}t_p^{-\beta}\xi^{-1} = {}^{\xi}\big(\mathfrak{h}_{\mathcal{O}} \cap \mathfrak{n}_{\mathcal{O}}\big) \oplus p^{\beta}\,{}^{\xi}\mathfrak{u}_{\mathcal{O}}.$$

Applying Proposition 2.4(ii) to the first summand and Proposition 2.4(i) to the second we obtain

$$\xi t_p^{\beta}\mathfrak{n}_{\mathcal{O}}t_p^{-\beta}\xi^{-1} \subseteq \big([\mathfrak{h},\mathfrak{h}]_{\mathcal{O}} + p^{\beta}\mathfrak{h}_{\mathcal{O}}\big) + \big({}^{\xi}\mathfrak{n}^{-}_{\mathcal{O}} + p^{\beta}\,{}^{\xi}\mathfrak{b}^{-}_{\mathcal{O}}\big).$$

One concludes again by the Poincaré–Birkhoff–Witt theorem, because the sums within the parentheses on the right-hand side are Lie $\mathcal{O}$-algebras. □

2.4.2 *Lattices and the projection formula.* Recall from (5) the lowest weight vector $v_0 \in V_E^{\mu}$ and the $G(\mathcal{O})$-lattice $V_{\mathcal{O}}^{\mu} = \mathcal{U}(\mathfrak{g}_{\mathcal{O}}) \cdot v_0 = \mathcal{U}(\mathfrak{n}_{\mathcal{O}}) \cdot v_0$. Recall also the $\bullet$-action of the semi-group Λ_p on $V_{\mathcal{O}}^{\mu}$ as in (12).

Given $j \in \mathrm{Crit}(\mu)$ recall from § 2.2.3 the map $\kappa_j : V_{\mathcal{O}}^{\mu} \to V_{\mathcal{O}}^{(j,\mathsf{w}-j)}$. By Corollary 2.5(i)

$$V_{\mathcal{O}}^{\mu} = \mathcal{U}(\mathfrak{h}_{\mathcal{O}}) \cdot \xi v_0,$$

which implies that $\kappa_j(\xi v_0)$ is an $\mathcal{O}$-basis of $V_{\mathcal{O}}^{(j,\mathsf{w}-j)}$ yielding a surjective $\mathcal{O}$-linear map

$$\kappa_j^{\circ} : V_{\mathcal{O}}^{\mu} \to \mathcal{O}, \quad \text{defined by } \kappa_j(v) = \kappa_j^{\circ}(v)\kappa_j(\xi v_0). \tag{40}$$

It is independent from the choice of κ_j because of (28), and $\kappa_j^{\circ}(\xi v_0) = 1$. We now come to the main technical result that is at the heart of our proof of the Manin relations.

Proposition 2.6. *For any $\beta \geqslant 0$, $v \in (\xi t_p^{\beta}) \bullet V_{\mathcal{O}}^{\mu} \subset V_{\mathcal{O}}^{\mu}$, and for all $j, j' \in \mathrm{Crit}(\mu)$, we have*

$$\kappa_j^{\circ}(v) \equiv \kappa_{j'}^{\circ}(v) \pmod{p^{\beta}}. \tag{41}$$

Proof. By (12) for $v \in (\xi t_p^{\beta}) \bullet V_{\mathcal{O}}^{\mu}$ there exists $m \in \mathcal{U}(\mathfrak{n}_{\mathcal{O}})$ with

$$v = \xi \cdot t_p^{\beta} \bullet (mv_0) = {}^{\xi t_p^{\beta}}m \cdot \xi(t_p^{\beta} \bullet v_0) = {}^{\xi t_p^{\beta}}m \cdot \xi v_0 \in \mathcal{U}({}^{\xi t_p^{\beta}}\mathfrak{n}_{\mathcal{O}}) \cdot \xi v_0.$$

By Corollary 2.5(ii) we can write

$${}^{\xi t_p^{\beta}}m = xy, \text{ with } x \in \mathcal{U}([\mathfrak{h},\mathfrak{h}]_{\mathcal{O}} + p^{\beta}\mathfrak{h}) \text{ and } y \in \mathcal{U}({}^{\xi}\mathfrak{n}^{-}_{\mathcal{O}} + p^{\beta\xi}\mathfrak{b}^{-}_{\mathcal{O}}).$$

Let $x_0, y_0 \in \mathcal{O}$ be the degree zero terms of x and y, respectively, and let

$$x_1 = x - x_0 \in ([\mathfrak{h},\mathfrak{h}]_{\mathcal{O}} + p^{\beta}\mathfrak{h}) \cdot \mathcal{U}([\mathfrak{h},\mathfrak{h}]_{\mathcal{O}} + p^{\beta}\mathfrak{h}),$$
$$y_1 = y - y_0 \in ({}^{\xi}\mathfrak{n}^{-}_{\mathcal{O}} + p^{\beta\xi}\mathfrak{b}^{-}_{\mathcal{O}}) \cdot \mathcal{U}({}^{\xi}\mathfrak{n}^{-}_{\mathcal{O}} + p^{\beta\xi}\mathfrak{b}^{-}_{\mathcal{O}}),$$

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be the higher degree terms in their respective enveloping algebras. Then

$$\begin{aligned}\kappa_j^\circ(v) &= \kappa_j^\circ(xy\cdot\xi v_0) = x\cdot\kappa_j^\circ(y\cdot\xi v_0) \quad (\text{because } \kappa_j^\circ \text{ is } H\text{-equivariant})\\ &\equiv x\cdot\kappa_j^\circ(y_0\cdot\xi v_0) \pmod{p^\beta} \quad (\text{because } {}^{\xi}\mathfrak{n}_{\mathcal{O}}^{-} \text{ acts trivially on } \xi v_0)\\ &\equiv x_0\cdot\kappa_j^\circ(y_0\cdot\xi v_0) \pmod{p^\beta} \quad (\text{because } [\mathfrak{h},\mathfrak{h}]_{\mathcal{O}} \text{ acts trivially on a line})\\ &= x_0y_0\cdot\kappa_j^\circ(\xi v_0) = x_0y_0,\end{aligned}$$

which does not depend on j as claimed. □

2.4.3 *Proof of Theorem 2.3.* As $\mathcal{O}[[\mathscr{C}\ell_F^+(p^\infty)]] = \varprojlim_\beta(\mathcal{O}/p^\beta\mathcal{O})[\mathscr{C}\ell_F^+(p^\beta)]$ and because $\varepsilon \pmod{p^\beta}$ factors through $\mathscr{C}\ell_F^+(p^\beta)$, it is enough to check that given $\beta \geqslant 1$ and $[x] \in \mathscr{C}\ell_F^+(p^\beta)$ one has

$$\varepsilon([x])\mathcal{E}^{j,\eta}_{\beta,[x]}(\phi) \equiv \mathcal{E}^{j+1,\eta}_{\beta,[x]}(\phi) \pmod{p^\beta}.$$

As, by (32), one has $\mathrm{triv}^*_{[\delta(x,y)]} = \varepsilon(x^jy^{\mathsf{w}})\,\mathrm{triv}^*_{\delta(x,y)}$, it suffices to show that (see (36))

$$\mathcal{E}^{j,\eta}_{\beta,\delta}(\phi) = (-\cap\theta_{[\delta]})\circ\mathrm{triv}^*_\delta\circ\kappa_j\circ\tau^\circ_\beta\circ\iota^*_\beta(\phi) \equiv (-\cap\theta_{[\delta]})\circ\mathrm{triv}^*_\delta\circ\kappa^*_{j+1}\circ\tau^\circ_\beta\circ\iota^*_\beta(\phi) \pmod{p^\beta}.$$

Now, by definition, the homomorphism of sheaves τ°_β defined in § 2.2.2 factors as

$$\iota^*_\beta\mathcal{V}^\mu_{\mathcal{O}} \longrightarrow \iota^*\big((\xi t_p^\beta)\bullet\mathcal{V}^\mu_{\mathcal{O}}\big) \longrightarrow \iota^*\mathcal{V}^\mu_{\mathcal{O}}.$$

Hence, Proposition 2.6 translates to the statement that (for the choice of basis of $V_{\mathcal{O}}^{(j,\mathsf{w}-j)}$ as in (40)) $(-\cap\theta_{[\delta]})\circ\mathrm{triv}^*_\delta\circ\kappa_j\circ\tau^\circ_\beta\circ\iota^*_\beta(\phi) \pmod{p^\beta}$ is independent of $j \in \mathrm{Crit}(\mu)$. □

3. Local considerations

We delineate some local calculations that are needed in the global considerations of the next section. For only this section, F denotes a finite extension of $\mathbb{Q}_p$, $\mathcal{O}$ its ring of integers, $\mathcal{P}$ the maximal ideal, $\varpi \in \mathcal{P}$ a uniformizer, $q = \#(\mathcal{O}/\mathcal{P})$, and δ the valuation of the different. We use local notation corresponding to the global notation introduced at the beginning of § 1. For example, $G = \mathrm{GL}_{2n}(F) \supset H = \mathrm{GL}_n(F)\times\mathrm{GL}_n(F)$.

3.1 Parahoric invariants

Let $K = \mathrm{GL}_{2n}(\mathcal{O})$ be the standard maximal compact subgroup of G. Define the parahoric (respectively, Iwahori) subgroup J (respectively, I) of K consisting of matrices whose reduction modulo $\mathcal{P}$ belongs to $Q(\mathcal{O}/\mathcal{P})$ (respectively, to $B(\mathcal{O}/\mathcal{P})$). One has

$$J = \left\{ \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in K \;\middle|\; a,d \in \mathrm{GL}_n(\mathcal{O}), c \in M_n(\mathcal{P}), b \in M_n(\mathcal{O})\right\}. \tag{42}$$

Let Π be an algebraic unramified and generic representation of G. Then there exists an unramified character $\lambda = \otimes_{i=1}^{2n}\lambda_i : T \to \bar{\mathbb{Q}}^\times \xrightarrow{i_\infty} \mathbb{C}^\times$ such that

$$\Pi = \mathrm{Ind}_B^G(|\cdot|^{(2n-1)/2}\lambda), \tag{43}$$

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where the right-hand side is the normalized parabolic induction, which differs from the usual induction by $\delta_B^{1/2}$ where

$$\delta_B(t_1,t_2,\dots,t_{2n}) = |t_1|^{2n-1}|t_2|^{2n-3}\cdot\dots\cdot|t_{2n}|^{1-2n}.$$

Recall Jacquet's exact functor sending an admissible G-representation V to the space of its co-invariants of U defined as $V_U = V/\langle\{u\cdot v - v \mid u\in U, v\in V\}\rangle$, which is an admissible H-representation. The Weyl groups of $G\supset H$ are given by $\mathfrak{S}_{2n}\simeq W_G\supset W_H\simeq(\mathfrak{S}_n\times\mathfrak{S}_n)$. The group W_G acts on the right on characters of T. There is a natural bijection:

$$W_G/W_H \xrightarrow{\sim} \{\tau\subset\{1,2,\dots,2n\} | \#\tau = n\},\ \ \rho\mapsto\{\rho(1),\dots,\rho(n)\}. \tag{44}$$

Lemma 3.1. *The semi-simplification of the Jacquet module Π_U is isomorphic to*

$$\bigoplus_{\tau\in W_G/W_H}\delta_Q^{1/2}\cdot \mathrm{Ind}_{B\cap H}^H(|\cdot|^{(2n-1)/2}\lambda^\tau), \tag{45}$$

where $\delta_Q(t_1,t_2,\dots,t_{2n}) = |t_1\dots t_n\cdot t_{n+1}^{-1}\dots t_{2n}^{-1}|^n$. The semi-simplification can be omitted if Π is regular in the sense that $\alpha_i=\lambda_i(\varpi)$ are pairwise distinct for $1\leqslant i\leqslant 2n$.

The characteristic polynomial of the Hecke operator $U_{\mathcal{P}}=[Jt_\varpi J]$ acting on Π^J equals

$$\prod_{\tau\in W_G/W_H}\left(X - q^{n(1-n)/2}\prod_{i\in\tau}\alpha_i\right).$$

Proof. The semi-simplification of the Jacquet module Π_N with respect to B is given by

$$\bigoplus_{\rho\in W_G}\delta_B^{1/2}|\cdot|^{(2n-1)/2}\lambda^\rho. \tag{46}$$

As $\mathrm{Ind}_B^G = \mathrm{Ind}_Q^G\,\mathrm{Ind}_{B\cap H}^H$, Frobenius reciprocity implies that any irreducible sub-quotient of the Jacquet module of Π with respect to Q is isomorphic to one of the summands in (45). The first claim then follows by a simple dimension count based on (46) and the transitivity of the Jacquet functors. By Bruhat decomposition,

$$G=\coprod_{\rho\in W_G}B\rho I=\coprod_{\rho\in W_G/W_H}B\rho J,\quad K=\coprod_{\rho\in W_G}(B\cap K)\rho I=\coprod_{\rho\in W_G/W_H}(B\cap K)\rho J, \tag{47}$$

the dimension of Π^J is $\#(W_G/W_H)$. By the Iwasawa decomposition $H=(B\cap H)\cdot(H\cap J)$,

$$\big(\mathrm{Ind}_{B\cap H}^H(\delta_Q^{1/2}|\cdot|^{(2n-1)/2}\lambda^\tau)\big)^{H\cap J}$$

is a line on which the central element $\iota(\mathbf{1}_n,\varpi\mathbf{1}_n)$ acts by $q^{n(1-n)/2}\prod_{i\in\tau}\alpha_i$. Under the assumption that Π is regular, the image of Π^J by the Jacquet functor equals the direct sum of the above lines when τ runs over W_G/W_H, hence the second claim. The proof of the third claim is a standard double coset computation based on (47) (see also [Hid98]). ☐

3.2 Twisted local Shalika integrals

We review the theory of global Shalika models and L-functions in § 4.1. The computations in this section are needed in § 4.3 to evaluate the twisted local zeta integral.

Fix an additive character $\psi: F\to\mathbb{C}^\times$ of conductor $\varpi^{-\delta}$ and a multiplicative character $\eta: F^\times\to\mathbb{C}^\times$.

DEFINITION 3.2. We say that an admissible representation Π of G has a local (η,ψ)-Shalika model if there is a non-trivial (and, hence, injective) intertwining of $G=\mathrm{GL}_{2n}(F)$-modules

$$\mathcal{S}_\psi^\eta : \Pi \hookrightarrow \mathrm{Ind}_S^G(\eta\otimes\psi).$$

For any $W\in\mathrm{Ind}_S^G(\eta\otimes\psi)$ and for any quasi-character $\chi : F^\times\to\mathbb{C}^\times$ the zeta integral

$$\zeta(s;W,\chi)=\int_{\mathrm{GL}_n(F)} W\left(\begin{pmatrix} h & 0\\ 0 & 1_n\end{pmatrix}\right)\chi(\det(h))|\det(h)|^{s-1/2}\,dh \tag{48}$$

is absolutely convergent for $\Re(s)\gg 0$. The following result is due to Friedberg and Jacquet.

PROPOSITION 3.3 [FJ93, Propositions 3.1 and 3.2]. *Assume that Π has an (η,ψ)-Shalika model. Then for each $W\in\mathcal{S}_\psi^\eta(\Pi)$, there is a holomorphic function $P(s;W,\chi)$ such that*

$$\zeta(s;W,\chi)=L(s,\Pi\otimes\chi)P(s;W,\chi).$$

One may analytically continue $\zeta(s;W,\chi)$ by re-defining it as $L(s,\Pi\otimes\chi)P(s;W,\chi)$ for all $s\in\mathbb{C}$. Moreover, there exists a vector $W_\Pi\in\mathcal{S}_\psi^\eta(\Pi)$ such that all unramified quasi-characters $\chi : F^\times\to\mathbb{C}^\times$ and every $s\in\mathbb{C}$ one has

$$P(s;W_\Pi,\chi)=(q^{s-1/2}\chi(\varpi))^{\delta n}.$$

If Π is spherical, then W_Π can be taken to be the spherical vector $W_\Pi^\circ\in\mathcal{S}_\psi^\eta(\Pi)$ normalized by the condition $W_\Pi^\circ(\mathbf{1}_{2n})=1$.

For ramified twists, we need the following refinement of Proposition 3.3.

PROPOSITION 3.4. *Let $W\in\mathcal{S}_\psi^\eta(\Pi)$ be a parahoric invariant vector, that is,*

$$W\left(\begin{pmatrix} h & \\ & h\end{pmatrix}\begin{pmatrix}\mathbf{1}_n & X\\ & \mathbf{1}_n\end{pmatrix} gk\right)=\eta(\det h)\psi(\operatorname{tr} X)W(g), \tag{49}$$

for all $h\in\mathrm{GL}_n(F)$, $X\in M_n(F)$, $g\in G$ and $k\in J$. Then for every finite order character $\chi : F^\times\to\mathbb{C}^\times$ of conductor $\beta\geqslant 1$, and for all $s\in\mathbb{C}$ with $\Re(s)\gg 0$ one has

$$\zeta(s;W(-\cdot\xi t_\varpi^\beta),\chi)=\mathcal{G}(\chi)^n\cdot q^{\beta n(1-n)+(\beta+\delta)n(s-1/2)}W(t_\varpi^{-\delta}).$$

Proof. For any $h\in\mathrm{GL}_n(F)$ and $X\in M_n(\mathcal{O})$, the Shalika property (49) implies that

$$W\left(\begin{pmatrix} h & \\ & \mathbf{1}\end{pmatrix}\xi t_\varpi^\beta\right)=W\left(\begin{pmatrix} h & \\ & \mathbf{1}\end{pmatrix}\xi t_\varpi^\beta\begin{pmatrix}\mathbf{1} & X\\ & \mathbf{1}\end{pmatrix}\right)=\psi(\operatorname{tr}(h\varpi^\beta X w_n))\cdot W\left(\begin{pmatrix} h & \\ & \mathbf{1}\end{pmatrix}\xi t_\varpi^\beta\right),$$

hence, the zeta integral is supported over $\mathrm{GL}_n(F)\cap\varpi^{-\beta-\delta}M_n(\mathcal{O})$. In addition, for $h\in\mathrm{GL}_n(F)$:

$$W\left(\begin{pmatrix} h & \\ & \mathbf{1}\end{pmatrix}\xi t_\varpi^\beta\right)=W\left(\begin{pmatrix}\mathbf{1}_n & h\\ & \mathbf{1}_n\end{pmatrix}\begin{pmatrix} h & \\ & w_n\end{pmatrix}t_\varpi^\beta\right)=\psi(\operatorname{tr} h)\cdot W\left(\begin{pmatrix} h\varpi^\beta & \\ & \mathbf{1}_n\end{pmatrix}\right).$$

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Using this and changing the variable $h \mapsto h\varpi^{-\beta'}$ with $\beta' = \beta + \delta$ yields

$$\zeta(s; W(-\cdot \xi t_\varpi^\beta), \chi) = \int_{\mathrm{GL}_n(F)\cap M_n(\mathcal{O})} W\left(\begin{pmatrix} h\varpi^{-\delta} & \\ & \mathbf{1}_n \end{pmatrix}\right)\psi(\mathrm{tr}(h\varpi^{-\beta'}))(\chi|\cdot|^{s-1/2})(\det(h\varpi^{-\beta'}))\,dh. \quad (50)$$

Denote by $(e_{ij})_{1\leqslant i,j\leqslant n}$ the standard basis of $M_n(\mathcal{O})$. As W is parahoric invariant, for any $i \neq j$ and $c \in \mathcal{O}$, right translation by $\mathbf{1}_n + ce_{ij} \in \mathrm{SL}_n(\mathcal{O})$ in (50) yields

$$\zeta(s; W(-\cdot \xi t_\varpi^\beta), \chi) = \int dh\, W\left(\begin{pmatrix} h\varpi^{-\delta} & \\ & \mathbf{1}_n \end{pmatrix}\right)\psi(\mathrm{tr}(h\varpi^{-\beta'}))(\chi|\cdot|^{s-1/2})(\det(h\varpi^{-\beta'}))\int_{\mathcal{O}} dc\, \psi(ch_{ji}\varpi^{-\beta'}),$$

and observe that $\int_{\mathcal{O}} \psi(ch_{ji}\varpi^{-\beta'})\,dc = 0$ unless $h_{ji} \in \mathcal{P}^\beta$.

Similarly, right translation by $\mathbf{1}_n + (c-1)e_{ii}$ with $c \in \mathcal{O}^\times$ shows that (50) equals

$$\int W\left(\begin{pmatrix} h & \\ & \mathbf{1}_n \end{pmatrix}\right)\psi((\mathrm{tr}(h) - h_{ii})\varpi^{-\beta'})(\chi|\cdot|^{s-1/2})(\det(h\varpi^{-\beta'}))\left(\int_{\mathcal{O}^\times}\psi(ch_{ii}\varpi^{-\beta'})\chi(c)d^\times c\right)dh,$$

and $\int_{\mathcal{O}^\times}\psi(ch_{ii}\varpi^{-\beta'})\chi(c)\,d^\times c = 0$ unless $h_{ii} \in \mathcal{O}^\times$ as $\beta \geqslant 1$ equals the conductor of χ.

Therefore, one can further restrict the domain of integration in (50) to the congruence subgroup $\ker(\mathrm{GL}_n(\mathcal{O}) \to \mathrm{GL}_n(\mathcal{O}/\mathcal{P}^\beta)) \cdot T_n(\mathcal{O})$, which, by the Iwahori decomposition, may be identified to the product $N_n^-(\mathcal{P}^\beta) \times T_n(\mathcal{O}) \times N_n(\mathcal{P}^\beta)$, where T_n denotes the diagonal subgroup of GL_n and N_n denotes the unipotent radical of the standard Borel subgroup B_n. Hence,

$$\zeta(s; W(-\cdot \xi t_\varpi^\beta), \chi) = q^{\beta' n(s-1/2)} W(t_\varpi^{-\delta})\int_{N_n^-(\mathcal{P}^\beta)T_n(\mathcal{O})N_n(\mathcal{P}^\beta)} \psi(\mathrm{tr}(k\varpi^{-\beta'}))\chi(\det(k\varpi^{-\beta'}))\,dk,$$

which can be simplified as

$$\begin{aligned} &q^{\beta n(1-n)+\beta' n(s-1/2)} W(t_\varpi^{-\delta})\prod_{1\leqslant i\leqslant n}\int_{\mathcal{O}^\times}\psi(t_i\varpi^{-\beta'})\chi(t_i\varpi^{-\beta'})\,d^\times t_i \\ &\quad = q^{\beta n(1-n)+(\beta+\delta)n(s-1/2)} W(t_\varpi^{-\delta})\cdot \mathcal{G}(\chi)^n, \end{aligned}$$

as desired. □

3.3 Non-vanishing of a local twisted zeta integral

In order to ensure the non-vanishing of the local twisted Shalika integral in Proposition 3.4, which is crucial for our applications, one has to exhibit a parahoric-spherical Shalika function W on G such that $W(t_\varpi^{-\delta}) \neq 0$. Assume that Π is a spherical representation isomorphic to $\mathrm{Ind}_B^G(|\cdot|^{(2n-1)/2}\lambda)$ as in (43) and let $\alpha_i = \lambda_i(\varpi)$, $1 \leqslant i \leqslant 2n$. Consider an unramified character η of $F^\times$.

Definition 3.5. Let $\tau \in W_G/W_H$ thought of as an n-element subset of $\{1, \ldots, 2n\}$ (see (44)). We say that $\tilde{\Pi} = (\Pi, \tau)$ is Q-regular if it satisfies the following two conditions:

- (i) $q^{n(1-n)/2}\prod_{i\in\tau}\alpha_i$ is a simple eigenvalue for $U_{\mathcal{P}} = [Jt_\varpi J]$ acting on Π^J;
- (ii) there exists $\rho \in \mathfrak{S}_{2n}$ such that for all $i \in \tau$, $\rho(i) \notin \tau$ and $\alpha_i\alpha_{\rho(i)} = q^{2n-1}\eta(\varpi)$.

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Assume that $\tilde{\Pi} = (\Pi, \tau)$ is Q-regular. Then condition (i) together with Lemma 3.1 implies

$$\prod_{i\in\tau, j\notin\tau} (\alpha_i - \alpha_j) \neq 0, \tag{51}$$

whereas condition (ii) implies by [AG94, Proposition 1.3] that Π admits a (η, ψ)-Shalika model.

Without loss of generality assume from now on that $\tau = \{n+1, \ldots, 2n\}$ and that $\rho \in \mathfrak{S}_{2n}$ is the order 2 element such that $\rho(i) = n + i$ for all $1 \leqslant i \leqslant n$. In [AG94, (1.3)] the authors construct an (η, ψ)-Shalika functional on Π sending $f \in \mathrm{Ind}_{B_{2n}}^{\mathrm{GL}_{2n}}(|\cdot|^{(2n-1)/2}\lambda)$ to

$$\mathcal{S}(f)(g) = \int_{B_n\backslash \mathrm{GL}_n} \int_{M_n} f\left(\begin{pmatrix} & \mathbf{1}_n \\ \mathbf{1}_n & X \end{pmatrix} \begin{pmatrix} h & \\ & h \end{pmatrix} g \right) \eta^{-1}(\det(h)) \bar{\psi}(\mathrm{tr}(X))\, dX\, dh. \tag{52}$$

By [AG94, Lemma 1.5], this integral converges in a certain domain and, when multiplied by (51), can be analytically continued to $\mathbb{C}^{2n}$, thus makes sense whenever (51) is non-zero. Let $f_0 \in \mathrm{Ind}_B^G(|\cdot|^{(2n-1)/2}\lambda)$ be the unique parahoric-spherical function supported on $Bw_{2n}J$ and characterized by $f_0\big(\left(\begin{smallmatrix} \mathbf{1}_n & \\ & \varpi^{-\delta}\mathbf{1}_n \end{smallmatrix}\right) w_{2n}\big) = q^{-\delta n^2}$. The following analogue of [AG94, Lemma 1.4] holds.

LEMMA 3.6. *Let $W = \mathcal{S}(f_0)$. Then $W(t_\varpi^{-\delta}) = 1$. Moreover, $U_{\mathcal{P}} \cdot f_0 = q^{n(1-n)/2}(\prod_{i=n+1}^{2n} \alpha_i) f_0$.*

Proof. By the Iwasawa decomposition $\mathrm{GL}_n = B_n K_n$ and as $\iota(K_n, K_n) \subset J$ we see that

$$\begin{aligned} W(t_\varpi^{-\delta}) = \mathcal{S}(f_0)(t_\varpi^{-\delta}) &= \int_{M_n} f_0\left(\begin{pmatrix} & \mathbf{1}_n \\ \mathbf{1}_n & X \end{pmatrix} t_\varpi^{-\delta} \right) \bar{\psi}(\mathrm{tr}(X))\, dX \\ &= \int_{M_n} f_0\left(\begin{pmatrix} \mathbf{1}_n & \\ & \varpi^{-\delta}\mathbf{1}_n \end{pmatrix} \left(\begin{pmatrix} & \mathbf{1}_n \\ \mathbf{1}_n & \varpi^{\delta} X \end{pmatrix} \right) \right) \bar{\psi}(\mathrm{tr}(X))\, dX \\ &= f_0\left(\begin{pmatrix} \mathbf{1}_n & \\ & \varpi^{-\delta}\mathbf{1}_n \end{pmatrix} w_{2n} \right) q^{\delta n^2} \int_{M_n} f_0\left(\begin{pmatrix} & \mathbf{1}_n \\ \mathbf{1}_n & X \end{pmatrix} \right) \bar{\psi}(\mathrm{tr}(\varpi^{-\delta} X))\, dX = 1. \end{aligned}$$

One checks that $\left(\begin{smallmatrix} & \mathbf{1}_n \\ \mathbf{1}_n & X \end{smallmatrix}\right) \in Bw_{2n}J$ if and only if $X \in M_n(\mathcal{O})$, in which case $\psi(\mathrm{tr}(\varpi^{-\delta}X)) = 1$. The parahoric decomposition of $J = (J \cap U^-)(J \cap Q) = (J \cap U^-)(J \cap Q)$ implies

$$Jt_\varpi J = \bigsqcup_{m\in M_n(\mathcal{O}/\mathcal{P})} \begin{pmatrix} \mathbf{1}_n & m \\ & \mathbf{1}_n \end{pmatrix} t_\varpi J. \tag{53}$$

By (47) it suffices to compute $(U_{\mathcal{P}} \cdot f_0)(\rho)$ for all $\rho \in W_G$. By the previous decomposition,

$$(U_{\mathcal{P}} \cdot f_0)(\rho) = \sum_{m\in M_n(\mathcal{O})/M_n(\mathcal{P})} f_0\left(\rho \begin{pmatrix} \varpi\mathbf{1}_n & m \\ & \mathbf{1}_n \end{pmatrix} \right).$$

Note that $\rho\left(\begin{smallmatrix} \mathbf{1}_n & m \\ & \mathbf{1}_n \end{smallmatrix}\right) t_\varpi$ belongs to the support $Bw_{2n}J = Bw_{2n}t_\varpi J = Bw_{2n}J^- t_\varpi$ of f_0 if and only if $\rho\left(\begin{smallmatrix} \mathbf{1}_n & m \\ & \mathbf{1}_n \end{smallmatrix}\right) \in K \cap Bw_{2n}J^- = (K \cap B)w_{2n}J^- = w_{2n}J^-$ (see (47)), which implies $\rho = w_{2n}$ and $m \in M_n(\mathcal{P})$. Hence, $(U_{\mathcal{P}} \cdot f_0)(\rho) = 0$ for all $\rho \neq w_{2n}$, while $(U_{\mathcal{P}} \cdot f_0)(w_{2n}) = f_0\big(w_{2n}\left(\begin{smallmatrix} \varpi\mathbf{1}_n & \\ & \mathbf{1}_n \end{smallmatrix}\right)\big) = f_0\big(\left(\begin{smallmatrix} \mathbf{1}_n & \\ & \varpi\mathbf{1}_n \end{smallmatrix}\right) w_{2n}\big) = q^{n(1-n)/2}(\prod_{i=n+1}^{2n} \alpha_i) f_0(w_{2n})$. ∎

4. L-functions for GL_{2n}

4.1 Global Shalika models and periods

This subsection contains a brief review of the necessary ingredients from [GR14] and a discussion involving p-adically integrally refined Betti–Shalika periods. Henceforth, Π denotes a (not necessarily unitary) cuspidal automorphic representation of $G(\mathbb{A}) = \mathrm{GL}_{2n}(\mathbb{A}_F)$. Keeping multiplicity one for GL_{2n} in mind, we let Π also denote its representation space within the space of cusp forms for $G(\mathbb{A})$. Fix the non-trivial additive unitary character $\psi : \mathbb{A}_F/F \longrightarrow \mathbb{A}/\mathbb{Q} \longrightarrow \mathbb{C}^\times$ where the first map is the trace, whereas the second is the usual additive character ψ_0 on $\mathbb{A}/\mathbb{Q}$ characterized by $\ker(\psi_0|_{\mathbb{Q}_\ell}) = \mathbb{Z}_\ell$ for every prime number ℓ and $\psi_0|_{\mathbb{R}}(x) = \exp(2\pi i x)$. We remark that $(\varpi_v^{-\delta_v})$, where δ_v is the valuation at v of the different $\mathfrak{d}$ of F, is the largest ideal contained in $\ker(\psi_v)$. The discriminant of F is $\mathrm{N}_{F/\mathbb{Q}}(\mathfrak{d})$.

4.1.1 *Global Shalika models.* Let $\eta : F^\times\backslash\mathbb{A}_F^\times \to \mathbb{C}^\times$ be a Hecke character such that η^n equals the central character ω_Π of Π. We obtain an automorphic character:

$$\eta \otimes \psi : S(F)\backslash S(\mathbb{A}_F) \to \mathbb{C}^\times, \quad \begin{pmatrix} h & hX \\ 0 & h \end{pmatrix} \mapsto \eta(\det(h))\psi(Tr(X)).$$

For a cusp form $\varphi \in \Pi$ and $g \in G(\mathbb{A})$ consider the integral

$$W_\varphi^\eta(g) = \int_{Z(\mathbb{A})S(F)\backslash S(\mathbb{A}_F)} \varphi(sg)(\eta\otimes\psi)^{-1}(s)\, ds, \tag{54}$$

where Haar measures are normalized as in [GR14, § 2.8]. It is well-defined by the cuspidality of the function φ (see [JS90, § 8.1]) and, hence, yields a function $W_\varphi^\eta : G(\mathbb{A}) \to \mathbb{C}$ such that

$$W_\varphi^\eta(sg) = (\eta\otimes\psi)(s)\cdot W_\varphi^\eta(g),$$

for all $g \in G(\mathbb{A})$ and $s \in S(\mathbb{A})$. In particular, we obtain an intertwining of $G(\mathbb{A})$-modules

$$\mathcal{S}_\psi^\eta : \Pi \to \mathrm{Ind}_{S(\mathbb{A})}^{G(\mathbb{A})}(\eta\otimes\psi), \quad \varphi \mapsto W_\varphi^\eta. \tag{55}$$

The following theorem, due to Jacquet and Shalika, gives a necessary and sufficient conditions for the existence of a non-zero intertwining as in (55).

Theorem 4.1 [JS90, Theorem 1]. *The following assertions are equivalent.*

- (i) *There exists $\varphi \in \Pi$ such that $W_\varphi^\eta \neq 0$.*
- (ii) *There exists an injection of $G(\mathbb{A})$-modules $\Pi \hookrightarrow \mathrm{Ind}_{S(\mathbb{A})}^{G(\mathbb{A})}(\eta\otimes\psi)$.*
- (iii) *The twisted partial exterior square L-function $\prod_{v\notin\Sigma_\Pi} L(s, \Pi_v, \wedge^2\otimes\eta_v^{-1})$ has a pole at $s = 1$, where Σ_Π is the set of places where Π is ramified.*

This is proved in [JS90] for unitary representations and its extension to the non-unitary case is easy. If Π satisfies any one, and hence all, of the equivalent conditions of Theorem 4.1, then we say that Π has an (η,ψ)-*Shalika model*, and we call the isomorphic image $\mathcal{S}_\psi^\eta(\Pi)$ of Π under (55) a *global* (η,ψ)-*Shalika model* of Π. Then clearly $\Pi\otimes\chi$ has an $(\eta\chi^2,\psi)$-Shalika model for any Hecke character χ, by keeping the same model and only twisting the action.

The following proposition (see [AS14]) gives another equivalent condition for Π to have a global Shalika model.

PROPOSITION 4.2. *Let Π be a cuspidal automorphic representation of $\mathrm{GL}_{2n}(\mathbb{A}_F)$ with central character ω_Π. Then the following assertions are equivalent:*

(i) *Π has a global (η,ψ)-Shalika model for some character η satisfying $\eta^n = \omega_\Pi$;*
(ii) *Π is the transfer of a globally generic cuspidal automorphic representation π of $\mathrm{GSpin}_{2n+1}(\mathbb{A}_F)$.*

In particular, if any of these equivalent conditions is satisfied, then Π is essentially self-dual, that is, $\Pi \cong \Pi^\vee \otimes \eta$. The character η may be taken to be the central character of π.

4.1.2 *Period integrals and L-functions.* The following proposition, due to Friedberg and Jacquet, is crucial for much that follows. It relates the period integral over H of a cusp form φ of G to a certain zeta integral of the function W_φ^η in the Shalika model corresponding to φ over one copy of GL_n.

PROPOSITION 4.3 [FJ93, Proposition 2.3]. *Assume that Π has an (η,ψ)-Shalika model. For $\varphi \in \Pi$,*

$$\Psi(s,\varphi,\chi,\eta) = \int_{Z(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})} \varphi\left(\begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}\right)(\chi|\cdot|^{s-1/2})\left(\frac{\det(h_1)}{\det(h_2)}\right)\eta^{-1}(\det(h_2))\,dh_1\,dh_2$$

converges absolutely for all $s \in \mathbb{C}$. For $\Re(s) \gg 0$ it is equal to

$$\zeta(s;W_\varphi^\eta,\chi) = \int_{\mathrm{GL}_n(\mathbb{A}_F)} W_\varphi^\eta\left(\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}\right)\chi(\det(h))|\det(h)|^{s-1/2}\,dh,$$

thus providing an analytic continuation of $\zeta(s;W_\varphi^\eta,\chi)$ to all of $\mathbb{C}$.

Suppose the representation Π of $G(\mathbb{A}) = \mathrm{GL}_{2n}(\mathbb{A}_F)$ decomposes as $\Pi = \otimes'_v \Pi_v$, where Π_v is an irreducible admissible representation of $\mathrm{GL}_{2n}(F_v)$.

If Π has a global Shalika model, then $\mathcal{S}_\psi^\eta$ defines local Shalika models at every place (see Definition 3.2). The corresponding local intertwining operators are denoted by $\mathcal{S}_{\psi_v}^{\eta_v}$ and their images by $\mathcal{S}_{\psi_v}^{\eta_v}(\Pi_v)$, whence $\mathcal{S}_\psi^\eta(\Pi) = \otimes'_v \mathcal{S}_{\psi_v}^{\eta_v}(\Pi_v)$. We can now consider cusp forms φ such that the function $W_\varphi \in \mathcal{S}_\psi^\eta(\Pi)$ is factorizable as $W_\varphi = \otimes'_v W_{\varphi_v}$, where

$$W_{\varphi_v} \in \mathcal{S}_{\psi_v}^{\eta_v}(\Pi_v) \subset \mathrm{Ind}_{S(F_v)}^{\mathrm{GL}_{2n}(F_v)}(\eta_v \otimes \psi_v).$$

Then the following factorization holds for $\Re(s) \gg 0$:

$$\zeta(s;W_\varphi,\chi) = \prod_v \zeta_v(s;W_{\varphi_v},\chi_v), \tag{56}$$

where the non-Archimedean local zeta integrals $\zeta_v(s;W_{\varphi_v},\chi_v)$ are related to L-functions in Proposition 3.3.

Proposition 4.3 relates this Shalika zeta integral to a period integral over H, and the main thrust of [AG94], refined and generalized in [GR14], is that the period integral over H admits a cohomological interpretation, provided that Π is of cohomological type.

4.1.3 *Shalika models and cuspidal cohomology.* In this section, we recall some well-known facts from Clozel [Clo90, § 3] (see also [GR14, § 3.4]). Assume from now on that the cuspidal automorphic representation Π is cohomological with respect to a dominant integral weight $\mu \in X^*_+(T)$ (see (3)), that is,

$$\mathrm{H}^q(\mathfrak{g}_\infty, K^\circ_\infty; \Pi \otimes V^\mu_{\mathbb{C}}) = \mathrm{H}^q(\mathfrak{g}_\infty, K^\circ_\infty; \Pi_\infty \otimes V^\mu_{\mathbb{C}}) \otimes \Pi_f \neq 0$$

for some degree q. A necessary condition for the non-vanishing of this cohomology group is that the weight μ is pure, that is, $\mu \in X^*_0(T)$. For each archimedean place $\sigma \in \Sigma_\infty$, Π_σ can be described explicitly as follows. For any integer $\ell \geqslant 1$ consider the unitary discrete series representation $D(\ell)$ of $\mathrm{GL}_2(\mathbb{R})$ of lowest non-negative SO_2-type $\ell+1$ and central character $\mathrm{sgn}^{\ell+1}$. Let P be the parabolic subgroup of GL_{2n} with Levi factor $\prod_{i=1}^n \mathrm{GL}_2$. Then

$$\Pi_\sigma \simeq \mathrm{Ind}_{P(\mathbb{R})}^{\mathrm{GL}_{2n}(\mathbb{R})}\left(\bigotimes_{i=1}^{n} D(2(\mu_{\sigma,i}+n-i)+1-\mathsf{w}) \otimes |\det|^{-\mathsf{w}/2}\right),$$

in particular $\omega_{\Pi_\sigma} = |\cdot|^{-n\mathsf{w}}$. The highest degree supporting cuspidal cohomology of G is $t = |\Sigma_\infty|(n^2+n-1)$. For any character ϵ of K_∞/K°_∞ the ϵ-eigenspace of

$$\mathrm{H}^t(\mathfrak{g}_\infty, K^\circ_\infty; \Pi_\infty \otimes V^\mu_{\mathbb{C}}) = \mathrm{Hom}_{K^\circ_\infty}\left(\wedge^t(\mathfrak{g}_\infty/\mathfrak{k}_\infty), \Pi_\infty \otimes V^\mu_{\mathbb{C}}\right), \tag{57}$$

is a line. If, in addition, Π admits an (η,ψ)-Shalika model, then η is forced to be algebraic of the form $\eta = \eta_0|\cdot|_F^{-\mathsf{w}}$ with η_0 of finite order, w is the purity weight of μ (see [GR13, Theorem 5.3]). Using the multiplicity one theorem for local Shalika models [Nie09] (see also [CS20]), one deduces that, for any character ϵ of K_∞/K°_∞,

$$\mathrm{H}^t(\mathfrak{g}_\infty, K^\circ_\infty; \mathcal{S}^{\eta_\infty}_{\psi_\infty}(\Pi_\infty) \otimes V^\mu_{\mathbb{C}})[\epsilon],$$

is a line, a basis Ξ^ϵ_∞ of which we fix in way compatible with twisting (see [GR14, Lemma 5.1.1]). The relative Lie algebra cohomology of Π as previously is a summand of the cuspidal cohomology, which, in turn, injects into the cohomology with compact supports (see [GR13, § 2])

$$\mathrm{H}^t(\mathfrak{g}_\infty, K^\circ_\infty; \Pi^K \otimes V^\mu_{\mathbb{C}}) \hookrightarrow \mathrm{H}^t_{\mathrm{cusp}}(S^G_K, \mathcal{V}^\mu_{\mathbb{C}}) \hookrightarrow \mathrm{H}^t_c(S^G_K, \mathcal{V}^\mu_{\mathbb{C}}). \tag{58}$$

We define an isomorphism Θ^ϵ of $G(\mathbb{A}_f)$-modules as the composition

$$\begin{aligned}\mathcal{S}^{\eta_f}_{\psi_f}(\Pi_f) &\xrightarrow{\sim} \mathcal{S}^{\eta_f}_{\psi_f}(\Pi_f) \otimes \mathrm{H}^t(\mathfrak{g}_\infty, K^\circ_\infty; \mathcal{S}^{\eta_\infty}_{\psi_\infty}(\Pi_\infty) \otimes V^\mu_{\mathbb{C}})[\epsilon] \\ &\xrightarrow{\sim} \mathrm{H}^t(\mathfrak{g}_\infty, K^\circ_\infty; \mathcal{S}^{\eta}_{\psi}(\Pi) \otimes V^\mu_{\mathbb{C}})[\epsilon] \xrightarrow{\sim} \mathrm{H}^t(\mathfrak{g}_\infty, K^\circ_\infty; \Pi \otimes V^\mu_{\mathbb{C}})[\epsilon],\end{aligned} \tag{59}$$

where the first map is $W_f \mapsto W_f \otimes \Xi^\epsilon_\infty$, the second map is the natural one and the third map is the map induced in cohomology by $(\mathcal{S}^\eta_\psi)^{-1}$ from (55). Taking K-invariants in (59) and composing with (58) yields a Hecke equivariant embedding

$$\Theta^\epsilon_K : \mathcal{S}^{\eta_f}_{\psi_f}(\Pi_f)^K \hookrightarrow \mathrm{H}^t_c(S^G_K, \mathcal{V}^\mu_{\mathbb{C}})[\epsilon]. \tag{60}$$

The reader should appreciate that the analytic condition on Π of admitting a Shalika model and the algebraic condition of contributing to the cuspidal cohomology of G are of an entirely different nature. One may construct examples of representations satisfying only one of these conditions and not the other (see [GR14, § 3.5]).

4.2 Ordinarity and regularity

For $\mathfrak{p}$ dividing p, we let $\varpi_\mathfrak{p}$ denote a uniformizer of $F_\mathfrak{p}$ and let $q_\mathfrak{p} = |\varpi_\mathfrak{p}|_p^{-1}$ denote the cardinality of its residue field. Let $\Sigma_\infty = \coprod_{\mathfrak{p}|p} \Sigma_\mathfrak{p}$ be the partition induced by $i_p : \bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$, where $\Sigma_\mathfrak{p} = \{\sigma : F_\mathfrak{p} \hookrightarrow \bar{\mathbb{Q}}_p\}$.

Recall from § 1.1 that a weight $\mu \in X_+^*(T)$ yields a rational character $\mu : T = \mathrm{Res}_{F/\mathbb{Q}} T_{2n} \to \mathrm{GL}_1$, therefore induces a character $\mu_p = \otimes_{\mathfrak{p}|p} \mu_\mathfrak{p}$ of $T(\mathbb{Q}_p) = \prod_{\mathfrak{p}|p} T_{2n}(F_\mathfrak{p})$, where

$$\begin{aligned} &\mu_\mathfrak{p} : T_{2n}(F_\mathfrak{p}) \to \bar{\mathbb{Q}}_p^\times \text{ is given by } (\mu_\sigma)_{\sigma\in\Sigma_\mathfrak{p}} \text{ subject to the dominance condition} \\ &\mu_{\sigma,1} \geqslant \mu_{\sigma,2} \geqslant \cdots \geqslant \mu_{\sigma,2n}, \text{ for all } \sigma \in \Sigma_\mathfrak{p}. \end{aligned} \tag{61}$$

Recall the maximal (n, n)-parabolic subgroup $Q \subseteq G$. Given a cuspidal automorphic representation Π of $G(\mathbb{A})$ that is cohomological with respect to the weight $\mu \in X_+^*(T)$, we say that $\Pi_\mathfrak{p}$ is *Q-ordinary* (respectively, *B-ordinary*) as in [Hid95, Hid98].

Assume from now on that $\Pi_\mathfrak{p}$ is unramified for all $\mathfrak{p} \mid p$. As Π is cohomological, there exists an unramified *algebraic* character $\lambda_\mathfrak{p} : T_{2n}(F_\mathfrak{p}) \to \bar{\mathbb{Q}}^\times \subset \mathbb{C}^\times$ such that (see (43))

$$\Pi_\mathfrak{p} = \mathrm{Ind}_{B_{2n}(F_\mathfrak{p})}^{\mathrm{GL}_{2n}(F_\mathfrak{p})}(|\cdot|^{(2n-1)/2}\lambda_\mathfrak{p}). \tag{62}$$

Using $i_p : \mathbb{C} \xrightarrow{\sim} \bar{\mathbb{Q}}_p$ allows us to see the Hecke parameters $\alpha_{\mathfrak{p},i} = \lambda_{\mathfrak{p},i}(\varpi_\mathfrak{p})$, $1 \leqslant i \leqslant 2n$, as elements of $\bar{\mathbb{Q}}_p^\times$. Then $\Pi_\mathfrak{p}$ is B-ordinary relative to an ordering of its Hecke parameters $\alpha_{\mathfrak{p},i}$ if and only if

$$\left|\mu_{\mathfrak{p},i}^\vee(\varpi_\mathfrak{p}) \cdot q_\mathfrak{p}^{1-i}\alpha_{\mathfrak{p},2n+1-i}\right|_p = 1, \quad \text{for all } 1 \leqslant i \leqslant 2n. \tag{63}$$

where $|\cdot|_p$ denotes the p-adic norm. The B-dominance condition (2) then implies that

$$|\alpha_{\mathfrak{p},1}|_p < |\alpha_{\mathfrak{p},2}|_p < \cdots < |\alpha_{\mathfrak{p},2n}|_p, \tag{64}$$

hence there exists at most one ordering of the Hecke parameters for which $\Pi_\mathfrak{p}$ is B-ordinary. Moreover, this implies that a B-ordinary $\Pi_\mathfrak{p}$ is necessarily regular, that is, the $\alpha_{\mathfrak{p},i}$ are pairwise distinct.

Similarly, $\Pi_\mathfrak{p}$ is Q-ordinary relative to $\tau \in W_G/W_H$ if and only if

$$\prod_{i\in\tau} |\alpha_{\mathfrak{p},i}|_p = \left|q_\mathfrak{p}^{n(n-1)/2}\mu_\mathfrak{p}^\vee(\iota(\varpi_\mathfrak{p}^{-1}\mathbf{1}_n, \mathbf{1}_n))\right|_p. \tag{65}$$

We make a key observation that Q-ordinarity implies Q-regularity (see Definition 3.5).

LEMMA 4.4. *Assume that the cuspidal automorphic representation Π of $G(\mathbb{A})$ is cohomological with respect to μ and admits an (η, ψ)-Shalika model. For $\mathfrak{p}$ dividing p, if $\Pi_\mathfrak{p}$ is spherical and Q-ordinary, then*

$$v_p(\alpha_{\mathfrak{p},i}) < |\Sigma_\mathfrak{p}|\frac{\mathsf{w} + 2n - 1}{2} < v_p(\alpha_{\mathfrak{p},i'}), \quad \textit{for all } i \in \tau,\ i' \notin \tau.$$

In particular, $\Pi_\mathfrak{p}$ is Q-ordinary only relative to τ. Moreover, $\tilde{\Pi}_\mathfrak{p} = (\Pi_\mathfrak{p}, \tau)$ is Q-regular, that is, $q_\mathfrak{p}^{n(1-n)/2}\prod_{i\in\tau}\alpha_{\mathfrak{p},i}$ is a simple eigenvalue of $U_\mathfrak{p}$ acting on $\Pi_\mathfrak{p}^{J_\mathfrak{p}}$ and $\lambda_{\mathfrak{p},i} \neq \lambda_{\mathfrak{p},i'}$ for all $i \in \tau, i' \notin \tau$.

Proof. Consider the Hecke operators $U_{\mathfrak{p},n-1} = [I_\mathfrak{p} t_{\mathfrak{p},n-1} I_\mathfrak{p}]$ acting on $\Pi_{\mathfrak{p},N}^{I_\mathfrak{p}}$, where $t_{\mathfrak{p},n-1} = \mathrm{diag}(\varpi_\mathfrak{p}\mathbf{1}_{n-1}, \mathbf{1}_{n+1}) \in \mathrm{GL}_{2n}(F_\mathfrak{p})$. As $\mu_\mathfrak{p}^\vee(t_{\mathfrak{p},n-1}) \cdot U_{\mathfrak{p},n-1}$ preserves p-integrality its eigenvalues on

$\Pi_{\mathfrak{p},N}^{I_\mathfrak{p}}$ are p-integral, in particular for any $i \in \tau$ we have

$$\prod_{i \neq i' \in \tau} |\alpha_{\mathfrak{p},i'}|_p \leqslant \big|\mu_\mathfrak{p}^\vee(t_{\mathfrak{p},n-1}^{-1}) \cdot q_\mathfrak{p}^{(n-1)(n-2)/2}\big|_p. \tag{66}$$

Together with (65) this implies that $|\alpha_{\mathfrak{p},i}|_p \geqslant |\mu_{\mathfrak{p},n}^\vee(\varpi_\mathfrak{p}^{-1}) q_\mathfrak{p}^{n-1}|_p = |\mu_{\mathfrak{p},n+1}(\varpi_\mathfrak{p}) q_\mathfrak{p}^{n-1}|_p$, that is,

$$v_p(\alpha_{\mathfrak{p},i}) \leqslant \sum_{\sigma \in \Sigma_\mathfrak{p}} (n - 1 + \mu_{\sigma,n+1}). \tag{67}$$

The existence of $(\eta_\mathfrak{p}, \psi_\mathfrak{p})$-Shalika model for $\Pi_\mathfrak{p}$ gives by [AG94, Proposition 1.3] an $i' = \rho(i)$ so that

$$v_p(\alpha_{\mathfrak{p},i}) + v_p(\alpha_{\mathfrak{p},i'}) = |\Sigma_\mathfrak{p}|(2n - 1 + \mathsf{w}). \tag{68}$$

The latter equality together with (3) and (67) yields, for all $i \in \tau$,

$$v_p(\alpha_{\mathfrak{p},i}) \leqslant \sum_{\sigma \in \Sigma_\mathfrak{p}} (n - 1 + \mu_{\sigma,n+1}) < |\Sigma_\mathfrak{p}| \frac{\mathsf{w} + 2n - 1}{2} < \sum_{\sigma \in \Sigma_\mathfrak{p}} (n + \mu_{\sigma,n}) \leqslant v_p(\alpha_{\mathfrak{p},\rho(i)}).$$

All claims then follow easily as clearly $\rho(\tau) \cap \tau = \varnothing$ as required by Definition 3.5. □

4.3 *p*-adic interpolation of critical values

We suppose in the sequel that Π is a cuspidal automorphic representation of $G(\mathbb{A})$ that is cohomological with respect to a pure weight $\mu \in X_0^*(T)$ and that Π admits an (η, ψ)-Shalika model. Assume further that for all $\mathfrak{p} \mid p$, $\Pi_\mathfrak{p}$ is spherical and that $\tilde{\Pi}_\mathfrak{p} = (\Pi_\mathfrak{p}, \tau)$ is *Q-regular* for $\tau = \{n+1, \dots, 2n\}$ in the sense of Definition 3.5.

4.3.1 *Choice of local Shalika vectors.* For $v \nmid p\infty$ we recall the vector $W_{\Pi_v} \in \mathcal{S}_{\psi_v}^{\eta_v}(\Pi_v)$ from Proposition 3.3.

For $\mathfrak{p} \mid p$, $\Pi_\mathfrak{p}$ is spherical and $\tilde{\Pi}_\mathfrak{p} = (\Pi_\mathfrak{p}, \{n+1, \dots, 2n\})$ is *Q-regular*, in particular, $\alpha_\mathfrak{p} = \prod_{n+1 \leqslant i \leqslant 2n} \alpha_{\mathfrak{p},i}$ is a simple eigenvalue for the Hecke operator $U_\mathfrak{p}$ acting on $\Pi_\mathfrak{p}^{J_\mathfrak{p}}$. By Lemma 3.6 there exists a unique $W_{\tilde{\Pi}_\mathfrak{p}}$ on the line $\mathcal{S}_{\psi_\mathfrak{p}}^{\eta_\mathfrak{p}}(\Pi_\mathfrak{p})^{J_\mathfrak{p}}[U_\mathfrak{p} - \alpha_\mathfrak{p}]$ normalized so that $W_{\tilde{\Pi}_\mathfrak{p}}(t_\mathfrak{p}^{-\delta_\mathfrak{p}}) = 1$. Let

$$W_{\tilde{\Pi}_f} = \otimes_{\mathfrak{p} \mid p} W_{\tilde{\Pi}_\mathfrak{p}} \bigotimes \otimes'_{v \nmid p\infty} W_{\Pi_v} \in \mathcal{S}_{\psi_f}^{\eta_f}(\Pi_f). \tag{69}$$

In addition to conditions (K1) and (K2) on K (see § 2), we henceforth assume that:

(K3) K fixes $W_{\tilde{\Pi}_f}$ and η is trivial on $I(\mathfrak{m})$, hence can be seen as a character of $\mathscr{C}\ell_F^+(\mathfrak{m})$.

For the local vectors at infinity, given any character ϵ of $K_\infty / K_\infty^\circ$ we recall the basis Ξ_∞^ϵ of the line $\mathrm{H}^t(\mathfrak{g}_\infty, K_\infty^\circ; \mathcal{S}_{\psi_\infty}^{\eta_\infty}(\Pi_\infty) \otimes V_\mathbb{C}^\mu)[\epsilon]$ from § 4.1.3. As in [GR14, § 4.1], we define the following.

Definition 4.5. Fix a basis $\{e_\alpha\}$ of $V_\mathbb{C}^\mu$ and a basis $\{\omega_i\}$ of $(\mathfrak{g}_\infty / \mathfrak{k}_\infty)^\vee$. For $\underline{i} = (i_1, \dots, i_t)$, with $1 \leqslant i_1 < \dots < i_t \leqslant \dim(\mathfrak{g}_\infty / \mathfrak{k}_\infty)$, we let $\omega_{\underline{i}} = \omega_{i_1} \wedge \dots \wedge \omega_{i_t} \in \bigwedge^t (\mathfrak{g}_\infty / \mathfrak{k}_\infty)^\vee$.

Using (57), Ξ_∞^ϵ can be written as a K_∞°-invariant element

$$\Xi_\infty^\epsilon = \sum_{\underline{i},\alpha} \omega_{\underline{i}} \otimes W_{\infty,\underline{i},\alpha}^\epsilon \otimes e_\alpha \in \wedge^t (\mathfrak{g}_\infty / \mathfrak{k}_\infty)^\vee \otimes \mathcal{S}_{\psi_\infty}^{\eta_\infty}(\Pi_\infty) \otimes V_\mathbb{C}^\mu, \tag{70}$$

for a unique choice of $W_{\infty,\underline{i},\alpha}^\epsilon \in \mathcal{S}_{\psi_\infty}^{\eta_\infty}(\Pi_\infty)$ called 'cohomological vectors at infinity'.

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For $\underline{i}$ and α as previously, let $\varphi^{\epsilon}_{\underline{i},\alpha} \in \Pi$ be the unique vector whose image under (55) equals

$$W^{\eta}_{\varphi^{\epsilon}_{\underline{i},\alpha}} = W_{\tilde{\Pi}_f} \otimes W^{\epsilon}_{\infty,\underline{i},\alpha} \in \mathcal{S}^{\eta}_{\psi}(\Pi).$$

For each character ϵ of K_∞/K_∞° the embedding Θ^{ϵ}_K defined in (60) yields

$$\mathcal{S}^{\eta_f}_{\psi_f}(\Pi_f) \xrightarrow{\Theta^{\epsilon}_K} \mathrm{H}^t_c(S^G_K, \mathcal{V}^{\mu}_{\mathbb{C}})[\epsilon] \xrightarrow{i_p^*} \mathrm{H}^t_c(S^G_K, \mathcal{V}^{\mu}_{\bar{\mathbb{Q}}_p})[\epsilon] \xrightarrow{(g,v)\mapsto(g,g^{-1}\cdot v)} \mathrm{H}^t_c(S^G_K, \mathcal{V}^{\mu}_{\bar{\mathbb{Q}}_p})[\epsilon].$$

The image $\phi^{\epsilon}_{\tilde{\Pi}}$ of $W_{\tilde{\Pi}_f}$ under the composition of these three maps belongs to $\mathrm{H}^t_c(S^G_K, \mathcal{V}^{\mu}_{E})[\epsilon]$ for some finite extension E of $\mathbb{Q}_p$, and after possibly rescaling the maps Θ^{ϵ}, that is, rescaling the basis elements Ξ^{ϵ}_{∞}, one can render the cohomology class $\mathcal{O}$-integral:

$$\phi^{\epsilon}_{\tilde{\Pi}} \in \mathrm{H}^t_c(S^G_K, \mathcal{V}^{\mu}_{\mathcal{O}})[\epsilon]. \tag{71}$$

Recall that the Hecke operator $U^{\circ}_{p^\beta}$ defines an endomorphism of $\mathrm{H}^t_c(S^G_K, \mathcal{V}^{\mu}_{\mathcal{O}})[\epsilon]$. As $W_{\tilde{\Pi}_f}$ is an U_{p^β}-eigenvector with eigenvalue $\alpha_{p^\beta} = \prod_{\mathfrak{p}|p} \alpha_{\mathfrak{p}}^{\beta_{\mathfrak{p}}}$, it follows (after possibly additionally rescaling by a power of p killing the torsion in $\mathrm{H}^t_c(S^G_K, \mathcal{V}^{\mu}_{\mathcal{O}})$) that one can assume $\phi^{\epsilon}_{\tilde{\Pi}}$ is an $U^{\circ}_{p^\beta}$-eigenvector with eigenvalue $\alpha^{\circ}_{p^\beta} = \mu^\vee(t_p^\beta)\alpha_{p^\beta}$.

4.3.2 *Interpolation formula at critical points.* In this section, we relate the image of $\phi^{\epsilon}_{\tilde{\Pi}}$ defined in (71) by the evaluation map $\mathcal{E}^{j,\eta}_{\beta,[\delta]}$ from (33), to the Friedberg–Jacquet integral from Proposition 4.3.

PROPOSITION 4.6. *For any character ϵ of K_∞/K_∞° and any $[\delta] \in \mathscr{C}\ell^+_F(p^\beta\mathfrak{m}) \times \mathscr{C}\ell^+_F(\mathfrak{m})$ we have*

$$i_p^{-1}\big(\mu^\vee(t_p^{-\beta}) \cdot \mathcal{E}^{j,\mathsf{w}}_{\beta,[\delta]}(\phi^{\epsilon}_{\tilde{\Pi}})\big) = \int_{\tilde{S}^H_{L_\beta}[\delta]} \varphi^{\epsilon}_{\tilde{\Pi},j}(h\xi t_p^\beta)|\det(h_1^j h_2^{\mathsf{w}-j})|_F \, dh,$$

where $\varphi^{\epsilon}_{\tilde{\Pi},j} = \sum_{\underline{i}} \sum_{\alpha} a^{\epsilon}_{\underline{i},\alpha,j} \cdot \varphi^{\epsilon}_{\underline{i},\alpha}$ for suitable $a^{\epsilon}_{\underline{i},\alpha,j} \in \mathbb{C}$.

In this proposition, and henceforth, $\sum_{\underline{i}}$ denotes summing over all $\underline{i} = (i_1, \dots, i_t)$ and $\sum_\alpha$ denotes summing over all $1 \leqslant \alpha \leqslant \dim(V^\mu)$, as in Definition 4.5. A more careful choice of the bases $\{e_\alpha\}$ and $\{\omega_i\}$ as in [Jan16, § 7] yields algebraic coefficients $a^{\epsilon}_{\underline{i},\alpha,j} \in \bar{\mathbb{Q}}$.

Proof. We follow closely the proof of [BDJ, Proposition 4.1]. Consider the commutative diagram

$$\begin{array}{ccc}
\mathrm{H}^t_c(S^G_K, \mathcal{V}^{\mu}_{E}) & \xrightarrow[\sim]{(g,v)\mapsto(g,g^{-1}\cdot v)} & \mathrm{H}^t_c(S^G_K, \mathcal{V}^{\mu}_{E}) \\
\downarrow{\scriptstyle \kappa_j \circ \mathcal{T}_\beta} & & \downarrow{\scriptstyle \kappa_j \circ \tau_\beta \circ \iota_\beta^*} \\
\mathrm{H}^t_c(\tilde{S}^H_{L_\beta}, \mathcal{V}^{(j,\mathsf{w}-j)}_{E}) & \xrightarrow[\sim]{(h,v)\mapsto(h,h^{-1}\cdot v)} & \mathrm{H}^t_c(\tilde{S}^H_{L_\beta}, \mathcal{V}^{(j,\mathsf{w}-j)}_{E}) \\
\downarrow{\scriptstyle (-\cap\theta_{[\delta]})\circ \mathrm{triv}'^{\,*}_\delta} & & \downarrow{\scriptstyle (-\cap\theta_{[\delta]})\circ \mathrm{triv}^*_\delta} \\
E & \xrightarrow[\sim]{\mathrm{N}_{F_p/\mathbb{Q}_p}\left(\det(\delta_{1,p}^j \delta_{2,p}^{\mathsf{w}-j})\right)} & E
\end{array}$$

where $\tau_\beta = \mu^\vee(t_p^{-\beta})\tau^\circ_\beta$ is defined in (27), the horizontal maps are induced from the morphisms of local systems written above them, the map $\mathcal{T}_\beta$ is induced from the morphisms of local systems

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$(h, v) \mapsto (h\xi t_p^\beta, v)$, and triv'_δ is induced from the morphisms of local systems:

$$H(\mathbb{Q})\delta L_\beta H_\infty^\circ \times V_{\mathbb{Q}(\tilde{\Pi})}^{(j,\mathsf{w}-j)} \to \mathcal{V}_{\mathbb{Q}(\tilde{\Pi})|\tilde{S}_{L_\beta}^H[\delta]}^{(j,\mathsf{w}-j)}, \quad (\gamma\delta\ell h_\infty, v) \mapsto (\gamma\delta\ell h_\infty, \gamma^{-1} \cdot v).$$

As $\mathcal{E}_{\beta,[\delta]}^{j,\mathsf{w}} = \varepsilon(\det(\delta_1^j \delta_2^{\mathsf{w}-j}))\mathcal{E}_{\beta,\delta}^{j,\mathsf{w}}$, with $\varepsilon_f = |\cdot|_{F,f}\, \mathrm{N}_{F_p/\mathbb{Q}_p}$, the previous diagram shows that the proposition is equivalent to

$$|\det(\delta_{1,f}^j \delta_{2,f}^{\mathsf{w}-j})|_F \big(-\cap\theta_{[\delta]}) \circ \mathrm{triv}_\delta^* \circ \kappa_j \circ \mathcal{T}_\beta\big)(\phi_{\tilde{\Pi}}^\epsilon) = \int_{\tilde{S}_{L_\beta}^H[\delta]} \varphi_{\tilde{\Pi},j}^\epsilon(h\xi t_p^\beta)|\det(h_1^j h_2^{\mathsf{w}-j})|_F\, dh, \qquad (72)$$

the left-hand side being considered over $\mathbb{C}$ via $i_p^{-1} : \bar{\mathbb{Q}}_p \xrightarrow{\sim} \mathbb{C}$. By Definition 4.5

$$\phi_{\tilde{\Pi}}^\epsilon = (\mathcal{S}_\psi^\eta)^{-1}(W_{\tilde{\Pi}_f} \otimes \Xi_\infty^\epsilon) = \sum_{\underline{i}} \sum_\alpha \omega_{\underline{i}} \otimes \varphi_{\underline{i},\alpha}^\epsilon \otimes e_\alpha \in \big(\wedge^t (\mathfrak{g}_\infty/\mathfrak{k}_\infty)^\vee \otimes \Pi \otimes V_{\mathbb{C}}^\mu\big)^{K_\infty^\circ},$$

yielding a $V_{\mathbb{C}}^\mu$-valued differential t-form on $G_\infty^\circ/K_\infty^\circ$. Now, recall the basis κ_j of the one-dimensional $\mathrm{Hom}_H(V^\mu, V^{(j,\mathsf{w}-j)})$ from (28) and consider the map

$$\kappa_j \circ \iota^* : \mathrm{H}^t(\mathfrak{g}_\infty, K_\infty^\circ; \Pi_\infty \otimes V_{\mathbb{C}}^\mu) \to \mathrm{H}^t(\mathfrak{h}_\infty, L_\infty^\circ; \Pi_\infty \otimes V_{\mathbb{C}}^{(j,\mathsf{w}-j)}).$$

We obtain

$$(\kappa_j \circ \mathcal{T}_\beta)(\phi_{\tilde{\Pi}}^\epsilon) = \sum_{\underline{i}} \sum_\alpha \iota^*\omega_{\underline{i}} \otimes \varphi_{\underline{i},\alpha}^\epsilon(-\cdot \xi t_p^\beta) \otimes \kappa_j(e_\alpha) \in \big(\wedge^t (\mathfrak{h}_\infty/\mathfrak{l}_\infty)^\vee \otimes \Pi \otimes V_{\mathbb{C}}^{(j,\mathsf{w}-j)}\big)^{L_\infty^\circ}.$$

First, let $\kappa_j^\circ : V_{\mathbb{C}}^{(j,\mathsf{w}-j)} \xrightarrow{\sim} \mathbb{C}$ be the scalar extension of (40), and so $\kappa_j(e_\alpha)$ corresponds to a complex number. Next, after the discussion in the paragraph following (23), we can fix a basis for the top exterior $\wedge^t(\mathfrak{h}_\infty/\mathfrak{l}_\infty)^\vee$ corresponding to the Haar measure dh_∞; hence $\iota^*\omega_{\underline{i}}$ is a scalar multiple of dh_∞. Putting both together, the restriction to $\tilde{S}_{L_\beta}^H[\delta]$ of $\kappa_j(\mathcal{T}_\beta(\phi_{\tilde{\Pi}}^\epsilon))$ can be seen as $V_{\mathbb{C}}^{(j,\mathsf{w}-j)}$-valued top-degree differential form on $H_\infty^\circ/L_\infty^\circ$ given by

$$\sum_{\underline{i}} \sum_\alpha a_{\underline{i},\alpha,j}^\epsilon \cdot \varphi_{\underline{i},\alpha}^\epsilon(\delta h_\infty \xi t_p^\beta) \det(h_{1,\infty}^j h_{2,\infty}^{\mathsf{w}-j})\, dh_\infty$$

for suitable $a_{\underline{i},\alpha,j}^\epsilon \in \mathbb{C}$. Writing $h = \gamma\delta l h'_\infty \in H(\mathbb{Q})\delta L_\beta H_\infty^\circ \subset H(\mathbb{A})$, and using $\mathrm{triv}'_\delta(\gamma\delta\ell h'_\infty, v) = (\gamma\delta\ell h'_\infty, \det(\gamma_{1,\infty}^{-j}\gamma_{2,\infty}^{j-\mathsf{w}})v)$ one obtains (72) and the proposition from

$$\begin{aligned}|\det(\delta_{1,f}^j \delta_{2,f}^{\mathsf{w}-j})|_F \det(\gamma_{1,\infty}^{-j}\gamma_{2,\infty}^{j-\mathsf{w}}) \det(h_{1,\infty}^j h_{2,\infty}^{\mathsf{w}-j}) &= |\det(\delta_{1,f}^j \delta_{2,f}^{\mathsf{w}-j})|_F \det(h_{1,\infty}^{\prime j} h_{2,\infty}^{\prime \mathsf{w}-j}) \\ &= |\det(h_1^j h_2^{\mathsf{w}-j})|_F. \qquad \square\end{aligned}$$

4.3.3 *p-adic distributions attached to* $\tilde{\Pi}$. Recall that Π is a cuspidal automorphic representation of $G(\mathbb{A})$ admitting a global (ψ, η)-Shalika model, which is cohomological with respect to a pure dominant integral weight μ. Recall also that $\Pi_{\mathfrak{p}}$ is spherical for all $\mathfrak{p} \mid p$ and that $\tilde{\Pi}_{\mathfrak{p}} = (\Pi_{\mathfrak{p}}, \{n+1, \ldots, 2n\})$ is *Q-regular* (see Definition 3.5), which by Lemma 4.4 is automatically fulfilled if $\Pi_{\mathfrak{p}}$ is Q-ordinary. In all cases $\Pi_{\mathfrak{p}}^{J_{\mathfrak{p}}}$ contains a unique line on which $U_{\mathfrak{p}}$ acts by $\alpha_{\mathfrak{p}}$.

Finally recall the $U_{\mathfrak{p}}^{\circ}$-eigenvectors $\phi_{\tilde{\Pi}}^{\epsilon}$ constructed in (71). Then

$$\phi_{\tilde{\Pi}} = \sum_{\epsilon: F_\infty^\times / F_\infty^{\times\circ} \to \{\pm 1\}} \phi_{\tilde{\Pi}}^{\epsilon} \tag{73}$$

is an $U_{\mathfrak{p}}^{\circ}$-eigenvector with same eigenvalue. When $\tilde{\Pi}_p$ is Q-ordinary, consider the element

$$\boldsymbol{\mu}_{\tilde{\Pi}}^{\eta} = \boldsymbol{\mu}_{\phi_{\tilde{\Pi}}}^{\eta} = \varepsilon_{\mathrm{cyc}}^{-j}(\boldsymbol{\mu}_{\phi_{\tilde{\Pi}}}^{j,\eta}) \in \mathcal{O}[[\mathscr{C}\ell_F^+(p^\infty)]], \tag{74}$$

constructed in (37) and (39), which defines a measure $d\boldsymbol{\mu}_{\tilde{\Pi}}^{\eta}$ on $\mathscr{C}\ell_F^+(p^\infty)$.

4.3.4 *Main theorem on p-adic interpolation.* Fix any character ϵ of K_∞/K_∞° and any $j \in \mathrm{Crit}(\mu)$. Consider the following cohomological test vector:

$$W_{\Pi_\infty,j}^{\epsilon} = \sum_{\underline{i},\alpha} a_{\underline{i},\alpha,j}^{\epsilon} W_{\infty,\underline{i},\alpha}^{\epsilon}. \tag{75}$$

A crucial result of Sun [Sun19, Theorem 5.5] asserts the following non-vanishing:

$$\zeta_\infty\big(j + \tfrac{1}{2}; W_{\Pi_\infty,j}^{\epsilon}\big) \in \mathbb{C}^\times. \tag{76}$$

Note that because $\Pi_\infty \otimes \mathrm{sgn} = \Pi_\infty$ we have suppressed χ_∞ from the notation.

In fact, using Künneth's theorem, it is easy to see that $\zeta_\infty(j + \frac{1}{2}; W_{\Pi_\infty,j}^{\epsilon})$ is a product of similar quantities parsed over the archimedean places. As we have multiplicity one for cuspidal cohomology in top-degree (see (57)), we can only change the class Ξ_∞^{ϵ} by a non-zero scalar, which correspondingly scales the cohomological test vector and so also the zeta integral. The variation of the complex period $\zeta_\infty(j + \frac{1}{2}; W_{\Pi_\infty,j}^{\epsilon})$ in j is studied in [Jan16]. The reader is also referred to the discussion around [GR14, Theorem 6.6.2].

Recall the auxiliary ideal $\mathfrak{m}$ from (L1) in § 2.1 and, for brevity, let us define

$$\gamma = \#\mathscr{C}\ell_F^+(\mathfrak{m}) \cdot \#\,\mathrm{GL}_n(\mathcal{O}_F/\mathfrak{m}) \cdot \#\,\mathrm{PGL}_n(\mathcal{O}_F/\mathfrak{m}) \cdot \prod_{\mathfrak{p}|p} \big(q_{\mathfrak{p}}^{-n^2} \cdot \#\,\mathrm{GL}_n(\mathcal{O}_F/\mathfrak{p})\big) \in \mathbb{Q}^\times. \tag{77}$$

THEOREM 4.7. *Let Π be a cuspidal automorphic representation of GL_{2n}/F admitting a (ψ,η)-Shalika model and such that Π_∞ is cohomological of weight μ. Assume that for all $\mathfrak{p} \mid p$, $\Pi_{\mathfrak{p}}$ is spherical and admits a Q-regular refinement $\tilde{\Pi}_{\mathfrak{p}}$. Then for any $j \in \mathrm{Crit}(\mu)$ and for any finite order character χ of $\mathscr{C}\ell_F^+(p^\infty)$ of conductor $\beta_{\mathfrak{p}} \geqslant 1$ at $\mathfrak{p} \mid p$:*

$$\begin{aligned} i_p^{-1}\bigg(\int_{\mathscr{C}\ell_F^+(p^\infty)} \chi(x)\, d\boldsymbol{\mu}_{\phi_{\tilde{\Pi}}}^{\eta,j}(x)\bigg) &= \gamma \cdot \mathrm{N}_{F/\mathbb{Q}}^{jn}(\mathfrak{d}) \cdot \prod_{\mathfrak{p}|p} \big(\alpha_{\mathfrak{p}}^{-1} q_{\mathfrak{p}}^{n(j+1)}\big)^{\beta_{\mathfrak{p}}} \\ &\quad \cdot \mathcal{G}(\chi_f)^n \cdot L\Big(j + \frac{1}{2}, \Pi_f \otimes \chi_f\Big) \zeta_\infty\Big(j + \frac{1}{2}; W_{\Pi_\infty,j}^{(\varepsilon^j \chi\eta)_\infty}\Big). \end{aligned}$$

Proof. Using (37) and (35) we find that $\int_{\mathscr{C}\ell_F^+(p^\infty)} \chi(x)\, d\boldsymbol{\mu}_{\phi_{\tilde{\Pi}}}^{\eta,j}(x)$ equals

$$(\alpha_{p^\beta}^{\circ})^{-1} \sum_{[x] \in \mathscr{C}\ell_F^+(p^\beta \mathfrak{m})} \chi([x]) \mathcal{E}_{\beta,[x]}^{j,\eta}(\phi_{\tilde{\Pi}}) = \alpha_{p^\beta}^{-1} \cdot \mu^\vee(t_p^{-\beta}) \sum_{\substack{[x] \in \mathscr{C}\ell_F^+(p^\beta \mathfrak{m}) \\ [y] \in \mathscr{C}\ell_F^+(\mathfrak{m})}} \chi([x]) \eta_0([y]) \mathcal{E}_{\beta,[\delta(x,y)]}^{j,\mathsf{w}}(\phi_{\tilde{\Pi}}).$$

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As $\pi_0(\tilde{S}^H_{L_\beta}) \simeq \mathscr{C}\ell_F^+(p^\beta\mathfrak{m}) \times \mathscr{C}\ell_F^+(\mathfrak{m})$ by (73) and Proposition 4.6 the integral equals

$$\alpha_{p^\beta}^{-1} \cdot \sum_{\epsilon:\{\pm 1\}^{\Sigma_\infty}\to\{\pm 1\}} \int_{\tilde{S}^H_{L_\beta}} \varphi^{\epsilon}_{\tilde{\Pi},j}(h\xi t_p^\beta)\chi\left(\frac{\det(h_1)}{\det(h_2)}\right)\left|\frac{\det(h_1)}{\det(h_2)}\right|_F^j \eta^{-1}(\det(h_2))\, dh.$$

Note that the integrand is $L^\circ_\infty Z(\mathbb{A}_f)$-invariant and $L_\infty/L^\circ_\infty Z_\infty$ acts on it by $\epsilon\varepsilon^j_\infty\chi_\infty\eta_\infty$, hence the integral vanishes unless $\epsilon = (\varepsilon^j\chi\eta)_\infty$. As $Z(\mathbb{A}_f)\cap L_\beta$ is independent of β, after some volume computation, one further finds

$$i_p^{-1}\left(\int_{\mathscr{C}\ell_F^+(p^\infty)} \varepsilon^j(x)\chi(x)\, d\boldsymbol{\mu}^\eta_{\tilde{\Pi}}(x)\right) = \gamma\cdot\Psi\left(j+\frac{1}{2}, \varphi^{(\varepsilon^j\chi\eta)_\infty}_{\tilde{\Pi},j}(-\cdot\xi t_p^\beta), \chi, \eta\right)\prod_{\mathfrak{p}\mid p}(\alpha_{\mathfrak{p}}^{-1}q_{\mathfrak{p}}^{n^2})^{\beta_{\mathfrak{p}}}.$$

By Proposition 4.3 the Friedberg–Jacquet integral has an Euler product for $\Re(s)\gg 0$:

$$\Psi(s,\varphi^{\epsilon}_{\tilde{\Pi},j}(-\cdot\xi t_p^\beta),\chi,\eta) = \prod_{v\nmid p\infty}\zeta_v(s;W_{\Pi_v},\chi_v)\cdot\prod_{\mathfrak{p}\mid p}\zeta_{\mathfrak{p}}(s;W_{\tilde{\Pi}_{\mathfrak{p}}}(-\cdot\xi t_{\mathfrak{p}}^{\beta_{\mathfrak{p}}}),\chi_{\mathfrak{p}})\cdot\zeta_\infty(s;W^{\epsilon}_{\Pi_\infty,j},\chi_\infty).$$

As $L(s,\Pi\otimes\chi)$ has trivial Euler factors at all places $\mathfrak{p}\mid p$ (as $\Pi_{\mathfrak{p}}$ is spherical while $\chi_{\mathfrak{p}}$ is ramified), Proposition 3.3 implies that

$$\prod_{v\nmid p\infty}\zeta_v\left(j+\frac{1}{2};W_{\Pi_v},\chi_v\right) = \mathrm{N}^{jn}_{F/\mathbb{Q}}(\mathfrak{d}^{(p)})\chi(\mathfrak{d}^{-1})^n L\left(j+\frac{1}{2},\Pi_f\otimes\chi_f\right).$$

The factor at $\mathfrak{p}\mid p$ is computed in Proposition 3.4, which together with Lemma 3.6 gives

$$q_{\mathfrak{p}}^{\beta_{\mathfrak{p}}n^2}\zeta_{\mathfrak{p}}(j+\tfrac{1}{2};W_{\tilde{\Pi}_{\mathfrak{p}}},\chi_{\mathfrak{p}}) = \mathcal{G}(\chi_{\mathfrak{p}})^n\cdot q_{\mathfrak{p}}^{\beta_{\mathfrak{p}}n+(\beta_{\mathfrak{p}}+\delta_{\mathfrak{p}})jn}W_{\tilde{\Pi}_{\mathfrak{p}}}(\mathbf{1}_{2n}) = \mathrm{N}^{jn}_{F/\mathbb{Q}}(\mathfrak{d}_p)\mathcal{G}(\chi_{\mathfrak{p}})^n\cdot q_{\mathfrak{p}}^{\beta_{\mathfrak{p}}n(j+1)}.$$

As $\mathcal{G}(\chi_f) = \chi(\mathfrak{d}^{-1})\prod_{\mathfrak{p}\mid p}\mathcal{G}(\chi_{\mathfrak{p}})$ and $\mathfrak{d} = \mathfrak{d}_p\mathfrak{d}^{(p)}$ we obtain the desired formula. □

Proof. By Lemma 4.4, for each $\mathfrak{p}\mid p$, the Q-ordinary refinement $\tilde{\Pi}_{\mathfrak{p}}$ of $\Pi_{\mathfrak{p}}$ is Q-regular, hence Theorem 4.7 applies. The interpolation formula in Theorem B then follows immediately because by Theorem 2.3 one has

$$\int_{\mathscr{C}\ell_F^+(p^\infty)}\varepsilon^j(x)\chi(x)\, d\boldsymbol{\mu}^\eta_{\tilde{\Pi}}(x) = \int_{\mathscr{C}\ell_F^+(p^\infty)}\chi(x)\, d\boldsymbol{\mu}^{\eta,j}_{\phi_{\tilde{\Pi}}}(x).$$ □

4.4 Non-vanishing of twists

4.4.1 *The main theorem.* Having established the Manin relations in our context (see Theorem 2.3), we can now prove a non-vanishing result for twisted L-functions using a method that goes back to Manin and Greenberg. Such a technique to prove non-vanishing of twists has also been used recently by Januszewski [Jan18] for Rankin–Selberg L-functions and Eischen [Eis] for L-functions of unitary groups. However, our results, Theorem 4.8 and Corollary 4.10, are not only independent of these other recent works but are also beyond the scope of results in both [Jan18] and [Eis] as well as previous results obtained by analytic number theoretic methods [Roh89, BR94, Luo05, CFH05].

Theorem 4.8. *Let μ be a pure dominant integral weight such that*

$$\mu_{\sigma,n} > \mu_{\sigma,n+1},\quad \textit{for all } \sigma\in\Sigma_\infty. \tag{78}$$

Let Π be a cuspidal automorphic representation that is cohomological with respect to the weight μ and admitting an (η, ψ)-Shalika model. Assume that for all primes $\mathfrak{p}$ above a prime number p, $\Pi_{\mathfrak{p}}$ is unramified and Q-ordinary. Then for all $j \in \mathrm{Crit}(\mu)$ and for all but finitely many Dirichlet characters χ of F of p-power conductor, we have

$$L\left(\frac{1}{2} + j, \Pi \otimes (\chi \circ \mathrm{N}_{F/\mathbb{Q}})\right) \neq 0.$$

We begin with a few comments. As $\Pi^\circ = \Pi \otimes |\cdot|^{\mathsf{w}/2}$ is a unitary cuspidal automorphic representation, we see that $(1+\mathsf{w})/2$ is the center of symmetry for the L-function of Π:

$$L\left(\frac{1+\mathsf{w}}{2}, \Pi \otimes \chi\right) = L\left(\frac{1}{2}, \Pi^\circ \otimes \chi\right).$$

By regularity one knows that $\mathrm{Crit}(\mu)$ is non-empty and condition (78) is equivalent to assuming that $\mathrm{Crit}(\mu)$ has at least two elements. If Π is unitary, then $\mathsf{w} = 0$ and $\frac{1}{2} \in \mathrm{Crit}(\Pi \otimes \chi) = \mathrm{Crit}(\mu)$, whence Theorem A is a particular case of Theorem 4.8.

If Leopoldt's conjecture holds for F at p, then one readily obtains a statement for all but finitely many p-power conductor Hecke characters, as opposed to Dirichlet characters.

We show non-vanishing of critical values of twisted L-functions by showing non-vanishing statement about distributions on the cyclotomic $\mathbb{Z}_p$-extension of F. Recall the p-adic cyclotomic character

$$\varepsilon : \mathcal{C}\ell^+_{\mathbb{Q}}(p^\infty) \xrightarrow{\sim} \mathbb{Z}_p^\times = \mu_{2p} \times (1 + 2p\mathbb{Z}_p)$$

the first component of which is given by the Teichmüller character ω, whereas the fixed field of the kernel of the second component $\varepsilon\omega^{-1}$ is the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. Then by a well-known result due to Serre there is an isomorphism $\mathcal{O}[[1+2p\mathbb{Z}_p]] \simeq \mathcal{O}[[T]]$ sending $1+2p$ to $1+T$. Composing with the norm map $\mathrm{N}_{F/\mathbb{Q}} : \mathcal{C}\ell^+_F(p^\infty) \to \mathcal{C}\ell^+_{\mathbb{Q}}(p^\infty)$ allows us to lift Dirichlet characters to Hecke characters over F, thus to push-forward of a measure on $\mathcal{C}\ell^+_F(p^\infty)$, such as $\mu_{\tilde{\Pi}}$, to a measure on $\mathcal{C}\ell^+_{\mathbb{Q}}(p^\infty)$. Further composing with $\omega^m : \mu_{2p} \to \mathcal{O}^\times$ for $0 \leqslant m \leqslant p-1$ allows us to define a measure on $1 + 2p\mathbb{Z}_p$, that is, an element $\omega^m(\mu_{\tilde{\Pi}}) \in \mathcal{O}[[T]]$.

Proof. We first show that $\omega^m(\mu_{\tilde{\Pi}}) \neq 0$ for all $m \in \mathbb{Z}$. By the interpolation property in Theorem 4.7, the measure $\omega^m(\mu_{\tilde{\Pi}})$ interpolates the algebraic parts of $L(\frac{1}{2}+j, \Pi \otimes \omega^{m-j}\chi)$ for $j \in \mathrm{Crit}(\mu)$ and χ runs over all Dirichlet characters of (non-trivial) p-power order and conductor. Our hypothesis (78) implies that we find $j \in \mathrm{Crit}(\mu)$ satisfying $j > \mathsf{w}/2$, hence $\frac{1}{2}+j$ lies outside the interior of the critical strip $w/2 < \Re(s) < w/2 + 1$ for $L(s, \Pi)$ and, thus, $L(\frac{1}{2}+j, \Pi \otimes \omega^{m-j}\chi) \neq 0$. Therefore, $\omega^m(\mu_{\tilde{\Pi}}) \neq 0$ as claimed.

By the Weierstrass preparation theorem, a non-zero element of $\mathcal{O}[[T]]$ admits only finitely many zeros in $\bar{\mathbb{Z}}_p$. Again by Theorem 4.7 this means that, given *any* $j \in \mathrm{Crit}(\mu)$ and m, there are at most finitely many Dirichlet characters χ of p-power order and conductor such that $L(\frac{1}{2}+j, \Pi \otimes \omega^{m-j}\chi) = 0$. As any p-power conductor Dirichlet character is of that form for some $0 \leqslant m \leqslant p-1$, the theorem follows. □

4.4.2 *Variations.*

Corollary 4.9 Nearly ordinary case. *Under the hypotheses of Theorem 4.8, let ν be a finite order character of $\mathcal{C}\ell_F^+(p^\infty)$. Then for all but finitely many Dirichlet characters χ of finite order and with p-power conductor, we have*

$$L\left(\frac{\mathsf{w}+1}{2}, \Pi \otimes \nu\chi\right) \neq 0.$$

Proof. Use the twisted norm map $[x] \mapsto \nu(x)[\mathrm{N}_{F/\mathbb{Q}}\, x]$ to push-forward $\mu_{\tilde{\Pi}}$ to a measure on $\mathcal{C}\ell_{\mathbb{Q}}^+(p^\infty)$. Then proceed *mutatis mutandis* as in the proof of Theorem 4.8. □

This result is slightly stronger because the representation $\Pi \otimes \nu$, even though of cohomological type and admitting a Shalika model, is no longer ordinary at p, nor spherical.

The following corollary of Theorem 4.8 follows from the fact that we have non-vanishing *for all but finitely many* Dirichlet characters χ of finite order and with p-power conductor.

Corollary 4.10 Simultaneous non-vanishing. *For $1 \leqslant k \leqslant r$ fix $n_k \in \mathbb{Z}_{>0}$ and let μ_k be a pure dominant integral weight for GL_{2n_k} over F. Suppose that each μ_k satisfies the regularity condition in (78) and that its purity weight w_k is even. Let Π_k be a cuspidal automorphic representation of $\mathrm{GL}_{2n_k}(\mathbb{A}_F)$ of cohomological weight μ_k admitting a Shalika model. For a prime number p, suppose that each Π_k is unramified and Q-ordinary at p. Then, for all but finitely many Dirichlet characters χ of p-power conductor, we have*

$$L\left(\frac{\mathsf{w}_1+1}{2}, \Pi_1 \otimes \chi\right) L\left(\frac{\mathsf{w}_2+1}{2}, \Pi_2 \otimes \chi\right) \cdots L\left(\frac{\mathsf{w}_r+1}{2}, \Pi_r \otimes \chi\right) \neq 0.$$

Let us note that this is a simultaneous non-vanishing result at the central point. We leave it to the reader to formulate the stronger version of simultaneous non-vanishing combining Corollaries 4.9 and 4.10.

As a very concrete example illustrating an application of simultaneous non-vanishing to algebraicity results, let us consider the unitary cuspidal automorphic representation $\pi(\Delta)$ of $\mathrm{GL}_2(\mathbb{A})$ associated to the Ramanujan Δ-function. A particular case of Corollary 4.10 gives infinitely many Dirichlet characters χ such that

$$L(17, \mathrm{Sym}^3(\Delta) \otimes \chi) L(6, \Delta \otimes \chi) = L\big(\tfrac{1}{2}, \mathrm{Sym}^3(\pi(\Delta)) \otimes \chi\big) L\big(\tfrac{1}{2}, \pi(\Delta) \otimes \chi\big) \neq 0.$$

For such a character we obtain from [Rag10, Corollary 5.2] the following identity of L-values:

$$L\big(\tfrac{1}{2}, \mathrm{Sym}^5(\pi(\Delta)) \otimes \chi\big) = \frac{L\big(\tfrac{1}{2}, \mathrm{Sym}^3(\pi(\Delta)) \times \mathrm{Sym}^2(\pi(\Delta))\big)}{L\big(\tfrac{1}{2}, \mathrm{Sym}^3(\pi(\Delta)) \otimes \chi\big) L\big(\tfrac{1}{2}, \pi(\Delta) \otimes \chi\big)}.$$

Using the rationality result in [Rag10, Theorem 1.1] for the L-value in the numerator of the right-hand side, and the [Rag10, Theorem 1.3] for the L-values in the denominator of the right-hand side, we obtain a new rationality result for

$$L\big(\tfrac{1}{2}, \mathrm{Sym}^5(\pi(\Delta)) \otimes \chi\big) = L\big(28, \mathrm{Sym}^5(\Delta) \otimes \chi\big).$$

Similarly, using the results of [Rag16], and simultaneous non-vanishing for the central values of the first and third symmetric power L-functions of a Hilbert cusp form, one may now generalize

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this to get new rationality results for the symmetric fifth power L-functions of a Hilbert cusp form.

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