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Eigenmodes in the water-wave problems for infinite pools with cone-shaped bottom

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In the framework of the assumptions of the linearized theory of small-amplitude water waves, the eigenfunctions of the point spectrum are studied for boundary-value problems in infinite domains. Special types of three-dimensional infinite water pools characterised by cone-shaped bottoms are considered. By means of an incomplete separation of variables and exploiting the Mellin transform, we reduce construction of the eigenmodes to the study and solution of the problems for some functional difference equations with meromorphic coefficients. The behaviour of the eigenmodes at a singular point of the boundary and the rate of their decay at infinity are also examined.

Key words: surface gravity waves, waves/free-surface flows

1. Introduction

The questions of the correct formulation of the boundary-value problems in the linear theory of water waves have attracted the attention of researchers due to their numerous applications in naval architecture, ocean engineering, geophysical hydrodynamics and acoustics (see Kuznetsov, Maz'ya & Vainberg 2002). It is well known that the study of the uniqueness of a solution is connected with the description of the point spectrum for water-wave problems in infinite domains. Important results in this direction have been obtained more than sixty years ago by Jones (1953), see also Ursell (1987), Kuznetsov *et al.* (2002) and the references therein. In this respect, the results on uniqueness by Kuznetsov *et al.* (2002, § 3.2.2.4) for the axially symmetric solutions are of particular interest and will be discussed in the context of our work in the § 8. Conditions for the existence of the discrete spectrum located on the left-hand side of the cutoff point of the continuous spectrum are studied in the papers by Ursell (1987), Evans, Levitin & Vassiliev (1994), Nazarov (2010) and in many others. The corresponding trapped waves exponentially vanish at infinity so that the eigenmodes are strongly localized. The recently developed variational formulation of the problems in Nazarov (2008) enables one to study the existence conditions for trapped waves in water-wave channels on a systematic basis.

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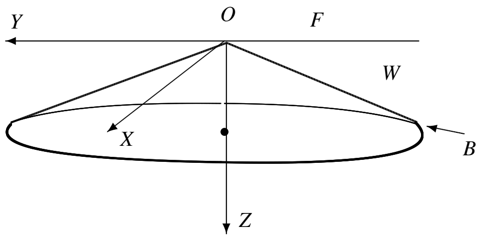

FIGURE 1. An axially symmetric infinite pool with the conical bottom B.

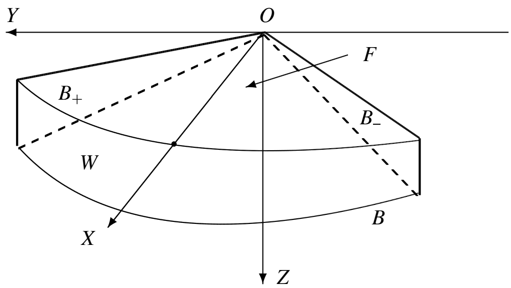

FIGURE 2. An infinite pool with the conical bottom B and vertical walls $B_\pm$.

Linton & McIver (2007) noticed that it was much more difficult to establish the existence of the trapped modes for parameters which also admitted the existence of eigenmodes corresponding to the continuous spectrum. Nevertheless, they discussed several interesting situations of that type. It is worth mentioning, however, that the results by Nazarov (2008, 2010) indicate the possibility of obtaining such important examples of the problems for water-wave channels.

There are some situations for which the solutions of the spectral problems can be derived explicitly, e.g. in quadratures. In particular, the edge waves on a sloping beach (Stokes' eigensolutions) can be written down as elementary functions, which is discussed in Kuznetsov *et al.* (2002, pp. 220–223). A systematic way to determine such solutions has been proposed in the work by Roseau (1958). In particular, this author has solved functional difference equations in order to obtain such solutions. At the same time, Malyuzhinets (1958) has obtained his important results in solving the problem of diffraction by an impedance wedge (see also Williams 1959; Lyalinov & Zhu 2012). He made use of a solution of the functional equations by means of the special meromorphic function, which is now referred to as the Malyuzhinets function. It is worth noting that, together with our studies in this work, the use of functional difference equations has also become commonplace in some of the problems of spectral theory (see e.g. Buslaev & Fedotov 1997; Fedotov & Sandomirskiy 2015), including in diffraction (Williams 1959; Lyalinov & Zhu 2003, 2012), in water-wave problems (Roseau 1958; Lawrie & King 1994) and in quantum theory (Faddeev, Kashaev & Volkov 2001). In particular, some analogues of the special Malyuzhinets function have been used in these works.

In this work we investigate spectral water-wave problems in the specific infinite three-dimensional (3-D) conical domains W of two types (see figures 1 and 2) with a cone-shaped bottom B. A more detailed description of the pools of both types is given in the following section. We consider linear time-harmonic water-wave processes described by the first-order velocity potential $U(X, Y, Z, t)$, see e.g. Kuznetsov *et al.* (2002, pp. 9–11). The complex velocity potential u is connected with the non-stationary harmonic one by

$$U(X, Y, Z, t) = \mathrm{Re}\{\exp(-\mathrm{i}\Omega t)u(X, Y, Z)\}, \tag{1.1}$$

where Ω is the circular frequency. The potential is governed by the Laplace equation. The boundary condition with the spectral parameter K is valid on the free water surface F (spectral Steklov condition), whereas the bottom B of the conical shape is rigid with a zero normal component of the velocity on it and also on the vertical side walls $B_{\pm}$. The boundary condition on the free water surface F arises from the non-stationary one, see Kuznetsov *et al.* (2002, p. 9, I.25). The latter condition follows from an asymptotics analysis of the corresponding original nonlinear problem in the small-amplitude wave approximation.

From the geometry of water domains it is natural to use spherical coordinates and separation of the variables in order to construct the eigensolutions of the problems at hand. To that end, we look for the eigensolutions by use of the Mellin transform with respect to the radial variable and reduce the problems to those in the domains on the unit sphere with the corresponding boundary conditions. The latter problems are then solved in spherical functions, whereas the unknown coefficients are determined from a system of functional difference equations which is an equivalent of the conditions on the boundary of the spherical domains. We solve the functional equations in a special class of meromorphic functions introducing a new, special meromorphic solution (double special function) of an auxiliary functional difference equation. As a result, the integral representations for the eigenfunctions of the point spectrum ($K > 0$) are given. The behaviour of the eigensolutions at infinity and at the conical singularity of the boundary are also addressed.

Although the problems under consideration deal with very special geometries, they can be solved in quadratures by means of the reduction to the problems for some new functional difference equations. On one hand, study and solution of these equations are challenging and require extensive use of the complex and asymptotic analysis as well as of the classical special functions. On the other hand, however, the solutions of the original water-wave problems can be represented in an explicit compact form of the Mellin integral which admits further efficient elaboration.

2. Formulation

Introduce the spherical coordinates

$$X = r\cos\varphi\sin\theta, \quad Y = r\sin\varphi\sin\theta, \quad Z = r\cos\theta \tag{2.1a–c}$$

instead of the Cartesian coordinates X, Y, Z with the axis OZ directed vertically downwards as shown in figure 1. Let the water medium occupy the conical domain

$$W = \left\{(r, \theta, \varphi) : r > 0, -\pi < \varphi \leqslant \pi, \theta_1 < \theta < \frac{\pi}{2}\right\}, \tag{2.2}$$

and $\theta = \theta_1$ is the equation of the bottom B, $0 < \theta_1 < \pi/2$.

The sought for potential satisfies the Laplace equation

$$\nabla^2 u(r, \theta, \varphi) = 0 \quad \text{in } W, \tag{2.3}$$

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{1}{r^2}\nabla^2_\omega, \quad \nabla^2_\omega = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}, \tag{2.4a,b}$$

the boundary condition on the free surface F ($\theta = \pi/2$)

$$\left.\left(\frac{1}{r}\frac{\partial}{\partial\theta}u(r, \theta, \varphi) - Ku(r, \theta, \varphi)\right)\right|_{\theta=\pi/2} = 0 \quad \text{on } F, \tag{2.5}$$

with the spectral parameter $K = \Omega^2/g$ (g the gravitational acceleration) and

$$\left.\frac{1}{r}\frac{\partial}{\partial\theta}u(r, \theta, \varphi)\right|_{\theta=\theta_1} = 0 \quad \text{on } B. \tag{2.6}$$

Remark that $(1/r)(\partial u/\partial\theta)|_{\theta=\pi/2} = -\partial u/\partial Z|_F$. The classical solution of (2.3)–(2.6) is 2π-periodic in φ,

$$u(r, \theta, \varphi) = u(r, \theta, \varphi + 2\pi) \tag{2.7}$$

and fulfils the Meixner's type conditions at the vertex O of the conical domain W. These conditions might be written in the form

$$|u(r, \theta, \varphi)| \leqslant \text{const.}\, r^\delta, \quad r|\nabla u(r, \theta, \varphi)| \leqslant \text{const.}\, r^\delta, \tag{2.8a,b}$$

which is valid uniformly with respect to the angular variables, for some $\delta > -1/2$.

We introduce the functional of energy

$$E(u) = \int_W |\nabla u|^2 \, \mathrm{d}X\, \mathrm{d}Y\, \mathrm{d}Z + K\int_F |u|^2 \, \mathrm{d}X\, \mathrm{d}Y, \tag{2.9}$$

where the first and second integrals in (2.9) are usually associated with the kinetic and potential energy. It is worth remarking that the spectral parameter K is in the boundary condition, which is, in general, not traditional for the spectral analysis. For this kind of problem, however, we accept a standard terminology and call a non-trivial solution u of (2.3)–(2.8) the eigenfunction of the point spectrum with eigenvalue K provided

$$E(u) < \infty. \tag{2.10}$$

In view of the definition (2.9), the condition (2.10) implies that u and $|\nabla u|$ decay sufficiently rapidly at infinity so that, at least, the estimate

$$|u(r, \theta, \varphi)| \leqslant \text{const.}\, r^{-\varkappa}, \quad r \to \infty \tag{2.11}$$

is valid uniformly with respect to the angular variables $\omega := (\theta, \varphi)$ for some $\varkappa > 1$. However, for some given K a solution u of the homogeneous problem may decay slower than that in (2.11) or even grow at most polynomially so that the integrals in (2.9) diverge. We consider $K > 0$ implying that $K = 0$ is non-physical.

The domain shown in figure 1 might serve to simulate water-wave processes near a shallow or sandbank in the ocean. For this reason the problem above may be called the linear water-wave problem for a sandbank or, for brevity, the sandbank water-wave problem.

2.1. *Problems for the domain of the second type*

We also consider a domain of the second type (figure 2). It is assumed that, contrary to that in figure 1 for the sandbank water-wave problem, the domain is not axially symmetric and is bounded in φ

$$W = \left\{(r, \theta, \varphi) : r > 0, -\Phi < \varphi < \Phi, \theta_1 < \theta < \frac{\pi}{2}\right\}, \tag{2.12}$$

with $0 < \Phi < \pi$. On the side walls $B_\pm = \{(r, \theta, \varphi) : r > 0, \theta_1 < \theta < \pi/2, \varphi = \pm\Phi\}$ of the pool the boundary conditions

$$\left.\frac{\partial}{\partial\varphi} u(r, \theta, \varphi)\right|_{\varphi=\pm\Phi} = 0 \quad \text{on } B_\pm \tag{2.13}$$

are implied. Now the condition (2.7) in the formulation should be changed by those in (2.13) and we are looking for classical solutions of the homogeneous problem (2.3)–(2.6), (2.13), (2.8) as well as the conditions of Meixner's type at the edges.

One might use the problem at hand in order to describe linear water-wave processes in a bay (or near a cliff) of a special conical shape (figure 2) formed by the coastal line of the ocean. So, such a problem could be naturally called a bay water-wave problem if $2\Phi < \pi$ and a cliff water-wave problem if $2\Phi > \pi$. In what follows, we shall mainly discuss the sandbank water-wave problem. In view of the direct similarity with those for the axially symmetric problem, only some basic derivations and results will be given for the bay or cliff water-wave problems.

3. Separation of the radial variable and the problem on the unit sphere

With the aim of separating the radial variable, the solution of the sandbank water-wave problem is sought in the form of the Mellin integral

$$u(r, \theta, \varphi) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} u_\nu(\omega) r^{\nu-1/2}\, d\nu, \tag{3.1}$$

where c is some constant, $\omega = (\theta, \varphi)$ and its inverse is

$$u_\nu(\omega) = \int_0^\infty u(r, \omega) r^{-\nu-1/2}\, dr. \tag{3.2}$$

Since we expect that the representation (3.1) satisfies Meixner's conditions, e.g. $u(r, \theta, \varphi) = O(r^a)$ as $r \to 0$, and decays at infinity, i.e. $u(r, \theta, \varphi) = O(r^{-b})$ as $r \to \infty$, the Mellin transformant u_ν in (3.2) is regular in the strip

$$\tfrac{1}{2} - b < \mathrm{Re}(\nu) < \tfrac{1}{2} + a, \tag{3.3}$$

where it is assumed that $a > -b$ with $a > c - 1/2 > -b$. (It is obvious that one might take $b \geqslant \varkappa$ (see (2.11)) and $a \geqslant \delta$, see (2.8).) The constants will be specified below, in particular, we may expect that $a \geqslant 0, b \geqslant 1$. The function $u(\cdot, \omega)$ is locally integrable on $[0, \infty)$. The transformant u_ν is sought in the class of meromorphic functions of ν for any $\omega \in \Sigma$, where Σ is the spherical layer on the unit sphere S^2 with the centre at the origin O (figure 3*a*) such that

$$\Sigma = \{\omega : \omega \in S^2 \cap W\}. \tag{3.4}$$

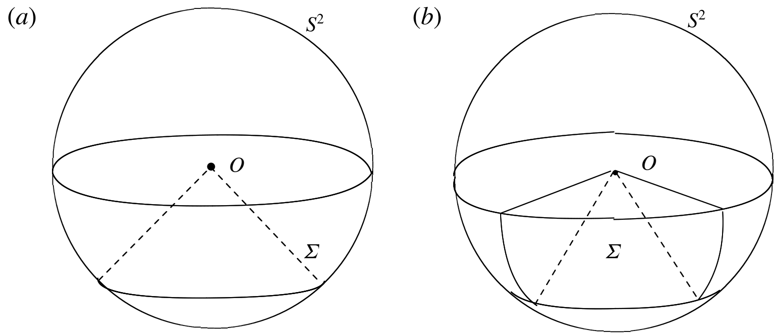

FIGURE 3. Spherical domains Σ on the unit sphere S^2 for both types of the problem.

Notice that the boundary of Σ consists of two circles on S^2 with the equations $\theta=\theta_1$ and $\theta=\pi/2$. We also require that

$$u_\nu(\omega)\to 0, \quad |\nu|\to\infty \tag{3.5a,b}$$

as ν belongs to the strip (3.3), $\omega\in\overline{\Sigma}$.

We substitute the representation (3.1) into the (2.3) and take into account the simple reduction

$$\begin{aligned}\nabla^2 u &= \left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\nabla^2_\omega\right)\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} u_\nu(\omega) r^{\nu-1/2}\,d\nu \\ &= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\left(\nabla^2_\omega u_\nu(\omega)+\left[\nu^2-\frac{1}{4}\right]u_\nu(\omega)\right) r^{\nu-5/2}\,d\nu=0.\end{aligned} \tag{3.6}$$

We now conclude that, provided the transformant $u_\nu(\cdot)$ fulfils the equation,

$$\left(\nabla^2_\omega+\nu^2-\tfrac{1}{4}\right)u_\nu(\omega)=0 \quad \text{on } \Sigma, \tag{3.7}$$

and the original u solves the Laplace equation (2.3) in W. We now turn to the boundary conditions for $u_\nu(\cdot)$. Substituting the representation (3.1) into the boundary condition (2.5) on F, one has

$$\begin{aligned}&\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\left(\frac{\partial u_\nu(\omega)}{\partial\theta} r^{\nu-3/2}-K u_\nu(\omega) r^{\nu-1/2}\right)d\nu\bigg|_{\theta=\pi/2} \\ &= \left\{\frac{1}{2\pi i}\int_{-i\infty-1}^{+i\infty-1}\frac{\partial u_{\nu+1}(\omega)}{\partial\theta} r^{\nu-1/2}\,d\nu-\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} K u_\nu(\omega) r^{\nu-1/2}\,d\nu\right\}\bigg|_{\theta=\pi/2}=0,\end{aligned} \tag{3.8}$$

where we made the change of the integration variable $\nu-1\to\nu$ in the first integral of the last line. Then we deform the straight line contour $(-i\infty-1, +i\infty-1)$ in this integral back onto the imaginary axis assuming that singularities of $(\partial u_{\nu+1}(\omega))/\partial\theta|_{\theta=\pi/2}$ are not intersected in this process, which might be justified *a posteriori*. Thus we obtain

$$\frac{1}{2\pi i}\int_{-i\infty}^{+i\infty}\left(\frac{\partial u_{\nu+1}(\omega)}{\partial\theta}-K u_\nu(\omega)\right)\bigg|_{\theta=\pi/2} r^{\nu-1/2}\,d\nu=0, \tag{3.9}$$

which is valid provided the condition

$$\left(\frac{\partial u_{\nu+1}(\omega)}{\partial \theta} - K u_\nu(\omega) \right) \bigg|_{\theta=\pi/2} = 0 \tag{3.10}$$

is satisfied. It is worth noting that the boundary condition (3.10) is non-local with respect to ν. This circumstance is a 'price' for our intention to separate the radial variable in the mixed boundary condition (2.5). In the same manner, from the condition (2.6) we have

$$\frac{\partial u_\nu(\omega)}{\partial \theta} \bigg|_{\theta=\theta_1} = 0. \tag{3.11}$$

In the following section we look for the solution of the problem (3.7)–(3.11) in terms of the spherical functions.

4. Reduction to the solution of the functional difference equations

The equation (3.7) has a set of the linear independent solutions

$$\mathrm{e}^{-\mathrm{i}n\varphi} P_{\nu-1/2}^{-|n|}(\pm \cos\theta), \quad n = 0, \pm 1, \pm 2, \ldots, \tag{4.1}$$

with the separated angular variables θ and φ. Here $P_{\nu-1/2}^{-|n|}(\cdot)$ is the associated Legendre function, see e.g. Gradshteyn & Ryzhik (1980, f. 8.704). It is convenient to represent the solution of (3.7) in the form

$$u_\nu(\omega) = \mathrm{e}^{-\mathrm{i}n\varphi} \left(A_n(\nu) \frac{P_{\nu-1/2}^{-|n|}(\cos\theta)}{\mathrm{d}_{\theta_1} P_{\nu-1/2}^{-|n|}(\cos\theta_1)} + B_n(\nu) \frac{P_{\nu-1/2}^{-|n|}(-\cos\theta)}{\mathrm{d}_{\theta_1} P_{\nu-1/2}^{-|n|}(-\cos\theta_1)} \right), \tag{4.2}$$

where we use the notation

$$\mathrm{d}_\theta P_{\nu-1/2}^{-|n|}(\cos\theta) := \frac{\mathrm{d}}{\mathrm{d}\theta} P_{\nu-1/2}^{-|n|}(\cos\theta) = -\sin\theta \, \frac{\mathrm{d}}{\mathrm{d}x} P_{\nu-1/2}^{-|n|}(x) \bigg|_{x=\cos\theta}. \tag{4.3}$$

It is worth remarking, however, that any linear combination of solutions (4.2) with the summation with respect to n can be also considered as a solution of the homogeneous problem. For this reason, the dependence on the parameter n is omitted in the left-hand side of (4.2) so that we make use of the notation $u_\nu(\omega)$ instead of $u_\nu^n(\omega)$, which should not be misleading.

The coefficients $A_n(\cdot)$ and $B_n(\cdot)$ are still unknown and should be found from the boundary conditions. Making use of the condition (3.11), we arrive at

$$A_n(\nu) + B_n(\nu) = 0. \tag{4.4}$$

The condition (3.10) leads to

$$\begin{aligned} & A_n(\nu+1) \frac{\mathrm{d}_\theta P_{\nu+1/2}^{-|n|}(\cos\theta)|_{\theta=\pi/2}}{\mathrm{d}_{\theta_1} P_{\nu+1/2}^{-|n|}(\cos\theta_1)} + B_n(\nu+1) \frac{\mathrm{d}_\theta P_{\nu+1/2}^{-|n|}(-\cos\theta)|_{\theta=\pi/2}}{\mathrm{d}_{\theta_1} P_{\nu+1/2}^{-|n|}(-\cos\theta_1)} \\ & \quad - K \left(A_n(\nu) \frac{P_{\nu-1/2}^{-|n|}(\cos\theta)|_{\theta=\pi/2}}{\mathrm{d}_{\theta_1} P_{\nu-1/2}^{-|n|}(\cos\theta_1)} + B_n(\nu) \frac{P_{\nu-1/2}^{-|n|}(-\cos\theta)|_{\theta=\pi/2}}{\mathrm{d}_{\theta_1} P_{\nu-1/2}^{-|n|}(-\cos\theta_1)} \right) = 0. \end{aligned} \tag{4.5}$$

By means of the equation (4.4) we eliminate $B_n(\nu) = -A_n(\nu)$ from equation (4.5) and thus obtain

$$A_n(\nu+1)\left.\frac{\mathrm{d}}{\mathrm{d}x}P^{-|n|}_{\nu+1/2}(x)\right|_{x=0}\left(\frac{(-1)}{\mathrm{d}_{\theta_1}P^{-|n|}_{\nu+1/2}(\cos\theta_1)}+\frac{(-1)}{\mathrm{d}_{\theta_1}P^{-|n|}_{\nu+1/2}(-\cos\theta_1)}\right) - KA_n(\nu)P^{-|n|}_{\nu-1/2}(0)\left(\frac{1}{\mathrm{d}_{\theta_1}P^{-|n|}_{\nu-1/2}(\cos\theta_1)}-\frac{1}{\mathrm{d}_{\theta_1}P^{-|n|}_{\nu-1/2}(-\cos\theta_1)}\right)=0. \tag{4.6}$$

The solution $A_n(\cdot)$ of the functional difference equation is sought in a class of meromorphic functions, regular in the strip $-C < \mathrm{Re}(\nu) < C$ with some positive C. The behaviour of $A_n(\nu)$ as $\nu \to \pm\mathrm{i}\infty$ in this strip should be prescribed such that the integral in (3.1) would converge.

4.1. *Functional difference equations in the bay (cliff) water-wave problem*

The solution of (2.3)–(2.6), (2.13), (2.8) is sought in the form

$$v(r,\theta,\varphi)=\frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} v_\nu(\omega)r^{\nu-1/2}\,\mathrm{d}\nu, \tag{4.7}$$

with the solution

$$v_\nu(\omega)=\cos(\mu_n[\varphi+\Phi])\left(\tilde{A}_n(\nu)\frac{P^{-\mu_n}_{\nu-1/2}(\cos\theta)}{\mathrm{d}_{\theta_1}P^{-\mu_n}_{\nu-1/2}(\cos\theta_1)}+\tilde{B}_n(\nu)\frac{P^{-\mu_n}_{\nu-1/2}(-\cos\theta)}{\mathrm{d}_{\theta_1}P^{-\mu_n}_{\nu-1/2}(-\cos\theta_1)}\right), \tag{4.8}$$

$\mu_n = \pi n/2\Phi$, $n = 0, 1, 2, \ldots$ of the problem on the domain Σ of the unit sphere S^2, figure 3(*b*), $\Sigma = \{\omega : \omega \in S^2 \cap W\}$. As in the previous section we have

$$\tilde{A}_n(\nu) = -\tilde{B}_n(\nu) \tag{4.9}$$

and

$$\tilde{A}_n(\nu+1)\left.\frac{\mathrm{d}}{\mathrm{d}x}P^{-\mu_n}_{\nu+1/2}(x)\right|_{x=0}\left(\frac{(-1)}{\mathrm{d}_{\theta_1}P^{-\mu_n}_{\nu+1/2}(\cos\theta_1)}+\frac{(-1)}{\mathrm{d}_{\theta_1}P^{-\mu_n}_{\nu+1/2}(-\cos\theta_1)}\right) - K\tilde{A}_n(\nu)P^{-\mu_n}_{\nu-1/2}(0)\left(\frac{1}{\mathrm{d}_{\theta_1}P^{-\mu_n}_{\nu-1/2}(\cos\theta_1)}-\frac{1}{\mathrm{d}_{\theta_1}P^{-\mu_n}_{\nu-1/2}(-\cos\theta_1)}\right)=0. \tag{4.10}$$

We remark that analysis and solution of equations (4.10) and (4.6) are conducted very similarly and we thus focus on equation (4.6).

5. Solution of the functional difference equations

Instead of $A_n(\cdot)$ in (4.6) it is helpful to introduce the unknown function $a_n(\cdot)$ by the formula

$$a_n(\nu)=A_n(\nu)\left.\frac{\mathrm{d}}{\mathrm{d}x}P^{-|n|}_{\nu-1/2}(x)\right|_{x=0}\left(\frac{(-1)}{\mathrm{d}_{\theta_1}P^{-|n|}_{\nu-1/2}(\cos\theta_1)}+\frac{(-1)}{\mathrm{d}_{\theta_1}P^{-|n|}_{\nu-1/2}(-\cos\theta_1)}\right). \tag{5.1}$$

The equation (4.6) is then written in the form

$$a_n(\nu+1) - Ka_n(\nu)\left(\frac{P_{\nu-1/2}^{-|n|}(0)}{\left.\dfrac{\mathrm{d}}{\mathrm{d}x}P_{\nu-1/2}^{-|n|}(x)\right|_{x=0}}\right) \times \left(\frac{\mathrm{d}_{\theta_1}P_{\nu-1/2}^{-|n|}(\cos\theta_1) - \mathrm{d}_{\theta_1}P_{\nu-1/2}^{-|n|}(-\cos\theta_1)}{\mathrm{d}_{\theta_1}P_{\nu-1/2}^{-|n|}(\cos\theta_1) + \mathrm{d}_{\theta_1}P_{\nu-1/2}^{-|n|}(-\cos\theta_1)}\right) = 0. \tag{5.2}$$

Meromorphic solution of the equation (5.2) can be determined in the form of the product

$$a_n(\nu) = \alpha_n(\nu)\beta_n(\nu)\gamma_n(\nu), \tag{5.3}$$

where the meromorphic functions $\alpha_n(\cdot)$, $\beta_n(\cdot)$, $\gamma_n(\cdot)$ in (5.3) fulfil the equations

$$\alpha_n(\nu+1) = K\alpha_n(\nu), \tag{5.4}$$

$$\beta_n(\nu+1) = -\beta_n(\nu)\left(\frac{P_{\nu-1/2}^{-|n|}(0)}{\left.\dfrac{\mathrm{d}}{\mathrm{d}x}P_{\nu-1/2}^{-|n|}(x)\right|_{x=0}}\right), \tag{5.5}$$

$$\gamma_n(\nu+1) = \gamma_n(\nu)\left(\frac{\mathrm{d}_{\theta_1}P_{\nu-1/2}^{-|n|}(-\cos\theta_1) - \mathrm{d}_{\theta_1}P_{\nu-1/2}^{-|n|}(\cos\theta_1)}{\mathrm{d}_{\theta_1}P_{\nu-1/2}^{-|n|}(\cos\theta_1) + \mathrm{d}_{\theta_1}P_{\nu-1/2}^{-|n|}(-\cos\theta_1)}\right). \tag{5.6}$$

The equation (5.4) has an entire solution

$$\alpha_n(\nu) = \exp(\nu\log K) \tag{5.7}$$

such that $|\alpha_n(\nu)| \leqslant \text{const.}$ as $\nu \in (-\mathrm{i}\infty, \mathrm{i}\infty)$, $|\nu| \to \infty$.

Solution of equation (5.5) requires more work. First, we consider an equivalent equation

$$\frac{\beta_n(\nu+1)}{\beta_n(\nu)} = -\frac{P_{\nu-1/2}^{-|n|}(0)}{\left.\dfrac{\mathrm{d}}{\mathrm{d}x}P_{\nu-1/2}^{-|n|}(x)\right|_{x=0}} = \frac{1}{2}\frac{\Gamma\left(\dfrac{\nu+|n|+\frac{1}{2}}{2}\right)\Gamma\left(\dfrac{-\nu+|n|+\frac{1}{2}}{2}\right)}{\Gamma\left(\dfrac{\nu+|n|+\frac{3}{2}}{2}\right)\Gamma\left(\dfrac{-\nu+|n|+\frac{3}{2}}{2}\right)}, \tag{5.8}$$

where we used the expressions (Gradshteyn & Ryzhik 1980, f. 8.756) written in the form

$$P_{\nu-1/2}^{-|n|}(0) = \frac{2^{-|n|}\sqrt{\pi}}{\Gamma\left(\dfrac{\nu+|n|-\frac{1}{2}}{2}+1\right)\Gamma\left(\dfrac{-\nu+|n|+\frac{3}{2}}{2}\right)}, \tag{5.9}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}P_{\nu-1/2}^{-|n|}(x)\right|_{x=0} = \frac{2^{1-|n|}(-\sqrt{\pi})}{\Gamma\left(\dfrac{\nu+|n|+\frac{1}{2}}{2}\right)\Gamma\left(\dfrac{-\nu+|n|+\frac{1}{2}}{2}\right)}, \tag{5.10}$$

where $\Gamma(\cdot)$ is Euler's gamma function. By means of the functional equation for the gamma function $\Gamma(z+1) = z\Gamma(z)$ one can represent (5.8) as follows

$$\frac{\beta_n(\nu+1)}{\beta_n(\nu)} = \frac{1}{2}\frac{\Gamma\left(\dfrac{\nu+|n|+\frac{1}{2}}{2}\right)\Gamma\left(\dfrac{-\nu+|n|+\frac{1}{2}}{2}\right)}{\Gamma\left(\dfrac{[\nu+1]+|n|+\frac{1}{2}}{2}\right)\Gamma\left(\dfrac{-[\nu+1]+|n|+\frac{1}{2}}{2}\right)}\frac{2}{(-\nu+|n|-\frac{1}{2})} \tag{5.11}$$

and then

$$\frac{\beta_n(\nu+1)}{\beta_n(\nu)} = -\frac{\Gamma\left(\dfrac{\nu+|n|+\frac{1}{2}}{2}\right)\Gamma\left(\dfrac{-\nu+|n|+\frac{1}{2}}{2}\right)}{\Gamma\left(\dfrac{[\nu+1]+|n|+\frac{1}{2}}{2}\right)\Gamma\left(\dfrac{-[\nu+1]+|n|+\frac{1}{2}}{2}\right)} \times \frac{\Gamma\left(\nu-|n|+\frac{1}{2}\right)}{\Gamma\left([\nu+1]-|n|+\frac{1}{2}\right)}. \tag{5.12}$$

The equation (5.12) has an obvious solution regular in the strip $|\mathrm{Re}(\nu)| < 1/2$

$$\beta_n(\nu) = \frac{\left[\Gamma\left(\dfrac{\nu+|n|+\frac{1}{2}}{2}\right)\Gamma\left(\dfrac{-\nu+|n|+\frac{1}{2}}{2}\right)\Gamma\left(\nu-|n|+\dfrac{1}{2}\right)\right]^{-1}}{\cos(\pi\nu)}, \tag{5.13}$$

where $1/\cos(\pi\nu)$ is a meromorphic solution of the equation $\beta(\nu+1) = -\beta(\nu)$. It is used in (5.13) in order to compensate for the exponential growth of the inverse to the product of the gamma functions so that

$$|\beta_n(\nu)| = O\left(\sqrt{|\nu|}\right) \quad \text{as } \nu \to \pm\mathrm{i}\infty, \tag{5.14}$$

$|\mathrm{Re}(\nu)| < 1/2$. The singularities of $\beta_n(\cdot)$ are the real poles which are due to the zeros of $\cos(\pi\nu)$.

Now we turn to the solution of the equation (5.6) and introduce an auxiliary function $s_n(\cdot)$ by the equality

$$\gamma_n(\nu) = s_n\left(\nu - \tfrac{1}{2}\right). \tag{5.15}$$

6. Solution of the auxiliary equation

Consider the equation for $s_n(\cdot)$

$$s_n\left(\nu+\frac{1}{2}\right) = \frac{h_n^-(\nu)}{h_n^+(\nu)} s_n\left(\nu-\frac{1}{2}\right), \tag{6.1}$$

$$h_n^{\pm}(\nu) = \mathrm{d}_{\theta_1} P_{\nu-1/2}^{-|n|}(-\cos\theta_1) \pm \mathrm{d}_{\theta_1} P_{\nu-1/2}^{-|n|}(\cos\theta_1). \tag{6.2}$$

Solution of (6.1) is sought in a class of the meromorphic functions which are regular in some strip $|\mathrm{Re}(\nu)| < \nu_*(\theta_1)$ with some positive $\nu_*(\theta_1)$ and bounded as $|\nu| \to \infty$

in this strip. The solution of (6.1) can be naturally called a double special function because the meromorphic coefficient of the functional equation is expressed in terms of the associated Legendre functions, which is a new circumstance. To our knowledge, until the present time, all special solutions of the functional difference equations, like the Malyuzhinets function, have solved the equations with the coefficients expressed in elementary functions e.g. by means of the rational functions of sine and cosine. It seems that the only exception to this observation is, perhaps, the double gamma function $G(\cdot):G(z+1)=\Gamma(z)G(z)$.

It is sufficient to have such a solution in the strip $|\mathrm{Re}(\nu)|<1/2$ then its meromorpphic continuation on the complex plane is performed by means of the functional equation (6.1). To this end, we study some properties of the entire function $h_n^{\pm}(\cdot)$.

6.1. *Properties of the functions $h_n^{\pm}(\cdot)$ and of their ratios*

By use of the asymptotics for the associated Legendre functions (Gradshteyn & Ryzhik 1980, f. 8.721(3)) we have the estimate

$$\frac{h_n^-(\nu)}{h_n^+(\nu)}=\frac{1-\mathrm{d}_{\theta_1}P_{\nu-1/2}^{-|n|}(\cos\theta_1)[\mathrm{d}_{\theta_1}P_{\nu-1/2}^{-|n|}(-\cos\theta_1)]^{-1}}{1+\mathrm{d}_{\theta_1}P_{\nu-1/2}^{-|n|}(\cos\theta_1)[\mathrm{d}_{\theta_1}P_{\nu-1/2}^{-|n|}(-\cos\theta_1)]^{-1}}=1+O(\mathrm{e}^{\pm\mathrm{i}\nu[\pi-2\theta_1]}) \tag{6.3}$$

as $\nu\to\pm\mathrm{i}\infty$, $|\mathrm{Re}(\nu)|<1/2$. The most important information for construction of the meromorphic solution for (6.1) deals with the location of zeros $\nu_m^{\pm}(|n|)$, $m=0,1,\ldots$ of the entire function $h_n^{\pm}(\cdot)$ located outside the strip $|\mathrm{Re}(\nu)|<\nu_*^{\pm}(\theta_1)$ and, therefore, with the positions of the poles and zeros for the ratio h_n^-/h_n^+.

Let $n=0$, then the zeros of $h_0^{\pm}(\cdot)$ are on the real axis. Indeed, consider a regular Sturm–Liouville problem ($x=\cos\theta$)

$$-\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}y^+(x)=\lambda_\nu^+y^+(x),\quad x\in(0,\cos\theta_1), \tag{6.4}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}y^+(x)\right|_{x=0}=0,\quad \left.\frac{\mathrm{d}}{\mathrm{d}x}y^+(x)\right|_{x=\cos\theta_1}=0, \tag{6.5a,b}$$

where $y^+(x)=P_{\nu-1/2}(-x)+P_{\nu-1/2}(x)$, $\lambda_\nu^+=\nu_+^2-1/4$ is the spectral parameter. The operator $\mathscr{L}^+$ corresponding to the Sturm–Liouville problem is symmetric and self-adjoint if it is appropriately defined. Then its spectrum is non-negative and discrete $\lambda_{\nu_m}^+=(\nu_m^+)^2-1/4$, $m=0,1,2\ldots$. As a result, the zeros ν_m^+ of the equation

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}(P_{\nu-1/2}(-x)+P_{\nu-1/2}(x))\right|_{x=\cos\theta_1}=0 \tag{6.6}$$

are on the real axis and $|\nu_m^+|\geqslant 1/2$. In a similar manner, one can prove that the zeros ν_m^- of the equation

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}(P_{\nu-1/2}(-x)-P_{\nu-1/2}(x))\right|_{x=\cos\theta_1}=0 \tag{6.7}$$

are real and $|\nu_m^-| \geqslant 1/2$. The asymptotics of the zeros for large numbers can be computed by means of the asymptotics of $P_{\nu-1/2}(x)$ as $\nu \to \infty$,

$$\left.\begin{aligned} \nu_{m_k}^- &= \left(\frac{1}{2}+2m_k\right)(1+o(1)), \quad |m_k| \gg 1, \quad \nu_{l_k}^- = \frac{\pi(2l_k+1)}{\pi-2\theta_1}(1+o(1)), \quad |l_k| \gg 1, \\ \nu_{p_k}^+ &= \left(\frac{1}{2}+2p_k+1\right)(1+o(1)), \quad |p_k| \gg 1, \quad \nu_{q_k}^+ = \frac{2\pi q_k}{\pi-2\theta_1}(1+o(1)), \quad |q_k| \gg 1, \end{aligned}\right\} \tag{6.8}$$

where $\nu_{m_k}^{\pm}, \nu_{l_k}^{\pm}, \ldots; k=0, 1, 2\ldots$ are subsequences of $\nu_m^{\pm}$. (We should also mention the zeros ν_n which are close to $-n$ for large $n>0$.)

The ratio h_0^-/h_0^+ has no poles and zeros in the strip $|\mathrm{Re}(\nu)| < C_0$ with $C_0 = 3/2$ and is regular there. In other words, the closest to the origin zeros ν_*^-, $-\nu_*^-$ or poles denoted by ν_*^+, $-\nu_*^+$ are real and such that

$$|\nu_*^{\pm}| \geqslant \tfrac{3}{2}. \tag{6.9}$$

To prove this we make use of the formula from Gradshteyn & Ryzhik (1980, f. 8.842(1))

$$P_{\nu-1/2}(\cos\theta) = \frac{2\cos\pi\nu}{\pi}\int_0^{\infty}\frac{\cosh(\nu\tau)\,\mathrm{d}\tau}{\sqrt{2(\cosh\tau+\cos\theta)}}, \quad |\mathrm{Re}(\nu)| < \frac{1}{2}, \tag{6.10}$$

thus having

$$h_0^{\pm}(\nu) = \pm\frac{\sin\theta_1\cos\pi\nu}{\sqrt{2}\pi}\int_0^{\infty}\left(\frac{\cosh(\nu\tau)}{\sqrt{(\cosh\tau+\cos\theta_1)}^3} \mp \frac{\cosh(\nu\tau)}{\sqrt{(\cosh\tau-\cos\theta_1)}^3}\right)\mathrm{d}\tau, \tag{6.11}$$

$|\mathrm{Re}(\nu)| < 3/2$, where the integrand does not change its sign for real ν.

Remark. $h_0^-(\nu)$ has non-zero limits as $\nu \to 3/2$ or $\nu \to -3/2$ from the strip $|\mathrm{Re}(\nu)| < 3/2$, which means that the ratio h_0^-/h_0^+ has poles in the closed strip $|\mathrm{Re}(\nu)| \leqslant 3/2$ at the points $\nu = \pm 3/2$. The latter implies that $|\nu_*^-| > 3/2$. We also have that $|\nu_*^+| = 3/2$.

Let us now consider the case $|n| \gg 1$. By use of simple argumentation this assumption enables us to demonstrate that there is a sufficiently wide strip on the complex ν plane which is free of zeros and poles of the ratio $\{h_n^-(\nu)/h_n^+(\nu)\}$, see (figure 4). We assume that ν belongs to a compact domain of the complex plane which contains the origin. Taking into account the asymptotics of the associated Legendre function

$$P_{\nu-1/2}^{-|n|}(\cos\theta_1) = \frac{\left[\tan\dfrac{\theta_1}{2}\right]^{|n|}}{\Gamma(|n|+1)}\left(1+O\left(\frac{1}{|n|}\right)\right), \tag{6.12}$$

see e.g. Babich & Cherednichenko (1997). After simple calculations we find that

$$h_n^{\pm}(\nu) = \left(\frac{\left[\cot\dfrac{\theta_1}{2}\right]^{|n|}}{\sin\theta_1\Gamma(|n|)} \pm \frac{\left[\tan\dfrac{\theta_1}{2}\right]^{|n|}}{\sin\theta_1\Gamma(|n|)}\right)\left(1+O\left(\frac{1}{|n|}\right)\right), \quad |n| \to \infty. \tag{6.13}$$

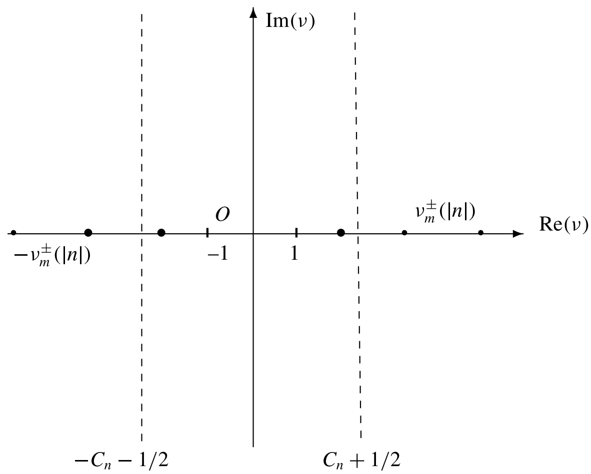

FIGURE 4. Regularity strip of $s_n(\cdot)$ and zeroes of the functions $h_n^{\pm}(\nu)$ for $n = 1, 2, \ldots$.

The latter asymptotics shows that $h_n^{\pm}(\cdot)$ have no zeros in any compact domain of complex ν containing the origin for the sufficiently large $|n|$. Recalling the asymptotics (6.3) for $\nu \to \pm i\infty$ for any fixed $|n|$, we can assert that the ratio $\{h_n^-(\nu)/h_n^+(\nu)\}$ is free of zeros and poles in the strip

$$|\mathrm{Re}(\nu)| < C_n, \tag{6.14}$$

where C_n is large enough as $|n| \gg 1$.

Some additional information on the location of zeros of $h_n^{\pm}(\cdot)$ for all values of $|n|$ is given in appendix A so that the zeros $\pm\nu_m^-(|n|)$ and poles $\pm\nu_m^+(|n|)$ of the ratio $\{h_n^-(\nu)/h_n^+(\nu)\}$ are on the real axis and $|\nu_m^{\pm}(|n|)| > \sqrt{1/4 + |n|^2}$ for any m and $|n|$ with $C_n = \min_m\{|\nu_m^+(|n|)|, |\nu_m^-(|n|)|\}$.

6.2. *Integral representation of the solution to (6.1)*

For any ν from the strip $|\mathrm{Re}(\nu)| < C_n$ we fix the branch of the function

$$\log\left\{\frac{h_n^-(\nu)}{h_n^+(\nu)}\right\} = \log\left\{\frac{1 - \mathrm{d}_{\theta_1} P_{\nu-1/2}^{-|n|}(\cos\theta_1)[\mathrm{d}_{\theta_1} P_{\nu-1/2}^{-|n|}(-\cos\theta_1)]^{-1}}{1 + \mathrm{d}_{\theta_1} P_{\nu-1/2}^{-|n|}(\cos\theta_1)[\mathrm{d}_{\theta_1} P_{\nu-1/2}^{-|n|}(-\cos\theta_1)]^{-1}}\right\} \tag{6.15}$$

by the condition

$$\log\left\{\frac{h_n^-(\nu)}{h_n^+(\nu)}\right\} = O(\exp[-i\nu(\pi - 2\theta_1)]), \quad \nu \to -i\infty \tag{6.16}$$

and conduct the branch cuts from the zeros of $h_n^{\pm}$ to ∞ and $-\infty$ along the real axis so that the function $\log\{h_n^-(\cdot)/h_n^+(\cdot)\}$ is regular in $|\mathrm{Re}(\nu)| < C_n$. The parameter $|n|$ is arbitrarily fixed. Recall that $C_0 = 3/2$.

Consider an auxiliary difference equation

$$t_n\left(\nu + \frac{1}{2}\right) - t_n\left(\nu - \frac{1}{2}\right) = \log\left\{\frac{h_n^-(\nu)}{h_n^+(\nu)}\right\}, \tag{6.17}$$

where

$$t_n(\nu) = \log s_n(\nu). \tag{6.18}$$

In order to derive the solution of (6.17) and hence of (6.1) we introduce

$$\Lambda_n(\xi) = \int_{-\mathrm{i}\infty}^{\xi} \mathrm{d}\tau \log\left\{\frac{h_n^-(\tau)}{h_n^+(\tau)}\right\}, \tag{6.19}$$

where the integration is performed along the imaginary axis. The function $\Lambda_n(\cdot)$ is regular in $|\mathrm{Re}(\xi)| < C_n$ and the integral exponentially converges. In particular, $\lambda_n := \Lambda_n(\mathrm{i}\infty)$ is finite. It is worth noting that

$$\left.\begin{aligned} \Lambda_n(\xi) &= O(\exp[-\mathrm{i}\xi(\pi - 2\theta_1)]), \quad \xi \to -\mathrm{i}\infty, \\ \Lambda_n(\xi) &= \lambda_n + O(\exp[\mathrm{i}\xi(\pi - 2\theta_1)]), \quad \xi \to \mathrm{i}\infty. \end{aligned}\right\} \tag{6.20}$$

By making use of the results in appendix B we find a particular solution of (6.17) in the form

$$t_n(\nu) = \frac{\pi}{2\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\Lambda_n(\xi)\,\mathrm{d}\xi}{\cos^2(\pi[\xi - \nu])}, \tag{6.21}$$

which is regular in $|\mathrm{Re}(\nu)| < C_n + 1/2$ and is estimated as

$$\left.\begin{aligned} t_n(\nu) &= O(\exp[-\mathrm{i}\nu(\pi - 2\theta_1)]), \quad \nu \to -\mathrm{i}\infty, \\ t_n(\xi) &= \lambda_n + O(\exp[\mathrm{i}\nu(\pi - 2\theta_1)]), \quad \nu \to \mathrm{i}\infty. \end{aligned}\right\} \tag{6.22}$$

Remark. Integration by parts in (6.21) enables one to write

$$t_n(\nu) = \frac{\mathrm{i}}{2} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \log\left\{\frac{h_n^-(\xi)}{h_n^+(\xi)}\right\} (\tan(\pi[\xi - \nu]) - \mathrm{i})\,\mathrm{d}\xi. \tag{6.23}$$

As a result, we find a particular solution to (6.1)

$$s_n(\nu) = \exp(t_n(\nu)) = \exp\left\{\frac{\pi}{2\mathrm{i}} \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\Lambda_n(\xi)\,\mathrm{d}\xi}{\cos^2(\pi[\xi - \nu])}\right\}, \quad |\mathrm{Re}(\nu)| < C_n + \frac{1}{2}. \tag{6.24a,b}$$

The solution (6.24) is meromorphically continued on the whole complex plane by means of (6.1). One also has

$$s_n(\nu) = s_n^{\pm}(1 + o(1)) \quad \nu \to \pm\mathrm{i}\infty, \tag{6.25}$$

with some constants $s_n^{\pm}$. In particular, as $n = 0$ the solution $s_0(\cdot)$ is regular in the strip

$$|\mathrm{Re}(\nu)| < 2. \tag{6.26}$$

Therefore,

$$\gamma_0(\nu) = s_0\left(\nu - \tfrac{1}{2}\right) \tag{6.27}$$

is regular and bounded in the strip

$$-\tfrac{3}{2} < \mathrm{Re}(\nu) < \tfrac{5}{2}. \tag{6.28}$$

In the same manner we have that the meromophic function $\gamma_n(\cdot) = s_n(\cdot - 1/2)$ is regular and bounded in the strip

$$-C_n < \mathrm{Re}(\nu) < C_n + 1, \tag{6.29}$$

where C_n are sufficiently large for large $|n|$. For other finite $|n| = 1, 2, \ldots$ we can write the estimate $C_n > 1$ (see appendix A for the estimates of the roots of $h_n^{\pm}$).

7. Integral representations for the eigenmodes

We turn to the explicit integral formulae for the eigenmodes of the point spectrum. The expression (5.13) is reduced to an equivalent form

$$\beta_n(\nu) = \frac{(-1)2^{|n|-1}}{\sqrt{\pi}} \left. \frac{\mathrm{d}}{\mathrm{d}x} P^{-|n|}_{\nu-1/2}(x) \right|_{x=0} \frac{\left\{ \Gamma\left(\nu - |n| + \frac{1}{2}\right)\right\}^{-1}}{\cos(\pi\nu)}. \tag{7.1}$$

As a result, exploiting the identity $\Gamma(1/2 - z)\Gamma(1/2 + z) = \pi/\cos(\pi z)$, from (5.3) and (5.13) we arrive at a countable number of solutions satisfying (2.3), (2.5) and (2.6), (2.7)

$$\begin{aligned} U_n(r, \theta, \varphi) &= \frac{\mathrm{e}^{-\mathrm{i}n\varphi}}{2\pi\mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}\nu r^{\nu-1/2} \left(\frac{P^{-|n|}_{\nu-1/2}(\cos\theta)}{\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-1/2}(\cos\theta_1)} - \frac{P^{-|n|}_{\nu-1/2}(-\cos\theta)}{\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-1/2}(-\cos\theta_1)} \right) \\ &\quad \times \frac{\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-1/2}(\cos\theta_1) \exp(\nu \log K) \Gamma\left(-\nu + |n| + \frac{1}{2}\right) s_n\left(\nu - \frac{1}{2}\right)}{[1 + \mathrm{d}_{\theta_1} P^{-|n|}_{\nu-1/2}(\cos\theta_1)[\mathrm{d}_{\theta_1} P^{-|n|}_{\nu-1/2}(-\cos\theta_1)]^{-1}]}, \end{aligned} \tag{7.2}$$

where the constant factor $((-1)^{|n|+1}2^{|n|-1})/\sqrt{\pi}^3$ in the integrand has been omitted.

First, consider $n = 0$ in (7.2). We represent a part of the integrand as

$$\begin{aligned} &\left(P_{\nu-1/2}(\cos\theta)\, \mathrm{d}_{\theta_1} P_{\nu-1/2}(-\cos\theta_1) - P_{\nu-1/2}(-\cos\theta)\, \mathrm{d}_{\theta_1} P_{\nu-1/2}(\cos\theta_1)\right) \\ &\quad \times \frac{s_0\left(\nu - \frac{1}{2}\right) \exp(\nu \log K) \Gamma\left(-\nu + \frac{1}{2}\right)}{[\mathrm{d}_{\theta_1} P_{\nu-1/2}(\cos\theta_1) + \mathrm{d}_{\theta_1} P_{\nu-1/2}(-\cos\theta_1)]} \end{aligned} \tag{7.3}$$

and conclude that it is regular in the strip

$$-\tfrac{3}{2} < \mathrm{Re}(\nu) < \tfrac{1}{2}, \tag{7.4}$$

because the ratio

$$\frac{\mathrm{d}_{\theta_1} P_{\nu-1/2}(\pm\cos\theta_1)}{[\mathrm{d}_{\theta_1} P_{\nu-1/2}(\cos\theta_1) + \mathrm{d}_{\theta_1} P_{\nu-1/2}(-\cos\theta_1)]} \tag{7.5}$$

is regular in the strip $-3/2 < \mathrm{Re}(\nu) < 3/2$. (The points $\nu = \pm 1/2$ are removable singularities for the latter ratio.)

In the same manner, analysis of the location of poles in the meromorphic integrand of (7.2) as $|n| = 1, 2, \ldots$ enables one to conclude that the integrand is regular in the strip

$$\tfrac{1}{2} - \varkappa_n < \mathrm{Re}(\nu) < \tfrac{1}{2}, \tag{7.6}$$

$|n| = 0, 1, 2\ldots$ so that $\varkappa_n \geqslant 3/2 + \varepsilon$ for some positive ε. The latter inequalities mean that

$$|U_n(r, \theta, \varphi)| \leqslant \mathrm{const.}\, r^{-\varkappa_n} \quad \text{as } r \to \infty, \tag{7.7}$$

and

$$|U_n(r, \theta, \varphi)| \leqslant \mathrm{const.} \quad \text{as } r \to 0, \tag{7.8}$$

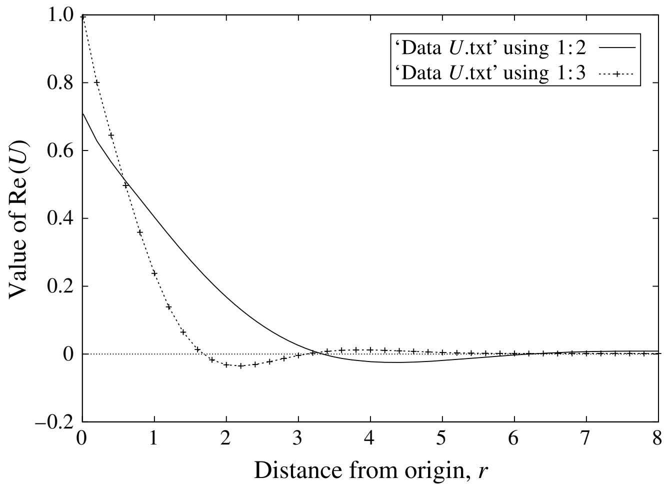

FIGURE 5. The values of $\mathrm{Re}\, U_0(r, \theta)|_{\theta=\pi/3}$ (solid line) for $K = 1$ and (dotted line with crosses) for $K = 2$, $\theta_1 = \pi/4$.

where const. does not depend on (θ, φ). It is worth commenting that deforming the contour of integration in (7.2) appropriately, one can easily obtain that $U_0(r, \theta, \varphi) = \text{const.} + O(r^{\delta_0})$ with some $\delta_0 > 0$ and $U_n(r, \theta, \varphi) = O(r^{\delta_n})$, $\delta_n > 0$, $r \to 0$.

Remark that $\partial u^n_{\nu+1}/\partial\theta|_{\theta=\pi/2}$, where u^n_ν is the meromorphic integrand of (7.2), has no poles in the strip $-1 \leqslant \mathrm{Re}(\nu) \leqslant 0$ as required. Indeed, in view of the boundary condition (3.10) the latter follows from that $u^n_\nu|_{\theta=\pi/2}$ has no poles in this strip.

The variation of the real part of $U_0(\cdot, \theta)|_{\theta=\pi/3}$ as a function of the distance r from the origin is shown for two spectral values $K = 1$ and $K = 2$ in (figure 5), $\theta_1 = \pi/4$.

It is worth noting that the expression (7.2) enables one to derive the exact power series asymptotics of $U_n(\cdot, \theta, \varphi)$ as $r \to \infty$ or $r \to 0$. The integral in (7.2) exponentially converges as $\theta \in (\theta_1, \pi/2)$ for $r > 0$. The solutions (7.2) have finite energy $E(U_n)$ and, therefore, are the eigenfunctions of the point spectrum for any $K > 0$. The point spectrum is of the infinite multiplicity.

7.1. *Eigenmodes in the bay (cliff) water-wave problems*

We now turn to the representation (4.7) with the integrand (4.8). In a formal way the construction is very similar to that for (7.2). Thus we have, verifying that the conditions (2.13) are satisfied,

$$V_n(r, \theta, \varphi) = \frac{\cos(\mu_n[\varphi + \Phi])}{2\pi \mathrm{i}} \int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty} \mathrm{d}\nu\, r^{\nu-1/2} \times \left(\frac{P^{-\mu_n}_{\nu-1/2}(\cos\theta)}{\mathrm{d}_{\theta_1} P^{-\mu_n}_{\nu-1/2}(\cos\theta_1)} - \frac{P^{-\mu_n}_{\nu-1/2}(-\cos\theta)}{\mathrm{d}_{\theta_1} P^{-\mu_n}_{\nu-1/2}(-\cos\theta_1)} \right) \times \frac{\mathrm{d}_{\theta_1} P^{-\mu_n}_{\nu-1/2}(\cos\theta_1) \exp(\nu \log K) \tilde{s}_n\left(\nu - \frac{1}{2}\right)}{\Gamma\left(\nu - \mu_n + \frac{1}{2}\right) \cos(\pi\nu)[1 + \mathrm{d}_{\theta_1} P^{-\mu_n}_{\nu-1/2}(\cos\theta_1)[\mathrm{d}_{\theta_1} P^{-\mu_n}_{\nu-1/2}(-\cos\theta_1)]^{-1}]}, \quad (7.9)$$

$\mu_n = \pi n/2\Phi$, where the constant $-2^{\mu_n}/\sqrt{\pi}$ is omitted in the integrand. $\tilde{s}_n(\cdot)$ solves the equation

$$\tilde{s}_n\left(\nu+\frac{1}{2}\right)=\left\{\frac{\tilde{h}_n^-(\nu)}{\tilde{h}_n^+(\nu)}\right\}\tilde{s}_n\left(\nu-\frac{1}{2}\right), \tag{7.10}$$

$$\tilde{h}_n^{\pm}(\nu)=\mathrm{d}_{\theta_1}P_{\nu-1/2}^{-\mu_n}(-\cos\theta_1)\pm\mathrm{d}_{\theta_1}P_{\nu-1/2}^{-\mu_n}(\cos\theta_1) \tag{7.11}$$

and has the integral representation

$$\tilde{s}_n(\nu)=\exp(t_n(\nu))=\exp\left\{\frac{\pi}{2\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\tilde{\Lambda}_n(\xi)\,\mathrm{d}\xi}{\cos^2(\pi[\xi-\nu])}\right\},\quad |\mathrm{Re}(\nu)|<\tilde{C}_n+\frac{1}{2},\quad (7.12a,b)$$

with

$$\tilde{\Lambda}_n(\xi)=\int_{-\mathrm{i}\infty}^{\xi}\log\left\{\frac{\tilde{h}_n^-(\tau)}{\tilde{h}_n^+(\tau)}\right\}\mathrm{d}\tau. \tag{7.13}$$

The expression (7.12) is meromorphically continued on the whole complex plane by means of (7.10). One also has

$$\tilde{s}_n(\nu)=\tilde{s}_n^{\pm}(1+o(1)),\quad \nu\to\pm\mathrm{i}\infty, \tag{7.14}$$

with some constants $\tilde{s}_n^{\pm}$. In particular, as $n=0$ the solution $\tilde{s}_0(\cdot)$ is regular in the strip $|\mathrm{Re}(\nu)|<2$. Therefore,

$$\tilde{\gamma}_0(\nu)=\tilde{s}_0\left(\nu-\tfrac{1}{2}\right) \tag{7.15}$$

is regular and bounded in the strip $-3/2<\mathrm{Re}(\nu)<5/2$.

Consider $n=0$ and $\mu_0=0$ then the expression (7.9) is indeed the solution of the problem because in the strip $-1\leqslant\mathrm{Re}(\nu)\leqslant 0$ there are no poles of $(\partial v_{\nu+1}^0(\omega))/\partial\theta|_{\theta=\pi/2}$, where v_ν^0 is the meromorphic integrand of (7.9). In this case we have

$$\begin{aligned}V_0(r,\theta,\varphi) &= \frac{1}{2\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{+\mathrm{i}\infty}\mathrm{d}\nu r^{\nu-1/2}\left(\frac{P_{\nu-1/2}(\cos\theta)}{\mathrm{d}_{\theta_1}P_{\nu-1/2}(\cos\theta_1)}-\frac{P_{\nu-1/2}(-\cos\theta)}{\mathrm{d}_{\theta_1}P_{\nu-1/2}(-\cos\theta_1)}\right)\\ &\quad\times\frac{\mathrm{d}_{\theta_1}P_{\nu-1/2}(\cos\theta_1)\exp(\nu\log K)\Gamma\left(-\nu+\frac{1}{2}\right)\tilde{s}_0\left(\nu-\frac{1}{2}\right)}{[1+\mathrm{d}_{\theta_1}P_{\nu-1/2}(\cos\theta_1)[\mathrm{d}_{\theta_1}P_{\nu-1/2}(-\cos\theta_1)]^{-1}]}.\end{aligned} \tag{7.16}$$

One can obtain the asymptotics

$$V_0(r,\theta,\varphi)\sim c_0(\theta,\varphi)r^{-\nu_0}\quad\text{as } r\to\infty, \tag{7.17}$$

and

$$V_0(r,\theta,\varphi)\sim\text{const.}\quad\text{as } r\to 0, \tag{7.18}$$

where const. does not depend on (θ,φ) and $c_0(\theta,\varphi)$ is specified by the residue of the integrand in the nearest to the origin negative pole at $-\nu_0$, $\nu_0>3/2$. It is useful to note that $V_0(r,\theta,\varphi)=U_0(r,\theta,\varphi)$ from (7.2) as $n=0$.

For any $K>0$ the expression (7.16) specifies an eigenfunction of the point spectrum. The multiplicity of each eigenvalue is equal to one provided $\pi/2\Phi$ is not rational, which means that Φ is not a rational multiple of $\pi/2$. However, if $\Phi=\pi p/2q$, where p, q are positive integers, the multiplicity of the eigenvalue becomes infinite. The eigenfunctions $V_{kp}(\cdot)$, $k=1, 2\ldots$ corresponding to $\mu_n=\pi n/2\Phi=kq$ with $n=kp$ are then described by (7.2). In this case, the eigenfunctions can be expressed in terms of those from the set (7.2) for the axially symmetric problem. This circumstance obviously shows the interconnection of the constructed solutions for the sandbank and bay (cliff) water-wave problems. The eigenfunction in the bay (cliff) water-wave problem for Φ being a rational multiple of $\pi/2$ may be used for construction of the eigenfunction for the axially symmetric sandbank water-wave problem. To this end, one may use the even with respect to φ continuation of the eigenfunction across the side walls $B_{\pm}$.

8. Concluding remarks

In this work we studied the eigensolutions of the point spectrum for a special class of the water-wave problems in the 3-D infinite domains. The point spectrum covers the semi-axis $K>0$ and has infinite multiplicity or the eigenvalues may be simple for the bay (cliff) water-wave problem if $\pi/2\Phi$ is not rational. The corresponding eigensolutions are represented in an integral Mellin form with the integrand depending on a double special function. The eigensolutions decay at infinity as some powers of the distance from the conical point so they are not so strongly localized as those with exponential decay in some other problems with a point spectrum. Nevertheless, such waves might be called cone-vertex trapped modes by analogy with the edge waves in the sloping beach domain. The solutions are also bounded at the origin.

It is now worth commenting on the uniqueness results for the axially symmetric problem in deep water discussed by Kuznetsov *et al.* (2002, § 3.2.2.4). Only axially symmetric solutions (with no dependence on φ) of the homogeneous problem are discussed in § 3.2.2.4, which corresponds to the case $n=0$ in the set of solutions (7.2). It seems that applicability of the method represented in § 3.2.2.4 and of the results require additional consideration and adaptation to the axially symmetric problem with the conical bottom. On a formal level, the uniqueness results are valid for the axially symmetric solutions for the water domains with no bottom and with bounded submersed bodies (as is claimed at the beginning of § 3.2.2.1). Nevertheless, one could try to apply the method of § 3.2.2.4 to our problem for the axially symmetric water domain (figure 1). As we assert there is no uniqueness in the axisymmetric problem under consideration which is shown by the construction of the eigenmodes (7.2) including the case $n=0$. Perhaps, in order to have a correctly formulated problem (in the sense of Hadamard) one might consider an 'imperfect' boundary condition on the conical bottom, e.g. a condition on the bottom with infiltration.

It is also useful to remark that the problem under consideration admits obvious rescaling of the coordinate $r\rightarrow Kr$, which means that it is sufficient to compute an eigenfunction for any fixed value of K. For the other values of K the eigenfunction can be computed by a simple rescale of the coordinates.

It may be of interest to have additional numerical results representing the forms of the corresponding eigenoscillations. On the other hand, completeness of the constructed eigensolutions requires a study. The numerical and additional analytical elaboration of the double special function $s_n(\cdot)$ as well as that of the location of zeros for the functions $h_n^{\pm}(\cdot)$ are useful, which may be a subject of the further work.

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Appendix A. Some estimates for the roots of the equations $h_n^{\pm}(\nu)=0$

Consider regular problems ($|n|=1, 2, \ldots, x=\cos\theta$)

$$-\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}Y^+(x)+\frac{|n|^2}{1-x^2}Y^+(x)=\lambda_\nu^+ Y^+(x), \quad x\in(0,x_1), x_1=\cos\theta_1, \tag{A 1}$$

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}Y^+(x)\right|_{x=0}=0, \quad \left.\frac{\mathrm{d}}{\mathrm{d}x}Y^+(x)\right|_{x=\cos\theta_1}=0, \tag{A 2}$$

where $Y^+(x)=P_{\nu-1/2}^{-|n|}(-x)+P_{\nu-1/2}^{-|n|}(x)$ and $\lambda_\nu^+(|n|)=\nu_+^2(|n|)-1/4$ is the spectral parameter depending on $|n|$. The differential operator of the first summand in the left-hand side of the equation is denoted $\mathscr{L}^+$. By use of the standard calculations, integrating by parts, we arrive at

$$\int_0^{x_1}\mathscr{L}^+Y^+(x)\overline{Y^+(x)}\,\mathrm{d}x=\int_0^{x_1}(1-x^2)\left|\frac{\mathrm{d}Y^+(x)}{\mathrm{d}x}\right|^2\mathrm{d}x \\ =\int_0^{x_1}\left[\left(\nu_+^2-\frac{1}{4}\right)|Y^+(x)|^2-\frac{|n|^2}{1-x^2}|Y^+(x)|^2\right]\mathrm{d}x. \tag{A 3}$$

As a result, any $\nu_m^+(|n|)$ corresponding to the non-trivial function $Y^+(\cdot)$ is real and satisfies the inequality

$$(\nu_m^+(|n|))^2\geqslant\frac{1}{4}+|n|^2+|n|^2\int_0^{x_1}\frac{x^2|Y^+(x)|^2}{1-x^2}\mathrm{d}x\left[\int_0^{x_1}|Y^+(x)|^2\mathrm{d}x\right]^{-1}. \tag{A 4}$$

It also is a root of the equation

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}(P_{\nu-1/2}^{-|n|}(-x)+P_{\nu-1/2}^{-|n|}(x))\right|_{x=\cos\theta_1}=0 \tag{A 5}$$

subjected to the limitations

$$|\nu_+(|n|)|>\sqrt{\frac{1}{4}+|n|^2}. \tag{A 6}$$

In a similar manner, one can prove that the zeros $\nu_m^-(|n|)$ of the equation

$$\left.\frac{\mathrm{d}}{\mathrm{d}x}(P_{\nu-1/2}^{-|n|}(-x)-P_{\nu-1/2}^{-|n|}(x))\right|_{x=\cos\theta_1}=0 \tag{A 7}$$

are real and satisfy the inequality

$$|\nu_-(|n|)|>\sqrt{\frac{1}{4}+|n|^2}. \tag{A 8}$$

Appendix B. Solution of some functional difference equations

We discuss construction of the integral representation for $s_n(\cdot)$. To this end, we consider a simple proposition dealing with the functional difference equation

$$t(\nu+h)-t(\nu-h)=l(\nu), \tag{B 1}$$

with $|l(\nu)|\leqslant C\exp(-d|\mathrm{Im}\,\nu|)$ as $\nu\to\pm\mathrm{i}\infty$, $d>0$. $l(\cdot)$ is a regular function in the complex plane C_* with the branch cuts $\{(-\infty,-\nu_*]\cup[\nu_*,\infty)\}$. Remark that $h=1/2$ in our case. Introduce a regular in C_* function

$$\Lambda(\nu)=\int_{-\mathrm{i}\infty}^{\nu} l(\tau)\,\mathrm{d}\tau. \tag{B 2}$$

Consider

$$t_1(\nu)=\frac{(-1)}{4\pi\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} l(\tau)\frac{\sin\left(\dfrac{\pi}{h}[\tau-\nu]\right)}{1+\cos\left(\dfrac{\pi}{h}[\tau-\nu]\right)}\,\mathrm{d}\tau=\frac{(-1)}{4h\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} l(\tau)\tan\left(\frac{\pi}{2h}[\tau-\nu]\right)\,\mathrm{d}\tau. \tag{B 3}$$

The function $t_1(\cdot)$ is regular in the strip $-\nu_*-h<\mathrm{Re}(\nu)<\nu_*+h$. On the other hand, the integrand in (B 3) at the point $\nu+h$ is identical with that at the point $\nu-h$ due to the periodicity of $\tan(\pi/2h[\tau-\cdot])$. The difference $t_1(\nu+h)-t_1(\nu-h)$ is then specified by the integral over the closed contour $(-\mathrm{i}\infty+h,+\mathrm{i}\infty+h)\cup(\mathrm{i}\infty-h,-\mathrm{i}\infty-h)$, containing the pole at ν in the interior, which is passed around in the right direction. As a result, one has

$$t_1(\nu+h)-t_1(\nu-h)=(-\pi/2h)\,\mathrm{res}_{\tau=\nu}\frac{l(\tau)\sin\left(\dfrac{\pi}{h}[\tau-\nu-h]\right)}{1+\cos\left(\dfrac{\pi}{h}[\tau-\nu-h]\right)}=l(\nu). \tag{B 4}$$

LEMMA 1. *The integral representation (B 3) for $t_1(\cdot)$ is a particular solution of the equation (B 1) that is regular in the strip $|\mathrm{Re}(\nu)|<\nu_*+h$.*

Remark that the general solution of (B 1) can be represented as a sum of a solution $t_0(\cdot)$ of the homogeneous equation

$$t_0(\nu+h)-t_0(\nu-h)=0 \tag{B 5}$$

and the particular solution $t_1(\cdot)$ in (B 3). Integration by parts enables us to write

$$t_1(\nu)=\frac{(-1)}{4h\mathrm{i}}\Lambda(\tau)\tan\left(\frac{\pi}{2h}[\tau-\nu]\right)\Bigg|_{-\mathrm{i}\infty}^{\mathrm{i}\infty}+\frac{\pi}{8h^2\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\Lambda(\tau)\,\mathrm{d}\tau}{\cos^2\left(\dfrac{\pi}{2h}[\tau-\nu]\right)}, \tag{B 6}$$

where the constant $\Lambda(\mathrm{i}\infty)/4h$ is a solution of the homogeneous equation. Thus a particular solution can be written in the form.

$$t(\nu)=\frac{\pi}{8h^2\mathrm{i}}\int_{-\mathrm{i}\infty}^{\mathrm{i}\infty}\frac{\Lambda(\tau)\,\mathrm{d}\tau}{\cos^2\left(\dfrac{\pi}{2h}[\tau-\nu]\right)}. \tag{B 7}$$

The latter representation enables us to write the solution (6.21) and then (6.24).

Remark. One may obtain the analytic continuation of the expression in (B 3) from the strip $|\mathrm{Re}(\nu)| < \nu_* + h$ on any wider one to the left-hand side from this strip. To this end, the contour of integration is deformed so that $\mathscr{G} = G_1 \cup G_2 \cup G_3$ on the left-hand side from $(-\mathrm{i}\infty, \mathrm{i}\infty)$ so that it consists of $G_1 = (-\mathrm{i}\infty, -a - \mathrm{i}0)$, $G_3 = (a + \mathrm{i}0, \mathrm{i}\infty)$ and G_2 which comprises the part $[-a, -\nu_*]$ of the branch cut $(-\infty, -\nu_*]$. The similar deformation in the opposite direction is used for the continuation on the right-hand side from the strip. Such a continuation is equivalent to making use of the functional equation (B 1).

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