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On the Hardy–Sobolev equation

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In this paper we study the problem

$$\left.\begin{aligned} -\Delta u - \frac{\lambda}{|x|^2}u = u^p \quad \text{in } \Omega, \\ u \geqslant 0 \quad \text{in } \Omega, \end{aligned}\right\} \tag{1}$$

where $\Omega = \mathbb{R}^N$ or $\Omega = B_1$, $N \geqslant 3$, $p > 1$ and $\lambda < \frac{1}{4}(N-2)^2$. Using a suitable map we transform problem (1) into another one without the singularity $1/|x|^2$. Then we obtain some bifurcation results from the radial solutions corresponding to some explicit values of λ.

Keywords: semilinear elliptic equations; positive solutions; bifurcation theory; Hardy inequality; Sobolev inequality

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1. Introduction, statement of the main results and sketch of the proofs

In this paper we consider the following problem:

$$\left.\begin{aligned} -\Delta u - \frac{\lambda}{|x|^2}u = u^p \quad \text{in } \Omega, \\ u \geqslant 0 \quad \text{in } \Omega, \end{aligned}\right\} \tag{1.1}$$

where $\Omega \subseteq \mathbb{R}^N$, $N \geqslant 3$, $p > 1$ and $\lambda < \frac{1}{4}(N-2)^2$. We shall focus on the case where either $\Omega = \mathbb{R}^N$ or $\Omega = B_1$ with p suitably chosen. By solutions we mean weak solutions, so we shall require that $u \in D^{1,2}(\mathbb{R}^N)$, where

$$D^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) \text{ such that (s.t.) } |\nabla u| \in L^2(\mathbb{R}^N)\}$$

in the first case, or $u \in H_0^1(B_1)$ in the case of the ball. These problems have been much studied using both variational and moving-plane methods or the finite-dimensional reduction of Lyapunov and Schmidt.

In this paper we follow a different approach that will allow us to obtain, among other results, a higher number of solutions. The main ingredient of our proofs is given by the following map:

$$\mathcal{L}_p : C(0,+\infty) \to C(0,+\infty),$$

defined as

$$v(r) = \mathcal{L}_p(u(r)) = r^a u(r^b) \quad \text{for } r > 0,\ p > 1, \tag{1.2}$$

with

$$a = 2\frac{(N-2)(1-\nu_\lambda)}{(p-1)(N-2)(\nu_\lambda - 1) + 4} \tag{1.3}$$

and

$$b = \frac{4}{(p-1)(N-2)(\nu_\lambda - 1) + 4}, \tag{1.4}$$

where

$$\nu_\lambda = \sqrt{1 - \frac{4\lambda}{(N-2)^2}}. \tag{1.5}$$

Let $(0,T) \subset \mathbb{R}$ be an interval ($T = +\infty$ is allowed) and let

$$\mathcal{D}^{1,2}((0,T), r^{N-1}\,\mathrm{d}r) = \left\{u : (0,T) \to \mathbb{R} \text{ such that } \int_0^T |u'|^2 r^{N-1}\,\mathrm{d}r < +\infty\right\}.$$

The following proposition highlights the main properties of the operator $\mathcal{L}_p$.

PROPOSITION 1.1. *Let $p > 1$, let $\lambda < \frac{1}{4}(N-2)^2$ and let u be a function satisfying*

$$-u'' - \frac{N-1}{r}u' - \frac{\lambda}{r^2}u = u^p \quad \text{in } (0,T) \tag{1.6}$$

with $T \in (0,+\infty]$. Then, the function $v(r) = \mathcal{L}_p(u(r))$ satisfies

$$-v'' - \frac{M-1}{r}v' = A(\lambda,p)v^p \quad \text{in } (0,T^{1/b}), \tag{1.7}$$

where

$$M - 1 = \frac{(p+3)(N-2)(\nu_\lambda - 1) + 4(N-1)}{(p-1)(N-2)(\nu_\lambda - 1) + 4} \tag{1.8}$$

and

$$A(\lambda,p) = b^2 = \left(\frac{4}{(p-1)(N-2)(\nu_\lambda - 1) + 4}\right)^2. \tag{1.9}$$

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Letting

$$T = +\infty \quad \textit{when } p = \frac{N+2}{N-2} \tag{1.10}$$

or

$$T = 1 \quad \textit{when } 1 < p < \frac{N+2}{N-2}, \tag{1.11}$$

we then have

$$\mathcal{L}_p\colon \mathcal{D}^{1,2}((0,T), r^{N-1}\,\mathrm{d}r) \to \mathcal{D}^{1,2}((0,T), r^{M-1}\,\mathrm{d}r) \tag{1.12}$$

and

$$\|\mathcal{L}_p u\|^2_{\mathcal{D}^{1,2}((0,T),r^{M-1}\,\mathrm{d}r)} = \frac{1}{\nu_\lambda}\int_0^T \left(u'(s)^2 - \frac{\lambda}{s^2}u^2(s)\right)s^{N-1}\,\mathrm{d}s. \tag{1.13}$$

This proposition establishes a one-to-one relationship between the radial solutions to (1.1) and the ordinary differential equation (ODE) (1.7). This allows us to find some old and new results about radial solutions to (1.1).

On the other hand, we stress that the map $\mathcal{L}_p$ will also be used to establish the existence of *non-radial* solutions.

In this paper we analyse two different situations: either

$$p = \frac{N+2}{N-2} \quad \text{and} \quad \Omega = \mathbb{R}^N \tag{1.14}$$

or

$$1 < p < \frac{N+2}{N-2} \quad \text{and} \quad \Omega = B_1. \tag{1.15}$$

Let us start by considering $p = (N+2)/(N-2)$ so that (1.1) becomes

$$\left.\begin{aligned} -\Delta u - \frac{\lambda}{|x|^2}u = C(\lambda)u^{(N+2)/(N-2)} \quad \text{in } \mathbb{R}^N,\\ u \geqslant 0, \quad u \in D^{1,2}(\mathbb{R}^N), \end{aligned}\right\} \tag{1.16}$$

where $N \geqslant 3$ and

$$C(\lambda) = N(N-2)\left(1 - \frac{4\lambda}{(N-2)^2}\right)$$

(we have added the constant $C(\lambda)$ just to have a simpler expression of the explicit radial solutions).

Our starting point is the paper [19] by Terracini. For the radial case, Terracini shows that the unique radial solutions of (1.16) in $D^{1,2}(\mathbb{R}^N)$ are given by the functions

$$u_{\delta,\lambda}(r) = \frac{r^{(N-2)(\nu_\lambda-1)/2}\delta^{(N-2)/2}}{(1+\delta^2 r^{2\nu_\lambda})^{(N-2)/2}} \tag{1.17}$$

with ν_λ as in (1.5). Moreover, she proved the following result.

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THEOREM (Terracini [19]). *Let $\lambda \in [0, \frac{1}{4}(N-2)^2)$. Then problem (1.16) has a unique (up to rescaling) solution in $D^{1,2}(\mathbb{R}^N)$. Moreover, there exists $\lambda^* < 0$ such that for $\lambda < \lambda^*$ problem (1.16) admits at least two positive solutions in $D^{1,2}(\mathbb{R}^N)$ that are distinct modulo rescaling. One is radial but the other is not.*

Another existence result was obtained some years later by Jin *et al*. [10], who proved the existence of singular solutions of the form $u(r,\theta) = r^{(2-N)/2}g(\theta)$, with (r,θ) polar coordinates in $\mathbb{R}^N$. These solutions do not belong to $D^{1,2}(\mathbb{R}^N)$. Finally, we recall a result by Musso and Wei [13], who proved the existence of infinitely many solutions for any $\lambda < 0$. Note that the *energy* of these solutions, namely the quantity

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left(|\nabla u|^2 - \frac{\lambda}{|x|^2}u^2\right)\mathrm{d}x - \frac{N-2}{2N}\int_{\mathbb{R}^N}|u|^{2N/(N-2)}\,\mathrm{d}x,$$

is large.

The results in [19] are based on the moving-plane method (when $\lambda > 0$) and on the analysis of the radially symmetric case in the phase space.

Using the map $\mathcal{L}_{(N+2)/(N-2)}$ we give another proof of some results in [19] in the radial case. In our opinion this approach is simpler. Actually, as shown in proposition 1.1, since $M = N$ in this case the map $\mathcal{L}_{(N+2)/(N-2)}$ reduces the study of the radial solutions of (1.16) to the well-known problem

$$\left.\begin{aligned} -\Delta U = N(N-2)U^{2^*-1} \quad \text{in } \mathbb{R}^N,\\ U \geqslant 0, \quad U \in D^{1,2}(\mathbb{R}^N). \end{aligned}\right\} \tag{1.18}$$

Solutions of (1.18) have been completely classified by Caffarelli *et al*. [3], who proved that the solutions are given by

$$U_\delta(r) = \frac{\delta^{(N-2)/2}}{(1+\delta^2 r^2)^{(N-2)/2}} \tag{1.19}$$

with $\delta > 0$, and that they are extremal functions for the well-known Sobolev inequality

$$\int_{\mathbb{R}^N}|\nabla u|^2\,\mathrm{d}x \geqslant S\left(\int_{\mathbb{R}^N}|u|^{2N/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N} \tag{1.20}$$

for $u \in D^{1,2}(\mathbb{R}^N)$ and S the best Sobolev constant (see [18]).

In this way we derive that the functions $u_{\delta,\lambda}$ in (1.17) are the unique radial solutions to (1.16) (see corollary 3.1) and some inequalities involving the Hardy norm (see proposition 3.2) (these results were proved in [19, § 4] using the phase plane).

As we have pointed out, the role of map $\mathcal{L}_{(N+2)/(N-2)}$ is not restricted to the radial case. Indeed, it can be used to characterize *all* solutions of the linearized problem at $u_{\delta,\lambda}$, namely

$$\left.\begin{aligned} -\Delta v - \frac{\lambda}{|x|^2}v = N(N+2)\nu_\lambda^2 u_{\delta,\lambda}^{4/(N-2)}v \quad \text{in } \mathbb{R}^N,\\ v \in D^{1,2}(\mathbb{R}^N). \end{aligned}\right\} \tag{1.21}$$

Our next result classifies the solution to (1.21).

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LEMMA 1.2. *Let* $\lambda < \frac{1}{4}(N-2)^2$ *and*

$$\lambda_j = \frac{(N-2)^2}{4}\left(1 - \frac{j(N-2+j)}{N-1}\right), \quad j \in \mathbb{N}. \tag{1.22}$$

If $\lambda \neq \lambda_j$, *then the space of solutions of (1.21)* (*with* $\delta = 1$) *has dimension* 1 *and it is spanned by*

$$Z_\lambda(x) = \frac{|x|^{(N-2)(\nu_\lambda-1)/2}(1-|x|^{2\nu_\lambda})}{(1+|x|^{2\nu_\lambda})^{N/2}}, \tag{1.23}$$

where ν_λ *is as defined in (1.5).*

Otherwise, if $\lambda = \lambda_j$ *for some* $j = 1, \dots,$ *then the space of solutions of (1.21)* (*with* $\delta = 1$) *has dimension*

$$1 + \frac{(N+2j-2)(N+j-3)!}{(N-2)!j!}$$

and it is spanned by

$$Z_{\lambda_j}(x), Z_{j,i}(x) = \frac{|x|^{N\nu_{\lambda_j}/2-(N-2)/2}Y_{j,i}(x)}{(1+|x|^{2\nu_{\lambda_j}})^{N/2}}, \tag{1.24}$$

where $\{Y_{j,i}\}$, $i = 1, \dots, (N+2j-2)(N+j-3)!/(N-2)!j!$, *form a basis of* $\mathbb{Y}_j(\mathbb{R}^N)$, *the space of all homogeneous harmonic polynomials of degree* j *in* $\mathbb{R}^N$.

A consequence of the above result is the computation of the Morse index of $u_\lambda = u_{1,\lambda}$.

PROPOSITION 1.3. *Let* $u_\lambda := u_{1,\lambda}$ *be the radial solution of (1.16). Then its Morse index* $m(\lambda)$ *is equal to*

$$m(\lambda) = \sum_{\substack{0 \leqslant j < \frac{2-N}{2} + \frac{1}{2}\sqrt{N^2 - \frac{16(N-1)\lambda}{(N-2)^2}},\\ j \text{ integer}}} \frac{(N+2j-2)(N+j-3)!}{(N-2)!j!}. \tag{1.25}$$

In particular, the Morse index of u_λ *changes as* λ *crosses the values* λ_j *and* $m(\lambda) \to +\infty$ *as* $\lambda \to -\infty$.

Next aim is to obtain multiplicity results of non-radial solutions as $\lambda < 0$ bifurcating from the radial solution $u_\lambda := u_{1,\lambda}$ in (1.17).

For any $h \in \mathbb{N}$ we let $O(h)$ be the orthogonal group in $\mathbb{R}^h$. Our main result for problem (1.16) is as follows (see (3.16) for the definition of the space X).

THEOREM 1.4. *Let fix* $j \in \mathbb{N}$ *and let* λ_j *as in (1.22). Then we have the following.*

(i) *If* j *is odd, there exists at least one continuum of non-radial weak solutions to (1.16), invariant with respect to* $O(N-1)$, *bifurcating from the pair* $(\lambda_j, u_{\lambda_j})$ *in* $(-\infty, 0) \times X$.

(ii) *If* j *is even, there exist at least* $[N/2]$ *continua of non-radial weak solutions to (1.16) bifurcating from the pair* $(\lambda_j, u_{\lambda_j})$ *in* $(-\infty, 0) \times X$. *The first branch is* $O(N-1)$ *invariant; the second is* $O(N-2) \times O(2)$ *invariant, and so on.*

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Moreover, all these solutions v_λ *satisfy*

$$\sup_{x\in\mathbb{R}^N}(1+|x|)^\gamma|v_\lambda(x)|\leqslant C,$$

where $\gamma\in\mathbb{R}$ *satisfies* $\frac{1}{2}(N-2)<\gamma<N-2$.

Remark 1.5. The solutions of theorem 1.4 are different from the non-radial solutions found in [19]. Indeed, the non-radial solution $\bar u$ in [19] satisfies

$$\frac{\int_{\mathbb{R}^N}(|\nabla\bar u|^2-(\lambda/|x|^2)\bar u^2)\,\mathrm{d}x}{(\int_{\mathbb{R}^N}|\bar u|^{2N/(N-2)}\,\mathrm{d}x)^{(N-2)/N}}<k^{2/N}S<\left(1-\frac{4\lambda}{(N-2)^2}\right)^{(N-1)/N}S \tag{1.26}$$

for some integer k and λ large enough.

On the other hand, since

$$\frac{\int_{\mathbb{R}^N}(|\nabla u_{\lambda_j}|^2-(\lambda/|x|^2)u_{\lambda_j}^2)\,\mathrm{d}x}{(\int_{\mathbb{R}^N}|u_{\lambda_j}|^{2N/(N-2)}\,\mathrm{d}x)^{(N-2)/N}}=\left(1-\frac{4\lambda_j}{(N-2)^2}\right)^{(N-1)/N}S, \tag{1.27}$$

our continuum of solutions does not contain $\bar u$, at least in a region 'close' to the radial branch u_λ.

This also applies to the solutions found in [13], since their energy is greater than nS for some large integer n.

Remark 1.6. A consequence of the preceding remark is that, for λ close to λ_j and j large, problem (1.16) has at least *three* solutions. One is the radial function u_λ in (1.17) and the others are non-radial functions. Moreover, if j is a sufficiently large even number, we have at least $[N/2]+2$ solutions. This shows that the bifurcation diagram of the solutions to (1.16) is very complicated for $\lambda<0$ and, of course, quite difficult to describe.

We now consider the case $1<p<(N+2)/(N-2)$ and the problem

$$\left.\begin{aligned} -\Delta u-\frac{\lambda}{|x|^2}u=u^p &\quad \text{in } B_1,\\ u>0 &\quad \text{in } B_1,\\ u\in H_0^1(B_1), & \end{aligned}\right\} \tag{1.28}$$

where B_1 is the unit ball in $\mathbb{R}^N$. A complete description of (1.28) for $\lambda\geqslant 0$ was given by Chaves and Garcia-Azorero [4], who proved that problem (1.28) admits a unique, radial solution. We are not aware of any result for $\lambda<0$. However, since the problem is subcritical, the existence of a radial solution is a straightforward consequence of the mountain-pass theorem. Analogously to the previous case, proposition 1.1 shows that the map $\mathcal{L}_p$ provides an equivalence between the radial solutions to (1.28) and the solutions of

$$\left.\begin{aligned} -v''-\frac{M-1}{r}v'=A(\lambda,p)v^p &\quad \text{in } (0,1),\\ v>0 &\quad \text{in } (0,1),\\ v'(0)=v(1)=0, & \end{aligned}\right\} \tag{1.29}$$

with M and $A(\lambda,p)$ as in (1.8), (1.9) (see § 4 for a discussion about the boundary conditions). It is known that this problem has a unique solution, which we call v_λ. Then we get the following result.

THEOREM 1.7. *Let $\lambda < \frac{1}{4}(N-2)^2$ and $1 < p < (N+2)/(N-2)$. Then problem (1.28) admits only one radial solution, $u_\lambda(r)$. Moreover,*

$$r^{(N-2)(1-\nu_\lambda)/2}u_\lambda(r) \to C \quad \text{as } r \to 0^+, \tag{1.30}$$

where $C > 0$.

This result extends the uniqueness result of [4] to the case $\lambda < 0$ and shows that radial solutions to (1.28) satisfy $u_\lambda(0) = 0$ for $\lambda < 0$ differently from the case $\lambda \geqslant 0$, where $u_\lambda \notin L^\infty(B_1)$. Then the monotonicity properties of the Gidas–Ni–Nirenberg theorem cannot hold when $\lambda < 0$.

Once we have this branch of the radial solution u_λ for any $\lambda < 0$ we can look for non-radial solutions that arise by bifurcation. The strategy to obtain multiplicity results for $\lambda < 0$ is the same as in the case $p = (N+2)/(N-2)$. First, we prove non-degeneracy of u_λ in the space of radial functions. Then we characterize the values of λ for which the linearized operator at the radial solution u_λ is non-invertible and we compute the change in the Morse index of the radial solutions at these points. These values of λ are related to a weighted eigenvalue for problem (1.29). To this end we let $v_\lambda \in H^1((0,1), r^{M-1}\,\mathrm{d}r)$ be the unique solution to (1.29) and set

$$\Lambda(\lambda) = \inf_{w \in H^1((0,1), r^{M-1}\,\mathrm{d}r), w(1)=0} \frac{\int_0^1 (|w'|^2 - pA(\lambda,p)v_\lambda^{p-1}w^2)r^{M-1}\,\mathrm{d}r}{\int_0^1 w^2 r^{M-3}\,\mathrm{d}r}. \tag{1.31}$$

The infimum $\Lambda(\lambda)$ is well defined by the Hardy inequality, but it is not clear if it is achieved. Indeed, the embedding of $H^1((0,1), r^{M-1}\,\mathrm{d}r) \hookrightarrow L^2((0,1), r^{M-3}\,\mathrm{d}r)$ is not compact. However, the crucial information that the infimum (1.31) is strictly negative implies that it is attained. This is proved in propositions A.4 and A.8 (see the appendix) in a more general case and relies on a careful study of some weighted problem given in [9, § 2].

Now we can state the following result.

THEOREM 1.8. *For any $j \in \mathbb{N}$, $j \geqslant 1$ there exist a value λ_j that satisfies*

$$\left(\frac{4}{(p-1)(2-N+\sqrt{(N-2)^2-4\lambda})+4}\right)^2 j(N-2+j) = -\Lambda(\lambda) \tag{1.32}$$

and an interval $I_j \subset (-\infty, 0)$ such that $\lambda_j \in I_j$ and such that a non-radial bifurcation occurs at (λ, u_λ) for $\lambda \in I_j$.

Moreover, if j is even, there exist at least $[N/2]$ continua of non-radial solutions to (1.28) bifurcating from (λ, u_λ) for $\lambda \in I_j$. The first branch is $O(N-1)$ invariant; the second is $O(N-2) \times O(2)$ invariant, and so on.

The bifurcation result in theorem 1.8 is more delicate than the one obtained in theorem 1.4. This is due to the fact that we cannot solve (1.32) explicitly, as in theorem 1.4. Also, in this case it seems very difficult to deduce whether the set of solutions λ of (1.32) is given by isolated points (for example, it is not clear if the function $-\Lambda$ is analytic). These difficulties will be overcome by showing that is possible to select small intervals containing solutions to (1.32) where there is a change of the Morse index (see propositions 4.9 and 4.10). This implies a change in the degree of the solution, which is the crucial tool to get the bifurcation result.

The paper is organized as follows: in § 2 we show the main properties of the map $\mathcal{L}_p$. In § 3 we consider the case $p = (N+2)/(N-2)$, and in § 4 we consider the subcritical case $1 < p < (N+2)/(N-2)$. In the appendix we prove some technical results.

2. Main properties of the map $\mathcal{L}_p$

In this section we give the proof of proposition 1.1.

Proof of proposition 1.1. Let $u(r)$ be a solution of (1.6). Then a straightforward computation shows that $v(r) = \mathcal{L}_p(u(r))$ is a solution to (1.7) with M and $A(\lambda, p)$ satisfying (1.8) and (1.9), respectively.

Now let us show (1.13). We just consider the case $p = (N+2)/(N-2)$ and $T = +\infty$ (the subcritical case $1 < p < (N+2)/(N-2)$ is similar and easier).

Note that in this case we have

$$a = \frac{(N-2)(1-\nu_\lambda)}{2\nu_\lambda}, \qquad b = \frac{1}{\nu_\lambda}$$

and

$$v(r) = r^{(N-2)(1-\nu_\lambda)/2\nu_\lambda} u(r^{1/\nu_\lambda}) \quad \text{for } r > 0. \tag{2.1}$$

First of all, we observe that, since $u \in \mathcal{D}^{1,2}((0,+\infty), r^{N-1}\,\mathrm{d}r)$,

$$\int_0^{+\infty} u(s)^{2N/(N-2)} s^{N-1}\,\mathrm{d}s < +\infty,$$

and so there exist sequences $\delta_n \to 0$ and $M_n \to +\infty$ such that

$$u(\delta_n)\delta_n^{(N-2)/2} \to 0 \quad \text{and} \quad u(M_n)M_n^{(N-2)/2} \to 0.$$

By using (2.1), we derive that

$$\frac{r^{N/\nu_\lambda - N}}{\nu_\lambda^2}(u'(r^{1/\nu_\lambda}))^2 = \frac{(N-2)^2(1-\nu_\lambda)^2}{4\nu_\lambda^2}\frac{v^2(r)}{r^2} - \frac{(N-2)(1-\nu_\lambda)}{\nu_\lambda}\frac{v(r)v'(r)}{r} + (v'(r))^2.$$

Choosing $\varepsilon_n = \delta_n^{\nu_\lambda}$ and $R_n = M_n^{\nu_\lambda}$ and integrating, we get

$$\begin{aligned}\frac{1}{\nu_\lambda^2}\int_{\varepsilon_n}^{R_n} (u'(r^{1/\nu_\lambda}))^2 r^{N/\nu_\lambda - 1}\,\mathrm{d}r &= \frac{(N-2)^2(1-\nu_\lambda)^2}{4\nu_\lambda^2}\int_{\varepsilon_n}^{R_n} v^2(r) r^{N-3}\,\mathrm{d}r \\ &\quad - \frac{(N-2)(1-\nu_\lambda)}{\nu_\lambda}\int_{\varepsilon_n}^{R_n} v(r)v'(r)r^{N-2}\,\mathrm{d}r + \int_{\varepsilon_n}^{R_n}(v'(r))^2 r^{N-1}\,\mathrm{d}r.\end{aligned} \tag{2.2}$$

Then, again using (2.1), we have

$$
\begin{aligned}
\int_{\varepsilon_n}^{R_n} & v(r)v'(r)r^{N-2}\,\mathrm{d}r \\
&= \frac{r^{N-2}v^2(r)}{2}\bigg|_{\varepsilon_n}^{R_n} - \frac{N-2}{2}\int_{\varepsilon_n}^{R_n} v^2(r)r^{N-3}\,\mathrm{d}r \\
&= \frac{(u(M_n)M_n^{(N-2)/2})^2}{2} - \frac{(u(\delta_n)\delta_n^{(N-2)/2})^2}{2} - \frac{N-2}{2}\int_{\varepsilon_n}^{R_n} v^2(r)r^{N-3}\,\mathrm{d}r. \quad (2.3)
\end{aligned}
$$

Hence, by the choice of ε_n and R_n we deduce that

$$
\int_0^{+\infty} v(r)v'(r)r^{N-2}\,\mathrm{d}r = -\frac{N-2}{2}\int_0^{+\infty} v^2(r)r^{N-3}\,\mathrm{d}r.
$$

So, (2.2) becomes

$$
\begin{aligned}
\frac{1}{\nu_\lambda}\int_0^{+\infty} & (u'(s))^2 s^{N-1}\,\mathrm{d}s \\
&= \frac{(N-2)^2(1-\nu_\lambda)}{4\nu_\lambda^2}(1+\nu_\lambda)\int_0^{+\infty} v^2(r)r^{N-3}\,\mathrm{d}r + \int_0^{+\infty}(v'(r))^2 r^{N-1}\,\mathrm{d}r \\
&\qquad\qquad \text{(recalling the definition of } \nu_\lambda) \\
&= \frac{\lambda}{\nu_\lambda}\int_0^{+\infty} u^2(s)s^{N-3}\,\mathrm{d}s + \int_0^{+\infty}(v'(r))^2 r^{N-1}\,\mathrm{d}r, \quad (2.4)
\end{aligned}
$$

which gives the claim. □

3. The critical case $p = (N+2)/(N-2)$

3.1. Basic properties and the main inequality

In this section we consider problem (1.16). First let us observe that if we put $p = (N+2)/(N-2)$ in (1.2)–(1.5) we get

$$
a = \frac{(N-2)(1-\nu_\lambda)}{2\nu_\lambda} \quad (3.1)
$$

and

$$
b = \frac{1}{\nu_\lambda}. \quad (3.2)
$$

Moreover, if $u \in \mathcal{D}^{1,2}((0,+\infty), r^{N-1}\,\mathrm{d}r)$ satisfies

$$
-u'' - \frac{N-1}{r}u' - \frac{\lambda}{r^2}u = C(\lambda)u^{2^*-1} \quad \text{in } (0,+\infty) \quad (3.3)
$$

in the weak sense, where $C(\lambda) = N(N-2)\nu_\lambda^2$ and ν_λ as in (1.5), then proposition 1.1 shows that $v \in \mathcal{D}^{1,2}((0,+\infty), r^{N-1}\,\mathrm{d}r)$ weakly solves

$$
-v'' - \frac{N-1}{r}v' = N(N-2)v^{2^*-1} \quad \text{in } (0,+\infty). \quad (3.4)
$$

We start with a very simple result.

Proposition 3.1. *All the radial solutions in $D^{1,2}(\mathbb{R}^N)$ of (1.16) are given by the functions $u_{\delta,\lambda}(r)$ in (1.17).*

Proof. This follows directly by (3.3) and (3.4). Since all solutions to (3.4) are given by

$$v_\delta(r) = \frac{\delta^{(N-2)/2}}{(1+\delta^2 r^2)^{(N-2)/2}},$$

by the definition of $\mathcal{L}_p$ we deduce that

$$u_{\delta,\lambda}(r) = r^{-a} v_\delta(r^{1/b}) = r^{(N-2)(\nu_\lambda-1)/2\nu_\lambda} v_\delta(r^{\nu_\lambda}) = \frac{r^{(N-2)(\nu_\lambda-1)/2\nu_\lambda}\delta^{(N-2)/2}}{(1+\delta^2 r^{2\nu_\lambda})^{(N-2)/2}}, \tag{3.5}$$

which gives the claim. □

Now we prove an interesting inequality. In the case $\lambda \geqslant 0$, this is basically contained in [19]. For $\lambda < 0$, we could not find any references in the literature, although this can be shown using (for example) the concentration–compactness principle of Lions (see [11, 12]). Nevertheless, we think that the interest of the next proposition lies in its proof, which reduces the Hardy inequality to the classical Sobolev imbedding.

Proposition 3.2. *Let $\lambda < \frac{1}{4}(N-2)^2$. Then, for any radial function $u \in D^{1,2}(\mathbb{R}^N)$,*

$$\int_{\mathbb{R}^N} \left(|\nabla u|^2 - \frac{\lambda}{|x|^2}u^2\right) \mathrm{d}x \geqslant \left(1 - \frac{4\lambda}{(N-2)^2}\right)^{(N-1)/N} S\left(\int_{\mathbb{R}^N} |u|^{2N/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N}, \tag{3.6}$$

where S is the best Sobolev constant. Moreover, the above inequality is achieved only for $u(r) = u_{\delta,\lambda}(r)$.

If $\lambda > 0$, then (3.6) holds for any $u \in D^{1,2}(\mathbb{R}^N)$.

Proof. Let v be as in (2.1). Then by (1.13) we get

$$\begin{aligned}
\int_{\mathbb{R}^N} \left(|\nabla u|^2 - \frac{\lambda}{|x|^2}u^2\right) \mathrm{d}x
&= \nu_\lambda \int_{\mathbb{R}^N} |\nabla v(|x|)|^2\,\mathrm{d}x \geqslant \nu_\lambda S\left(\int_{\mathbb{R}^N} |v|^{2N/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N} \\
&= \nu_\lambda S\left(\omega_N \int_0^{+\infty} |v(r)|^{2N/(N-2)} r^{N-1}\,\mathrm{d}r\right)^{(N-2)/N} \\
&= \nu_\lambda^{2(N-1)/N} S\left(\omega_N \int_0^{+\infty} |u(s)|^{2N/(N-2)} s^{N-1}\,\mathrm{d}s\right)^{(N-2)/N} \\
&= \left(1 - \frac{4\lambda}{(N-2)^2}\right)^{(N-1)/N} S\left(\int_{\mathbb{R}^N} |u|^{2N/(N-2)}\,\mathrm{d}x\right)^{(N-2)/N},
\end{aligned} \tag{3.7}$$

which gives the claim. Note that the previous inequality becomes an identity if and only if

$$\int_{\mathbb{R}^N} |\nabla v(|x|)|^2 \, \mathrm{d}x = S\biggl(\int_{\mathbb{R}^N} |v|^{2N/(N-2)} \, \mathrm{d}x\biggr)^{(N-2)/N}.$$

It is well known (see, for example, [18]) that this implies

$$v(x) = \frac{\delta^{(N-2)/2}}{(1+\delta^2 r^2)^{(N-2)/2}}$$

for some positive δ. Recalling (see proposition 3.1) that

$$u(r) = r^{-(N-2)(1-\nu_\lambda)/2} v(r^{\nu_\lambda}),$$

we obtain the uniqueness of the minimizer.

Let us show that if $\lambda > 0$, then (3.6) holds for any $u \in D^{1,2}(\mathbb{R}^N)$. This follows by the classical spherical rearrangement theory. Indeed, denoting by $u^* = u^*(|x|)$ the classical Schwarz rearrangement, we have

$$\int_{\mathbb{R}^N} |\nabla u^*|^2 \, \mathrm{d}x \leqslant \int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x,$$

$$\int_{\mathbb{R}^N} |u^*|^{2N/(N-2)} \, \mathrm{d}x = \int_{\mathbb{R}^N} |u|^{2N/(N-2)} \, \mathrm{d}x,$$

$$\int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, \mathrm{d}x \leqslant \int_{\mathbb{R}^N} \frac{|u^*|^2}{|x|^2} \, \mathrm{d}x.$$

Hence, if $\lambda > 0$, we get

$$\frac{\int_{\mathbb{R}^N} (|\nabla u^*|^2 - (\lambda/|x|^2)|u^*|^2) \, \mathrm{d}x}{(\int_{\mathbb{R}^N} |u^*|^{2N/(N-2)} \, \mathrm{d}x)^{(N-2)/N}} \leqslant \frac{\int_{\mathbb{R}^N} (|\nabla u|^2 - (\lambda |x|^2/u^2)) \, \mathrm{d}x}{(\int_{\mathbb{R}^N} |u|^{2N/(N-2)} \, \mathrm{d}x)^{(N-2)/N}},$$

which implies the claim. □

3.2. The linearized operator

In this section we linearize problem (1.16) at the radial solution $u_\lambda := u_{1,\lambda}$ given in (1.17), and we look for the degeneracy points. This is equivalent to finding non-trivial solutions for the linearized problem (1.21).

Proof of lemma 1.2. Using (1.17), we rewrite (1.21) as follows:

$$-\Delta v - \frac{\lambda}{|x|^2} v = N(N+2)\nu_\lambda^2 \frac{|x|^{2(\nu_\lambda - 1)}}{(1+|x|^{2\nu_\lambda})^2} v \quad \text{in } \mathbb{R}^N, \quad v \in D^{1,2}(\mathbb{R}^N). \tag{3.8}$$

We solve (3.8) using the decomposition along the spherical harmonic functions and we write

$$v(r,\theta) = \sum_{j=0}^{\infty} \psi_j(r) Y_j(\theta), \quad \text{where } r = |x|, \ \theta = \frac{x}{|x|} \in S^{N-1}$$

and

$$\psi_j(r) = \int_{S^{N-1}} V(r,\theta) Y_j(\theta) \, \mathrm{d}\theta.$$

Here $Y_j(\theta)$ denotes the jth spherical harmonics, i.e. it satisfies

$$-\Delta_{S^{N-1}} Y_j = \mu_j Y_j,$$

where $\Delta_{S^{N-1}}$ is the Laplace–Beltrami operator on S^{N-1} with the standard metric and μ_j is the jth eigenvalue of $-\Delta_{S^{N-1}}$. It is known that

$$\mu_j = j(N-2+j), \quad j = 0, 1, 2, \dots, \tag{3.9}$$

whose multiplicity is

$$\frac{(N+2j-2)(N+j-3)!}{(N-2)!j!}, \tag{3.10}$$

and that

$$\operatorname{Ker}(\Delta_{S^{N-1}} + \mu_j) = \mathbb{Y}_j(\mathbb{R}^N)|_{S^{N-1}},$$

where $\mathbb{Y}_j(\mathbb{R}^N)$ is the space of all homogeneous harmonic polynomials of degree j in $\mathbb{R}^N$. The function v is a weak solution of (3.8) if and only if $\psi_j(r)$ is a weak solution of

$$\left.\begin{aligned} -\psi_j''(r) - \frac{N-1}{r}\psi_j'(r) + \frac{\mu_j - \lambda}{r^2}\psi_j(r) = N(N+2)\nu_\lambda^2 \frac{r^{2(\nu_\lambda - 1)}}{(1+r^{2\nu_\lambda})^2}\psi_j \quad \text{in } (0,\infty), \\ \psi_j'(0) = 0 \text{ if } j = 0 \quad \text{and} \quad \psi_j(0) = 0 \text{ if } j \geqslant 1, \\ \psi_j \in D^{1,2}((0,+\infty), r^{N-1}\,\mathrm{d}r). \end{aligned}\right\} \tag{3.11}$$

Letting,

$$\hat{\psi}_j(r) = r^{(N-2)(1-\nu_\lambda)/2\nu_\lambda}\psi_j(r^{1/\nu_\lambda}),$$

as in (1.2), $\hat{\psi}_j$ weakly solves

$$\left.\begin{aligned} -\hat{\psi}_j''(r) - \frac{N-1}{r}\hat{\psi}_j'(r) + \frac{\mu_j}{\nu_\lambda^2 r^2}\hat{\psi}_j(r) = N(N+2)\frac{1}{(1+r^2)^2}\hat{\psi}_j \quad \text{in } (0,\infty), \\ \hat{\psi}_j \in D^{1,2}((0,+\infty), r^{N-1}\,\mathrm{d}r). \end{aligned}\right\} \tag{3.12}$$

Problem (3.12) is well known: it comes from the linearization of the solution U_1 in (1.19) to problem (1.18). This equation has a non-trivial solution (since $\mu_j/\nu_\lambda^2 \geqslant 0$) if and only if one of the following holds:

(i) $\mu_j/\nu_\lambda^2 = 0$,

(ii) $\mu_j/\nu_\lambda^2 = N-1$.

Case (i) corresponds to the radial degeneracy, i.e. $j = 0$. Equation (3.12) has the solution

$$\hat{\psi}_0(r) = \frac{1-r^2}{(1+r^2)^{N/2}},$$

and, returning to (3.11), we get

$$\psi_0(r) = \frac{r^{(N-2)(\nu_\lambda - 1)/2}(1 - r^{2\nu_\lambda})}{(1+r^{2\nu_\lambda})^{N/2}},$$

which is a solution to (3.11) for any $\lambda < \frac{1}{4}(N-2)^2$. This proves (1.23).

When $\mu_j/\nu_\lambda^2 = N-1$, (3.12) has the solution

$$\hat{\psi}_j(r) = \frac{r}{(1+r^2)^{N/2}}.$$

Returning to (3.11), we find it has the solution

$$\psi_j(r) = \frac{r^{((N-2)/2)(\nu_\lambda - 1)+\nu_\lambda}}{(1+r^{2\nu_\lambda})^{N/2}}$$

when (ii) is satisfied. Then (ii) implies that (3.11) has the solution $\psi_j(r)$ if and only if $\lambda = \lambda_j$:

$$\lambda_j = \frac{(N-2)^2}{4}\left(1 - \frac{\mu_j}{N-1}\right)$$

as in (1.22). This proves (1.24) and completes the proof of lemma 1.2. □

A first consequence of lemma 1.2 is the computation of the Morse index of the solution u_λ given in proposition 1.3.

Proof of proposition 1.3. As shown in the appendix (corollary A.7), the Morse index of the radial solution u_λ is given by the number of negative values Λ_i counted with multiplicity such that the problem

$$\left.\begin{aligned} -\Delta w - \frac{\lambda}{|x|^2}w - N(N+2)\nu_\lambda^2\frac{|x|^{2\nu_\lambda}}{(1+|x|^{2\nu_\lambda})^2}w = \frac{\Lambda}{|x|^2}w \quad \text{in } \mathbb{R}^N,\\ w \in D^{1,2}(\mathbb{R}^N), \end{aligned}\right\} \tag{3.13}$$

admits a weak solution w_i. We denote by w_i the solution of (3.13) related to a negative value Λ_i. We argue as before, setting

$$w_{i,j}(r) = \int_{S^{N-1}} w_i(r,\theta)Y_j(\theta)\,\mathrm{d}\theta$$

and $\hat{w}_{i,j}(r) = r^{(N-2)(1-\nu_\lambda)/2\nu_\lambda}w_{i,j}(r^{1/\nu_\lambda})$. Then $\hat{w}_{i,j}(r)$ weakly satisfies

$$\left.\begin{aligned} -\hat{w}_{i,j}''(r) - \frac{N-1}{r}\hat{w}_{i,j}'(r) - N(N+2)\frac{1}{(1+r^2)^2}\hat{w}_{i,j}&\\ = \frac{\Lambda_i - \mu_j}{\nu_\lambda^2 r^2}\hat{w}_{i,j}(r) \quad \text{in } (0,\infty),&\\ \hat{w}_{i,j} \in D^{1,2}((0,+\infty), r^{N-1}\,\mathrm{d}r).& \end{aligned}\right\} \tag{3.14}$$

Since the problem

$$\begin{gathered} -\eta''(r) - \frac{N-1}{r}\eta'(r) - N(N+2)\frac{1}{(1+r^2)^2}\eta = \frac{\nu}{r^2}\eta \quad \text{in } (0,\infty),\\ \eta \in D^{1,2}((0,+\infty), r^{N-1}\,\mathrm{d}r), \end{gathered}$$

admits only one negative eigenvalue, which is $1-N$, we can derive that (3.14) has a non-trivial solution corresponding to a $\Lambda_i < 0$ if and only if

$$1 - N = \frac{\Lambda_i - \mu_j}{\nu_\lambda^2}.$$

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So the indices j that contribute to the Morse index of the solution u_λ are those that satisfy $\Lambda_i = \nu_\lambda^2(1-N) + \mu_j < 0$ and this implies (recalling the value of μ_j given in (3.9))

$$j < \frac{2-N}{2} + \frac{1}{2}\sqrt{N^2 - \frac{16(N-1)\lambda}{(N-2)^2}}.$$

Finally, as the dimension of the eigenspace of the Laplace–Beltrami operator on S^{N-1} related to μ_j is given in (3.10), equation (1.25) follows. □

REMARK 3.3. Reasoning as in the proof of the previous corollary, it is easy to see that any eigenfunction of (3.13) corresponding to an eigenvalue $\Lambda < 0$ can be written

$$w(r,\theta) = r^{(N-2)(\nu_\lambda - 1)/2}\frac{r^{\nu_\lambda}}{(1+r^{2\nu_\lambda})^{N/2}}Y_j(\theta),$$

where $Y_j(\theta)$ is a spherical harmonic related to the eigenvalue μ_j.

3.3. The bifurcation result

In this section we shall start to prove theorem 1.4 using the bifurcation theory.

To this end let us give some definitions. Let $\gamma > 0$ be such that $\frac{1}{2}(N-2) < \gamma < N-2$. For every $g \in L^\infty(\mathbb{R}^N)$ we define the weighted norm

$$\|g\|_\gamma := \sup_{x\in\mathbb{R}^N} (1+|x|)^\gamma |g(x)| \tag{3.15}$$

and the space $L^\infty_\gamma(\mathbb{R}^N) := \{g \in L^\infty(\mathbb{R}^N) \text{ such that } \exists C > 0 \text{ and } \|g\|_\gamma < C\}$. Set

$$X = D^{1,2}(\mathbb{R}^N) \cap L^\infty_\gamma(\mathbb{R}^N). \tag{3.16}$$

X is a Banach space with the norm

$$\|g\|_X := \max\{\|g\|_{1,2}, \|g\|_\gamma\}, \tag{3.17}$$

where $\|g\|_{1,2}$ denotes the usual norm in $D^{1,2}(\mathbb{R}^N)$, i.e.

$$\|g\|_{1,2} = \left(\int_{\mathbb{R}^N} |\nabla g|^2 \,\mathrm{d}x\right)^{1/2}.$$

To apply the standard bifurcation theory we have to define a compact operator T from $(-\infty, 0) \times X$ into X and to compute its Leray–Schauder degree in 0 in a suitable neighbourhood of the radial solution (λ, u_λ), at least for the values $\lambda \neq \lambda_j$. This seems difficult since the linearized operator (see (1.21)) is not invertible due to the radial degeneracy of u_λ for every λ proved in lemma 1.2. To this end we define

$$K_\lambda := \left\{v \in D^{1,2}(\mathbb{R}^N) \text{ such that } \int_{\mathbb{R}^N} u_\lambda^{2^*-2} v Z_\lambda \,\mathrm{d}x = 0\right\} \tag{3.18}$$

with Z_λ as defined in (1.23). Since $u_\lambda^{2^*-2} \in L^{N/2}(\mathbb{R}^N)$ and $v, Z_\lambda \in L^{2^*}(\mathbb{R}^N)$, K_λ is a linear closed subspace of $D^{1,2}(\mathbb{R}^N)$. We let P_λ be orthogonal the projection of $D^{1,2}(\mathbb{R}^N)$ on K_λ.

Now we define the operator $T(\lambda, v)\colon (-\infty, 0) \times X \to K_\lambda \cap X$ as

$$T(\lambda, v) = P_\lambda\biggl(\biggl(-\Delta - \frac{\lambda}{|x|^2}I\biggr)^{-1}(C(\lambda)(v^+)^{2^*-1})\biggr) \tag{3.19}$$

and look for zeros of the operator $I-T(\lambda, v)$. A function $v \in X$ is a zero of $I-T(\lambda, v)$ if $v \in K_\lambda \cap X$ and v is a weak solution of

$$-\Delta v - \frac{\lambda}{|x|^2}v - C(\lambda)v^{2^*-1} = LC(\lambda)\frac{N+2}{N-2}u_\lambda^{2^*-2}Z(x) \quad \text{in } \mathbb{R}^N, \tag{3.20}$$

where $L = L(v) \in \mathbb{R}$ (L is the Lagrange multiplier). The final step will be to show that $L = 0$ so that v is indeed a weak solution of (1.16). This will be done in § 3.4.

Before proving our bifurcation result we need some technical results.

LEMMA 3.4. *The operator $T(\lambda, v)$ is well defined from $(-\infty, 0) \times X$ into $K_\lambda \cap X$.*

Proof. It is enough prove that the operator

$$\tilde{T}(\lambda, v) = \biggl(-\Delta - \frac{\lambda}{|x|^2}I\biggr)^{-1}(C(\lambda)(v^+)^{2^*-1}) \tag{3.21}$$

is well defined from $(-\infty, 0) \times X$ in X.

Since $(v^+)^{2^*-1} \in L^{2N/(N+2)}(\mathbb{R}^N)$, there exists a unique $g \in D^{1,2}(\mathbb{R}^N)$ such that $g = \tilde{T}(\lambda, v)$ (see lemma A.2 in the appendix), i.e. g is a weak solution to

$$-\Delta g - \frac{\lambda}{|x|^2}g = C(\lambda)(v^+)^{2^*-1} \quad \text{in } \mathbb{R}^N.$$

Then, the comparison theorem for functions in $D^{1,2}(\mathbb{R}^N)$ yields

$$|g(x)| \leqslant C|w(x)| \quad \text{almost everywhere (a.e.) in } \mathbb{R}^N,$$

where w is the unique weak solution to

$$-\Delta w - \frac{\lambda}{|x|^2}w = \frac{1}{(1+|x|)^{\gamma(N+2)/(N-2)}} \quad \text{in } \mathbb{R}^N. \tag{3.22}$$

We shall prove that

$$(1+r)^\gamma w(r) \leqslant C. \tag{3.23}$$

To do this let $\bar{w}(r) = r^k w(r)$, where $k = \frac{1}{2}(N-2)(1-\nu_\lambda)$ and ν_λ is as defined in (1.5). The function $\bar{w}$ weakly satisfies

$$-\bar{w}'' - \frac{N-1-2k}{r}\bar{w}' = \frac{r^k}{(1+r)^{\gamma(N+2)/(N-2)}} \quad \text{in } (0, +\infty).$$

Integrating, we get

$$-r^{N-1-2k}\bar{w}'(r) = C + \int_{r_0}^{r} \frac{s^{N-1-k}}{(1+s)^{\gamma(N+2)/(N-2)}}\,\mathrm{d}s$$

for any $r_0 > 0$. Consequently, $-\bar{w}'(r) \leqslant Cr^{1-N+2k} + Cr^{1-N+2k+N-k-\gamma(N+2)/(N-2)}$ (we assume that $N-1-k-\gamma(N+2)/(N-2) \neq -1$; the case $N-k-\gamma(N+2)/(N-2) = 0$ follows in a very similar way).

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Since $w \in D^{1,2}(\mathbb{R}^N)$, from Ni's radial lemma (see [14]) we know that $w(r) \leqslant Cr^{(2-N)/2}$ for any r, so that $\bar{w}(r) \leqslant Cr^{k-(N-2)/2} = Cr^{-(N-2)\nu_\lambda/2}$. Then $\bar{w}(r) \to 0$ as $r \to +\infty$. Integrating $\bar{w}'(r)$ from r to $+\infty$ yields

$$\bar{w}(r) \leqslant Cr^{2-N+2k} + Cr^{2+k-\gamma(N+2)/(N-2)}.$$

Since, by assumption, $\frac{1}{2}(N-2) < \gamma < N-2$ and $k < 0$ for any $\lambda < 0$, this implies that

$$(1+r)^\gamma w(r) \leqslant Cr^{\gamma+2-N+k} + Cr^{\gamma+2-\gamma(N+2)/(N-2)} \leqslant C \tag{3.24}$$

for r large enough.

To complete the proof of (3.23) we need to prove that $|w(x)|$ is bounded in a neighbourhood of the origin. To this end we set

$$\tilde{w}(r) = \frac{1}{r^{N-2k-2}}\bar{w}\left(\frac{1}{r}\right).$$

The function $\tilde{w}$ is the Kelvin transform of $\bar{w}$, and so it satisfies

$$-\tilde{w}'' - \frac{N-1-2k}{r}\tilde{w}' = \frac{1}{r^{N-2k+2}}\frac{r^{-k}}{(1+1/r)^{\gamma(N+2)/(N-2)}} \quad \text{in } \mathbb{R}^N \setminus \{0\}.$$

Reasoning as before and integrating from r_0 to r, we get

$$-r^{N-1-2k}\tilde{w}'(r) \leqslant C + C\int_{r_0}^{r} s^{-3-k}\,\mathrm{d}s$$

and, assuming $-3-k > -1$ (observe that the case $-3-k < -1$ is easier and the case $-3-k = -1$ follows the reasoning in the first case),

$$-\tilde{w}'(r) \leqslant Cr^{1-N+2k} + Cr^{1-N+2k-2-k}$$

for r large enough. Then, using that $w(r) \to 0$ as $r \to +\infty$,

$$\tilde{w}(r) \leqslant Cr^{2-N+2k} + Cr^{-N+k}$$

for r large enough. This implies that

$$\bar{w}(r) = \frac{1}{r^{N-2-2k}}\tilde{w}\left(\frac{1}{r}\right) \leqslant Cr^{2+k} + C$$

for r small enough. Finally, using that $w(r) = r^{-k}\bar{w}(r)$, we obtain

$$w(r) \leqslant C \begin{cases} r^{-k} + r^2 & \text{if } k < -2, \\ r^2(1-\log r) & \text{if } k = -2, \\ r^{-k} & \text{if } -2 < k < 0, \end{cases} \tag{3.25}$$

for r small enough. Estimates (3.24) and (3.25) imply that

$$\sup_{x\in\mathbb{R}^N}(1+|x|)^\gamma|w(x)| \leqslant C$$

so that w and hence g belong to $L^\infty_\gamma(\mathbb{R}^N)$, concluding the proof. □

PROPOSITION 3.5.

(i) *The operator* $T(\lambda, v)\colon (-\infty, 0) \times X \to K_\lambda \cap X$ *defined in (3.19) is continuous with respect to* λ *and it is compact from* X *into* $K_\lambda \cap X$ *for any* $\lambda \in (-\infty, 0)$ *fixed.*

(ii) *The linearized operator* $I - T_v(\lambda, u_\lambda)I$ *is invertible for any* $\lambda \neq \lambda_j$*, where* λ_j *are as defined in (1.22).*

Proof.
(i) The operator $T(\lambda, v)$ is clearly continuous with respect to λ. As in the proof of lemma 3.4, we shall prove that the operator $\tilde{T}$, defined in (3.21), is compact from X into X for every λ fixed. This implies in turn that T is compact for every λ fixed. To this end let v_n be a sequence in X such that $\|v_n\|_X \leqslant C$ and let $g_n = \tilde{T}(\lambda, v_n)$. Then $g_n \in X$ and, by lemma 3.4, is a weak solution to

$$-\Delta g_n - \frac{\lambda}{|x|^2} g_n = C(\lambda)(v_n^+)^{2^*-1}. \tag{3.26}$$

Since v_n is bounded in X, we have $|v_n(x)| \leqslant C(1+|x|)^{-\gamma}$ and v_n is uniformly bounded in $D^{1,2}(\mathbb{R}^N)$. Then, up to a subsequence, $v_n \rightharpoonup \bar{v}$ weakly in $D^{1,2}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. Multiplying (3.26) by g_n and integrating, we get

$$\int_{\mathbb{R}^N} |\nabla g_n|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} \frac{\lambda}{|x|^2} g_n^2 \,\mathrm{d}x = C(\lambda) \int_{\mathbb{R}^N} (v_n^+)^{2^*-1} g_n \,\mathrm{d}x. \tag{3.27}$$

Then the Hardy and Sobolev inequalities imply that

$$c_\lambda \int_{\mathbb{R}^N} |\nabla g_n|^2 \,\mathrm{d}x \leqslant C \|v_n^{2^*-1}\|_{2N/(N+2)} \|g_n\|_{1,2},$$

where c_λ is as in lemma A.1 and $\|\cdot\|_q$ denotes the usual norm in $L^q(\mathbb{R}^N)$. Then

$$\|g_n\|_{1,2} \leqslant C,$$

so that, up to a subsequence, $g_n \rightharpoonup \bar{g}$ weakly in $D^{1,2}(\mathbb{R}^N)$ and a.e. in $\mathbb{R}^N$. Passing to the limit in (3.26), we get that $\bar{g}$ is a weak solution of

$$-\Delta \bar{g} - \frac{\lambda}{|x|^2} \bar{g} = C(\lambda)(\bar{v}^+)^{2^*-1} \quad \text{in } \mathbb{R}^N.$$

Moreover, reasoning exactly as in the proof of lemma 3.4 we also get $|g_n(x)| \leqslant C(1+|x|)^{-\gamma}$ for any n. This estimate allow us to pass to the limit in (3.27), to obtain that

$$\begin{aligned}\int_{\mathbb{R}^N} |\nabla g_n|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} \frac{\lambda}{|x|^2} g_n^2 \,\mathrm{d}x &= C(\lambda) \int_{\mathbb{R}^N} (v_n^+)^{2^*-1} g_n \,\mathrm{d}x \\ &\to C(\lambda) \int_{\mathbb{R}^N} (\bar{v}^+)^{2^*-1} \bar{g} \,\mathrm{d}x = \int_{\mathbb{R}^N} |\nabla \bar{g}|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} \frac{\lambda}{|x|^2} \bar{g}^2 \,\mathrm{d}x.\end{aligned}$$

By lemma A.1 this implies that $g_n \to g$ strongly in $D^{1,2}(\mathbb{R}^N)$. To finish the proof we need to show that $\|g_n - g\|_\gamma < \varepsilon$ if n is large enough. To this end, observe that

$g_n - g$ weakly solves

$$-\Delta(g_n - \bar g) - \frac{\lambda}{|x|^2}(g_n - \bar g) = C(\lambda)((v_n^+)^{2^*-1} - (\bar v^+)^{2^*-1}) \quad \text{in } \mathbb{R}^N,$$

and since v_n and $\bar v$ are uniformly bounded in $L^\infty_\gamma(\mathbb{R}^N)$ as in lemma 3.4 we obtain $|g_n - \bar g| \leqslant Cw$, where w is defined in (3.22). Then, by (3.24) there exists $R_0 > 0$ such that $(1+|x|)^\gamma |g_n(x) - \bar g(x)| \leqslant \frac{1}{3}\varepsilon$ in $\mathbb{R}^N \setminus B_{R_0}$ uniformly in n. Using (3.25) instead, we get that there exists r_0 such that $(1+|x|)^\gamma |g_n(x) - \bar g(x)| \leqslant \frac{1}{3}\varepsilon$ in B_{r_0} uniformly in n. Finally, since, $v_n \to \bar v$ in $L^\infty(B_{R_0} \setminus B_{r_0})$, we get that $(1+|x|)^\gamma |g_n - \bar g| < \frac{1}{3}\varepsilon$ for n large enough in $B_{R_0} \setminus B_{r_0}$, and the proof of (i) is complete.

(ii) Let us consider the linearized operator of $I - T$ in (λ, u_λ). We have that

$$w - T_v(\lambda, u_\lambda)w = w - P_{K_\lambda}\left(\left(-\Delta - \frac{\lambda}{|x|^2} I\right)^{-1}\left(C(\lambda)\frac{N+2}{N-2} u_\lambda^{2^*-2} w\right)\right)$$

so that $w - T_v(\lambda, u_\lambda)w = 0$ if and only if $w \in K_\lambda \cap X$ satisfies

$$-\Delta w - \frac{\lambda}{|x|^2} w - C(\lambda)\frac{N+2}{N-2} u_\lambda^{2^*-2} w = LC(\lambda)\frac{N+2}{N-2} u_\lambda^{2^*-2} Z_\lambda$$

weakly in $D^{1,2}(\mathbb{R}^N)$ for some $L = L(w) \in \mathbb{R}$. Multiplying by Z_λ and recalling the equation satisfied by Z_λ, we get

$$0 = LC(\lambda)\frac{N+2}{N-2}\int_{\mathbb{R}^N} u_\lambda^{2^*-2} Z_\lambda^2 \, \mathrm{d}x,$$

and this implies $L = 0$. Then $w \in K_\lambda$ is a weak solution of

$$-\Delta w - \frac{\lambda}{|x|^2} w - C(\lambda)\frac{N+2}{N-2} u_\lambda^{2^*-2} w = 0 \quad \text{in } \mathbb{R}^N. \tag{3.28}$$

Using lemma 1.2 we then get that if $\lambda \neq \lambda_n$, (3.28) has only one solution Z_λ that is not in K_λ. This means that in K_λ (3.28) has only the solution $w = 0$ and the operator $I - T_v(\lambda, u_\lambda)I$ is indeed invertible, concluding the proof. □

To prove the bifurcation result (theorem 1.4) we need to exploit some of the symmetries of problem (1.16). So, we define the subspace $\mathcal{H}$ of X as

$$\mathcal{H} := \{v \in X \text{ s.t. } v(x_1, \dots, x_N) = v(g(x_1, \dots, x_{N-1}), x_N) \text{ for any } g \in O(N-1)\}.$$

Now let us consider the subgroups $\mathcal{G}_h$ of $O(N)$ defined by

$$\mathcal{G}_h = O(h) \times O(N-h) \quad \text{for } 1 \leqslant h \leqslant [N/2],$$

where $[a]$ stands for the integer part of a. We consider also the subspaces $\mathcal{H}^h$ of X of functions invariant by the action of $\mathcal{G}_h$.

The results of Smoller and Wasserman [15, 16] imply that, for any j, the eigenspace of the Laplace–Beltrami operator related to μ_j (see § 3.2) contains only one eigenfunction that is $O(N-1)$-invariant (or that is invariant by the action of $\mathcal{G}_h$). Then, corollary 1.3 implies that

$$m_{\mathcal{H}}(\lambda_j - \varepsilon) - m_{\mathcal{H}}(\lambda_j + \varepsilon) = 1$$

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if ε is small enough, where $m_{\mathcal{H}}$ denotes the Morse index of u_λ in the space $\mathcal{H}$ (or $\mathcal{H}^h$).

The change in the Morse index of u_λ is a good clue that we have the bifurcation, but since u_λ is radially degenerate we have to use the projection P_λ, changing problem (1.16) to problem (3.20).

What we can do at this step is to prove a bifurcation result for problem (3.20). To prove this we need that the Morse index of u_λ as a solution of problem (3.20) is the same as $m(\lambda)$, and this is proved in the following proposition.

Proposition 3.6. *The number of eigenvalues of $T_v(\lambda, u_\lambda)$ counted with multiplicity in $(1, +\infty)$ coincides with the Morse index $m(\lambda)$ of u_λ.*

Proof. Λ is an eigenvalue for the linear operator $T_v(\lambda, u_\lambda)I$ if and only if

$$\Lambda I - T_v(\lambda, u_\lambda)I = 0$$

has a non-trivial solution in $X \cap K_\lambda$. This means that we have to find a $w \in X \cap K_\lambda$, $w \neq 0$, which verifies

$$-\Delta w - \frac{\lambda}{|x|^2} w = \frac{1}{\Lambda} C(\lambda) \frac{N+2}{N-2} u_\lambda^{2^*-2} w + \frac{L}{\Lambda} C(\lambda) \frac{N+2}{N-2} u_\lambda^{2^*-2} Z_\lambda \quad \text{in } \mathbb{R}^N \tag{3.29}$$

for some $L = L(w) \in \mathbb{R}$ and $1/\Lambda \in (0, 1)$.

Observe that, since $\Lambda \neq 1$, the function $w_1 = LZ_\lambda/(\Lambda - 1)$ is always a solution of (3.29) (which does not belong to K_λ) and all the other solutions of (3.29) are given by $w = w_1 + \tilde{w}$, where $\tilde{w} \in X \cap K_\lambda$ satisfies

$$-\Delta \tilde{w} - \frac{\lambda}{|x|^2} \tilde{w} = \frac{1}{\Lambda} C(\lambda) \frac{N+2}{N-2} u_\lambda^{2^*-2} \tilde{w} \quad \text{in } \mathbb{R}^N. \tag{3.30}$$

Now, if $1/\Lambda$ is not an eigenvalue of (3.30), then $\tilde{w} = 0$ and (3.29) has only the solution w_1. But w_1 is not in K_λ, so $w = 0$ and $L = 0$ in (3.29).

Otherwise, if $1/\Lambda$ is an eigenvalue of (3.30) and $\tilde{w}$ is a corresponding eigenfunction, we can use $\tilde{w}$ as a test function in (3.29) and Z_λ as a test function in (3.30), obtaining that

$$\int_{\mathbb{R}^N} u_\lambda^{2^*-2} Z_\lambda \tilde{w} \, \mathrm{d}x = 0,$$

so $\tilde{w} \in K_\lambda$. Since $w_1 \notin K_\lambda$, this implies $L = 0$ in (3.29), so (3.29) coincides with (3.30).

So far, we have shown that the number of eigenvalues of $T_v(\lambda, u_\lambda)$ counted with multiplicity in $(1, +\infty)$ is equal to the number of eigenvalues of (3.30) counted with multiplicity in $(0, 1)$, and this is the Morse index of u_λ. □

From proposition 3.6 we know that the number of eigenvalues of $T_v(\lambda, u_\lambda)$ counted with multiplicity in $(1, +\infty)$ decreases by 1 going from $\lambda_j - \varepsilon$ to $\lambda_j + \varepsilon$ with ε small enough in the space $\mathcal{H}$ (or $\mathcal{H}^h$), and this is sufficient to have the bifurcation.

We shall not give the details of the global bifurcation result for problem (3.20); we only sketch the proof of the local bifurcation result in order to have an idea how to use the results of propositions 3.5 and 3.6. Then the global bifurcation result will follow the reasoning in [7, theorem 3.3] (see also [1]).

Proposition 3.7. *The points $(\lambda_j, u_{\lambda_j})$ are non-radial bifurcation points for the curve (λ, u_λ) of radial solutions of (3.20).*

Proof. Assume by contradiction that $(\lambda_j, u_{\lambda_j})$ is not a bifurcation point for (3.20) for some j. Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ and for all $c \in (0, \varepsilon_0)$

$$I - T(\lambda, v) \neq 0$$

for any $\lambda \in (\lambda_j - \varepsilon, \lambda_j + \varepsilon) \subset (-\infty, 0)$ and for any $v \in \mathcal{H}$ (or in $\mathcal{H}^h$) such that $\|v - u_\lambda\|_X \leqslant c$ and $v \neq u_\lambda$.

Let $\Gamma := \{(\lambda, v) \in (\lambda_j - \varepsilon, \lambda_j + \varepsilon) \times \mathcal{H}\colon \|v - u_\lambda\|_X \leqslant c\}$ and $\Gamma_\lambda := \{v \in \mathcal{H}$ s.t. $(\lambda, v) \in \Gamma\}$. The homotopy invariance of the Leray–Schauder degree then implies

$$\deg(I - T(\lambda, \cdot), \Gamma_\lambda, 0) \text{ is constant on } (\lambda_j - \varepsilon, \lambda_j + \varepsilon). \tag{3.31}$$

Since the linearized operator is invertible for $\lambda = \lambda_j - \varepsilon$ and $\lambda = \lambda_j + \varepsilon$, we can compute the Leray–Schauder degree, which is given by

$$\deg(I - T(\lambda_j \pm \varepsilon, \cdot), \Gamma_{\lambda_j \pm \varepsilon}, 0) = (-1)^{\beta(\lambda_j \pm \varepsilon)},$$

where $\beta(\lambda)$ is the number of eigenvalues of $T_v(\lambda, u_\lambda)$ counted with multiplicity in $(1, +\infty)$ (see [1, theorem 3.20]). Then proposition 3.6 implies that

$$\deg(I - T(\lambda_j - \varepsilon, \cdot), \Gamma_{\lambda_j - \varepsilon}, 0) = -\deg(I - T(\lambda_j + \varepsilon, \cdot), \Gamma_{\lambda_j + \varepsilon}, 0),$$

contradicting (3.31). Then $(\lambda_j, u_{\lambda_j})$ is a bifurcation point for (3.20), and the bifurcating solutions are non-radial since u_λ is radially non-degenerate in K_λ. □

Finally, we can state the global bifurcation result for (3.20).

Proposition 3.8. *Let fix $j \in \mathbb{N}$ and let λ_j be as defined in (1.22). Then we have the following.*

(i) *If j is odd, there exists at least one continuum of non-radial solutions to (3.20), invariant with respect to $O(N-1)$, bifurcating from the pair $(\lambda_j, u_{\lambda_j})$.*

(ii) *If j is even, there exist at least $[N/2]$ continua of non-radial solutions to (3.20) bifurcating from the pair $(\lambda_j, u_{\lambda_j})$. The first branch is $O(N-1)$ invariant; the second is $O(N-2) \times O(2)$ invariant, and so on.*

Proof. The proof is standard and follows from proposition 3.7 using, for example, [1, theorem 4.8]. □

The final step of the proof of theorem 1.4 is to show that the solutions we have found in proposition 3.8 are indeed solutions of (1.16). This will be done in the next subsection.

3.4. The Lagrange multiplier is zero

In the previous subsection we proved the existence of solutions $u_{\varepsilon,n}$ and parameters $\lambda_{\varepsilon,n}, L_\varepsilon$ verifying

$$-\Delta u_{\varepsilon,n} - \frac{\lambda_{\varepsilon,n}}{|x|^2} u_{\varepsilon,n} - C(\lambda_{\varepsilon,n}) u_{\varepsilon,n}^{2^*-1} = L_\varepsilon C(\lambda_{\varepsilon,n}) \frac{N+2}{N-2} u_{\varepsilon,n}^{2^*-2} Z_{\varepsilon,n} \quad \text{in } \mathbb{R}^N, \tag{3.32}$$

where $Z_{\varepsilon,n} = Z_{\lambda_{\varepsilon,n}}$, with $\lambda_{\varepsilon,n}$, $u_{\varepsilon,n}$ and L_ε such that $\lambda_{0,n} = \lambda_n$, $u_{\varepsilon,n} = u_{\lambda_n}$ and $L_0 = 0$.

In the following we denote by C a generic constant (independent of n and ε) that can change from line to line. First we prove a bound on L_ε.

LEMMA 3.9. *Let L_ε be the Lagrange multiplier in (3.32). Then, for ε small enough,*

$$|L_\varepsilon| \leqslant C.$$

Proof. Using $Z_{\varepsilon,n}$ as a test function in (3.32) we get

$$\begin{aligned} L_\varepsilon C(\lambda_{\varepsilon,n})\frac{N+2}{N-2}\int_{\mathbb{R}^N} u_{\lambda_{\varepsilon,n}}^{2^*-2} Z_{\varepsilon,n}^2 \,\mathrm{d}x \\ = \int_{\mathbb{R}^N} \nabla u_{\varepsilon,n}\cdot\nabla Z_{\varepsilon,n}\,\mathrm{d}x - \int_{\mathbb{R}^N}\frac{\lambda_{\varepsilon,n}}{|x|^2}u_{\varepsilon,n}Z_{\varepsilon,n}\,\mathrm{d}x - C(\lambda_{\varepsilon,n})\int_{\mathbb{R}^N} u_{\varepsilon,n}^{2^*-1}Z_{\varepsilon,n}\,\mathrm{d}x. \end{aligned} \tag{3.33}$$

Using lemma A.1 and the Hölder and Sobolev inequalities, we get

$$L_\varepsilon C(\lambda_{\varepsilon,n})\frac{N+2}{N-2}\int_{\mathbb{R}^N} u_{\lambda_{\varepsilon,n}}^{2^*-2} Z_{\varepsilon,n}^2 \,\mathrm{d}x \leqslant C\|u_{\varepsilon,n}\|_{1,2}\|Z_{\varepsilon,n}\|_{1,2},$$

so the claim follows. □

PROPOSITION 3.10. *Let $u_{\varepsilon,n}$ be the solution of (3.32). Then $L_\varepsilon = 0$ in (3.32) for ε small enough.*

Proof. Applying the Pohozaev identity (A 4) with

$$f(x,u) = \frac{\lambda_{\varepsilon,n}}{|x|^2}u + C(\lambda_{\varepsilon,n})u^{2^*-1} + L_\varepsilon C(\lambda_{\varepsilon,n})\frac{N+2}{N-2}u^{2^*-2}Z_{\varepsilon,n},$$

we get

$$\begin{aligned} \int_{\mathbb{R}^N}|\nabla u_{\varepsilon,n}|^2\,\mathrm{d}x - \frac{N}{N-2}\int_{\mathbb{R}^N}\frac{\lambda_{\varepsilon,n}}{|x|^2}u_{\varepsilon,n}^2\,\mathrm{d}x - C(\lambda_{\varepsilon,n})\int_{\mathbb{R}^N}u_{\varepsilon,n}^{2^*}\,\mathrm{d}x \\ - \frac{2N}{N-2}L_\varepsilon C(\lambda_{\varepsilon,n})\int_{\mathbb{R}^N}u_{\varepsilon,n}^{2^*-1}Z_{\varepsilon,n}\,\mathrm{d}x + \frac{2}{N-2}\int_{\mathbb{R}^N}\frac{\lambda_{\varepsilon,n}}{|x|^2}u_{\varepsilon,n}^2\,\mathrm{d}x \\ - \frac{2}{N-2}L_\varepsilon C(\lambda_{\varepsilon,n})\int_{\mathbb{R}^N}u_{\varepsilon,n}^{2^*-1}x\cdot\nabla Z_{\varepsilon,n}\,\mathrm{d}x = 0. \end{aligned} \tag{3.34}$$

Using $u_{\varepsilon,n}$ as a test function in (3.32), we then get

$$L_\varepsilon\int_{\mathbb{R}^N}(u_{\varepsilon,n}^{2^*-1}Z_{\varepsilon,n} + u_{\varepsilon,n}^{2^*-1}x\cdot\nabla Z_{\varepsilon,n})\,\mathrm{d}x = 0,$$

and this implies $L_\varepsilon = 0$ if we show that the integral is non-zero. Recall that $u_{\varepsilon,n} \to u_{\lambda_n}$, $Z_{\varepsilon,n} \to Z_{\lambda_n}$ as $\varepsilon \to 0$. Moreover, by the definition of $Z_{\varepsilon,n}$ and since $\lambda_n < 0$, we get $Z_{\varepsilon,n} = O((1+|x|)^{2-N})$ and $|\nabla Z_{\varepsilon,n}| = O((1+|x|)^{1-N})$. Finally, since $u_{\varepsilon,n} \in X$, we have $u_{\varepsilon,n} \in X$ and then $u_{\varepsilon,n} = O((1+|x|)^\gamma)$ with $\frac{1}{2}(N-2) < \gamma < N-2$. So, by the dominated convergence theorem, we derive that

$$\int_{\mathbb{R}^N} (u_{\varepsilon,n}^{2^*-1} Z_{\varepsilon,n} + u_{\varepsilon,n}^{2^*-1} x \cdot \nabla Z_{\varepsilon,n})\,\mathrm{d}x \to \int_{\mathbb{R}^N} u_{\lambda_n}^{2^*-1} x \cdot \nabla Z_{\lambda_n}\,\mathrm{d}x \neq 0,$$

so $L_\varepsilon = 0$ if ε small enough, concluding the proof. □

Now we can prove theorem 1.4.

Proof of theorem 1.4. This follows from propositions 3.8 and 3.10. □

4. The subcritical case $1 < p < (N+2)/(N-2)$

Let us start this section by recalling some facts. The following theorem collects some results from the literature (see [6, 17]).

THEOREM 4.1. *Let $1 < p < (L+2)/(L-2)$ and let v_L be the unique positive solution of*

$$\left.\begin{aligned} -v'' - \frac{L-1}{r}v' = v^p \quad &\text{in } (0,1),\\ v > 0 \quad &\text{in } (0,1),\\ v'(0) = v(1) = 0, & \end{aligned}\right\} \tag{4.1}$$

where L is a real number greater than 1. Then v_L is non-degenerate and its Morse index is 1.

REMARK 4.2. Theorem 4.1 allows us to establish the existence of the branch of radial solutions u_λ as stated in theorem 1.7. Moreover, using the transformation $\mathcal{L}_p$, we are able to find the behaviour of the radial solution u_λ at zero (see (1.30)).

REMARK 4.3. Problem (1.28) as $\lambda > 0$ was studied by Chaves and Garcia-Azorero [4], who proved the existence of a unique radial solution u_λ and its behaviour near the origin, which is exactly the same as in (1.30). Their proof relies on the moving-plane method (which ensures that every positive solution is radial) and on the phase-plane analysis of the radial solutions. Both steps strongly rely on the hypothesis that $\lambda > 0$ and cannot be extended to $\lambda \leqslant 0$. Using the map $\mathcal{L}_p$, we easily obtain a new proof of the results of [4] and we extend them to the case $\lambda < 0$.

REMARK 4.4. The non-degeneracy result in theorem 4.1 and the implicit function theorem imply that the function $\lambda \to v_\lambda$ is C^1.

Proof of theorem 1.7. Let $u(r)$ be a radial solution to (1.28). Let $v(r) = \mathcal{L}_p(u(r))$, as defined in (1.2). From proposition 1.1 we know that the transformed function $v(r)$ satisfies

$$\left.\begin{aligned} -v'' - \frac{M-1}{r}v' = A(\lambda,p)v^p \quad &\text{in } (0,1),\\ v > 0 \quad &\text{in } (0,1),\\ v(1) = 0, & \end{aligned}\right\} \tag{4.2}$$

with M as in (1.8) and $A(\lambda, p)$ as in (1.9).

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Moreover, a straightforward computation shows that if $1 < p < (N+2)/(N-2)$, then $1 < p < (M+2)/(M-2)$. Then, (1.13) implies

$$\int_0^1 r^{M-1}(v'(r))^2\,\mathrm{d}r = \frac{1}{\nu_\lambda}\int_0^1 s^{N-1}\left(u'(s)^2 - \frac{\lambda}{s^2}u^2(s)\right)\mathrm{d}s \leqslant C. \tag{4.3}$$

We want to use (4.3) to prove that the function v also satisfies $v'(0) = 0$. This will imply the existence and uniqueness result using theorem 4.1 with $L = M$.

To this end we let

$$\tilde{v}(r) = \frac{1}{r^{M-2}}v\left(\frac{1}{r}\right).$$

The function $\tilde{v}$ solves the equation

$$-(r^{M-1}\tilde{v}'(r))' = A(\lambda,p)r^{(M-2)p-3}\tilde{v}^p(r) \quad \text{in } (1,+\infty), \qquad \tilde{v}(1) = 0, \tag{4.4}$$

and satisfies

$$\int_1^{+\infty} r^{M-1}(\tilde{v}'(r))^2\,\mathrm{d}r \leqslant C.$$

Reasoning exactly as in the radial lemma of Ni [14], we then get that

$$\tilde{v}(r) \leqslant Cr^{(2-M)/2}, \tag{4.5}$$

so $\tilde{v}(r) \to 0$ as $r \to +\infty$ since $M > 2$. Let r_0 be a maximum point for $\tilde{v}$ in $(1,+\infty)$. Integrating (4.4) in (r_0, r), we get

$$-r^{M-1}\tilde{v}'(r) = A(\lambda,p)\int_{r_0}^r s^{(M-2)p-3}\tilde{v}^p(s)\,\mathrm{d}s.$$

Using estimate (4.5), we have

$$\left|\int_{r_0}^r s^{(M-2)p-3}\tilde{v}^p(s)\,\mathrm{d}s\right| \leqslant C\left|\int_{r_0}^r s^{(M-2)p/2-3}\,\mathrm{d}s\right|. \tag{4.6}$$

This implies that

$$|\tilde{v}'(r)| \leqslant \begin{cases} Cr^{1-M} & \text{when } p < \dfrac{4}{M-2}, \\ Cr^{1-M}\log r & \text{when } p = \dfrac{4}{M-2}, \\ Cr^{-1-M+(M-2)p/2} & \text{when } p > \dfrac{4}{M-2}. \end{cases} \tag{4.7}$$

When $p < 4/(M-2)$ (4.7) produces the optimal decay for $\tilde{v}'(r)$. Otherwise, if $p \geqslant 4/(M-2)$, we need to repeat the procedure, again estimating the integral in (4.6) using (4.7). In any case, after a finite number of steps we get that

$$|\tilde{v}'(r)| \leqslant Cr^{1-M}, \tag{4.8}$$

and this implies that

$$\tilde{v}(r) \leqslant Cr^{2-M}. \tag{4.9}$$

Returning to the function v, we get that

$$v(r) \leqslant C \quad \text{in } [0,1].$$

Furthermore, using the definition of $\tilde{v}$ and estimates (4.8) and (4.9) we have

$$\lim_{r\to 0} r^{M-1} v'(r) = \lim_{r\to 0} -\frac{1}{r}\tilde{v}'\left(\frac{1}{r}\right) + (2-M)\tilde{v}\left(\frac{1}{r}\right) = 0,$$

since $M > 2$. Integrating (4.2), we then obtain that

$$-r^{M-1}v'(r) = A(\lambda, p)\int_0^r s^{M-1} v^p(s)\,\mathrm{d}s, \tag{4.10}$$

so $v'(r) < 0$ in $(0,1)$ and $\lim_{r\to 0} v(r)$ exists and it is finite, showing that v is continuous at the origin. Using (4.10) again, we have

$$\lim_{r\to 0} v'(r) = -A(\lambda,p)\lim_{r\to 0}\frac{1}{r^{M-1}}\int_0^r s^{M-1}v^p(s)\,\mathrm{d}s = -A(\lambda,p)\lim_{r\to 0}\frac{r}{M-1}v^p(r) = 0.$$

This shows that the transformed function $v(r)$ has to be a solution of (4.1), since the constant $A(\lambda, p)$ can be merged into the equation. Theorem 4.1 then implies the existence and uniqueness of the radial solution. The final estimate follows by inverting the transformation $\mathcal{L}_p$ and using the continuity of $v(r)$ at 0. □

Corollary 4.5. *The radial solution to (1.28), u_λ, satisfies $u_\lambda(0) = 0$ if $\lambda < 0$.*

Proof. It is sufficient to note that $\nu_\lambda > 1$ as $\lambda < 0$. Then the claim follows by (1.30). □

In the rest of this section we shall denote by u_λ the unique radial solution to (1.28) and set

$$H = \{u \in H^1((0,1), r^{M-1}\,\mathrm{d}r) \text{ such that } u(1) = 0\}.$$

Set $v_\lambda(r)$ as in (1.2) and let $\Lambda(\lambda)$ be as defined in (1.31). Although the embedding of $H \hookrightarrow L^2((0,1), r^{M-3}\,\mathrm{d}r)$ is not compact, $\Lambda(\lambda)$ is achieved. This is a consequence of proposition A.8 (see the appendix), whose proof is basically the same as [8, proposition A.1]. Then we have the following result.

Corollary 4.6. *The first eigenvalue $\Lambda(\lambda)$ defined in (1.31) is achieved.*

Proof. Since

$$\Lambda(\lambda) \leqslant (1-p)\frac{\int_0^1 v_\lambda^{p+1} r^{M-1}\,\mathrm{d}r}{\int_0^1 v_\lambda^2 r^{M-3}\,\mathrm{d}r} < 0,$$

the claim follows by proposition A.8. □

As in the previous section we study the linearized operator at the solution u_λ and we recall that u_λ is non-degenerate if the linear problem

$$\left.\begin{aligned} -\Delta w - \frac{\lambda}{|x|^2}w = pu_\lambda^{p-1}w \quad \text{in } B_1,\\ w \in H_0^1(B_1), \end{aligned}\right\} \tag{4.11}$$

admits only the trivial solution.

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THEOREM 4.7. *Let $k \in \mathbb{N}$, $k \geqslant 1$ and $\lambda \leqslant \frac{1}{4}(N-2)^2$. The linearized equation at the radial solution u_λ, i.e. (4.11), admits a solution if and only if λ satisfies*

$$-\Lambda(\lambda) = \frac{16k(N-2+k)}{[(p-1)(2-N+\sqrt{(N-2)^2-4\lambda})+4]^2} \tag{4.12}$$

for some $k \geqslant 1$. Moreover, the space of solutions of (4.11), corresponding to a value of λ that satisfies (4.12) related to some k, has dimension

$$\frac{(N+2k-2)(N+k-3)!}{(N-2)!k!}$$

and is spanned by

$$Z_{k,i,\lambda}(x) = \frac{1}{|x|^{a/b}}\psi_1(|x|^{1/b})Y_{k,i}(x),$$

where ψ_1 is the positive eigenfunction associated with $\Lambda(\lambda)$ and $\{Y_{k,i}\}$, $i = 1, \dots, (N+2k-2)(N+k-3)!/(N-2)!k!$, form a basis of $\mathbb{Y}_k(\mathbb{R}^N)$, the space of all homogeneous harmonic polynomials of degree k in $\mathbb{R}^N$.

Finally, for every $k \geqslant 1$ there exists at least one value of λ that satisfies (4.12), and if λ is not a solution to (4.12), then the solution u_λ is non-degenerate.

Proof. The beginning of the proof is basically the same as lemma 1.2. Let v be a solution to (4.11). Then, decomposing v along spherical harmonics, we obtain the following ODE:

$$-\psi_k''(r) - \frac{N-1}{r}\psi_k'(r) + \frac{\mu_k - \lambda}{r^2}\psi_k(r) = pu_\lambda^{p-1}(r)\psi_k(r) \quad \text{in } (0,1),$$
$$\psi_k(1) = 0, \quad \int_0^1 r^{N-1}(\psi_k'(r))^2 \, dr < \infty,$$

where $\mu_k = k(N-2+k)$. Setting $\hat{\psi}_k(r) = r^a\psi_k(r^b)$ again, we have that $\hat{\psi}_k$ solves

$$\left.\begin{aligned} -\hat{\psi}_k''(r) - \frac{M-1}{r}\hat{\psi}_k'(r) + \frac{b^2\mu_k}{r^2}\hat{\psi}_k(r) = pv_\lambda^{p-1}(r)\hat{\psi}_k \quad \text{in } (0,1), \\ \hat{\psi}_k(1) = 0, \quad \hat{\psi}_k \in H. \end{aligned}\right\} \tag{4.13}$$

Note that, since v_λ is non-degenerate by theorem 4.1, the above problem cannot have solutions for $k = 0$. So we assume that $k \geqslant 1$.

By theorem 4.1 we get that (4.13) has a non-trivial solution belonging to the space H if and only if $-b^2\mu_k = \Lambda(\lambda)$, which is the unique negative eigenvalue. Moreover, by lemma A.9 we get that $\hat{\psi}_k \in L^\infty(0,1)$. Recalling (1.4), we get that (4.11) admits a solution if and only if

$$-\Lambda(\lambda) = \frac{16k(N-2+k)}{[(p-1)(2-N+\sqrt{(N-2)^2-4\lambda})+4]^2} \tag{4.14}$$

for some $k \geqslant 1$. Since the solution u_λ is not explicitly known, we have to show that (4.14) has at least one solution. Let us consider the two *limit* cases $\lambda = 0$ and $\lambda = -\infty$. By remark 4.4 we may derive that $\Lambda(\lambda)$ is a continuous function of λ.

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CASE 1 ($\lambda = 0$). First let us study the limit of the solution v_λ to (1.29) as λ goes to zero. By the uniqueness result of theorem 4.1 we know that v_λ can be characterized as

$$\inf_{\int_0^1 v(r)^{p+1} r^{M-1}\,\mathrm{d}r=1} \int_0^1 (v'(r))^2 r^{M-1}\,\mathrm{d}r, \quad v \in H.$$

It is easy to see that this infimum is attained when $v = v_\lambda$, and then

$$\int_0^1 (v_\lambda'(r))^2 r^{M-1}\,\mathrm{d}r \leqslant C$$

for some positive constant C independent of λ. So $v_\lambda \rightharpoonup v_0$, where, by remark 4.4, v_0 satisfies

$$\begin{aligned} -v'' - \frac{N-1}{r} v' &= v^p \quad \text{in } (0,1), \\ v &> 0 \quad \text{in } (0,1), \\ v'(0) = v(1) &= 0, \end{aligned}$$

since $A(0,p) = 1$. Then, for any $k \geqslant 1$, we get

$$\begin{aligned} \lim_{\lambda\to 0} &\left(\Lambda(\lambda) + \frac{16k(N-2+k)}{[(p-1)(2-N+\sqrt{(N-2)^2-4\lambda})+4]^2} \right) \\ &= \inf_{v\in H} \frac{\int_0^1 (|v'|^2 - p v_0^{p-1} v^2) r^{N-1}\,\mathrm{d}r}{\int_0^1 v^2 r^{N-3}\,\mathrm{d}r} + k(N-2+k) \\ &> 0 \end{aligned}$$

because

$$\inf_{v\in H} \frac{\int_0^1 (|v'|^2 - p v_0^{p-1} v^2) r^{N-1}\,\mathrm{d}r}{\int_0^1 v^2 r^{N-3}\,\mathrm{d}r} > 1 - N$$

by comparing the eigenfunction that gives $\Lambda(0)$ with v_0' and using the maximum principle.

CASE 2 ($\lambda = -\infty$). For any $k \geqslant 1$, testing $\Lambda(\lambda)$ with v_λ, we get

$$\begin{aligned} \Lambda(\lambda) &+ \frac{16k(N-2+k)}{[(p-1)(2-N+\sqrt{(N-2)^2-4\lambda})+4]^2} \\ &\leqslant (1-p)A(\lambda,p) \frac{\int_0^1 v_\lambda^{p+1} r^{M-1}\,\mathrm{d}r}{\int_0^1 v_\lambda^2 r^{M-3}\,\mathrm{d}r} + o(1) \end{aligned} \tag{4.15}$$

for λ large enough. Formula (4.2) implies

$$\int_0^1 |v_\lambda'|^2 r^{M-1}\,\mathrm{d}r = A(\lambda,p) \int_0^1 v_\lambda^{p+1} r^{M-1}\,\mathrm{d}r$$

and, using the Hardy inequality for the radial function (see [5]),

$$\int_0^1 v^2 r^{M-3}\,\mathrm{d}r \leqslant \left(\frac{M-2}{2}\right)^2 \int_0^1 |v'|^2 r^{M-1}\,\mathrm{d}r,$$

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we get that (4.15) becomes

$$\Lambda(\lambda)+\frac{16k(N-2+k)}{[(p-1)(2-N+\sqrt{(N-2)^2-4\lambda})+4]^2} \leqslant \frac{1-p}{((M-2)/2)^2}+o(1)=\frac{1-p}{(2/(p-1))^2}+o(1)<0$$

for λ large enough. Using both cases we can derive that, for any $k \geqslant 1$, there exists at least one value of λ that solves (4.14). □

Corollary 4.8. *The Morse index $m(\lambda)$ of u_λ is equal to*

$$m(\lambda)=\sum_{\substack{0\leqslant j<\frac{2-N}{2}+\frac{1}{2}\sqrt{(N-2)^2-4\frac{\Lambda(\lambda)}{b^2}},\\ j \text{ integer}}} \frac{(N+2j-2)(N+j-3)!}{(N-2)!j!}.$$

In particular, $m(\lambda)\to+\infty$ as $\lambda\to-\infty$.

Proof. Reasoning exactly as in the proof of proposition 1.3, we consider the weighted eigenvalue problem and we call the corresponding eigenvalues Γ_i. Then, the linearized equation has a negative eigenvalue with weight Γ_i if and only if

$$\Lambda(\lambda)=b^2(\Gamma_i-\mu_j)$$

for some $j\in\mathbb{N}$. So the indices j that contribute to the Morse index of the solution u_λ are those that satisfy

$$\Gamma_i=\frac{\Lambda(\lambda)}{b^2}+\mu_j<0 \tag{4.16}$$

for some $j\in\mathbb{N}$. Recalling the value of μ_j, this implies that

$$j<\frac{2-N}{2}+\frac{1}{2}\sqrt{(N-2)^2-4\frac{\Lambda(\lambda)}{b^2}}.$$

The claim follows by recalling the dimension of the eigenspace of the Laplace–Beltrami operator related to μ_j. The last assertion follows since $\Lambda(\lambda)\to-\infty$ for $\lambda\to-\infty$. □

Theorem 4.7 and corollary 4.8 imply that if λ^* satisfies (4.12) and the function $\Lambda(\lambda)+b^2\mu_k$ changes sign at the endpoints of a suitable interval containing λ^*, then the Morse index of the radial solution u_λ changes. This change in the Morse index is responsible for the bifurcation. From the continuity of $\Lambda(\lambda)$ we know that there should exist at least one value λ_k that satisfies (4.12) for every $k\geqslant 1$, but since we do not know whether the function $\Lambda(\lambda)$ is analytic we cannot say that these values λ_k are isolated. To overcame this problem, in the following proposition we construct an interval $I_k=[\alpha_k,\beta_k]$ that contains at least one of the points λ_k that satisfies (4.12) and at which the function $\Lambda(\lambda)+b^2\mu_k$ changes sign, and such that the Morse index of the radial solution u_λ at the values α_k and β_k differs from $(N+2k-2)(N+k-3)!/(N-2)!k!$, which is the dimension of the eigenspace of the Laplace–Beltrami operator related to the eigenvalue μ_k.

PROPOSITION 4.9. *There exist a sequence λ_k verifying*

$$-\Lambda(\lambda_k) = \frac{16k(N+k-2)}{[(p-1)(2-N+\sqrt{(N-2)^2-4\lambda_k})+4]^2} \tag{4.17}$$

and a sequence of intervals $I_k = [\alpha_k, \beta_k] \subset (-\infty, 0)$ with $\lambda_k \in I_k$ such that

$$\Lambda(\beta_k) > -\frac{16k(N+k-2)}{[(p-1)(2-N+\sqrt{(N-2)^2-4(\beta_k)})+4]^2}, \tag{4.18}$$

$$\Lambda(\alpha_k) < -\frac{16k(N+k-2)}{[(p-1)(2-N+\sqrt{(N-2)^2-4(\alpha_k)})+4]^2} \tag{4.19}$$

and

$$\Lambda(\beta_k) < -\frac{16h(N-2+h)}{[(p-1)(2-N+\sqrt{(N-2)^2-4\beta_k})+4]^2}, \tag{4.20}$$

for any $h < k$, while

$$\Lambda(\alpha_k) > -\frac{16j(N-2+j)}{[(p-1)(2-N+\sqrt{(N-2)^2-4\alpha_k})+4]^2} \tag{4.21}$$

for any $j > k$.

Proof. In order to simplify the notation we consider first the case $k = 1$. For $\lambda \leqslant 0$, set

$$L(\lambda) = \Lambda(\lambda)[(p-1)(2-N+\sqrt{(N-2)^2-4\lambda})+4]^2$$

and define λ_1 as

$$\lambda_1 = \sup_{\lambda \leqslant 0} I_{1,\lambda},$$

where

$$I_{1,\lambda} = \{\lambda \leqslant 0 \text{ such that } L(\lambda) = -16(N-1)\}.$$

By cases 1 and 2 in the proof of theorem 4.7, we get that $I_{1,\lambda} \neq \emptyset$, and since L is a continuous function there exists λ_1 such that

$$L(\lambda_1) = -16(N-1),$$

and any other point $\lambda^* \neq \lambda_1$ that satisfies

$$L(\lambda^*) = -16(N-1)$$

must verify

$$\lambda^* < \lambda_1.$$

Analogously, for $k \geqslant 2$ we define

$$\lambda_k = \sup_{\lambda \leqslant 0} I_{k,\lambda}, \tag{4.22}$$

where

$$I_{k,\lambda} = \{\lambda \leqslant 0 \text{ such that } L(\lambda) = -16k(N-2+k)\}.$$

As in the previous case, using the proof of theorem 4.7, there exists λ_k such that

$$L(\lambda_k) = -16k(N-2+k),$$

and λ_k satisfies (4.22).

Let us show that

$$\lambda_{k+1} < \lambda_k \quad \text{for any } k \geqslant 1.$$

Since the function $16k(N-2+k)$ is strictly increasing in k, we cannot have that $\lambda_{k+1} = \lambda_k$. So, by contradiction let us suppose that

$$\lambda_{k+1} > \lambda_k$$

for some $k \geqslant 1$. Then,

$$L(\lambda_{k+1}) = -16(k+1)(N-1+k) < -16k(N-2+k) \tag{4.23}$$

and by case 1 of the proof of theorem 4.7 we have

$$\begin{aligned}\lim_{\lambda\to 0} L(\lambda) &= \lim_{\lambda\to 0} \Lambda(\lambda)[(p-1)(2-N+\sqrt{(N-2)^2-4\lambda})+4]^2 \\ &= (16+o(1))\lim_{\lambda\to 0}\Lambda(\lambda) > -16(N-1) \geqslant -16k(N-2+k).\end{aligned}$$

By the intermediate value theorem for continuous functions we get that there exists $\tilde{\lambda}_k \geqslant \lambda_{k+1}$ such that

$$L(\tilde{\lambda}_k) = -16k(N-2+k),$$

and this contradicts the definition of λ_k.

So we have shown that

$$0 > \lambda_1 > \lambda_2 > \cdots > \lambda_k > \cdots .$$

Now we prove the claim: again, by case 1 of theorem 4.7, since

$$\lim_{\lambda\to 0} L(\lambda) > -16(N-1),$$

we get that there exists $\lambda_1 < \beta_1 < 0$ such that

$$L(\beta_1) > -16(N-1),$$

and this implies

$$\Lambda(\beta_1) > -\frac{16(N-1)}{[(p-1)(2-N+\sqrt{(N-2)^2-4(\beta_1)})+4]^2}.$$

This proves (4.18).

On the other hand, since $L(\lambda_2) = -32N < -16(N-1) = L(\lambda_1)$ by (4.23), there exists $\lambda_2 < \alpha_1 < \lambda_1$ such that $L(\alpha_1) < -16(N-1)$, which implies

$$\Lambda(\alpha_1) < -\frac{16(N-1)}{[(p-1)(2-N+\sqrt{(N-2)^2-4(\alpha_1)})+4]^2}.$$

This proves (4.19).

Finally, since $\sup_{k\geqslant 2}\lambda_k=\lambda_2<\alpha_1$, we have $L(\alpha_1)>-32N\geqslant -16j(N-2+j)$ for any $j>1$, so (4.21) follows.

Now we explain how to pass from $k=1$ to $k=2$. We take $\beta_2=\alpha_1\in(\lambda_2,\lambda_1)$. Then, by (4.19) and (4.21),

$$L(\beta_2)=L(\alpha_1)>-32N$$

and

$$L(\beta_2)=L(\alpha_1)<-16(N-2),$$

so (4.18) and (4.20) follow for $k=2$.

By (4.23), $L(\lambda_3)=-48(N+1)<-32N=L(\lambda_2)$. Then there exists $\lambda_3<\alpha_2<\lambda_2$ such that $-48(N+1)<L(\alpha_2)<-32N$, so $L(\alpha_2)<-32N$ and this proves (4.19) for $k=2$. Finally, by the choice of α_2, we have $L(\alpha_2)>-48(N+1)\geqslant -16j(N-2+j)$ for any $j>2$, so (4.21) is proved for $k=2$. The general case can be carried out with the same proof. □

As in § 3.3, one can define the operator

$$T(\lambda,v)\colon(-\infty,0)\times H_0^1(B_1)\cap L^\infty(B_1)\to H_0^1(B_1)\cap L^\infty(B_1)$$

as

$$T(\lambda,v)=\left(-\Delta-\frac{\lambda}{|x|^2}I\right)^{-1}((v^+)^p)$$

and look for zeros of $I-T(\lambda,v)$. Letting $X=H_0^1(B_1)\cap L^\infty(B_1)$ and reasoning as in the proof of lemma 3.4, we have that the operator T is well defined from $(-\infty,0)\times X$ into X. T is continuous with respect to λ and it is compact from X into X for any $\lambda\in(-\infty,0)$ fixed.

Moreover, the linearized operator $I-T_v(\lambda,u_\lambda)I$ is invertible for any value of λ that does not satisfy (4.12).

To prove the bifurcation we have to consider (as in the previous section) the subspace $\mathcal{H}$ of X of functions that are $O(N-1)$-invariant and the subspaces $\mathcal{H}^h$ of X of functions that are invariant by the action of $\mathcal{G}_h$.

Using these spaces, we deduce the following result by using theorem 4.7 and proposition 4.9.

Proposition 4.10. *For every $k\in\mathbb{N}$ the curve of the radial solution $(\lambda,u_\lambda)\in(-\infty,0)\times X$ contains a non-radial bifurcation point in the interval $I_k\times\mathcal{H}$, where I_k is as defined in proposition 4.9.*

Moreover, if k is even, for every $h=1,\dots,[N/2]$ there exists a continuum of non-radial solutions bifurcating from (λ,u_λ) in the interval $I_k\times\mathcal{H}^h$.

Proof. The proof is by contradiction. We consider only the case of the space $\mathcal{H}$. The other case follows very similarly.

Assume by contradiction that the curve (λ,u_λ) does not contain any bifurcation point in the interval $I_k\times\mathcal{H}$. Then there exists an $\varepsilon_0>0$ such that for $\varepsilon\in(0,\varepsilon_0)$ and every $c\in(0,\varepsilon_0)$ we have

$$v-T(\lambda,v)\neq 0\quad\forall\lambda\in I_k,\ \forall v\in X\text{ such that }0<\|v-u_\lambda\|_X\leqslant c.\tag{4.24}$$

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Let us consider the sets $\mathcal{C} := \{(\lambda, v) \in I_k \times X\colon \|v - u_\lambda\|_X < c\}$ and $\mathcal{C}_\lambda := \{v \in X$ such that $(\lambda, v) \in \mathcal{C}\}$. From (4.24) it follows that there exist no solutions of $v - T(\lambda, v) = 0$ on $\partial_{I_k \times X}\mathcal{C}$ different from u_λ. By the homotopy invariance of the degree, we get

$$\deg(I - T(\lambda, \cdot), \mathcal{C}_\lambda, 0) \quad \text{is constant on } I_k. \tag{4.25}$$

Moreover, from (4.12) and (4.18)–(4.21) the linearized operator $T_v(\lambda, u_\lambda)$ is invertible for $\lambda = \alpha_k$ and $\lambda = \beta_k$. Then

$$\deg(I - T(\beta_k, \cdot), \mathcal{C}_{\beta_k}, 0) = (-1)^{m_{\mathcal{H}}(\beta_k)}$$

and

$$\deg(I - T(\alpha_k, \cdot), \mathcal{C}_{\alpha_k}, 0) = (-1)^{m_{\mathcal{H}}(\alpha_k)},$$

where $m_{\mathcal{H}}(\lambda)$ denotes the Morse index of the radial solution u_λ in the space $\mathcal{H}$. By the choice of the space $\mathcal{H}$, we know that the eigenspace of the Laplace–Beltrami operator associated to μ_k is one dimensional. Then, repeating the proof of corollary 4.8 in the space $\mathcal{H}$ yields

$$m_{\mathcal{H}}(\lambda) = \begin{cases} 1 + \sup\{j \in \mathbb{N} \text{ s.t. } \Lambda(\lambda) + b^2\mu_j < 0\} & \text{if } \lambda \text{ does not satisfy (4.12)}, \\ \sup\{j \in \mathbb{N} \text{ s.t. } \Lambda(\lambda) + b^2\mu_j < 0\} & \text{if } \lambda \text{ satisfies (4.12)}. \end{cases}$$

Then, from (4.18)–(4.21), $m_{\mathcal{H}}(\beta_k) = 1 + (k - 1) = k$ and $m_{\mathcal{H}}(\alpha_k) = 1 + k$, so that

$$\deg(I - T(\beta_k, \cdot), \mathcal{C}_{\beta_k}, 0) = -\deg(I - T(\alpha_k, \cdot), \mathcal{C}_{\alpha_k}, 0),$$

contradicting (4.25). Then, in the interval $I_k \times X$ there exists a bifurcation point for the curve (λ, u_λ), and the bifurcating solutions are non-radial since u_λ is radially non-degenerate. □

We are now in a position to prove theorem 1.8.

Proof of theorem 1.8. This follows from theorem 4.7 and propositions 4.9 and 4.10. □

Appendix A.

In this appendix we show some properties of the Hardy operator. Let us start with some classical results.

LEMMA A.1. *Let* $\lambda \in (-\infty, \frac{1}{4}(N - 2)^2)$. *Then*

$$\left(\int_{\mathbb{R}^N} |\nabla v|^2 \,\mathrm{d}x - \int_{\mathbb{R}^N} \frac{\lambda}{|x|^2} v^2 \,\mathrm{d}x\right)^{1/2} \tag{A 1}$$

is a norm on $D^{1,2}(\mathbb{R}^N)$, *which is equivalent to the standard one.*

Proof. This follows by the Hardy inequality, distinguishing the two different cases $\lambda > 0$ and $\lambda \leqslant 0$. □

Lemma A.2. *Let $f(x) \in L^{2N/(N+2)}(\mathbb{R}^N)$ and let $\lambda \in (-\infty, \frac{1}{4}(N-2)^2)$. Then the equation*

$$-\Delta v - \frac{\lambda}{|x|^2} v = f \quad \text{in } \mathbb{R}^N \tag{A 2}$$

has a unique weak solution in $D^{1,2}(\mathbb{R}^N)$.

Proof. It follows by the Hardy inequality and the coercivity of the functional

$$J(v) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla v|^2 \, \mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^N} \frac{\lambda}{|x|^2} v^2 \, \mathrm{d}x - \int_{\mathbb{R}^N} f v \, \mathrm{d}x.$$

□

Next we state the Pohozaev identity for a weak solution of

$$-\Delta u = f(x, u) \quad \text{in } \mathbb{R}^N. \tag{A 3}$$

Lemma A.3. *Let $u \in D^{1,2}(\mathbb{R}^N)$ be a weak solution of (A 3) and let*

$$F(x, u) = \int_0^u f(x, t) \, \mathrm{d}t.$$

Assume furthermore that $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ and $F(x, u), x \cdot F_x(x, u) \in L^1(\mathbb{R}^N)$, where F_x is the gradient of F with respect to x. Then u satisfies

$$\int_{\mathbb{R}^N} |\nabla u|^2 \, \mathrm{d}x - \frac{2N}{N-2}\int_{\mathbb{R}^N} F(x, u) \, \mathrm{d}x - \frac{2}{N-2}\int_{\mathbb{R}^N} x \cdot F_x(x, u) \, \mathrm{d}x = 0. \tag{A 4}$$

Proof. We can proceed exactly as in the proof of [2, proposition 1]. There are only two differences: the first is the presence of the term $x \cdot F_x(x, u)$; the second is that the solution $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N \setminus \{0\})$ and so we have to integrate (A 3) in $B_R \setminus B_\rho$. These terms can be handled exactly as in the proof in [2]. □

Here we prove some results that deal with the infimum (1.31) and some other related results in the spirit of [9, § 2].

Proposition A.4. *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with $0 \in \Omega$. Moreover, assume that*

$$\nu_1 = \inf_{\substack{\eta \in H_0^1(\Omega),\\ \eta \not\equiv 0}} \frac{\int_\Omega |\nabla \eta|^2 \, \mathrm{d}x - \int_\Omega a(x)\eta^2 \, \mathrm{d}x}{\int_\Omega (\eta^2/|x|^2) \, \mathrm{d}x} < 0 \tag{A 5}$$

with $a(x) \in L^\infty(\Omega)$. Then ν_1 is achieved in a function ψ_1. The function ψ_1 is strictly positive in $\Omega \setminus \{0\}$ and satisfies

$$\int_\Omega \nabla\psi_1 \cdot \nabla\phi - a(x)\psi_1\phi \, \mathrm{d}x = \nu_1 \int_\Omega \frac{\psi_1\phi}{|x|^2} \, \mathrm{d}x \tag{A 6}$$

for any $\phi \in H_0^1(\Omega)$. The eigenvalue ν_1 is simple.

Proof. Consider a minimizing sequence $\eta_n \in H_0^1(\Omega)$ for ν_1, i.e.

$$\frac{\int_\Omega |\nabla \eta_n|^2 \, \mathrm{d}x - \int_\Omega a(x)\eta_n^2 \, \mathrm{d}x}{\int_\Omega (\eta_n^2/|x|^2) \, \mathrm{d}x} = \nu_1 + o(1). \tag{A 7}$$

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Let us normalize η_n such that

$$\int_\Omega \eta_n^2 \, \mathrm{d}x = 1. \tag{A 8}$$

Then, since $\nu_1 < 0$, by (A 7) we get

$$\int_\Omega |\nabla \eta_n|^2 \, \mathrm{d}x - \int_\Omega a(x)\eta_n^2 \, \mathrm{d}x \leqslant 0 \tag{A 9}$$

and then, since a is bounded and (A 8) holds, we deduce from (A 9) that

$$\int_\Omega |\nabla \eta_n|^2 \, \mathrm{d}x \leqslant C \int_\Omega \eta_n^2 \, \mathrm{d}x \leqslant C. \tag{A 10}$$

Hence, $\eta_n \rightharpoonup \eta$ weakly in $H_0^1(\Omega)$ and then it holds that

$$\int_\Omega |\nabla \eta|^2 \, \mathrm{d}x \leqslant \liminf_{n\to+\infty} \int_\Omega |\nabla \eta_n|^2 \, \mathrm{d}x, \tag{A 11}$$

$$\int_\Omega a(x)\eta_n^2 \, \mathrm{d}x \to \int_\Omega a(x)\eta^2 \, \mathrm{d}x. \tag{A 12}$$

So, we get

$$\int_\Omega |\nabla \eta|^2 \, \mathrm{d}x - \int_\Omega a(x)\eta^2 \, \mathrm{d}x \leqslant \liminf_{n\to+\infty} \int_\Omega |\nabla \eta_n|^2 \, \mathrm{d}x - \int_\Omega a(x)\eta_n^2 \, \mathrm{d}x + o(1), \tag{A 13}$$

which, since $\nu_1 < 0$ and

$$1 = \int_\Omega \eta_n^2 \, \mathrm{d}x \leqslant C \int_\Omega \frac{\eta_n^2}{|x|^2} \, \mathrm{d}x,$$

implies

$$\frac{\int_\Omega |\nabla \eta|^2 \, \mathrm{d}x - \int_\Omega a(x)\eta^2 \, \mathrm{d}x}{\limsup_{n\to+\infty} \int_\Omega (\eta_n^2/|x|^2) \, \mathrm{d}x} \leqslant \nu_1. \tag{A 14}$$

Then elementary properties of lim inf and lim sup imply

$$\limsup_{n\to+\infty} \frac{\int_\Omega |\nabla \eta|^2 \, \mathrm{d}x - \int_\Omega a(x)\eta^2 \, \mathrm{d}x}{\int_\Omega (\eta_n^2/|x|^2) \, \mathrm{d}x} \leqslant \nu_1. \tag{A 15}$$

Moreover, by Hardy's inequality,

$$\int_\Omega \frac{\eta_n^2}{|x|^2} \, \mathrm{d}x \leqslant \frac{(N-2)^2}{4} \int_\Omega |\nabla \eta_n|^2 \, \mathrm{d}x \leqslant C, \tag{A 16}$$

and so, by semicontinuity,

$$\int_\Omega \frac{\eta^2}{|x|^2} \, \mathrm{d}x \leqslant \liminf_{n\to+\infty} \int_\Omega \frac{\eta_n^2}{|x|^2} \, \mathrm{d}x. \tag{A 17}$$

Hence, again using that $\nu_1 < 0$, we get from (A 14) that

$$\int_\Omega |\nabla \eta|^2 \, \mathrm{d}x - \int_\Omega a(x)\eta^2 \, \mathrm{d}x < 0. \tag{A 18}$$

On the other hand, from (A 17) we get

$$\limsup_{n\to+\infty} \frac{1}{\int_\Omega(\eta_n^2/|x|^2)\,\mathrm{d}x} = \frac{1}{\liminf_{n\to+\infty}\int_\Omega(\eta_n^2/|x|^2)\,\mathrm{d}x} \leqslant \frac{1}{\int_\Omega(\eta^2/|x|^2)\,\mathrm{d}x} \tag{A 19}$$

and then

$$\frac{\int_\Omega |\nabla\eta|^2\,\mathrm{d}x - \int_\Omega a(x)\eta^2\,\mathrm{d}x}{\int_\Omega(\eta^2/|x|^2)\,\mathrm{d}x} \leqslant \liminf_{n\to+\infty} \frac{\int_\Omega |\nabla\eta|^2\,\mathrm{d}x - \int_\Omega a(x)\eta^2\,\mathrm{d}x}{\int_\Omega(\eta_n^2/|x|^2)\,\mathrm{d}x}. \tag{A 20}$$

Finally, by (A 14) we get

$$\frac{\int_\Omega |\nabla\eta|^2\,\mathrm{d}x - \int_\Omega a(x)\eta^2\,\mathrm{d}x}{\int_\Omega(\eta^2/|x|^2)\,\mathrm{d}x} \leqslant \nu_1, \tag{A 21}$$

which proves the first part of the proposition. The rest follows exactly as in proof of [9, proposition 2.1]. □

The same result also holds if we minimize the quadratic form (A 5) with some orthogonality conditions. To this end we say that ψ and η are orthogonal if they satisfy

$$\int_\Omega \frac{\psi\eta}{|x|^2}\,\mathrm{d}x = 0.$$

Indeed, we have the following.

Proposition A.5. *Assume Ω, ν_1, ψ_1 and $a(x)$ are as in proposition A.4. Then, if*

$$\nu_2 = \inf_{\substack{\eta\in H_0^1(\Omega),\\ \eta\perp\psi_1}} \frac{\int_\Omega |\nabla\eta|^2\,\mathrm{d}x - \int_\Omega a(x)\eta^2\,\mathrm{d}x}{\int_\Omega(\eta^2/|x|^2)\,\mathrm{d}x} < 0, \tag{A 22}$$

ν_2 is achieved. Moreover, the function $\psi_2 \in H_0^1(\Omega)$ that attains ν_2 satisfies

$$\int_\Omega \nabla\psi_2\cdot\nabla\phi - a(x)\psi_2\phi\,\mathrm{d}x = \nu_2\int_\Omega \frac{\psi_2\phi}{|x|^2}\,\mathrm{d}x$$

for any $\phi \in H_0^1(\Omega)$.

Similarly, for $i = 3, \dots, k$, if

$$\nu_i = \inf_{\substack{\eta\in H_0^1(\Omega),\\ \eta\perp\mathrm{span}\{\psi_1,\psi_2,\dots,\psi_{i-1}\}}} \frac{\int_\Omega |\nabla\eta|^2\,\mathrm{d}x - \int_\Omega a(x)\eta^2\,\mathrm{d}x}{\int_\Omega(\eta^2/|x|^2)\,\mathrm{d}x} < 0, \tag{A 23}$$

then ν_i is achieved and the functions $\psi_i \in H_0^1(\Omega)$ that attain ν_i satisfy

$$\int_\Omega \nabla\psi_i\cdot\nabla\phi - a(x)\psi_i\phi\,\mathrm{d}x = \nu_i\int_\Omega \frac{\psi_i\phi}{|x|^2}\,\mathrm{d}x \tag{A 24}$$

for any $\phi \in H_0^1(\Omega)$.

Proof. This is the same as the previous lemma. For any i let us consider a minimizing sequence $\eta_{i,n} \in H_0^1(\Omega)$ for ν_i. This sequence converges to a function η_i. This function achieves the infimum ν_i and weakly solves (A 24). □

Now, we use the above result to compute the Morse index of the radial solution u_λ to (1.1). We state the result in the case when $\Omega = B_1$.

LEMMA A.6. *Let u_λ be a solution to (1.28) whose Morse index is $M > 0$. Then there exist exactly M functions $\psi_i \in H_0^1(B_1)$ and M numbers $\nu_i < 0$ such that the problem*

$$-\Delta\psi_i - \frac{\lambda}{|x|^2}\psi_i - pu_\lambda^{p-1}\psi_i = \frac{\nu_i}{|x|^2}\psi_i \quad \text{in } B_1 \setminus \{0\}, \quad \psi_i \in H_0^1(B_1), \tag{A 25}$$

admits a weak solution. The functions ψ_i can be taken in such a way that they verify

$$\int_{B_1} \frac{\psi_i\psi_j}{|x|^2}\,\mathrm{d}x = 0 \quad \text{for } i \neq j. \tag{A 26}$$

The proof follows exactly as in the proof of [9, lemma 2.6] and we do not reproduce it here.

The results of propositions A.4 and A.5 and lemma A.6 also hold true if we let $\Omega = \mathbb{R}^N$ and substitute $H_0^1(\Omega)$ by $D^{1,2}(\mathbb{R}^N)$. Then we can state the following.

COROLLARY A.7. *The Morse index of the radial solution u_λ of (1.16) is given by the number of negative values Λ_i counted with multiplicity such that the problem*

$$-\Delta w - \frac{\lambda}{|x|^2}w - N(N+2)\nu_\lambda^2\frac{|x|^{2\nu_\lambda}}{(1+|x|^{2\nu_\lambda})^2}w = \frac{\Lambda_i}{|x|^2}w \quad \text{in } \mathbb{R}^N, \quad w \in D^{1,2}(\mathbb{R}^N), \tag{A 27}$$

admits a weak solution.

Proof. Let u_λ be the radial solution of (1.16). Then we can use the analogue of lemma A.6 in $\mathbb{R}^N$ to prove the claim. □

Finally, the result of proposition A.4 can also be used to prove that the first eigenvalue with weight (1.31) is attained. Indeed, we have the following.

PROPOSITION A.8. *Assume that*

$$\Lambda_1 = \inf_{\substack{\eta \in H^1((0,1), r^{M-1}\,\mathrm{d}r),\\ \eta(1)=0, \eta\not\equiv 0}} \frac{\int_0^1 r^{M-1}|\eta'|^2\,\mathrm{d}r - \int_0^1 r^{M-1}a(r)\eta^2\,\mathrm{d}r}{\int_0^1 r^{M-3}\eta^2\,\mathrm{d}r} < 0 \tag{A 28}$$

with $a \in L^\infty(0,1)$. Then Λ_1 is achieved.

Proof. The claim follows as in the proof of proposition A.4. □

LEMMA A.9. *Assume ψ is a solution to*

$$\left.\begin{aligned} -\psi'' - \frac{M-1}{r}\psi' + \beta^2\frac{\psi}{r^2} = h\psi \quad \text{in } (0,1), \\ \psi(1) = 0, \qquad \int_0^1 r^{M-1}(\psi')^2\,\mathrm{d}r < \infty, \end{aligned}\right\} \tag{A 29}$$

with $h \in L^\infty(0,1)$ and $\beta \neq 0$. Then $\psi \in L^\infty(0,1)$ and $\psi(0) = 0$.

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Proof. Let

$$\theta = \frac{2 - M + \sqrt{(M-2)^2 + 4\beta^2}}{2} > 0.$$

Since

$$\int_0^1 r^{M-1}(\psi')^2 \, \mathrm{d}r < +\infty,$$

by (A 29) we get that

$$\int_0^1 \psi^2 r^{M-3} \, \mathrm{d}r < +\infty. \tag{A 30}$$

Then there exists a sequence $r_n \to 0$ such that $r_n^{\theta+M-2}\psi(r_n) = o(1)$ as $n \to +\infty$ for any $\beta > 0$. Such a sequence exists because otherwise we get $\psi(r) \geqslant C/r_n^{\theta+M-2}$ in a suitable neighbourhood of 0 and this contradicts (A 30) (note that we have used that $\theta > \frac{1}{2}(2 - M)$).

Observe that the function $v(r) = r^\theta$ satisfies

$$-v'' - \frac{M-1}{r}v' + \frac{\beta^2}{r^2}v = 0 \quad \text{in } (0, +\infty), \qquad v(0) = 0. \tag{A 31}$$

From (A 31) and (A 29), integrating on (r_n, R), we obtain

$$\begin{aligned}\int_{r_n}^R s^{\theta+M-1}h(s)\psi(s) \, \mathrm{d}s = -R^{\theta+M-1}\psi'(R) + r_n^{\theta+M-1}\psi'(r_n) \\ + \theta R^{\theta+M-2}\psi(R) - \theta r_n^{\theta+M-2}\psi(r_n).\end{aligned} \tag{A 32}$$

We claim that

$$r_n^{\theta+M-1}\psi'(r_n) = o(1) \quad \text{as } n \to \infty. \tag{A 33}$$

Integrating (A 29), we get

$$r_n^{\theta+M-1}\psi'(r_n) = O(r_n^\theta) + r_n^\theta \int_{r_n}^1 s^{M-1}h(s)\psi(s) \, \mathrm{d}s - \beta^2 r_n^\theta \int_{r_n}^1 s^{M-3}\psi(s) \, \mathrm{d}s = o(1),$$

since

$$r_n^\theta \int_{r_n}^1 s^{M-3}\psi(s) \, \mathrm{d}s \leqslant r_n^\theta \biggl(\int_{r_n}^1 \frac{\psi^2(s)}{s^2} s^{M-1} \, \mathrm{d}s\biggr)^{1/2} \biggl(\int_{r_n}^1 s^{M-3} \, \mathrm{d}s\biggr)^{1/2} = o(1),$$

and this proves (A 33). Hence, (A 32) becomes

$$\int_0^R s^{\theta+M-1}h(s)\psi(s) \, \mathrm{d}s = -R^{\theta+M-1}\psi'(R) + \theta R^{\theta+M-2}\psi(R). \tag{A 34}$$

Then we deduce that

$$\frac{\psi(t)}{t^\theta} = \int_t^1 \frac{1}{R^{2\theta+M-1}} \biggl(\int_0^R s^{\theta+M-1}h(s)\psi(s) \, \mathrm{d}s\biggr) \mathrm{d}R. \tag{A 35}$$

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Now, since $h \in L^{\infty}(0,1)$ we get

$$\begin{aligned}\left|\int_0^R s^{\theta+M-1}h(s)\psi(s)\,\mathrm{d}s\right| &\leqslant C\int_0^R s^{\theta+(M+1)/2}\psi(s)s^{(M-3)/2}\,\mathrm{d}s\\ &\leqslant C\biggl(\int_0^R s^{2\theta+M+1}\,\mathrm{d}s\biggr)^{1/2}\biggl(\int_0^R \psi^2(s)s^{M-3}\,\mathrm{d}s\biggr)^{1/2}\\ &\leqslant CR^{\theta+(M+2)/2}. \end{aligned} \tag{A 36}$$

Finally, (A 35) becomes

$$\psi(t) = \begin{cases} O(t^{\theta}) & \text{if } \theta < -\tfrac{1}{2}M+3,\\ O(t^{-M/2+3}) & \text{if } \theta > -\tfrac{1}{2}M+3,\\ O(t^{-M/2+3}|\log t|) & \text{if } \theta = -\tfrac{1}{2}M+3. \end{cases} \tag{A 37}$$

Then, if $M < 6$, (A 37) implies that $\psi(0) = 0$, which gives the claim. If $M \geqslant 6$, instead we have $-\frac{1}{2}M+3 \leqslant 0 < \theta$ and so $\psi(t) = O(t^{-M/2+3})$. Plugging this estimate into (A 36), we get

$$\left|\int_0^R s^{\theta+M-1}h(s)\psi(s)\,\mathrm{d}s\right| \leqslant CR^{\theta+(M+6)/2}, \tag{A 38}$$

and (A 35) becomes

$$\psi(t) = \begin{cases} O(t^{\theta}) & \text{if } \theta < -\tfrac{1}{2}M+5,\\ O(t^{-M/2+5}) & \text{if } \theta \neq -\tfrac{1}{2}M+5,\\ O(t^{-M/2+5}|\log t|) & \text{if } \theta = -\tfrac{1}{2}M+5, \end{cases} \tag{A 39}$$

which gives the claim for $M < 10$. Iterating the procedure for a finite number of steps, we get that $\psi(t) = O(t^{\theta})$ as $t \to 0$, so $\psi(0) = 0$. This completes the proof. □

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