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DISTRIBUTION OF THE DIVISOR FUNCTION AT CONSECUTIVE INTEGERS

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Abstract

In this paper we sharpen Hildebrand's earlier result on a conjecture of Erdős on limit points of the sequence $\{d(n)/d(n+1)\}$.

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1. Introduction

The famous Erdős–Mirsky conjecture [1] asserts that $d(n) = d(n+1)$ infinitely often, where $d(n)$ is the divisor function. In 1983, Spiro [9] showed that $d(n) = d(n+5040)$ for infinitely many $n \in \mathbb{N}$. In 1984, Heath-Brown [5], using Spiro's argument, proved the conjecture of Erdős and Mirsky. Moreover, he showed that for large x,

$$D(x) = \#\{n \le x : d(n) = d(n+1)\} \ge \frac{x}{(\log x)^7}.$$

Hildebrand [6] improved the lower bound to $x/(\log\log x)^3$. Using a heuristic argument, Bateman and Spiro claimed that $D(x) \sim cx(\log\log x)^{-1/2}$ for some constant $c > 0$. As an application of their work in [3], Goldston *et al.* [4] have shown, among several other interesting results, that $d(n) = d(n+1) = 24$ for infinitely many integers n.

The Erdős–Mirsky conjecture is equivalent to the statement that $d(n)/d(n+1) = 1$ for infinitely many n. More generally, one can ask which numbers occur as limit points of the sequence $\{d(n)/d(n+1)\}_{n=1}^{\infty}$. Let $\mathbf{E}$ denote the set of limit points of the sequence $\{d(n)/d(n+1)\}$, and let $\mathbf{L}$ denote the set of limit points of $\{\log(d(n)/d(n+1))\}$. The Erdős–Mirsky conjecture implies that $1 \in \mathbf{E}$. Erdős conjectured that $\mathbf{E} = [0, \infty]$, or equivalently, $\mathbf{L} = [-\infty, \infty]$. Erdős *et al.* [2] proved that for any $\alpha \in \mathbb{R}^+$, at least one of the seven numbers $2^i\alpha$, $i \in \{0, \pm1, \pm2, \pm3\}$, belongs to $\mathbf{E}$. This result was improved by Kan and Shan [7, 8] who showed that for any real $\alpha > 0$, either α or 2α belongs to $\mathbf{E}$. On the other hand, it can be shown under the assumption of the prime k-tuple conjecture that for any $r \in \mathbb{Q}^+$ there exist infinitely many $n \in \mathbb{N}$ such that $d(n)/d(n+1) = r$.

Hildebrand proved that for $x > 0$

$$|\mathbf{L} \cap [0, x]| \geqslant \frac{x}{36}, \quad |\mathbf{L} \cap [-x, 0]| \geqslant \frac{x}{36}, \tag{1.1}$$

where $|\cdot|$ denotes the Lebesgue measure. The main result of this paper is the following improvement.

THEOREM 1.1. *For any number* $x \geqslant 0$,

$$|\mathbf{L} \cap [0, x]| \geqslant \frac{x}{3} \quad \textit{and} \quad |\mathbf{L} \cap [-x, 0]| \geqslant \frac{x}{3}. \tag{1.2}$$

Moreover, there exists a number $A \geqslant 0$ *such that, for any* $x \geqslant A$,

$$|\mathbf{L} \cap [0, x]| \geqslant \frac{x-A}{2} \quad \textit{and} \quad |\mathbf{L} \cap [-x, 0]| \geqslant \frac{x-A}{2}. \tag{1.3}$$

2. Preliminary lemmas

A triple of linear forms is called admissible if for every prime p, there is at least one m mod p such that $L_1(m)L_2(m)L_3(m) \not\equiv 0$ mod p. Numbers that are products of exactly two distinct primes are called E_2 numbers. Unconditionally, the following result holds.

LEMMA 2.1 [4]. *Let* $L_i(x) := a_i x + b_i$, $i = 1, 2, 3$, *be an admissible triple of linear forms, and let* r_1, r_2, r_3 *be coprime integers with* $(r_i, a_i) = 1$ *for each* i *and such that* $(r_i, a_i b_j - a_j b_i) = 1$ *for* $i \neq j$. *Then there exist* i, j *with* $1 \leqslant i < j \leqslant 3$ *such that there are infinitely many integers* n *for which* $L_k(n)$ *equals* r_k *times an* E_2 *number that is coprime to all primes less than or equal to* C, *for* $k = i, j$.

In the lemma, C can be any constant. Hildebrand deduced (1.1) from the fact that among any seven integers $a_1, \ldots, a_7$, there exists $i < j$ such that $d(n)/d(n+1) = a_i/a_j$ for infinitely many n. We can replace $x/36$ by $x/3$, in view of the following result.

LEMMA 2.2. *Let* a_1, a_2, a_3 *be positive integers. For some* $i < j$, *there are infinitely many integers* n *such that* $d(n)/d(n+1) = a_i/a_j$.

PROOF. Define a triple of linear forms

$$L_1(x) = 9x + 1, \quad L_2(x) = 8x + 1, \quad L_3(x) = 6x + 1,$$

and note the relations

$$8L_1(x) + 1 = 9L_2(x), \quad 2L_1(x) + 1 = 3L_3(x), \quad 3L_2(x) + 1 = 4L_3(x).$$

For given a_1, a_2, a_3 let

$$r_1 := 5^{a_1 - 1}, \quad r_2 := 3 \cdot 7^{a_2 - 1}, \quad r_3 := 11^{a_3 - 1}.$$

We check that the hypotheses of Lemma 2.1 are satisfied. First of all, the triple is admissible because we may take $m \equiv 0$ mod p for all primes p. We have $(r_i, r_j) = 1$ for

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$i \neq j$, and $(r_1, 9) = (r_1, 9 \cdot 1 - 8 \cdot 1) = (r_1, 9 \cdot 1 - 6 \cdot 1) = 1$, $(r_2, 8) = (r_2, 8 \cdot 1 - 9 \cdot 1) = (r_1, 8 \cdot 1 - 6 \cdot 1) = 1$ and $(r_3, 6) = (r_3, 6 \cdot 1 - 9 \cdot 1) = (r_3, 6 \cdot 1 - 8 \cdot 1) = 1$.

We put $C = 11$ in the lemma. Then for some $i < j$, there exist infinitely many integers m for which $L_k(m)$ equals r_k times an E_2 number, all of whose prime factors are greater than 11, for $k = i, j$. If the forms are $L_1(x)$ and $L_2(x)$, then for infinitely many m, we have E_2 numbers A_1 and A_2, such that $(2 \cdot 3 \cdot 5 \cdot 7 \cdot 11, A_1A_2) = 1$ and

$$\frac{d(8L_1(m))}{d(8L_1(m)+1)} = \frac{d(8L_1(m))}{d(9L_2(m))} = \frac{d(2^3 r_1 A_1)}{d(3^2 r_2 A_2)} = \frac{d(2^3)d(5^{a_1-1})d(A_1)}{d(3^3)d(7^{a_2-1})d(A_2)} = \frac{a_1}{a_2}.$$

Similarly, if $L_1(x)$ and $L_3(x)$ are the relevant forms, then we have E_2 numbers A_1, A_3 such that

$$\frac{d(2L_1(m))}{d(2L_1(m)+1)} = \frac{d(2L_1(m))}{d(3L_3(m))} = \frac{d(2r_1A_1)}{d(3r_3A_3)} = \frac{d(2)d(5^{a_1-1})d(A_1)}{d(3)d(11^{a_3-1})d(A_3)} = \frac{a_1}{a_3}.$$

Finally, if the forms are $L_2(x)$ and $L_3(x)$, then

$$\frac{d(3L_2(m))}{d(3L_2(m)+1)} = \frac{d(3L_2(m))}{d(4L_3(m))} = \frac{d(3r_2A_2)}{d(2^2 r_3 A_3)} = \frac{d(3^2)d(7^{a_2-1})d(A_2)}{d(2^2)d(11^{a_3-1})d(A_3)} = \frac{a_2}{a_3}. \qquad \square$$

Let $q_1 = b_1/b_2$, $q_2 = b_3/b_4$ be positive rational numbers. If we take $a_1 = b_1b_3$, $a_2 = b_2b_3$, $a_3 = b_2b_4$, then from Lemma 2.2, $d(n)/d(n+1) \in \{q_1, q_2, q_1q_2\}$ for every $q_1, q_2 \in \mathbb{Q}^+$. Since rational numbers are dense in $\mathbb{R}$ and every irrational number can be approximated by rationals, for every $r_1, r_2 \in \mathbb{R}^+$, either $r_1 \in \mathbf{E}$ or $r_2 \in \mathbf{E}$ or $r_1r_2 \in \mathbf{E}$.

3. Proof of the main theorem

We are now ready to improve Hildebrand's result. In this section, we shall prove Theorem 1.1.

PROOF OF THEOREM 1.1. Let

$$\mathbf{L}' = \left\{ \log \frac{r}{s} : r, s \in \mathbb{N}; \frac{d(n+1)}{d(n)} = \frac{r}{s} \text{ for infinitely many } n \in \mathbb{N} \right\}. \tag{3.1}$$

It is obvious that $\overline{\mathbf{L}'} \subset \mathbf{L}$. Lemma 2.2 shows that for any positive integers a_1, a_2, a_3, there exist indices $i < j$ such that $\log a_j/a_i \in \mathbf{L}$. From this it follows that, given any positive real numbers u_1, u_2, u_3,

$$u_j - u_i \in \overline{\mathbf{L}'} \subset \mathbf{L} \quad \text{for some } i < j. \tag{3.2}$$

Applying (3.2) with $u_i = iu$ for $i = 1, 2, 3$, for $u > 0$,

$$u \in \mathbf{L} \cup \frac{\mathbf{L}}{2}. \tag{3.3}$$

Now, using subadditivity and positive homogeneity properties of Lebesgue measure, for $x > 0$,

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$$x = \left|[0,x] \cap \left\{\mathbf{L} \cup \frac{\mathbf{L}}{2}\right\}\right| \le |\mathbf{L} \cap [0,x]| + \left|\frac{\mathbf{L}}{2} \cap [0,x]\right| \le \frac{3}{2}|\mathbf{L} \cap [0,2x]|$$

and therefore

$$|\mathbf{L} \cap [0,x]| \ge \frac{x}{3} \quad (x > 0).$$

A similar argument with $u_i = (4-i)u$ yields

$$|\mathbf{L} \cap [-x,0]| \ge \frac{x}{3} \quad (x > 0).$$

Hence (1.2) holds.

If $\mathbf{L} = \mathbb{R}$ we are done. Otherwise, there exists $A > 0$ such that $A \notin \mathbf{L}$. By Lemma 2.2, for every $x \in \mathbb{R}^+$, either A or x or $A + x \in \mathbf{L}$. Then

$$x = |\mathbf{L} \cap \{[0,A] \cup [A, A+x]\}| \le |\mathbf{L} \cap [0,A]| + |\mathbf{L} \cap [A, A+x]| \le 2|\mathbf{L} \cap [0, A+x]|$$

and therefore $|\mathbf{L} \cap [0,x]| \ge (x - A)/2$ for $x \ge A$. □

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