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Two-parameter right definite Sturm–Liouville problems with eigenparameter-dependent boundary conditions*

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Linked equations

$$-(p_i y_i')' + q_i y_i = \sum_{j=1}^{2} \lambda_j r_{ij} y_i, \quad i = 1, 2,$$

are studied on $[0, 1]$ subject to boundary conditions of the form

$$y_i(0)\cos\alpha_i = (p_i y_i')(0)\sin\alpha_i,$$
$$(a_i\lambda_i + b_i)y_i(1) = (c_i\lambda_i + d_i)(p_i y_i')(1).$$

Results are given on existence, location, asymptotics and perturbation of the eigenvalues λ_j and oscillation of the eigenfunctions y_i.

1. Introduction

One of the central results for the Sturm–Liouville equation

$$-(py')' + qy = \lambda ry \quad \text{on } [0, 1], \tag{1.1}$$

with p, q, r continuous and $p, r > 0$ subject to separated end conditions

$$y(0)\cos\alpha = (py')(0)\sin\alpha, \tag{1.2}$$
$$y(1)\cos\beta = (py')(1)\sin\beta, \tag{1.3}$$

is Sturm's oscillation theorem. In full generality this is an existence and uniqueness result for eigenvalues $\lambda = \lambda^n$ and (up to scalar multiples) the eigenfunctions y^n, the λ^n being ordered $\lambda^0 < \lambda^1 < \cdots$, accumulating at ∞, and the y^n possessing n zeros in $(0, 1)$ (see [8, § 8.3]). Of many related results we mention continuous and

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monotonic dependence on parameters and asymptotic expansions in n as relevant to the discussion below. We refer to such (oscillation, comparison, etc.) results as 'Sturm theory'.

Equation (1.1) has also been investigated subject to λ-dependent boundary conditions. See [11, 18] for extensive bibliographies on the case where (1.3) is replaced by

$$(a\lambda + b)y(1) = (c\lambda + d)(py')(1). \tag{1.4}$$

More recent reference lists, for what remains an active research area, can be found in [9] and [12]. Most authors have studied completeness and expansion theory, the extension of Sturm theory to (1.4) being comparatively recent. In the case $\delta = ad - bc > 0$, a modified oscillation theorem holds, where there is a unique eigenvalue for each oscillation count (except one, where there are two distinct eigenvalues); there are also parametric dependence results and asymptotic expansions (see [6] and § 2 below). We remark that (1.1), (1.2), (1.4) can be cast as an abstract equation $Ax = \lambda Bx$, where $B > 0$ provided $r > 0$ and $\delta > 0$ (cf. [18]), but alternative definiteness conditions are possible (cf. [4]).

Another generalization of Sturm theory is to linked two-parameter equations of the form,

$$-(p_i y_i')' + q_i y_i = \sum_{j=1}^{2} \lambda_j r_{ij} y_i \quad \text{on } [0,1], \quad i = 1, 2, \tag{1.5}$$

again subject to separated end conditions,

$$y_i(0)\cos\alpha_i = (p_i y_i')(0)\sin\alpha_i, \tag{1.6}$$

$$y_i(1)\cos\beta_i = (p_i y_i')(1)\sin\beta_i, \tag{1.7}$$

Under the condition $\det R > 0$ ($R = [r_{ij}]$), which is known as right definiteness (RD), Klein's oscillation theorem states that for each non-negative integer pair $\boldsymbol{n} = (n_1, n_2)$ there is a unique eigenvalue $\boldsymbol{\lambda}^{\boldsymbol{n}}$ in $\mathbb{R}^2$ and (up to scalar multiples) a unique pair of eigenfunctions $y_i^{\boldsymbol{n}}$ with n_i zeros in $(0, 1)$. A special case was proved by Klein, the general one (for continuous coefficients) by Ince [15]. Asymptotics are discussed in, for example, [7, 11, 16], parameter dependence in [2]. For weaker conditions on the coefficients, alternative definiteness conditions and completeness and expansion theory we refer to [17]. We remark that many of these works treat problems involving more than two parameters.

So far there seems to be no analysis of multiparameter Sturm theory with $\boldsymbol{\lambda}$-dependent boundary conditions, and it is our aim to start such a theory by considering a special case. We study (1.5) subject to (1.6) and

$$(a_i\lambda_i + b_i)y_i(1) = (c_i\lambda_i + d_i)(p_i y_i')(1), \quad i = 1, 2, \tag{1.8}$$

which generalizes (1.4). As an application of such a system, we can consider the following generalization of a problem from [13]. Fulton and Pruess discuss the temperature of a cylindrical bar immersed in liquid, with imperfect thermal contact. They obtain a singular one-parameter problem with a boundary condition of the form (1.4), for a cylinder of circular cross-section, after separating out the angular variable in polar coordinates. If we allow a general elliptical cross-section, then it

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is natural to use elliptical coordinates (cf. [5]) and then a corresponding separation yields a regular system of the form (1.5), (1.6) and (1.8).

In §§ 2 and 3 we study the eigencurves for (1.5) for each fixed i and we obtain expressions for the derivative $\mathrm{d}\lambda_2/\mathrm{d}\lambda_1$ along the eigencurves, and certain asymptotics. In § 4 we give the basic existence and uniqueness theorem for eigenvalues $\boldsymbol{\lambda}^n$ and we obtain an oscillation theorem (theorem 4.4) which generalizes all those mentioned above. In § 5 we refine the analysis, showing how to locate the $\boldsymbol{\lambda}^n$ in certain cones and establishing asymptotic and perturbation results. We conclude with an illustrative example.

2. Preliminaries

In this section we shall discuss existence and perturbation results for eigenvalues of Sturm–Liouville differential equations on the unit interval $[0,1]$ involving two eigenparameters λ_1 and λ_2, one equation at a time. We assume that the coefficients q_i and r_{ij} are real-valued continuous functions. By a transformation of the independent variable, we can assume without loss of generality that p_1 and p_2 are identically 1 (see [6, Appendix]). Then the differential equations become

$$-y_i'' + q_i y_i = (\lambda_1 r_{i1} + \lambda_2 r_{i2}) y_i, \tag{2.1}$$

with boundary conditions

$$\frac{y_i'}{y_i}(0) = \cot\alpha_i \quad \text{and} \quad \frac{y_i'}{y_i}(1) = \frac{a_i\lambda_i + b_i}{c_i\lambda_i + d_i}, \tag{2.2}$$

where $i = 1, 2$.

The angles α_1, α_2 are given constants in $(0,\pi)$ and a_i, b_i, c_i, d_i are real numbers satisfying $\delta_i = a_i d_i - b_i c_i > 0$ and $c_i \neq 0$ for each i. (If $\delta_i = 0$ or $c_i = 0$, analogous but slightly different results hold, cf. [6, § 5] for the one-parameter case.) By changing the $\boldsymbol{\lambda}$ origin to $(-d_1/c_1, -d_2/c_2)$, we may (and shall) assume in what follows that $d_1 = d_2 = 0$. Note that the above reductions (to $p_i = 1$, $d_i = 0$) do not affect the continuity of q_i and r_{ij}. We shall use r_{ij} also to denote the corresponding quadratic forms on $L_2[0,1]$, so

$$r_{ij}(y) = \int_0^1 r_{ij}|y|^2.$$

We shall assume the RD condition,

$$\det\begin{pmatrix} r_{11}(y_1) & r_{12}(y_1) \\ r_{21}(y_2) & r_{22}(y_2) \end{pmatrix} > 0 \quad \text{for all } y_1, y_2 \in L_2[0,1].$$

By continuity of r_{ij}, the determinant has a positive lower bound, so the RD is in fact uniform. It follows that the multiplication operator induced by r_{ij} can be given a prescribed sign, after an invertible linear eigenvalue transformation (see [1, lemma 2.3]). In the sequel we shall assume that this transformation has been performed in such a way that $(-1)^{i+j} r_{ij}(y) > 0$ for all $y \in L_2[0,1]$. By virtue of the continuity of r_{ij}, this is equivalent to $(-1)^{i+j} r_{ij}(x) > 0$ for $0 \leqslant x \leqslant 1$.

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Lemma 2.1. *Given the system (2.1), (2.2), we have two sequences* $\{\lambda^1_{2n}\}$ *and* $\{\lambda^2_{2n}\}$ *of continuous monotone increasing functions of* λ_1 *and two sequences of eigenfunctions* y_{1n} *and* y_{2n} *such that, for each integer* $n \geqslant 0$, *the pair* $(\lambda_1, \lambda^i_{2n}(\lambda_1))$ *and the function* y_{in} *satisfy the equations (2.1), (2.2). Moreover,* $\lambda^1_{20}(\lambda_1) > \lambda^1_{21}(\lambda_1) > \cdots$ *and* $\lambda^2_{20}(\lambda_1) < \lambda^2_{21}(\lambda_1) < \cdots$. *There are natural numbers* N_1 *and* N_2 *depending on* λ_2 *and* λ_1, *respectively, such that* y_{in} *has* n *zeros in* $(0,1)$ *for* $n \leqslant N_i$ *and* $n-1$ *zeros for* $n > N_i$, $i = 1, 2$.

Proof. Our method will rely on parametrized one-parameter equations. A one-parameter Sturm–Liouville differential equation $-y'' + qy = \lambda r y$ with boundary conditions

$$\frac{y'}{y}(0) = \cot\alpha \text{ (const.)} \quad \text{and} \quad \frac{y'}{y}(1) = \frac{a\lambda + b}{c\lambda + d}$$

is said to be *parametrized* if the coefficient functions q and r and the constants a, b, c, d involved in the boundary condition depend on a parameter t. For parametrized one-parameter equations, existence of eigenvalues, oscillation of eigenfunctions and variation of eigenvalues with respect to the parameter t were established in [6, theorems 3.1 and 3.2]. In the present context, we rewrite the second equation in (2.1) as

$$-y_2'' + (q_2 - \lambda_1 r_{21})y_2 = \lambda_2 r_{22} y_2,$$

with boundary conditions

$$\frac{y_2'}{y_2}(0) = \cot\alpha_2 \quad \text{and} \quad \frac{y_2'}{y_2}(1) = \frac{a_2\lambda_2 + b_2}{c_2\lambda_2 + d_2}.$$

This Sturm–Liouville problem involving one eigenparameter λ_2 is parametrized by λ_1. Only the coefficient of y_2 depends on the parameter and, since $r_{21}(x) < 0$, the function $q_2 - \lambda_1 r_{21}$ is increasing in λ_1. Then the above-mentioned two theorems imply that, for each λ_1, the eigenvalues $\lambda_{2n}(\lambda_1)$ can be ordered as

$$\lambda_{20}(\lambda_1) < \lambda_{21}(\lambda_1) < \lambda_{22}(\lambda_1) < \cdots$$

and there exists an integer $N_2 = N_2(\lambda_1)$ defined by

$$\lambda_{2,N_2-1} < 0 \leqslant \lambda_{2,N_2} \tag{2.3}$$

such that y_{2n} has n zeros for $n \leqslant N_2$ and $n-1$ zeros for $n > N_2$. Moreover, λ_{2n} is a continuous increasing function of λ_1 for each n.

Applying the same analysis to the second equation of (2.1), this time with λ_2 as the parameter, we get the existence of parametrized eigenvalues $\lambda_{10}(\lambda_2) < \lambda_{11}(\lambda_2) < \cdots$ and corresponding eigenfunctions y_{1n} which have the same properties as above. By virtue of the fact that $\lambda_{1n}(\lambda_2)$ is a continuous increasing function of λ_2, we can consider the inverse function which we denote by $\lambda^1_{2n}(\lambda_1)$ and which will satisfy $\lambda^1_{20}(\lambda_1) > \lambda^1_{21}(\lambda_1) > \cdots$. The superscript 1 denotes the fact that it is obtained from the first equation. Similarly, $\lambda_{2n}(\lambda_1)$, the parametrized eigenvalues obtained from the second equation, will henceforth be denoted by $\lambda^2_{2n}(\lambda_1)$. The integer N_1 in this case depends on λ_2 and satisfies the analogue of (2.3). Thus we have the result. □

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The next result is about continuous dependence of the eigenfunctions on the parameter. It will be an important tool for differentiating λ^i_{2n}, but has some independent interest also. The norm of any eigenfunction y will always mean the $L_2[0,1]$ norm

$$\|y\| = \left(\int_0^1 |y|^2\right)^{1/2}.$$

LEMMA 2.2. *Let $n \geqslant 0$ be any integer, y_{in} be as defined above with $\|y_{in}\| = 1$ and K_1 be a compact subset of $\mathbb{R}$. Then, for each x in $[0,1]$ and λ_1 in K_1,*

$$y_{in}(x, \lambda_1, \lambda^i_{2n}(\lambda_1)) \quad \textit{and} \quad y'_{in}(x, \lambda_1, \lambda^i_{2n}(\lambda_1))$$

are continuous functions of λ_1. Similarly, if K_2 is a compact subset of $\mathbb{R}$, then

$$y_{in}(x, (\lambda^i_{2n})^{-1}(\lambda_2), \lambda_2) \quad \textit{and} \quad y'_{in}(x, (\lambda^i_{2n})^{-1}(\lambda_2), \lambda_2)$$

are continuous functions of λ_2 in K_2 for each $x \in [0,1]$.

Proof. Let

$$Y_i = \begin{pmatrix} y_i \\ y'_i \end{pmatrix} \quad \text{and} \quad A_i(x, \lambda_1) = \begin{pmatrix} 0 & 1 \\ q_i - \lambda_1 r_{i1} - \lambda^i_{2n}(\lambda_1) r_{i2} & 0 \end{pmatrix}.$$

Then Y_i is a solution of

$$Y' = A_i(x, \lambda_1)Y.$$

Note that A_i is a continuous function of both x and λ_1. Thus, for λ_1 lying in a compact subset K, $\|A_i(x, \lambda_1)\|$ has an upper bound (which may depend on x). This will imply that the function $f_{\lambda_1} : \mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$f_{\lambda_1}(x, Y) = A_i(x, \lambda_1)Y \quad \text{for } x \in [0,1], \quad Y \in \mathbb{R}^2,$$

is *Lipschitz* in Y for any fixed x and the Lipschitz constant is independent of $\lambda_1 \in K$. Let

$$Z_1 = \begin{pmatrix} z_1 \\ z'_1 \end{pmatrix} \quad \text{and} \quad Z_2 = \begin{pmatrix} z_2 \\ z'_2 \end{pmatrix}$$

be two fundamental solutions of the differential equation $Y' = f_{\lambda_1}(x, Y)$ subject to boundary conditions $[Z_1(0)\ Z_2(0)] = I_2$. It is a standard result of the theory of linear differential equations that in such a case the solutions $Z_1(x)$ and $Z_2(x)$ depend continuously on the parameter λ_1 (see, for example, [14, theorem 3.2]). If $y_{in}(0) = 0$, then $y_{in} = z_2/\|z_2\|$ and if $y_{in}(0) \neq 0$, then $y_{in} = (z_1 + (\cot\alpha_i)z_2)/\|z_1 + (\cot\alpha_i)z_2\|$. Since norm is a continuous function, $y_{in}(x)$ and, similarly, $y'_{in}(x)$ are continuous functions of λ_1 on K_1. □

3. Eigenvalues for one equation in two parameters

The graphs of the functions λ^i_{2n} given by lemma 2.1 will be called the eigencurves of the ith equation (2.1). The first result of this section gives us the slopes of these eigencurves $\lambda^i_{2n}(\lambda_1)$ for $i = 1, 2$.

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THEOREM 3.1. *For $i = 1, 2$, let λ^i_{2n} and y_{in} be as above. Then*

$$\frac{\mathrm{d}\lambda^1_{2n}}{\mathrm{d}\lambda_1} = -(r_{12}(y_{1n}))^{-1}\biggl(\frac{\delta_1 y_{1n}(1)^2}{(c_1\lambda_1 + d_1)^2} + r_{11}(y_{2n})\biggr), \tag{3.1}$$

$$\frac{\mathrm{d}\lambda^2_{2n}}{\mathrm{d}\lambda_1} = -r_{21}(y_{2n})^{-1}\biggl(\frac{\delta_2 y_{2n}(1)^2}{(c_2\lambda^2_{2n}(\lambda_1) + d_2)^2} + r_{22}(y_{2n})\biggr)^{-1}. \tag{3.2}$$

Proof. For simplicity of notation, we fix n and suppress it. To calculate the derivative of λ^1_2, we start with (2.1) in the form,

$$-y_1'' + q_1 y_1 = (\lambda_1 r_{11} + \lambda^1_2(\lambda_1) r_{12}) y_1.$$

Choose any $\epsilon > 0$ and let z_1 be the eigenfunction corresponding to $\lambda_1 + \epsilon$, i.e.

$$-z_1'' + q_1 z_1 = ((\lambda_1 + \epsilon) r_{11} + \lambda^1_2(\lambda_1 + \epsilon) r_{12}) z_1.$$

Multiplying the first equation by z_1 and the second by y_1 and subtracting, we obtain

$$-z_1'' y_1 + y_1'' z_1 = (\epsilon r_{11} + (\lambda^1_2(\lambda_1 + \epsilon) - \lambda^1_2(\lambda_1)) r_{12}) y_1 z_1,$$

which yields

$$\frac{1}{\epsilon}(y_1' z_1 - y_1 z_1')|_0^1 = \int r_{11} y_1 z_1 + \frac{1}{\epsilon}(\lambda^1_2(\lambda_1 + \epsilon) - \lambda^1_2(\lambda_1)) \int r_{12} y_1 z_1. \tag{3.3}$$

Using the continuity established in the last lemma, we have

$$\lim_{\epsilon \to 0} z_1 = y_1.$$

So, in the limit, the right-hand side of (3.3) is

$$r_{11}(y_1) + \frac{\mathrm{d}\lambda^1_2}{\mathrm{d}\lambda_1}(\lambda_1) r_{12}(y_1).$$

Recall that y_1 and z_1 satisfy the same boundary condition at 0, but not at 1, where they are as follows:

$$\frac{y_1'}{y_1}(1) = \frac{a_1\lambda_1 + b_1}{c_1\lambda_1 + d_1} \quad \text{and} \quad \frac{z_1'}{z_1}(1) = \frac{a_1(\lambda_1 + \epsilon) + b_1}{c_1(\lambda_1 + \epsilon) + d_1}.$$

Thus the left-hand side of (3.3) is

$$\frac{1}{\epsilon}\biggl(-y_1(1) z_1(1)\biggl(\frac{a_1(\lambda_1 + \epsilon) + b_1}{c_1(\lambda_1 + \epsilon) + d_1} - \frac{a_1\lambda_1 + b_1}{c_1\lambda_1 + d_1}\biggr)\biggr),$$

which in the limit tends to

$$-y_1(1)^2 \frac{\delta_1}{(c_1\lambda_1 + d_1)^2}.$$

Thus

$$-y_1(1)^2 \frac{\delta_1}{(c_1\lambda_1 + d_1)^2} = r_{11}(y_1) + \frac{\mathrm{d}\lambda^1_2}{\mathrm{d}\lambda_1}(\lambda_1) r_{12}(y_1),$$

and the desired result follows.

For the other derivative, one has to carry out the same analysis with the roles of λ_1 and λ_2 interchanged and then take the reciprocal. This is due to the fact that the boundary condition in this case is λ_2 dependent. □

REMARK 3.2. Our analysis so far has depended on the signs of the r_{ij}, but not on the values of δ_i. In the case of eigenparameter-independent boundary conditions, $a_i = c_i = 0$ so that one has $\delta_i = 0$. Then the derivatives in (3.1) and (3.2) simplify to

$$\frac{\mathrm{d}\lambda_{2n}^1}{\mathrm{d}\lambda_1} = -\frac{r_{11}(y_{1n})}{r_{12}(y_{1n})} \quad \text{and} \quad \frac{\mathrm{d}\lambda_{2n}^2}{\mathrm{d}\lambda_1} = -\frac{r_{21}(y_{2n})}{r_{22}(y_{2n})}.$$

To conclude this section we give two asymptotic results which will also be used to analyse the eigencurves in subsequent sections. In the next result we do use $\delta_i > 0$.

LEMMA 3.3. *$\lambda_{20}^2(\lambda_1)$ is always negative and $\lim_{\lambda_1\to\infty}\lambda_{20}^2(\lambda_1) = 0$. On the other hand, the graph of λ_{20}^1 lies in the left half-plane and $\lim_{\lambda_1\nearrow 0}\lambda_{20}^1(\lambda_1) = \infty$.*

Proof. For any given λ_1, $\mu = \cot\theta_2(1,\lambda_1,\lambda_2)$ decreases continuously on its leftmost branch B_0 (see [6] for more details). Thus the intersection of B_0 with the hyperbola $(a_2\lambda_2+b_2)/c_2\lambda_2$ is in the left half-plane. It follows that $\lambda_{20}^2(\lambda_1) < 0$, and since λ_{20}^2 is increasing, we can let $\lim_{\lambda_1\to\infty}\lambda_{20}^2(\lambda_1) = l$. To show that $l = 0$, it is enough to show that $\lim_{\lambda_1\to\infty}\cot\theta_2(1,\lambda_1,\lambda_{20}^2(\lambda_1)) = \infty$. Choose $\eta > 0$ such that $\eta < \pi - \alpha_2$ and $2\eta \leqslant \pi$. Then, for $\eta \leqslant \theta_2 \leqslant \pi - \eta$ and $\lambda_1 > 0$, we have $\lambda_1 r_{21}\sin^2\theta \leqslant \lambda_1 r_{21}\sin^2\eta$, $\lambda_{20}^2(\lambda_1)r_{22}\sin^2\theta \leqslant \lambda_1 r_{22}\sin^2\eta$ and $q_2\sin^2\theta \leqslant |q_2|\sin^2\theta$, so

$$\begin{aligned}\theta_2' &= \cos^2\theta_2 + (\lambda_1 r_{21} + \lambda_{20}^2(\lambda_1)r_{22} - q_2)\sin^2\theta_2 \\ &< 1 + (\lambda_1 r_{21} + l r_{22})\sin^2\eta + |q_2|.\end{aligned}$$

Thus $\theta_2' < 0$ for large λ_1 at $\theta_2 = \eta$, whence $\theta_2 \leqslant \eta$. Since η was arbitrary, we are done. The proof for the other assertion is similar. □

LEMMA 3.4. *If*

$$M_i = \sup\{-r_{i1}(x)/r_{i2}(x) : 0 \leqslant x \leqslant 1\}$$

and

$$m_i = \inf\{-r_{i1}(x)/r_{i2}(x) : 0 \leqslant x \leqslant 1\}$$

for $i = 1, 2$, then $M_1 < \infty$ and $m_2 > 0$. Moreover,

$$\lim_{\lambda_1\to\infty}\frac{\lambda_{2n}^2(\lambda_1)}{\lambda_1} = m_2 \quad \textit{for } n > 0$$

and

$$\lim_{\lambda_1\to\infty}\frac{\lambda_{2m}^1(\lambda_1)}{\lambda_1} = M_1 \quad \textit{for } m > 0.$$

Proof. $M_1 < \infty$ and $m_2 > 0$ because they are extrema of positive continuous functions on the compact unit interval.

For $i = 2$, let $\lambda^{2A}_{2n}(\lambda_1)$ be the eigenvalues of the equation (2.1) with asymptotic boundary conditions

$$\frac{y'(0)}{y(0)} = \cot\alpha_2 \quad \text{and} \quad \frac{y'(1)}{y(1)} = \frac{a_2}{c_2}.$$

Then

$$\lim_{\lambda_1\to\infty} \frac{\lambda^{2A}_{2n}(\lambda_1)}{\lambda_1} = m_2 \quad \text{by [3, theorem 3.1]}$$

and

$$\lambda^{2A}_{2(n-1)}(\lambda_1) < \lambda^2_{2n}(\lambda_1) < \lambda^{2A}_{2n}(\lambda_1) \quad \text{for } n > 0 \quad \text{by [6, theorem 3.3]}.$$

This gives the first result and the proof for the second one is similar. □

4. Existence and oscillation

Before examining the intersections of the eigencurves for the two simultaneous equations, we detail one more result about their separate behaviours. In the case where the boundary conditions are independent of $\boldsymbol{\lambda}$, it is a well-known fact (proved, for example, via the Prüfer angle) that eigenfunction oscillation counts are constant along eigencurves. In the present situation, this result fails, but the following is a suitable analogue.

LEMMA 4.1. *If $\boldsymbol{\lambda}$ is the nth eigencurve for the ith equation, i.e. $\lambda_2 = \lambda^i_{2n}(\lambda_1)$, then y_{in} has n (respectively, $n-1$) zeros in $]0,1[$ if $\lambda_i < 0$ (respectively, $\lambda_i \geqslant 0$).*

Proof. Consider $i = 2$. If λ_1 is such that $\lambda_2 = \lambda^2_{2n}(\lambda_1) < 0$, then $n < N_2(\lambda_1)$ by (2.3), so the oscillation count of y_{2n} is n by lemma 2.1. By the same reasoning, if $\lambda_2 \geqslant 0$, then $n \geqslant N_2(\lambda_1)$, so y_{2n} has $n-1$ zeros. The argument for $i = 1$ is similar. □

REMARK 4.2. This result is consistent with the continuity expressed in lemma 2.2, since the discontinuity in oscillation count occurs at $\lambda_i = 0$, and this is precisely where $y_{in}(1) = 0$.

THEOREM 4.3. *The system (2.1), (2.2) has countably many two-parameter eigenvalues. For each non-negative integer pair $\boldsymbol{n} = (n_1, n_2)$, there is a unique two-parameter eigenvalue $\boldsymbol{\lambda}^{\boldsymbol{n}}$ on the n_ith eigencurve of equation i ($i = 1, 2$).*

Proof. By virtue of the fact that $r_{ij}(y) < 0$ for $y \in L_2[0,1]$ and $i \neq j$, we see from the expressions of the derivatives of the eigencurves in theorem 3.1 that

$$\frac{\mathrm{d}\lambda^1_{2n_1}}{\mathrm{d}\lambda_1} \geqslant -\frac{r_{11}(y_{1n_1})}{r_{12}(y_{1n_1})} \quad \text{and} \quad \frac{\mathrm{d}\lambda^2_{2n_2}}{\mathrm{d}\lambda_1} \leqslant -\frac{r_{21}(y_{2n_2})}{r_{22}(y_{2n_2})} \quad \text{for all } n_1, n_2 \geqslant 0.$$

Since RD holds, there are constants γ and η so that

$$\frac{\mathrm{d}\lambda^1_{2n_1}}{\mathrm{d}\lambda_1} \geqslant \gamma > \eta \geqslant \frac{\mathrm{d}\lambda^2_{2n_2}}{\mathrm{d}\lambda_1} \quad \text{for all } n_1, n_2 \geqslant 0. \tag{4.1}$$

Thus, if we plot $\lambda^1_{2n_1}$ and $\lambda^2_{2n_2}$ against λ_1, then these two curves meet exactly once, say, at $\lambda^{\boldsymbol{n}}_1$. We denote the point $\lambda^1_{2n_1}(\lambda^{\boldsymbol{n}}_1) = \lambda^2_{2n_2}(\lambda^{\boldsymbol{n}}_1)$ by $\lambda^{\boldsymbol{n}}_2$. Then it follows

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from lemma 2.1 that $(\lambda_1^{\boldsymbol{n}}, \lambda_2^{\boldsymbol{n}})$ satisfy (2.1) and (2.2), with eigenfunctions given by $y_1(x) = y_{1n_1}(x, \boldsymbol{\lambda}^{\boldsymbol{n}})$ and $y_2(x) = y_{2n_2}(x, \boldsymbol{\lambda}^{\boldsymbol{n}})$. To complete the proof of uniqueness, we note that two eigencurves from the same equation (say, $i = 2$) cannot intersect. For if they did at $\boldsymbol{\lambda}$, say, then, by lemma 4.1, there would be two distinct oscillation counts (and hence two linearly independent eigenfunctions) corresponding to the same boundary value problem (2.1), (2.2) and this is impossible. □

Although the above result resembles Klein's oscillation theorem, it says nothing directly about eigenfunction oscillation. To obtain a genuine oscillation theorem, we proceed as follows. By the oscillation count of an eigenvalue $\boldsymbol{\lambda}$ of (2.1), (2.2), we mean the pair $\boldsymbol{n} = (n_1, n_2)$, where n_i is the number of zeros of y_i in $]0, 1[$. Thus each eigenvalue has a unique oscillation count and the following result addresses to what extent the converse is true. From now on, we regard the first quadrant Q_1 (respectively, third quadrant Q_3) as closed (respectively, open) and the second quadrant Q_2 (respectively, fourth quadrant Q_4) as containing the negative λ_1- (respectively, λ_2-) axis.

Theorem 4.4. *With the exceptions below, each oscillation count corresponds to one eigenvalue. Let*

$$M_1 = \min\{n_1 : \lambda_2^{(n_1,0)} \in Q_4 \text{ and } \lambda_2^{(n_1,1)} \in Q_1\}$$

and

$$M_2 = \min\{n_2 : \lambda_1^{(0,n_2)} \in Q_3 \text{ and } \lambda_1^{(1,n_2)} \in Q_1\}$$

(by lemmas 3.3, 3.4, M_1 and M_2 are well defined). Then we have the following.

(a) *For $n_1 \geqslant M_1$ and $n_2 \geqslant M_2$, each of the oscillation counts $(n_1, 0)$ and $(0, n_2)$ correspond to exactly two eigenvalues.*

(b) *For $n_1 < M_1$ and $n_2 < M_2$, the oscillation count $\boldsymbol{n} = (n_1, n_2)$ corresponds to at most four eigenvalues.*

Proof. The oscillation count of the eigenvalue $\boldsymbol{\lambda}^{\boldsymbol{n}}$ is $(n_1 - 1, n_2 - 1)$ (respectively, $(n_1, n_2 - 1)$, (n_1, n_2), $(n_1 - 1, n_2)$) if $\boldsymbol{\lambda} \in Q_1$ (respectively, Q_2, Q_3, Q_4). Thus an oscillation count can correspond to two or more eigenvalues only if they are in separate quadrants. For $n_1 \geqslant M_1$, the oscillation count $(n_1, 0)$ occurs twice—once each in the fourth and first quadrant corresponding to $\boldsymbol{\lambda}^{(n_1+1,0)}$ and $\boldsymbol{\lambda}^{(n_1+1,1)}$. Similarly, when $n_2 \geqslant M_2$, the oscillation count $(0, n_2)$ occurs once each in the second and first quadrants corresponding to $\boldsymbol{\lambda}^{(0,n_2+1)}$ and $\boldsymbol{\lambda}^{(1,n_2+1)}$.

Let $\varGamma^{\boldsymbol{n}}$ denote the curvilinear cell defined by the vertices

$$\boldsymbol{\lambda}^{\boldsymbol{n}}, \qquad \boldsymbol{\lambda}^{(n_1+1,n_2)}, \qquad \boldsymbol{\lambda}^{(n_1+1,n_2+1)}, \qquad \boldsymbol{\lambda}^{(n_1,n_2+1)}$$

and the corresponding eigencurve sections as edges. Since the repeated oscillation counts must correspond to the vertices of some cell, there can be at most four occurrences of a particular oscillation count. For $n_1 \geqslant M_1$ and $n_2 \geqslant M_2$, the cell $\varGamma^{\boldsymbol{n}}$ is contained in the first quadrant. So, except as in (a) above, repetitions can occur only for $n_1 < M_1$ and $n_2 < M_2$. □

Remark 4.5. Let us discuss case (b) above in more detail. The cell $\Gamma^{\boldsymbol{n}}$ contains two eigencurve segments corresponding to equation 1 (respectively, 2) of (2.1), (2.2) and we refer to them as 1-edges (respectively, 2-edges). By (4.1), 1-edges have steeper slope than 2-edges. Whenever there are two adjacent vertices joined by an i-edge of $\Gamma^{\boldsymbol{n}}$ in adjacent quadrants separated by $\lambda_i = 0$, those two vertices have the same oscillation count. Thus the oscillation count (n_1, n_2) corresponds to four eigenvalues if and only if all four vertices of $\Gamma^{\boldsymbol{n}}$ are in separate quadrants. (This forces $(0,0) \in \Gamma^{\boldsymbol{n}}$, and so at most one oscillation count corresponds to four eigenvalues.) If exactly two (respectively, three) adjacent vertices are in adjacent quadrants (as above), then the oscillation count corresponds to exactly two (respectively, three) different eigenvalues.

5. Asymptotics of eigenvalues

In this section we locate the eigenvalues in certain cones, identify the asymptotic spectrum and refine the asymptotics of the eigencurves obtained in lemmas 3.3 and 3.4 to get asymptotic expansions of the eigenvalues.

Let C_{c} be the cone of all points in the $\boldsymbol{\lambda}$-plane such that $m_2 \leqslant \lambda_2/\lambda_1 \leqslant M_1$. This is called the *continuous cone*. The union of the two positive semi-axes is defined to be the *discrete cone* C_{d}. The union of C_{c} and C_{d} will be denoted by C. The *asymptotic spectrum*, denoted by AS, is the closure in S^1 of the set

$$\{\boldsymbol{\lambda}/\|\boldsymbol{\lambda}\| : \boldsymbol{\lambda} \text{ is an eigenvalue of the system (2.1), (2.2)}\},$$

where S^1 denotes the unit circle.

Theorem 5.1. $AS = C \cap S^1$.

Proof. The following assertions, which follow from lemmas 3.3 and 3.4, show that $AS \subseteq C \cap S^1$:

$$\lim_{n_2\to\infty} \frac{\boldsymbol{\lambda}^{(0,n_2)}}{\|\boldsymbol{\lambda}^{(0,n_2)}\|} = (0,1), \qquad \lim_{n_1\to\infty} \frac{\boldsymbol{\lambda}^{(n_1,0)}}{\|\boldsymbol{\lambda}^{(n_1,0)}\|} = (1,0)$$

and

$$\lim_{\|\boldsymbol{\lambda}^{\boldsymbol{n}}\|\to\infty} \frac{\boldsymbol{\lambda}^{\boldsymbol{n}}}{\|\boldsymbol{\lambda}^{\boldsymbol{n}}\|} \in C_{\mathrm{c}} \quad \text{for } n_1, n_2 > 0.$$

For the reverse inclusion, lemma 3.3 gives $C_{\mathrm{d}} \cap S^1 \subset AS$, and the proof of theorem 6.2 of [7] can be adapted to yield $C_{\mathrm{c}} \cap S^1 \subset AS$. □

Let μ_i^n denote the eigenvalues of the one-parameter problem $-y_i'' + q_i y_i = \lambda_i r_{ii} y_i$, with eigenparameter-dependent boundary condition

$$\cot \alpha_i(1) = (a_i\lambda_i + b_i)/(c_i\lambda_i + d_i).$$

It is known from [6, corollary 3.4] that

$$\mu_i^n = \left(\frac{n\pi}{\sigma_i}\right)^2 + o(n^2), \quad \text{where } \sigma_i = \int_0^1 r_{ii}^{1/2}. \tag{5.1}$$

The next two results give asymptotics of $\boldsymbol{\lambda}^{\boldsymbol{n}}$.

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THEOREM 5.2. *$\boldsymbol{\lambda}^{(n_1,0)} = \mu_1^{n_1} + O(n_1^{-1})$ and $\boldsymbol{\lambda}^{(0,n_2)} = \mu_2^{n_2} + O(n_2^{-1})$.*

Proof. Since $m_2 > 0$, we choose positive $\eta < m_2$ and let

$$-\gamma = \sup\{r_{21}(x) + \eta r_{22}(x) : 0 \leqslant x \leqslant 1\}.$$

For large λ_1, by lemma 3.3, $\lambda_{20}^2(\lambda_1)/\lambda_1 < \eta$. Let

$$\epsilon = \{\sup |q_2|(x) : 0 \leqslant x \leqslant 1\}.$$

Then

$$\begin{aligned}\theta_2' &= \cos^2\theta_2 + (\lambda_1 r_{21} + \lambda_{20}^2 r_{22} - q_2)\sin^2\theta_2 \\ &< \cos^2\theta_2 + (\lambda_1(r_{21} + (\lambda_{20}^2/\lambda_1)r_{22}) + \epsilon)\sin^2\theta_2 \\ &< \cos^2\theta_2 + (-\gamma\lambda_1 + \epsilon)\sin^2\theta_2.\end{aligned}$$

It follows that

$$\frac{\mathrm{d}}{\mathrm{d}\theta_2}\cot\theta_2 > -\cot^2\theta_2 + \gamma\lambda_1 - \epsilon.$$

Letting $\cot\theta_2 = u$ and $s^2 = \gamma\lambda_1 - \epsilon$ (> 0 for large positive λ_1), we have

$$u' > s^2 - u^2.$$

This differential inequality with the boundary condition $u(0) = \cot\alpha_2$ leads to

$$\cot\theta_2(1) \geqslant s\frac{\exp(2s)(s + \cot\alpha_2) - (s - \cot\alpha_2)}{\exp(2s)(s + \cot\alpha_2) + (s - \cot\alpha_2)}.$$

It follows that

$$\lim_{s\to\infty}\frac{\cot\theta_2(1)}{s} \geqslant 1.$$

Recalling that $\cot\theta_2(1) = (a_2\lambda_{20}^2(\lambda_1) + b_2)/(c_2\lambda_{20}^2(\lambda_1))$ and $s = \sqrt{\gamma\lambda_1 - \epsilon}$, we have

$$\lim_{\lambda_1\to\infty}\frac{b_2}{c_2\lambda_{20}^2(\lambda_1)\sqrt{\gamma\lambda_1 - \epsilon}} \geqslant 1.$$

Hence $\lambda_{20}^2(\lambda_1) = O(\lambda_1^{-1/2})$ as $\lambda_1 \to \infty$. Since the derivatives of the $\lambda_{2n_1}^1$ are bounded below by m_1, this implies that $|\boldsymbol{\lambda}^{(n_1,0)} - \mu_1^{n_1}| = O((\mu_1^{n_1})^{-1/2})$. In view of the asymptotic estimates (5.1), the proof of the first contention is complete, and the second follows analogously. □

The corresponding result for $n_j > 0$ is slightly better. Let $\boldsymbol{\lambda}^{A\boldsymbol{n}}$ denote the eigenvalues of the asymptotic two-parameter problem, i.e. (2.1) with boundary conditions

$$\frac{y_i'}{y_i}(0) = \cot\alpha_i, \qquad \frac{y_i'}{y_i}(1) = \frac{a_i}{c_i}. \tag{5.2}$$

THEOREM 5.3. *$\boldsymbol{\lambda}^{\boldsymbol{n}+(1,1)} = \boldsymbol{\lambda}^{A\boldsymbol{n}} + \boldsymbol{O}(\|\boldsymbol{n}\|^{-1})$ as $n_1 + n_2 \to \infty$.*

Proof. The proof depends on variation of the boundary condition at 1. For a fixed λ_1, consider the second equation with two different boundary conditions, namely, equations (2.2) and (5.2). If θ_2^A denotes the Prüfer angle for the asymptotic problem,

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then $\cot\theta_2(1) - \cot\theta_2^A(1) = O(\lambda_2^{-1})$ (cf. [6, eqn (2.8)]) $= O(\lambda_1^{-1})$ by lemma 3.4. It follows that

$$|\lambda_2(\theta_2(1)) - \lambda_2(\theta_2^A(1))| \leqslant \left|\frac{\mathrm{d}\lambda_2}{\mathrm{d}\theta_2(1)}\right| \sin^2\theta_2(1) O(\lambda_1^{-1}).$$

By corollary 5.2 of [2], we have

$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}\theta_2(1)} = -(r_{22}(y))^{-1},$$

where y is of norm 1. Hence

$$\left|\frac{\mathrm{d}\lambda_2}{\mathrm{d}\theta_2(1)}\right| \leqslant (\inf\{r_{22}(x) : 0 \leqslant x \leqslant 1\})^{-1}.$$

Thus we have $|\lambda_2(\theta_2(1)) - \lambda_2(\theta_2^A(1))| = O(\lambda_1^{-1})$. A similar analysis for the first equation yields $|\lambda_1(\theta_1(1)) - \lambda_1(\theta_1^A(1))| = O(\lambda_2^{-1}) = O(\lambda_1^{-1})$. Since

$$M_1 \geqslant \frac{\mathrm{d}\lambda^1_{2n_1}}{\mathrm{d}\lambda_1} \geqslant m_1 > M_2 \geqslant \frac{\mathrm{d}\lambda^2_{2n_2}}{\mathrm{d}\lambda_1} \geqslant m_2,$$

there is a constant c such that

$$\boldsymbol{\lambda}^{\boldsymbol{n}+(1,1)} - \boldsymbol{\lambda}^{A\boldsymbol{n}} \leqslant c(|\lambda_2(\theta_2(1)) - \lambda_2(\theta_2^A(1))| + |\lambda_1(\theta_1(1)) - \lambda_1(\theta_1^A(1))|) = \boldsymbol{O}(\lambda_1^{-1}). \tag{5.3}$$

Now we use the asymptotics for the eigenvalues of a two-parameter system obtained by Faierman in [10]. He partitioned the first quadrant into three polar sectors. In theorem 3.3 and eqn (4.2) in [10], he gives asymptotic formulas in all three sectors and thus for the whole of the first quadrant. These yield constants $k_j > 0$ satisfying

$$k_1 < \frac{\boldsymbol{\lambda}_1^{A\boldsymbol{n}}}{n_1^2 + n_2^2} < k_2 \quad \text{and} \quad k_3 < \frac{\boldsymbol{\lambda}_2^{A\boldsymbol{n}}}{\boldsymbol{\lambda}_1^{A\boldsymbol{n}}} < k_4$$

in all three sectors. Hence $\boldsymbol{\lambda}^{A\boldsymbol{n}} = \boldsymbol{O}(\|\boldsymbol{n}\|^2)$. Now, from (5.3), we obtain

$$\boldsymbol{\lambda}^{\boldsymbol{n}+(1,1)} = \boldsymbol{\lambda}^{A\boldsymbol{n}} + \boldsymbol{O}(\|\boldsymbol{n}\|^{-2}).$$

□

We conclude with an example which illustrates the above results.

EXAMPLE 5.4. Consider the equations

$$-y'' + y = \lambda y - \tfrac{1}{4}\mu y \tag{5.4}$$

and

$$-z'' + z = -\tfrac{1}{4}\lambda + \mu z, \tag{5.5}$$

with boundary conditions

$$y(0) = 0, \qquad y(1) + \lambda y'(1) = 0,$$
$$z(0) = 0, \qquad z(1) + \mu z'(1) = 0.$$

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When $\lambda = 0$, equation (5.4) is the Dirichlet problem for $-y'' + y = -\mu/4y$. So $-\frac{1}{4}\mu = n^2\pi^2 + 1$. Since $-\frac{1}{4}\mu_1^1(0)$ is the minimal positive eigenvalue of this problem, we have $\mu_1^1(0) = -4$. Now the solution for (5.5) with $\lambda = 0$ is $z = \sinh(x\sqrt{1-\mu})$. The boundary condition at 1 implies that $\sinh\sqrt{1-\mu} + \mu\sqrt{1-\mu}\cosh\sqrt{1-\mu} = 0$. The solution of this equation gives the following inequalities:

$$0 > \mu_0^2(0) > -3.$$

By theorem 3.1, each μ_n^i is increasing, so lemma 3.3 shows that the graphs of μ_1^1 and μ_0^2 meet in the fourth quadrant, i.e. $(\lambda,\mu)^{(1,0)}$ is in the fourth quadrant. By symmetry of the problem, $(\lambda,\mu)^{(0,1)}$ is in the second quadrant. Since the μ_n^i increase, we have four eigenvalues around the origin which have the same oscillation count $(0,0)$ by remark 4.5. These eigenvalues are $(\lambda,\mu)^{(0,0)}$, $(\lambda,\mu)^{(1,0)}$, $(\lambda,\mu)^{(0,1)}$ and $(\lambda,\mu)^{(1,1)}$.

From theorem 5.2, $(\lambda,\mu)^{(n_1,0)} = (n_1^2\pi^2, 0) + \boldsymbol{O}(n_1^{-1})$, with a similar formula for $(\lambda,\mu)^{(0,n_2)}$. These eigenvalues approach the positive half-axes asymptotically. From theorem 5.3,

$$(\lambda,\mu)^{(n_1+1,1)} = (\lambda,\mu)^{A(n_1,0)} + \boldsymbol{O}(n_1^{-2}). \tag{5.6}$$

The boundary conditions in (5.2) are Dirichlet at 0 and Neumann at 1. Thus $(\lambda,\mu)^{A(n_1,0)}$ is at the intersection of the lines

$$\lambda - \tfrac{1}{4}\mu = 1 + (n_1 + \tfrac{1}{2})^2\pi^2 \quad \text{and} \quad -\tfrac{1}{4}\lambda + \mu = 1 + \tfrac{1}{4}\pi^2.$$

The second of these lines is the asymptote for the eigenvalues $(\lambda,\mu)^{(n_1+1,1)}$ by (5.6). Following remark 4.5, we see that the eigenvalue $(\lambda,\mu)^{(n_1+1,1)}$ has the same oscillation count as $(\lambda,\mu)^{(n_1+1,0)}$, namely, $(n_1,0)$ for all $n_1 \geqslant 0$. Similar considerations hold for the eigenvalues $(\lambda,\mu)^{(0,n_2+1)}$ and $(\lambda,\mu)^{(1,n_2+1)}$ for all $n_2 \geqslant 0$.

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References

1 P. A. Binding. Multiparameter definiteness conditions. *Proc. R. Soc. Edinb.* A **89** (1981), 319–332.

2 P. A. Binding and P. J. Browne. Multiparameter Sturm theory. *Proc. R. Soc. Edinb.* A **99** (1984), 173–184.

3 P. A. Binding and P. J. Browne. Asymptotics of eigencurves for second-order ordinary differential equations. *J. Diff. Eqns* **127** (1990), 30–45.

4 P. A. Binding and P. J. Browne. Oscillation theory for indefinite Sturm–Liouville problems with eigenparameter-dependent boundary conditions. *Proc. R. Soc. Edinb.* A **127** (1997), 1123–1136.

5 P. A. Binding and H. Volkmer. Eigencurves for two-parameter Sturm–Liouville equations. *SIAM Rev.* **38** (1996), 27–48.

6 P. A. Binding, P. J. Browne and K. Seddighi. Sturm–Liouville problems with eigenparameter-dependent boundary conditions. *Proc. Edinb. Math. Soc.* **37** (1993), 57–72.

7 P. A. Binding, P. J. Browne and K. Seddighi. Two-parameter asymptotic spectra in the uniformly elliptic case. *Result. Math.* **31** (1997), 1–13.

8 E. A. Coddington and N. Levinson. *Theory of ordinary differential equations* (McGraw-Hill, 1955).

9 A. Dijksma and H. Langer. *Operator theory and ordinary differential operators.* Fields Institute Lectures. *Am. Math. Soc.* **3** (1996), 75–139.

10 M. Faierman. On the distribution of the eigenvalues of a two-parameter system of ordinary differential equations of second order. *SIAM J. Math. Analysis* **8** (1977), 854–870.

11 C. T. Fulton. Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions. *Proc. R. Soc. Edinb.* A **77** (1977), 293–308.

12 C. T. Fulton. Singular eigenvalue problems with eigenvalue parameter contained in the boundary conditions. *Proc. R. Soc. Edinb.* A **87** (1980/81), 1–34.

13 C. T. Fulton and S. Pruess. Numerical methods for a singular eigenvalue problems with eigenparameter contained in the boundary conditions. *J. Math. Analysis Appl.* **71** (1979), 431–462.

14 J. K. Hale. *Ordinary differential equations* (Wiley, 1969).

15 E. L. Ince. *Ordinary differential equations* (New York: Dover, 1956).

16 W. T. Reid. *Sturmian theory for ordinary differential equations* (Springer, 1980).

17 H. Volkmer. *Multiparameter eigenvalue problems and expansion theorems*, Lecture Notes in Mathematics, vol. 1356 (Springer, 1988).

18 J. Walter. Regular eigenvalue problems with eigenvalue parameter in the boundary conditions. *Math. Z.* **133** (1973), 301–312.

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