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Coalgebraic modal logic of finite rank

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This paper studies coalgebras from the perspective of finite observations. We introduce the notion of finite step equivalence and a corresponding category with finite step equivalence-preserving morphisms. This category always has a final object, which generalises the canonical model construction from Kripke models to coalgebras. We then turn to logics whose formulae are invariant under finite step equivalence, which we call logics of rank ω. For these logics, we use topological methods and give a characterisation of compact logics and definable classes of models.

1. Introduction

Coalgebras for an endofunctor T on Set encompass many types of state base systems, including Kripke models and frames, labelled transition systems, Moore and Mealy automata and deterministic systems – see, for example, Rutten (2000). The research on modal logics as specification languages for coalgebras began with Moss (1999) and was taken up in, for example, Kurz (2001c), Rößiger (2000a; 2001) and Jacobs (1999; 2001a).

The relationship between modal logic and coalgebras was explained in Kurz (2001a) as follows. If Z denotes the carrier of the final coalgebra, we can consider the semantics of a modal formula φ as the subset $[\![\varphi]\!] \subseteq Z$ of states that satisfy φ. Intuitively, the elements of Z are behaviours, and every modal formula φ determines a set of behaviours that satisfy φ. When the logic is *fully expressive* in the sense that it allows us to define *all* subsets of Z, we can identify modal formulae with subsets of Z, resulting in an algebraic approach to the investigation of modal logics (Kurz 2001a; Kurz 2001b).

In general, however, finitary modal logics are not fully expressive. It is the main issue of this paper to present a semantics that adapts the ‘formulae as subsets of the final coalgebra’ idea to the case of finitary logics. We use the so-called *terminal sequence* $(T^n 1)_{n\in\mathbb{N}}$ of the underlying endofunctor to capture the notion of finitely observable behaviour. The terminal sequence can be understood as approximating the final coalgebra (Adámek and Koubek 1995). Intuitively, the elements of the n-th approximant represent the behaviour that can be observed in n transition steps. Following Pattinson (2001; 2003), we represent the semantics of a modal formula φ of rank n as the subset $[\![\varphi]\!]_n \subseteq T^n 1$.

The terminal sequence also gives rise to a notion of *finite step equivalence*. Intuitively, two processes are n-step equivalent iff they show the same n-step behaviour, that is, if their projections into $T^n 1$ coincide. The main novelty of the paper is the introduction of the category $\mathsf{Beh}_\omega(T)$, which has coalgebras as objects and functions that preserve finite step behaviours as morphisms (Section 4). This paper argues that the role of $\mathsf{Beh}_\omega(T)$

for finitary logics is the same as that of $\mathsf{Coalg}(T)$ for fully expressive logics. In Section 5, we show that $\mathsf{Beh}_\omega(T)$ always has a final object, the subsets of which represent formulae of finitary logics. Moreover, we show that the final object in $\mathsf{Beh}_\omega(T)$ generalises the canonical model construction from Kripke models to coalgebras.

In Section 6, we begin the study of logics whose formulae are invariant under finite step equivalence. These logics are called *logics of rank* ω. When the semantics of every formula φ can be represented as a subset $[\![\varphi]\!]_n \subseteq T^n 1$, $n < \omega$, we speak of *logics of finite rank*. Whereas fully expressive modal logics allows us to express all predicates of the carrier of the final coalgebra, logics of rank ω do not, in general, allow us to express all predicates on the final object of $\mathsf{Beh}_\omega(T)$. This is the reason for considering topologies on coalgebras. The main idea here is that clopen subsets are precisely the predicates that can be expressed through a single formula.

This topology is then used to prove compactness and definability results. Section 7 shows that – assuming the induced sub-logics of formulae of finite rank to be compact and expressive – a logic of finite rank is compact iff the functor T weakly preserves the limit of the finite part $(T^n 1)_{n<\omega}$ of the terminal sequence. Section 8 characterises classes of coalgebras that are definable by a logic of rank ω as being closed under images, subcoalgebras, coproducts and topological closure.

2. Preliminaries and notation

Throughout this paper, T denotes an endofunctor on the category of sets and functions. A T-coalgebra is a pair (C, γ) where C is a set and $\gamma : C \to TC$ is a function. A coalgebra morphism $f : (C, \gamma) \to (D, \delta)$ is a function $f : C \to D$ such that $\delta \circ f = Tf \circ \gamma$. The category of T-coalgebras and coalgebra morphisms is denoted $\mathsf{Coalg}(T)$. Given two T-Coalgebras (C, γ) and (D, δ), two states $c \in C$ and $d \in D$ are called *behaviourally equivalent* if they can be identified by a morphism of coalgebras, that is, if there exists $(E, \epsilon) \in \mathsf{Coalg}(T)$, $f : (C, \gamma) \to (E, \epsilon)$ and $g : (D, \delta) \to (E, \epsilon)$ with $f(c) = g(d)$. If $\mathsf{Coalg}(T)$ has a final object (Z, ζ) and $!_C : (C, \gamma) \to (Z, \zeta)$ and $!_D : (D, \delta) \to (Z, \zeta)$ denote the unique morphisms into the final object, this is clearly equivalent to $!_C(c) = !_D(d)$. We think of a coalgebra (C, γ) together with a state c as a process and call $!_C(c)$ its *behaviour*.

Example 2.1 (Streams). For a set D consider $TX = D \times X$. Given a coalgebra $\gamma = \langle head, tail \rangle : C \to D \times C$ the behaviour of an element $c \in C$ is the infinite list $(head(c), head(tail(c)), head(tail(tail(c))), \ldots)$. Hence, the structure $(D^\omega, \langle head, tail \rangle)$ of infinite lists over D is final in $\mathsf{Coalg}(T)$.

Example 2.2 (Kripke models). Suppose Prop is a (usually countable) set and $TX = \mathscr{P}X \times \mathscr{P}\mathsf{Prop}$. Then T-coalgebras are in 1-1 correspondence with Kripke models and behavioural equivalence coincides with bisimilarity.

We have seen that the final coalgebra defines a notion of behaviour. In general, every state of the final coalgebra represents an infinite amount of information. This paper suggests a framework to study finitely observable properties of systems. Hence the final coalgebra (containing the infinite behaviours of all coalgebras) has to be replaced by finitary approximations. These approximations are provided by the (finitary part of the) so-called terminal sequence of the endofunctor T.

2.1. *The terminal sequence*

The terminal sequence can be thought of as approximating the final coalgebra. The following definition has been taken from Worrell (1999).

The *terminal sequence* of T is an ordinal indexed sequence of sets (Z_n) together with a family $(p^n_m)_{m \leqslant n}$ of functions $p^n_m : Z_n \to Z_m$ for all ordinals $m \leqslant n$ such that:

— $Z_{n+1} = TZ_n$ and $p^{n+1}_{m+1} = Tp^n_m$ for all $m \leqslant n$.
— $p^n_n = id_{Z_n}$ and $p^n_k = p^m_k \circ p^n_m$ for $k \leqslant m \leqslant n$.
— The cone $(Z_n, (p^n_m))_{m<n}$ is limiting whenever n is a limit ordinal.

Thinking of Z_n as the n-fold application of T to the limit $1 = \{0\}$ of the empty diagram, we write T^n1 for Z_n in the rest of this paper. Intuitively, T^n1 represents behaviour that can be observed in n steps. If, for example, $TX = D \times X$, then $T^n1 \cong D^n$ contains all lists of length n.

Note that every coalgebra (C, γ) gives rise to a cone $(C, (\gamma_n : C \to T^n1))$ over the terminal sequence.

Definition 2.3. If $(C, \gamma) \in \mathsf{Coalg}(T)$, define $\gamma_n : C \to T^n1$ to be $T\gamma_m \circ \gamma$ if $n = m + 1$ is a successor ordinal, and the unique map satisfying $\gamma_m = p^n_m \circ \gamma_n$ for all $m < n$ if n is a limit ordinal.

We will often use the following easy fact without further reference.

Proposition 2.4. Let n be an ordinal.

1 Let $f : (C, \gamma) \to (D, \delta)$ be a coalgebra morphism. Then $\delta_n \circ f = \gamma_n$.
2 Let $(C, \gamma) \in \mathsf{Coalg}(T)$. Then $p^{n+1}_n \circ T(\gamma_n) \circ \gamma = \gamma_n$.

3. Introductory examples

For illustration and motivation of the later development, we will now discuss two different logics in detail. The main claim that we want to substantiate is that modal formulae can be semantically represented as subsets of T^n1, where n is the rank of the formula.

3.1. *Propositional modal logic*

This section argues that modal formulae of finite rank, interpreted over coalgebras, have a natural representation as subsets of some T^n1, where $n \in \omega$ is a finite ordinal. We start by re-considering Example 2.2 and show that a formula of rank n can be represented as a subset of T^n1.

Suppose $TX = \mathscr{P}X \times \mathscr{P}\mathsf{Prop}$. Then T-coalgebras are Kripke models, which is why we use propositional modal logic to describe properties of T-coalgebras. We denote the

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language of propositional modal logic by $\mathscr{ML}$, that is, $\mathscr{ML}$ is the least set according to the grammar

$$\mathscr{ML} \ni \varphi, \psi ::= \mathrm{ff} \mid p \mid \varphi \to \psi \mid \Box\varphi$$

where $p \in \mathsf{Prop}$ ranges over the set of atomic propositions.

Given a T-coalgebra (C, γ), the *semantics* $[\![\varphi]\!] = [\![\varphi]\!]_C \subseteq C$ of a modal formula $\varphi \in \mathscr{ML}$ is then inductively defined by

— $[\![\mathrm{ff}]\!] = \varnothing$
— $[\![p]\!] = \{c \in C \mid p \in \pi_2 \circ \gamma(c)\}$
— $[\![\varphi \to \psi]\!] = (C \setminus [\![\varphi]\!]) \cup [\![\psi]\!]$
— $[\![\Box\varphi]\!] = \{c \in C \mid \pi_1 \circ \gamma(c) \subseteq [\![\varphi]\!]\}$.

This definition is a coalgebraic formulation of the standard semantics of propositional modal logic (*cf.*, for example, Goldblatt (1992)). Given a formula $\varphi \in \mathscr{ML}$, the *rank* of φ, which represents the nesting depth of $\Box$-operators, is then given inductively by $\mathit{rank}(\mathrm{ff}) = 0$, $\mathit{rank}(\varphi \to \psi) = \max\{\mathit{rank}(\varphi), \mathit{rank}(\psi)\}$, $\mathit{rank}(p) = 1$ for $p \in \mathsf{Prop}$, $\mathit{rank}(\Box\varphi) = \mathit{rank}(\varphi) + 1$.

Semantically, the rank can be thought of as the number of transition steps a formula contains information about. A similar intuition applies to the approximants $T^n 1$ of the endofunctor: we think of predicates on $T^n 1$ as representing behaviour that can be observed in n transition steps. The following proposition makes this relationship precise.

Proposition 3.1. Suppose $\varphi \in \mathscr{ML}$ has rank n. Then there exists $t \subseteq T^n 1$ such that $[\![\varphi]\!]_C = \gamma_n^{-1}(t)$ for all $(C, \gamma) \in \mathsf{Coalg}(T)$.

Proof. We use induction on the structure of formulae. For $\varphi = \mathrm{ff}$, evidently, $[\![\varphi]\!] = \gamma_0^{-1}(\varnothing)$. For the case $\varphi = p$ for $p \in \mathsf{Prop}$, let $t = \{(X, Y) \mid X \subseteq 1, Y \subseteq \mathsf{Prop}, p \in Y\} \subseteq T1$. Then $[\![p]\!] = \gamma_1^{-1}(t)$. If $\varphi, \psi \in \mathscr{ML}$ with $\mathit{rank}(\varphi) = n, \mathit{rank}(\psi) = m$, we put $k = \max\{n, m\}$ and assume that $[\![\varphi]\!] = \gamma_n^{-1}(t)$, $[\![\psi]\!] = \gamma_m^{-1}(s)$. For $u = (T^k 1 \setminus (p_n^k)^{-1}(t)) \cup (p_m^k)^{-1}(s) \subseteq T^k 1$, the fact that $(C, (\gamma_n))$ is a cone over the terminal sequence implies that $[\![\varphi \to \psi]\!] = \gamma_k^{-1}(u)$.

For the most interesting case $\varphi = \Box\psi$, consider the operation defined by

$$\lambda(X)(\mathfrak{x}) = \{(\mathfrak{x}', \mathfrak{a}) \in \mathscr{P}(\mathfrak{X}) \times \mathscr{P}(\mathsf{Prop}) \mid \mathfrak{x}' \subseteq \mathfrak{x}\}$$

where X is a set and $\mathfrak{x} \subseteq \mathfrak{X}$. An easy calculation shows that we can rephrase the semantics of the $\Box$-operator as $[\![\Box\psi]\!] = \gamma^{-1} \circ \lambda(C)([\![\psi]\!])$. Now assume that ψ has rank n with $[\![\psi]\!] = \gamma_n^{-1}(s)$. Put $t = \lambda(T^n 1)(s)$. Then $[\![\Box\psi]\!] = \gamma_{n+1}^{-1}(t)$ follows from the fact that $\lambda : 2 \to 2 \circ T$ is a natural transformation, where 2 denotes the contravariant powerset functor. $\Box$

Remark 3.2. We have seen that formulae of rank n correspond to subsets of $T^n 1$ of the terminal sequence of T. This generalises from $TX = \mathscr{P}X \times \mathscr{P}\mathsf{Prop}$ to arbitrary functors $T : \mathsf{Set} \to \mathsf{Set}$. Indeed, in the proof of the above proposition, we only used the fact that atomic propositions can be represented as subsets of $T^n 1$ and that the semantics of the $\Box$-operator can be formulated in terms of a natural transformation $2 \to 2 \circ T$. Such natural transformations are often called predicate liftings and have been used by a number of authors (Rößiger 2001; Rößiger 2000a; Jacobs 1999; Jacobs 2001a) to describe the semantics of modal logics over coalgebras. We thus obtain a wealth of examples for arbitrary endofunctors T if we consider modal logics where atomic propositions can be represented as subsets of $T1$ and modal operators are interpreted using predicate liftings (see Pattinson (2001) for more information).

3.2. *Linear temporal logic on streams*

Linear temporal logic $\mathscr{LTL}$ (see Kröger (1987) and Manna and Pnueli (1992), for example) is a temporal logic to describe properties of infinite runs of programs, that is, streams. We use T-coalgebras for $TX = X \times \mathscr{P}\mathsf{Prop}$ (*cf.* Example 2.1), with Prop countably infinite, as semantics. This is a slight deviation from the standard semantics, which is given in terms of infinite sequences of subsets of Prop. The language $\mathscr{LTL}$ of linear temporal logic is the least set according to the grammar

$$\mathscr{LTL} \ni \varphi, \psi ::= \mathrm{ff} \mid p \mid \varphi \to \psi \mid \bigcirc\varphi \mid \Box\varphi$$

where $p \in \mathsf{Prop}$ ranges over the set of atomic propositions. We read $\bigcirc$ as 'next' and $\Box$ as 'always'. Given a T-coalgebra (C, γ), we define the semantics $[\![\varphi]\!] = [\![\varphi]\!]_C$ of an $\mathscr{LTL}$-formula φ inductively by

— $[\![\bigcirc\varphi]\!] = \{c \in C \mid \pi_1 \circ \gamma(c) \in [\![\varphi]\!]\}$
— $[\![\Box\varphi]\!] = \bigcap_{n<\omega} [\![\bigcirc^n\varphi]\!]$

where $\bigcirc^n$ stands for a sequence of n '$\bigcirc$' and the semantics of boolean operators and atomic propositions is as in the previous example. In contrast to the previous example, not all formulae can be represented as subsets of some approximant T^n1.

Example 3.3. Let $p \in \mathsf{Prop}$ and $\varphi = \Box p$. Then there is no $n < \omega$ and $t \subseteq T^n1$ with $[\![\varphi]\!]_C = \gamma_n^{-1}(t)$ for all $(C, \gamma) \in \mathsf{Coalg}(T)$.

We can, however, represent every formula as a subset of $T^\omega 1 \cong (\mathscr{P}\mathsf{Prop})^\omega$:

Proposition 3.4. For all $\varphi \in \mathscr{LTL}$ there is $t \subseteq T^\omega 1$ such that $[\![\varphi]\!]_C = \gamma_\omega^{-1}(t)$.

Proof. Consider $(K, \kappa) = ((\mathscr{P}\mathsf{Prop})^\omega, \langle \mathit{head}, \mathit{tail}\rangle)$. Then we have $K \cong T^\omega 1$ and $[\![\varphi]\!]_C = \gamma_\omega^{-1}([\![\varphi]\!]_K)$. □

4. Finite step equivalence and the category $\mathsf{Beh}_\omega(T)$

The previous section showed that, for logics interpreted via predicate liftings as described in Remark 3.2, formulae of finite rank can be represented as subsets of the elements T^n1 of T's terminal sequence. For the remainder of the exposition, we take a semantical view and take subsets of the T^n1 as representing formulae of finite rank; this allows us to consider logics for coalgebras in broad generality without making a commitment to any particular syntax.

We begin by introducing a notion of equivalence on states that reflects the fact that two states cannot be distinguished by a predicate of finite rank.

Definition 4.1. Let n be an ordinal and suppose $(C,\gamma),=(D,\delta)\in\mathsf{Coalg}(T)$. For $c\in C$ we call $\gamma_n(c)$ the *n-step behaviour of c*.

1 Two states $(c,d)\in C\times D$ are called *n-step equivalent*, denoted $c\sim_n d$, if $\gamma_n(c)=\delta_n(d)$. We call c and d *finite step equivalent* if $c\sim_n d$ for all $n<\omega$.
2 The systems (C,γ) and (D,δ) are n-step equivalent, denoted $(C,\gamma)\sim_n(D,\delta)$, if $\gamma_n(C)=\delta_n(D)$. They are called *finite step equivalent*, denoted $(C,\gamma)\sim_{<\omega}(D,\delta)$, if $(C,\gamma)\sim_n(D,\delta)$ for all $n<\omega$.

Under the assumption that the final coalgebra exists, two states of coalgebras are behaviourally equivalent, if they are identified by the unique morphism into the final coalgebra. As shown in Adámek and Koubek (1995), this is equivalent to $\gamma_n(x)=\gamma_n(y)$ for all ordinals n. Finite step equivalence, as introduced above, restricts the validity of this equation to *finite* ordinals. Note that c,d are finite step equivalent iff $c\sim_\omega d$. In the context of modal logic, (that is, for $TX=\mathscr{P}X\times\mathscr{P}\mathsf{Prop}$), finite step equivalence is (a slight variation of) the bounded bisimulation of modal logic as studied in Gerbrandy (1999).

The next proposition clarifies the relationship between finite step equivalence and behavioural equivalence on states of coalgebras.

Proposition 4.2. Suppose $(C,\gamma),(D,\delta)\in\mathsf{Coalg}(T)$ and $(c,d)\in C\times D$.

1 If c and d are behaviourally equivalent, they are finite step equivalent.
2 If T is ω-accessible, c and d are behaviourally equivalent if and only if they are finite step equivalent.

Proof. The first claim is an easy induction, the second follows by terminal sequence induction, see Worrell (1999) or Pattinson (2004, Theorem 4.1). □

In order to obtain an example of two states which are finite step equivalent but not behaviourally equivalent, one therefore needs to consider a functor that is not ω-accessible. The following is a standard example.

Example 4.3. Let $TX=\mathscr{P}(X)$ and consider $C=\omega+2$, $\gamma(c)=c$. One can show by induction that $\gamma_n(c)=c\cap n$. Hence ω and $\omega+1$ are finite step equivalent. If they were behaviourally equivalent, one would obtain $\gamma_{\omega+1}(\omega)=\gamma_{\omega+1}(\omega+1)$, which is not the case.

While for states finite step equivalence and ω-step equivalence define the same notion of equivalence, for coalgebras, ω-step equivalence is, in general, *not* implied by finite step equivalence.

Example 4.4. Let $TX=\{a,b\}\times X$, (C,γ) be the final coalgebra with carrier $\{a,b\}^\omega$ and (D,δ) be the subcoalgebra with carrier $\{s\cdot a^\omega : s\in\{a,b\}^*\}$. Then (C,γ) and (D,δ) are finite step equivalent, but not ω-step equivalent.

In the category $\mathsf{Coalg}(T)$, morphisms are easily seen to preserve behavioural equivalence. We now introduce the category $\mathsf{Beh}_\omega(T)$, the morphisms of which are only required to preserve finite step equivalence. Recall that $\delta_\omega\circ f=\gamma_\omega$ iff $\delta_n\circ f=\gamma_n$ for all $n<\omega$ whenever $(C,\gamma),(D,\delta)\in\mathsf{Coalg}(T)$ and $f:C\to D$ is any function.

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Definition 4.5 ($\mathsf{Beh}_\omega(T)$). The category $\mathsf{Beh}_\omega(T)$ has T-coalgebras as objects. Morphisms $f : (C,\gamma) \to (D,\delta)$ of $\mathsf{Beh}_\omega(T)$ are functions $f : C \to D$ such that $\delta_\omega \circ f = \gamma_\omega$.

Remark 4.6. Clearly, every morphism of coalgebras $f : (C,\gamma) \to (D,\delta) \in \mathsf{Coalg}(T)$ is also a morphism $f \in \mathsf{Beh}_\omega(T)$. Hence, we obtain a functorial inclusion $\mathsf{Coalg}(T) \to \mathsf{Beh}_\omega(T)$. In order to explain the relationship of $\mathsf{Beh}_\omega(T)$ to $\mathsf{Coalg}(T)$, consider the following categories

$$\begin{array}{ccc} \mathsf{c\text{-}Beh}(T) & \hookrightarrow & \mathsf{Beh}(T) \\ \downarrow & & \downarrow \\ \mathsf{c\text{-}Beh}_\omega(T) & \hookrightarrow & \mathsf{Beh}_\omega(T) \end{array}$$

which all have coalgebras as objects and morphisms as follows. $f : (C,\gamma) \to (D,\delta)$ is a $\mathsf{Beh}(T)$-morphism iff $\gamma_n(c) = \delta_n(f(c))$ for all ordinals n and all $c \in C$. The definitions of $\mathsf{c\text{-}Beh}(T)$ and $\mathsf{c\text{-}Beh}_\omega(T)$ follow the same idea, but take colourings into account, giving $f : (C,\gamma) \to (D,\delta)$ is a $\mathsf{c\text{-}Beh}_\omega(T)$-morphism iff f is a $\mathsf{Beh}_\omega(T \times X)$-morphism $(C, \langle \gamma, v \circ f \rangle) \to (D, \langle \delta, v \rangle)$ for all $X \in \mathsf{Set}$ and $v : D \to X$.

If $\mathsf{Coalg}(T)$ has cofree coalgebras, then $\mathsf{c\text{-}Beh}(T) = \mathsf{Coalg}(T)$. If T is finitary (that is, ω-accessible), then $\mathsf{Beh}_\omega(T) = \mathsf{Beh}(T)$ and $\mathsf{c\text{-}Beh}_\omega(T) = \mathsf{c\text{-}Beh}(T)$. Whether the converse holds, that is, whether $\mathsf{c\text{-}Beh}_\omega(T) = \mathsf{c\text{-}Beh}(T)$ implies that T is finitary is an open question.

We conclude the section with a couple of simple properties of $\mathsf{Beh}_\omega(T)$, all of which are also true for $\mathsf{Coalg}(T)$.

Proposition 4.7. Let $U : \mathsf{Beh}_\omega(T) \to \mathsf{Set}$ be the forgetful functor.

1 $\mathsf{Beh}_\omega(T) \hookrightarrow \mathsf{Coalg}(T)$ preserves and reflects coproducts.
2 Injective and surjective morphisms form a factorisation system for $\mathsf{Beh}_\omega(T)$. In particular, every morphism $f \in \mathsf{Beh}_\omega(T)$ factors as $f = m \circ e$ with Um mono, Ue epi.

Proof. The claim for coproducts is immediate. For factorisations, let $f : (C,\gamma) \to (D,\delta)$ be a morphism in $\mathsf{Beh}_\omega(T)$ and $C \xrightarrow{e} I \xrightarrow{m} D$ be its epi-mono factorisation in Set. Choose h with $e \circ h = id_I$ and define $\iota : I \to TI$ as $Te \circ \gamma \circ h$. Assuming $\iota_n \circ e = \gamma_n$, we verify

$$\begin{aligned} \iota_{n+1} \circ e &= T\iota_n \circ \iota \circ e \\ &= \gamma_{n+1} \circ h \circ e \\ &= \delta_{n+1} \circ f \circ h \circ e \\ &= \delta_{n+1} \circ f \\ &= \gamma_{n+1}, \end{aligned}$$

showing that $e : (C,\gamma) \to (I,\iota)$ is a morphism, and hence also m. We have seen that factorisations exist in $\mathsf{Beh}_\omega(T)$. The remaining conditions on a factorisation system (see, for example, Adámek *et. al.* (1990)) are easy to check. □

5. Final and quasi-canonical models

When reasoning about behaviours, the final coalgebra plays a central role because, given the unique coalgebra morphism $!_C : (C, \gamma) \to (Z, \zeta)$ from a coalgebra (C, γ) into the final coalgebra (Z, ζ), for every element c of (the carrier of) (C, γ), we can consider $!_C(c)$ as the behaviour of c. Similarly, final objects of $\mathsf{Beh}_\omega(T)$ (*cf.* Definition 4.5) consist of the finite behaviours. This section shows that $\mathsf{Beh}_\omega(T)$ always has a final object, which generalises the canonical model construction from Kripke models to coalgebras.

5.1. *Final objects in* $\mathsf{Beh}_\omega(T)$

A final object of $\mathsf{Beh}_\omega(T)$ should 'realise' all n-step behaviours, $n < \omega$. Accordingly, the carrier of a final object in $\mathsf{Beh}_\omega(T)$ will be a subset of $T^\omega 1$.

Recall that, given any coalgebra (C, γ), we write γ_ω for the unique mediating map $\gamma_\omega : C \to T^\omega 1$. That is, all ω-step behaviours appear as some $\gamma_\omega(c)$ in $T^\omega 1$. On the other hand, it may happen that not every point $t \in T^\omega 1$ can be presented as $t = \gamma_\omega(c)$ by some structure (C, γ) and some $c \in C$. Consider, for example, the finite powerset functor $T = \mathscr{P}_\omega$. It was shown in Worrell (1999) that for the final T-coalgebra (Z, ζ) the morphism $\zeta_\omega : Z \to T^\omega 1$ is (injective but) not surjective. Hence we construct the carrier of the coalgebra final in $\mathsf{Beh}_\omega(T)$ by collecting all $t \in T^\omega 1$ that can be 'realised' by some structure, that is, for which there are $(C, \gamma) \in \mathsf{Coalg}(T)$ and $c \in C$ such that $\gamma_\omega(c) = t$. It then remains to find an appropriate coalgebra structure.

Throughout, we fix the set K of 'realisable' elements $t \in T^\omega 1$, which is given by

$$K = \{t \in T^\omega 1 \mid \exists (C, \gamma) \in \mathsf{Coalg}(T) \,.\, \exists c \in C \,.\, \gamma_\omega(c) = t\}.$$

For each $k \in K$, we can now choose $(C^k, \gamma^k) \in \mathsf{Coalg}(T)$ and $c^k \in C^k$ such that $\gamma^k_\omega(c_k) = k$. Note that K is a set, which enables us to consider

$$(C, \gamma) = \coprod_{k \in K} (C^k, \gamma^k)$$

where the coproduct is taken in $\mathsf{Coalg}(T)$. Denoting the coproduct injections by $in_k : C_k \to C$ (which, by the construction of coproducts in $\mathsf{Coalg}(T)$, are also coproduct injections in the category of sets), we are ready to note the following result.

Lemma 5.1. $\gamma_\omega \circ in_k(c) = \gamma^k_\omega(c)$ for all $k \in K$ and $c \in C_k$.

Proof. Since γ^k_ω is the unique mediating map into the limiting cone with vertex $T^\omega 1$, it suffices to prove that $\gamma_n \circ in_k(c) = \gamma^k_n(c)$ for all $n < \omega$. For $n = 0$, this is obvious. For the induction step, we calculate

$$\begin{aligned} \gamma_{n+1} \circ in_k(c) &= T\gamma_n \circ \gamma \circ in_k(c) \\ &= T\gamma_n \circ T in_k \circ \gamma^k(c) \\ &= T\gamma^k_n \circ \gamma^k(c) \\ &= \gamma^k_{n+1}(c). \end{aligned}$$

□

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We obtain the following immediate corollary.

Corollary 5.2. For all $k \in K$ there exists $c \in C$ with $\gamma_\omega(c) = k$.

In other words, γ_ω factors through K as $\gamma_\omega = m \circ e$, m injective, e surjective. Now consider the diagram

$$\begin{array}{ccccc} TT^\omega 1 & \xleftarrow{Tm} & TK & \xleftarrow{Te} & TC \\ & & \uparrow{\scriptstyle \kappa} & & \uparrow{\scriptstyle \gamma} \\ T^\omega 1 & \xleftarrow[m]{} & K & \underset{o}{\overset{e}{\rightleftarrows}} & C \end{array} \tag{1}$$

where o is any one-sided inverse of e, that is, $e \circ o = id_K$, the existence of which is guaranteed by e being a surjection. We let

$$\kappa = Te \circ \gamma \circ o.$$

Note that $\kappa : K \to TK$ makes K into a T-coalgebra. Recalling the notation for the limit projections $p_n^\omega : T^\omega 1 \to T^n 1$, we obtain the following lemma.

Lemma 5.3. For all $n < \omega$, $\kappa_n = p_n^\omega \circ m$, and hence $m = \kappa_\omega$.

Proof. We proceed by induction on n, where the case $n = 0$ is evident. We calculate

$$\begin{aligned} \kappa_{n+1} &= T\kappa_n \circ \kappa \\ &= T(p_n^\omega \circ m) \circ Te \circ \gamma \circ o \\ &= Tp_n^\omega \circ T(m \circ e) \circ \gamma \circ o \\ &= Tp_n^\omega \circ T\gamma_\omega \circ \gamma \circ o \\ &= T\gamma_n \circ \gamma \circ o \\ &= \gamma_{n+1} \circ o \\ &= p_{n+1}^\omega \circ \gamma_\omega \circ o \\ &= p_{n+1}^\omega \circ m \circ e \circ o \\ &= p_{n+1}^\omega \circ m \end{aligned}$$

for the induction step, as desired. □

The proof of the main theorem of this section is now straightforward.

Theorem 5.4. $\mathsf{Beh}_\omega(T)$ has a final object.

Proof. We show that (K, κ), as constructed above, is final in $\mathsf{Beh}_\omega(T)$. Take any object $(D, \delta) \in \mathsf{Beh}_\omega(T)$. Consider the mapping $\delta_\omega : D \to T^\omega 1$, which is the unique mediating map between the cones $(D, (\delta_n)_{n<\omega})$ and $(T^\omega 1, (p_n^\omega)_{n<\omega})$. By construction, δ_ω factors as $\delta_\omega = m \circ h$ where $m : K \to T^\omega 1$ is as above. By Lemma 5.3,

$$\delta_\omega = \kappa_\omega \circ h,$$

which implies that h is a $\mathsf{Beh}_\omega(T)$-morphism. h is unique since κ_ω is injective. □

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Using ! to denote the morphisms into the final $\mathsf{Beh}_\omega(T)$-object (K,κ), the fact that $\kappa_\omega : K \to T^\omega 1$ is injective gives us the following corollary.

Corollary 5.5. Let (C,γ), (D,δ) be T-coalgebras and $c \in C$, $d \in D$. Then c and d are finite step equivalent iff $!_C(c) = !_D(d)$.

Final objects in $\mathsf{Beh}_\omega(T)$ are not determined uniquely up to $\mathsf{Coalg}(T)$-isomorphism. This is due to the fact that not all $\mathsf{Beh}_\omega(T)$-morphisms are also coalgebra morphisms and that, accordingly, objects isomorphic in $\mathsf{Beh}_\omega(T)$ may fail to be isomorphic in $\mathsf{Coalg}(T)$. When $p_\omega^{\omega+1} : TT^\omega 1 \to T^\omega 1$ is surjective†, we can classify, up to coalgebra isomorphism, the final objects of $\mathsf{Beh}_\omega(T)$ as being given by the right inverses of $p_\omega^{\omega+1}$.

Corollary 5.6. Assume that $p_\omega^{\omega+1}$ is surjective. An object is final in $\mathsf{Beh}_\omega(T)$ iff it is isomorphic in $\mathsf{Coalg}(T)$ to some $(T^\omega 1, \theta)$ with $p^{\omega+1} \circ \theta = id_{T^\omega 1}$.

Proof.

— *If* : To show that $(T^\omega 1, \theta)$ is final, it suffices to observe that $\theta_\omega = id_{T^\omega 1}$. This follows from $\theta_n = p_n^\omega$, $n < \omega$, the inductive case being $\theta_{n+1} = T(\theta_n) \circ \theta = T(p_n^\omega) \circ \theta = p_{n+1}^\omega \circ p_n^{\omega+1} \circ \theta = p_{n+1}^\omega$.

— *Only if* : Let (C,γ) be final in $\mathsf{Beh}_\omega(T)$. Consider a final object (K,κ) as constructed in the proof of the theorem. Let $f : (C,\gamma) \to (K,\kappa)$ be the unique morphism. In particular, f is iso and $\kappa_\omega \circ f = \gamma_\omega$. Since κ_ω is injective, γ_ω is also. By Proposition 2.4(ii), $\gamma_\omega = p_\omega^{\omega+1} \circ T(\gamma_\omega) \circ \gamma$, so γ_ω is also surjective, and hence iso. Now define $\theta = T(\gamma_\omega) \circ \gamma \circ \gamma_\omega^{-1}$. □

We conclude with the useful observation that all $t \in T^n 1$, $n < \omega$, are realised as n-step behaviours in the final $\mathsf{Beh}_\omega(T)$-coalgebra. We first note that every element of an approximant $T^n 1$ is realised by a coalgebra.

Proposition 5.7. Let f be any mapping $1 \to T1$ and $(C,\gamma) = (T^n 1, T^n f)$. Then $\gamma_n = id_C$.

As a corollary, we get that the maps κ_n are surjections.

Corollary 5.8. Suppose (K,κ) is final in $\mathsf{Beh}_\omega(T)$ and $n < \omega$. Then κ_n is a surjection.

Proof. Let (C,γ) be given as in the above proposition. If $x \in T^n 1$, we have $x = \gamma_n(x) = \kappa_n \circ !(x)$, where $! : (C,\gamma) \to (K,\kappa)$ is the map given by finality. □

5.2. *The canonical model*

In this section we consider the functor $M = \mathscr{P} \times \mathscr{P}\mathsf{Prop}$ where Prop is a countably infinite set.

The **canonical model** (see, for example, Blackburn *et. al.* (2001) and Goldblatt (1992)) for the modal logic $\mathscr{ML}$ is the M-coalgebra $(L, \langle \lambda_R, \lambda_V \rangle)$

† With the exception of $T = \mathscr{P}_\omega$, this is the case for all examples in this paper. A sufficient condition for $p_\omega^{\omega+1}$ to be surjective is that T weakly preserves limits of ω^{op}-chains.

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$$\begin{array}{ll} L & \{\Phi \subseteq \mathscr{ML} : \Phi \text{ is maximally consistent}\} \\ \lambda_R : L \to \mathscr{P}L & \Phi \mapsto \{\Psi : \psi \in \Psi \Rightarrow \Diamond\psi \in \Phi\} \\ \lambda_V : L \to \mathscr{P}\mathsf{Prop} & \Phi \mapsto \Phi \cap \mathsf{Prop}. \end{array}$$

The canonical model is final in the category $\mathsf{Th}_{\mathscr{ML}}$ that has M-coalgebras as objects and whose morphisms $f : (C,\gamma) \to (D,\delta)$ are functions $f : C \to D$ such that for all $c \in C$, c and $f(c)$ have the same modal theory.

Proposition 5.9. $\mathsf{Beh}_\omega(M) \cong \mathsf{Th}_{\mathscr{ML}}$.

Proof. We have to show that for any coalgebras (C,γ), (D,δ) and any function $f : C \to D$,

$$\delta_\omega \circ f(c) = \gamma_\omega(c) \;\Leftrightarrow\; \mathrm{Th}(c) = \mathrm{Th}(f(c)),$$

which is equivalent to $[\forall n < \omega \,.\, \delta_n \circ f(c) = \gamma_n(c)] \;\Leftrightarrow\; [\forall n < \omega \,.\, \forall \varphi \in \mathscr{ML} \,.\, rank(\varphi) = n \Rightarrow (c \models \varphi \;\Leftrightarrow\; f(c) \models \varphi)]$, which can be shown using induction on n. □

It follows that the canonical model is final in $\mathsf{Beh}_\omega(M)$. We now show that, conversely, every final object in $\mathsf{Beh}_\omega(M)$ satisfies the so-called truth-lemma, which is the main property of the canonical model.

Definition 5.10. An M-coalgebra (L,λ) is called a *quasi-canonical model* if L is the set of maximal consistent sets of formulae and

$$(L,\lambda), \Phi \models \varphi \iff \varphi \in \Phi \tag{2}$$

for all $\Phi \in L$.

The canonical model is quasi-canonical. In fact, it is Property (2) that makes the canonical model useful. In other words, any quasi-canonical model can serve potentially the same purpose as the canonical model. The following theorem characterises the quasi-canonical models as – up to isomorphism of coalgebras – the final coalgebras constructed in the previous subsection. This gives a syntax-free description of the quasi-canonical models.

Theorem 5.11. Suppose (C,γ) is an M-coalgebra. Then (C,γ) is final in $\mathsf{Beh}_\omega(M)$ iff (C,γ) is $\mathsf{Coalg}(M)$-isomorphic to a quasi-canonical model.

Proof. First, every quasi-canonical model is easily seen to be final in $\mathsf{Th}_{\mathscr{ML}}$, and hence, by Proposition 5.9, final in $\mathsf{Beh}_\omega(M)$. Now suppose (C,γ) is final in $\mathsf{Beh}_\omega(M)$. Since the canonical model $(L, \langle\lambda_R, \lambda_V\rangle)$ is also final in $\mathsf{Beh}_\omega(M)$, the map $f : C \to L$, $c \mapsto \{\varphi \in \mathscr{ML} \mid c \models \varphi\}$ is a bijection. Let $\gamma' = Mf^{-1} \circ \gamma \circ f^{-1}$. Then $(C,\gamma) \cong (L,\gamma') \in \mathsf{Coalg}(M)$. It remains to show the truth lemma for (L,γ'):

$$\begin{aligned} (L,\gamma'), \Phi \models \varphi &\Leftrightarrow (C,\gamma), f^{-1}(\Phi) \models \varphi \\ &\Leftrightarrow \varphi \in f(f^{-1}(\Phi)) \\ &\Leftrightarrow \varphi \in \Phi. \end{aligned}$$

□

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Since the projection $p_\omega^{\omega+1} : MM^\omega 1 \to M^\omega 1$ is surjective, Corollary 5.6 shows that the choice of a transition relation on a quasi-canonical model corresponds to the choice of a right inverse of the projection $p_\omega^{\omega+1}$.

Corollary 5.12. There is a 1-1 correspondence between the set of quasi-canonical models and the set of right inverses of $p_\omega^{\omega+1}$.

6. Abstract logics and their topologies

In this section, we give an abstract account of the logics we are going to work with later: logics of finite rank and logics of rank ω. These logics are the subject of our study in the remainder of this paper, where we prove a compactness theorem and give a characterisation of definable classes of models. Both results rely on (and can be best explained in terms of) the topologies that are defined by the logics under consideration; we give a short account of these topologies.

6.1. *Logics of finite rank and logics of rank ω*

In Section 4, we introduced a notion of finite step equivalence between elements of coalgebras. This section starts the investigation of logics whose formulae are invariant under finite step equivalence. Since we do not want to commit ourselves to a particular syntax, we assume that a logic $\mathscr{L}$ for T-coalgebras already comes with an interpretation function $[\![\cdot]\!]_C : \mathscr{L} \to \mathscr{P}(C)$ for every T-coalgebra (C, γ) that maps a formula $\varphi \in \mathscr{L}$ to the set $[\![\varphi]\!] \subseteq C$ of states that satisfy φ.

Definition 6.1. An *abstract logic* for T-coalgebras is a pair $(\mathscr{L}, [\![\cdot]\!])$ where

— $\mathscr{L}$ is the set of formulae and
— $[\![\cdot]\!]$ is a family of mappings $[\![\cdot]\!]_C : \mathscr{L} \to \mathscr{P}(C)$ indexed by the T-coalgebras,

such that $\mathscr{L}$ has (classical) negation and conjunctions that are interpreted as complement and intersection, respectively.

Given an abstract logic $(\mathscr{L}, [\![\cdot]\!])$, $(C, \gamma) \in \mathsf{Coalg}(T)$ and $c \in C$, we write $c \models_C \varphi$ if $c \in [\![\varphi]\!]_C$ and $\mathrm{Th}(c) = \{\varphi \in \mathscr{L} \mid c \models \varphi\}$. Our interest in abstract logics lies in studying the properties that we now introduce.

Definition 6.2. Let $\varphi \in \mathscr{L}$ and $t \subseteq T^n 1$. We say that *t represents* φ iff $[\![\varphi]\!]_C = \gamma_n^{-1}(t)$ for all T-coalgebras (C, γ). In this case, φ has *rank n*. We say a logic $(\mathscr{L}, [\![\cdot]\!])$ is:

1 *of finite rank* iff every $\varphi \in \mathscr{L}$ has finite rank;
2 *of rank* ω if every $\varphi \in \mathscr{L}$ has rank ω;
3 *invariant under finite step equivalence* if $c \sim_n d$ for all $n \in \mathbb{N} \implies \mathrm{Th}(c) = \mathrm{Th}(d)$;
4 *finite step expressive* if for all $n < \omega$ and all $t \subseteq T^n 1$ there is $\varphi \in \mathscr{L}$ such that t represents φ.

Finite step expressive logics play a role similar to the fully expressive logics mentioned in the introduction.

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Example 6.3. Propositional modal logic is our prime example of a logic of finite rank (Proposition 3.1). A logic of finite rank is also of rank ω. Linear temporal logic is an example of a logic of rank ω that is not of finite rank (Example 3.3). For an endofunctor T, the coalgebraic logic of Moss (1999) associated with T is a logic of rank ω if T is ω-accessible.

We conclude with two characterisations of logics of rank ω.

First, when the final coalgebra (Z, ζ) exists, we can represent any logic $\mathscr{L}$ whose formulae are invariant under behavioural equivalence by $[\![\cdot]\!]_Z : \mathscr{L} \to \mathscr{P}Z$, the $[\![\cdot]\!]_C$ being determined by $[\![\cdot]\!]_C = !_C^{-1} \circ [\![\cdot]\!]_Z$, where $!_C : (C, \gamma) \to (Z, \zeta)$ is given by finality in $\mathsf{Coalg}(T)$. Similarly, a logic $\mathscr{L}$ of rank ω can be represented by $[\![\cdot]\!]_K$, where (K, κ) is the final object in $\mathsf{Beh}_\omega(T)$ and $!_C : (C, \gamma) \to (K, \kappa)$ is again given by finality.

Second, logics of rank ω are precisely those logics whose formulae are invariant under finite step equivalence.

Proposition 6.4. Suppose $\mathscr{L}$ is an abstract logic and (K, κ) is the final object of $\mathsf{Beh}_\omega(T)$. The following are equivalent:

1 $\mathscr{L}$ is of rank ω.
2 $[\![\varphi]\!]_C = !_C^{-1}([\![\varphi]\!]_K)$ for all $\varphi \in \mathscr{L}$ and all T-coalgebras (C, γ).
3 $\mathscr{L}$ is invariant under finite step equivalence.

Proof. First suppose that $\mathscr{L}$ is of rank ω and $\varphi \in \mathscr{L}$. By assumption, there is $t \subseteq T^\omega 1$ such that $[\![\varphi]\!]_C = \gamma_\omega^{-1}(t)$ for all T-coalgebras (C, γ). Since $!_C$ is a morphism of $\mathsf{Beh}_\omega(T)$, we have $\gamma_\omega = \kappa_\omega \circ !_C$. Thus $[\![\varphi]\!]_C = \gamma_\omega^{-1}(t) = !_C^{-1} \circ \kappa_\omega^{-1}(t) = !_C^{-1}([\![\varphi]\!]_K)$.

Next assume $[\![\varphi]\!]_C = !_C^{-1}([\![\varphi]\!]_K)$ for all $\varphi \in \mathscr{L}$. Furthermore, let (C, γ), (D, δ) be T-coalgebras and $(c, d) \in C \times D$ be such that $c \sim_n d$ for all $n \in \omega$. To show $\mathrm{Th}(c) = \mathrm{Th}(d)$, pick $\varphi \in \mathrm{Th}(c)$, that is, $!_C(c) \models_K \varphi$. From Corollary 5.5, we know $!_D(d) = !_C(c)$, so $!_D(d) \models_K \varphi$, and thus $d \models_D \varphi$ by assumption.

Finally, assume that $\mathscr{L}$ is invariant under finite step equivalence and $\varphi \in \mathscr{L}$. For $t = [\![\varphi]\!]_K \subseteq K \subseteq T^\omega 1$ and $(C, \gamma) \in \mathsf{Coalg}(T)$, we obtain $[\![\varphi]\!]_C = \gamma_\omega^{-1}(t)$. □

We conclude that logics of rank ω are precisely those logics whose formulae are invariant under finite step equivalence.

Corollary 6.5. A logic is of rank ω iff its formulae are invariant under finite step equivalence.

Proof. The statement follows from the above proposition and the observation that, given T-coalgebras $(C, \gamma), (D, \delta)$ and $(c, d) \in C \times D$, we have $c \sim_\omega d$ iff $!_C(c) = !_D(d)$, where $!_C$ and $!_D$ are the unique morphisms into the final object of $\mathsf{Beh}_\omega(T)$. □

6.2. *Topologies on coalgebras*

We now study logics for coalgebras from a topological perspective, where the topology on a model is generated by the set of denotations of logical formulae. We have seen that every formula of rank ω can be represented as a subset of the final object in $\mathsf{Beh}_\omega(T)$.

Topology comes into play since one cannot expect that all subsets of the final object can be represented in the logic (since the set of formulae of a logic is in general countable). For introductory material on the relation between logic and topology, refer to Smyth (1993) and Vickers (1998). For the rest of the paper, we assume that $(\mathscr{L}, [\![\cdot]\!])$ is an abstract logic (Definition 6.1). We begin with the definition of the topologies of interest.

Definition 6.6 (Topologies τ_C). Suppose (C, γ) is a T-coalgebra. The topology τ_C on C is generated by the basis $\{[\![\varphi]\!]_C \mid \varphi \in \mathscr{L}\}$.

Remark 6.7. Suppose $f : (C, \gamma) \to (D, \delta) \in \mathsf{Beh}_\omega(T)$. If $\mathscr{L}$ is of rank ω, the semantics of formulae is stable under $\mathsf{Beh}_\omega(T)$-morphisms (Corollary 6.5), hence $f : (C, \tau_C) \to (D, \tau_D)$ is continuous. Since every morphism of coalgebras qualifies as a $\mathsf{Beh}_\omega(T)$-morphism, we have a chain of functors $\mathsf{Coalg}(T) \to \mathsf{Beh}_\omega(T) \to \mathsf{Top}$, where Top is the category of topological spaces.

By definition, every formula of a logic of finite rank can be represented as a subset $t \subseteq T^n 1$ for some $n < \omega$. If the approximants $T^n 1$ are finite it is natural to assume that all subsets of $T^n 1$ can be expressed by a formula, that is, that $\mathscr{L}$ is finite step expressive (*cf.* Definition 6.2). Since this is not the case in general (see, for example, propositional modal logic over an infinite set of atomic propositions as discussed in Section 3.1), we also introduce topologies on the approximants $T^n 1$.

Definition 6.8 (Topologies τ_n, τ_C^ω). For $n < \omega$, the topology τ_n on $T^n 1$ is given by the basis $\{t \subseteq T^n 1 \mid \exists \varphi \in \mathscr{L} \,.\, t \text{ represents } \varphi\}$. If (C, γ) is a T-coalgebra, the topology τ_C^ω on C is given by the basis $\{\gamma_n^{-1}(U) \mid U \in \tau_n, n < \omega\}$.

The topology on the approximants would not be worth its salt if it did not turn the connecting morphisms $p_m^n : T^n 1 \to T^m 1$ into continuous functions.

Remark 6.9. Suppose $m \leqslant n < \omega$ and $t \subseteq T^n 1$ represents a formula φ of $\mathscr{L}$ (that is, t is a basic open of $(T^n 1, \tau_n)$). Then $(p_m^n)^{-1}(t)$ also represents φ, showing that p_m^n is continuous.

The following easy proposition is useful in that it allows us to compute the topologies τ_C via the topologies on the approximants $T^n 1$.

Proposition 6.10. Let $\mathscr{L}$ be a logic of finite rank and (C, γ) be a coalgebra. Then the topologies τ_C^ω and τ_C coincide.

The converse of the proposition only holds in compact spaces. Before we turn to compactness issues, we discuss an important special case.

Definition 6.11 (Cantor space topology). When $\mathscr{L}$ is finite step expressive, that is, when the topologies τ_n are discrete, we call τ_C^ω the Cantor space topology.

The terminology is motivated by the following example.

Example 6.12. Suppose $TX = 2 \times X$, where $2 = \{0, 1\}$. Consider the (final) T-coalgebra (C, γ) with $C = 2^\omega = \{f : \omega \to 2\}$ and $\gamma(f) = (f(0), \lambda n \,.\, f(n+1))$. Then (C, τ_C) is homeomorphic to the Cantor discontinuum $\mathbb{C}$ (also known as the middle-third set, see, for example, Jelley (1995)) via the mapping $2^\omega \to \mathbb{C}$, $f \mapsto \sum_{i=0}^{\infty} \frac{2}{3^{i+1}} \cdot f(i)$.

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Remark 6.13. Let $(C, \gamma) \in \mathsf{Coalg}(T)$, and let, for $c_0, c_1 \in C$, $d_C(c_0, c_1) = \inf\{2^{-n} : \forall k < n \,.\, \gamma_k(c_0) = \gamma_k(c_1)\}$. Then d_C is a pseudo-ultrametric on C, and d_C is an ultrametric if $\gamma_\omega : C \to T^\omega 1$ is injective. The Cantor space topology τ_C coincides with the topology induced by d_C, as studied in Barr (1993) and Worrell (2000).

In the remainder of the section we relate topological and logical notions. All of the results below are consequences of the following observations:

— The subsets expressible by single formulae form a basis.
— This basis is closed under complements (and finite unions).

The second point is due to the requirement that the logics are closed under boolean operators (Definition 6.1).

We shall often require our topologies to be compact and Hausdorff†. The relationship of these properties to logical issues becomes apparent in the context of final coalgebras in $\mathsf{Beh}_\omega(T)$.

Proposition 6.14. Suppose (K, κ) is final in $\mathsf{Beh}_\omega(T)$.

1 K is Hausdorff iff for all distinct $k_1, k_2 \in K$ there is $\varphi \in \mathscr{L}$ such that $k_1 \models_K \varphi$ and $k_2 \not\models_k \varphi$.
2 K is compact iff for all $\Phi \subseteq \mathscr{L}$ with $\Phi \models_K \varphi$ there is a finite subset $\Phi' \subseteq \Phi$ with $\Phi' \models \varphi$.

Logically speaking, K is Hausdorff iff $\mathscr{L}$ is expressive in the sense that every pair of different states can be separated by a formula. Compactness says that if φ is a consequence of a set Φ of formulae, there is a finite subset $\Phi' \subseteq \Phi$ such that Φ' already forces the validity of φ. For finitary logics with a sound and complete axiomatisation this is always the case, since a proof of φ from Φ can only use finitely many premises. Since we study logics without making any commitment to a particular syntax, this property is not guaranteed, and we have to require it for a number of results later in the paper.

The following are easy consequences of the definition of the topologies as generated by the semantics of modal formulae, where we call a T-coalgebra (C, γ) *logically compact* if every set Φ of formulae that is finitely satisfiable in (C, γ) (that is, for every finite subset $\Phi' \subseteq \Phi$ there exists $c \in C$ such that $c \models \Phi'$) is satisfiable in (C, γ) (that is, there exists $c \in C$ such that $c \models \Phi$).

Proposition 6.15. Let $(C, \gamma) \in \mathsf{Coalg}(T)$.

1 A subset of C is definable by a set of formulae iff it is closed with respect to τ_C.
2 If (C, τ_C) is compact, any clopen is expressible by a single formula.
3 (C, γ) is logically compact iff (C, τ_C) is topologically compact.

The proof is standard and therefore omitted. From a logical point of view, compactness corresponds to finiteness of proofs and is therefore not an issue for finitary logics that have a sound and complete axiomatisation. However, there are models that are not compact.

† A set is *compact* iff any open cover has a finite subcover. This is sometimes called quasi-compact. A space (X, τ) is *Hausdorff* iff $\forall x, y \in X \,.\, x \neq y \Rightarrow \exists\, U, V \in \tau \,.\, x \in U \wedge y \in V \wedge U \cap V = \varnothing$.

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Example 6.16. Let $TX = D \times X$ and consider the final coalgebra (Z, ζ) given by $Z = D^\omega$.

1 (Z, ζ) is compact in the Cantor space topology iff D is finite.
2 Suppose $D = \{a, b\}$. Then examples of non-compact coalgebras are given by the carriers $Z \setminus \{b^\omega\}$ and $\{s \cdot a^\omega : s \in \{a, b\}^*\}$ (and inheriting the structure from ζ).

Example 6.17. Let $TX = \{a, b\} \times X + 1$ and consider the final coalgebra (Z, ζ) with $Z = \{a, b\}^* \cup \{a, b\}^\omega$. Then Z is compact in the Cantor space topology (since the limit of compact Hausdorff spaces is compact Hausdorff, see Engelking (1989, 3.2.13)) and $\{a, b\}^*$ is not compact. The topology on Z is as follows. A subset of Z is open iff it is a subset of $\{a, b\}^*$ or of the form $V \cdot (\{a, b\}^* + \{a, b\}^\omega)$ for some $V \subseteq \{a, b\}^*$. In particular, every open cover of $\{a, b\}^\omega$ also covers $\{a, b\}^*$.

Another example where the final coalgebra is not compact is obtained for $TX = \mathscr{P}_\omega(X)$ by applying Proposition 6.15 (3).

Example 6.18. For finitely branching Kripke structures, that is, $T = \mathscr{P}_\omega$, it is not difficult to write down formulae φ_n that force any point satisfying φ_n to have at least n successors. The set $\Phi = \{\varphi_n \mid n < \omega\}$ is then finitely satisfiable, but not satisfiable by a $\mathscr{P}_\omega$-coalgebra.

7. Compactness for logics of rank ω

It is well known that (standard) model logic is compact. Generalising to coalgebras compactness may fail, for example, in the case of image-finite Kripke models (Example 6.18). Hence we are drawn to investigate sufficient and necessary conditions for the compactness theorem to hold.

Extending the terminology we introduced in Section 6.2 for the level of models, we say a set $\Phi \subseteq \mathscr{L}$ is *satisfiable* if there exists a T-coalgebra (C, γ) such that Φ is satisfiable in (C, γ). We say Φ is *finitely satisfiable* if every finite subset of Φ is satisfiable. Finally, a logic $\mathscr{L}$ is *compact* if every finitely satisfiable set of formulae is satisfiable. Using this terminology, we are in a position to present the first version of the compactness theorem.

Theorem 7.1. Suppose $\mathscr{L}$ is of rank ω. Then $\mathscr{L}$ is compact iff $\mathsf{Beh}_\omega(T)$ has a compact final object.

Proof.

— *Only if* : By Theorem 5.4, there exists a final object $(K, \kappa) \in \mathsf{Beh}_\omega(T)$. We show that (K, κ) is logically compact, from which the result then follows by Proposition 6.15. So, suppose $\Phi \subseteq \mathscr{L}$ is finitely satisfiable in (K, κ). By compactness, Φ is satisfiable. Hence there is (C, γ) and $c \in C$ such that $c \models_C \Phi$. Since (K, κ) is final in $\mathsf{Beh}_\omega(T)$, there is a mapping $u : (C, \gamma) \to (K, \kappa) \in \mathsf{Beh}_\omega(T)$. By definition of morphisms in $\mathsf{Beh}_\omega(T)$, we obtain $u(c) \models_K \Phi$. Hence Φ is satisfiable in (K, κ).

— *If* : Let (K, κ) be compact and final in $\mathsf{Beh}_\omega(T)$ and suppose $\Phi \subseteq \mathscr{L}$ is finitely satisfiable. Then, by finality and the definition of morphisms in $\mathsf{Beh}_\omega(T)$, we have Φ is finitely satisfiable in (K, κ), and hence satisfiable in (K, κ) by compactness and Proposition 6.15. $\square$

We now proceed to characterise those endofunctors T for which $\mathsf{Beh}_\omega(T)$ has a compact final object. Concerning the logics, we need to impose the following condition.

Condition 7.2. The topologies τ_n are compact and Hausdorff.

Logically speaking, this condition says that, for a given a logic $\mathscr{L}$, the induced sub-logics $\mathscr{L}_n$ of formulae of finite rank are compact and expressive. See Section 6.2 for a brief discussion of compactness and the Hausdorff property in the context of logics.

It will turn out that $\mathsf{Beh}_\omega(T)$ has a compact final object iff T weakly preserves the limit of its final sequence up to ω. More precisely, we say that T *weakly preserves* the limit of the sequence $(T^n1)_{n\in\omega}$, if the cone $(TT^\omega 1, (Tp_n^\omega)_{n\in\omega})$ is weakly limiting†. We first show that the carrier of a compact final object in $\mathsf{Beh}_\omega(T)$ is isomorphic to $T^\omega 1$. This is the crucial step in our proof.

Lemma 7.3. Assume Condition 7.2. If (K,κ) is compact and final in $\mathsf{Beh}_\omega(T)$, then $\kappa_\omega : K \to T^\omega 1$ is iso.

Proof. It follows from the construction of (K,κ) that κ_ω, called m in Diagram (1), is injective. To see that κ_ω is surjective, consider $t \in T^\omega 1$. The elements of the set $\mathscr{S} = \{\kappa_n^{-1}(\{p_n^\omega(t)\}) \mid n \in \omega\}$ are closed (since one-element sets are closed in a Hausdorff space) and non-empty (this follows from Corollary 5.8). It follows from $\kappa_n^{-1}(\{p_n^\omega(t)\}) \cap \kappa_m^{-1}(\{p_m^\omega(t)\}) = \kappa_{min(n,m)}^{-1}(\{p_{min(n,m)}^\omega(t)\})$ that $\mathscr{S}$ has the finite intersection property. By compactness, there is $k \in \bigcap \mathscr{S}$. Since $\kappa_n(k) = p_n^\omega(t)$ for all $n \in \omega$, it follows that $\kappa(k) = t$. □

We are now able to prove our second compactness theorem showing that, under suitable hypotheses, a logic of finite rank is compact iff T weakly preserves the limit of $(T^n1)_{n<\omega}$.

Theorem 7.4. Let $\mathscr{L}$ be a logic of finite rank satisfying Condition 7.2. The final object of $\mathsf{Beh}_\omega(T)$ is compact iff T weakly preserves the limit of $(T^n1)_{n<\omega}$.

Proof. Observe that T weakly preserves the limit of $(T^n1)_{n<\omega}$ iff $p_\omega^{\omega+1}$ has a one-sided inverse i, $p_\omega^{\omega+1} \circ i = id_{T^\omega 1}$.

- ⇒Let (K,κ) be final and compact in $\mathsf{Beh}_\omega(T)$. Due to the lemma above, we can define $i = T\kappa_\omega \circ \kappa \circ \kappa_\omega^{-1}$. It remains to check that, indeed, $p_\omega^{\omega+1} \circ i = p_\omega^{\omega+1} \circ T\kappa_\omega \circ \kappa \circ \kappa_\omega^{-1} = \kappa_\omega \circ \kappa_\omega^{-1} = id_{T^\omega 1}$.
- ⇐Let $p_\omega^{\omega+1} \circ i = id_{T^\omega 1}$. It was shown in Corollary 5.6 that $(T^\omega 1, i)$ is final in $\mathsf{Beh}_\omega(T)$. It is compact since $T^\omega 1$ is the limit of compact Hausdorff spaces and the induced topology on a limit of compact Hausdorff spaces is compact Hausdorff (Engelking 1989, 3.2.13). □

Remark 7.5. An inspection of the proof shows that '⇒' also holds for logics of rank ω. Moreover, for '⇒', we can weaken Condition 7.2 and only require that elements of T^n1, $n < \omega$, are closed with respect to τ_n. On the other hand, '⇐' does not hold for logics of rank

† A weak limit is defined like a limit, but the mediating morphism need not be unique.

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ω, as can be seen in the example of $\mathscr{LTL}$ (Section 3.2). Indeed, $\{\bigcirc^n p \mid n < \omega\} \cup \{\neg\Box p\}$ is finitely satisfiable but not satisfiable.

For the Cantor space topology, we have the following corollary.

Corollary 7.6. Let T map finite sets to finite sets. The final object of $\mathsf{Beh}_\omega(T)$ is compact in the Cantor space topology iff T weakly preserves the limit of $(T^n 1)_{n\in\mathbb{N}}$.

Note that in this case Condition 7.2 is automatically satisfied.

8. Definability for logics of rank ω

In this section we prove a characterisation result for classes of coalgebras definable by logics of rank ω. The main idea is again to replace $\mathsf{Coalg}(T)$ by $\mathsf{Beh}_\omega(T)$ and to reuse well-known techniques†. We begin by relating morphisms of $\mathsf{Beh}_\omega(T)$ and $\mathsf{Coalg}(T)$-morphisms.

Proposition 8.1. For any injective $\mathsf{Beh}_\omega(T)$-morphism $m : (C,\gamma) \to (D,\delta)$ there is δ' such that $m : (C,\gamma) \to (D,\delta')$ is a $\mathsf{Coalg}(T)$-morphism and $id_D : (D,\delta') \to (D,\delta)$ is a $\mathsf{Beh}_\omega(T)$-morphism.

Proof. Let L be the image of m, $m_0 : C \to Lp$ be the induced mapping, and $R = D \setminus L$. Define $\lambda : L \to TD$ as $Tm \circ \gamma \circ m_0^{-1}$ and $\delta' : D \cong L + R \to TD$ as $[\lambda, \delta \circ inr]$. Then m is a $\mathsf{Coalg}(T)$-morphism since $\delta' \circ m = \lambda \circ m_0 = Tm \circ \gamma$. To see that id_D is a $\mathsf{Beh}_\omega(T)$-morphism, assume $\delta'_n = \delta_n$ and consider the following two cases:

— For $d \in L$, we have $\delta'_{n+1}(d) = \gamma_{n+1}(m^{-1}(d)) = \delta_{n+1}(m(m^{-1}(d))) = \delta_{n+1}(d)$.
— For $d \in R$, we have $\delta'_{n+1}(d) = T\delta'_n \circ \delta'(d) = T\delta_n \circ \delta(d) = \delta_{n+1}(d)$. □

In addition to the classical closure operators, we need a further one to account for the restricted expressiveness of logics of rank ω.

Definition 8.2. Let $\mathscr{L}$ be a logic and (K,κ) be the final object in $\mathsf{Beh}_\omega(T)$. Define a relation $\sim^{\mathscr{L}}_\omega$ on coalgebras via

$$(C,\gamma) \sim^{\mathscr{L}}_\omega (D,\delta) \iff \mathsf{cl}(!_C(C)) = \mathsf{cl}(!_D(D))$$

where ! denotes the morphisms given by finality of (K,κ) and cl denotes the topological closure with respect to (K,τ_K) (Definition 6.6).

The theorem below parallels the definability theorem for infinitary modal logics, but adds $\sim^{\mathscr{L}}_\omega$ to the closure operators.

Theorem 8.3. Let $\mathscr{L}$ be a logic of rank ω. A class $\mathscr{B}$ of T-coalgebras is definable by a set of formulae iff $\mathscr{B}$ is closed under coproducts, subcoalgebras, and $\sim^{\mathscr{L}}_\omega$.

† See, for example, Adámek *et. al.* (1990, Chapter 16) for a textbook presentation and Kurz (2000, Chapter 2) for applications to modal logic.

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Proof. Since $\mathscr{L}$ is of rank ω, Proposition 6.4 shows that $\mathscr{L}$ can be represented by $[\![\cdot]\!]_K : \mathscr{L} \to \mathscr{P}K$, where (K, κ) denotes the final object of $\mathsf{Beh}_\omega(T)$. Recall that $(C, \gamma) \models \varphi \Leftrightarrow\ !_C(C) \subseteq [\![\varphi]\!]$ where $!_C$ is the morphism given by finality. The 'only if' direction follows easily from this observation (for closure under $\sim_\omega^{\mathscr{L}}$ recall Proposition 6.15).

For the 'if' direction, note first that the assumed closure conditions imply:

— $(C, \gamma) \to (D, \delta)$ is a surjective $\mathsf{Beh}_\omega(T)$-morphism only if $(C, \gamma) \in \mathscr{B} \Leftrightarrow (D, \delta) \in \mathscr{B}$; and
— $(C, \gamma) \to (D, \delta)$ is a $\mathsf{Beh}_\omega(T)$-morphism only if $(D, \delta) \in \mathscr{B} \Rightarrow (C, \gamma) \in \mathscr{B}$ (use Proposition 8.1); and
— for a class $\{f_i : (C, \gamma_i) \to (D, \delta) \mid i \in I\}$ of $\mathsf{Beh}_\omega(T)$-morphisms with $(C_i, \gamma_i) \in \mathscr{B}$, the union of the images of the f_i carries a coalgebra structure and is in $\mathscr{B}$ (use Proposition 4.7).

Let (S, σ) be the coalgebra given by the union of the images of all $!_D : (D, \delta) \to (K, \kappa)$, $(D, \delta) \in \mathscr{B}$. By Proposition 6.15, cl is expressible in $\mathscr{L}$ by a set of formulae Φ. We show that $\mathscr{B} = \mathrm{Mod}(\Phi)$. For $(D, \delta) \in \mathscr{B}$ we have, by the definition of S, $(D, \delta) \models \Phi$. To show $\mathscr{B} \supseteq \mathrm{Mod}(\Phi)$, define $(\bar{S}, \bar{\sigma})$ as the largest subcoalgebra of $\mathsf{cl}(S)$. Since $S \subseteq \bar{S} \subseteq \mathsf{cl}(S)$, it follows that $\mathsf{cl}(S) = \mathsf{cl}(\bar{S})$, and hence $(\bar{S}, \bar{\sigma}) \sim_\omega^{\mathscr{L}} (S, \sigma)$. Since $\mathscr{B}$ is closed under images and coproducts, $\mathscr{B}$ is also closed under unions, so $(S, \sigma) \in \mathscr{B}$, and hence $(\bar{S}, \bar{\sigma}) \in \mathscr{B}$. Now assume $(C, \gamma) \models \Phi$, that is, $!_C(C) \subseteq \mathsf{cl}(S)$, and hence $!_C(C) \subseteq \bar{S}$, that is, there is a morphism $(C, \gamma) \to (\bar{S}, \bar{\sigma})$. Since $\mathscr{B}$ is closed under domains of morphisms, $(C, \gamma) \in \mathscr{B}$. □

For the Cantor space, we obtain the following corollary.

Corollary 8.4. Let $\mathscr{L}$ be a finite step expressive logic of finite rank. A class $\mathscr{B}$ of T-coalgebras is definable by a set of formulae iff $\mathscr{B}$ is closed under coproducts, subcoalgebras, and $\sim_{<\omega}$.

9. Conclusions and related work

We have studied definability and compactness for finitary coalgebraic modal logic. The main instrument through which finitary logics have been studied is the terminal sequence and the shift from the category $\mathsf{Coalg}(T)$ to the category $\mathsf{Beh}_\omega(T)$.

In this category, points (or states) can be distinguished iff their *finite* behaviour differs. Also, $\mathsf{Beh}_\omega(T)$ provides the correct framework in which the construction of canonical models can be generalised to a coalgebraic setting. The main handle that allows us to formalise the finitary character of the logics considered is to identify finitary predicates with subsets of T^n1, where n is a finite ordinal. The idea of interpreting formulae on the elements T^n1 of the terminal sequence has already been used in Pattinson (2001). The same idea (without the restriction to finite ordinals) also prevails in Moss (1999). There, formulae are constructed using infinitary conjunctions (which do not change the degree of the formulae) and the application of the signature functor T (increasing the degree of the constructed formulae by 1).

The signature functors (and hence the logics) that have been discussed in the present paper are all one-sorted. The passage to multi-sorted signatures, that is, endofunctors

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$\mathsf{Set}^n \to \mathsf{Set}^n$ is standard and allows us to include the logics discussed in Rößiger (2000a) and Jacobs (2001a), which also rely on (syntactically defined) predicate liftings. Since the endofunctors discussed in *loc. cit*. are all ω-accessible, final coalgebras and canonical models coincide for these logics (which is also reflected by the fact that they are strong enough to characterise behavioural equivalence).

A coalgebraic representation of the Cantor discontinuum was also given in Pavlovic and Pratt (2000) in the category of posets. However, the cantor space topology discussed in the present paper arises in a different way: we start with a final coalgebra on the category of sets, which is then equipped with a natural topology.

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