Can. J. Math. Vol. 45 (3), 1993 pp. 612–625

SECOND-ORDER GÂTEAUX DIFFERENTIABLE BUMP FUNCTIONS AND APPROXIMATIONS IN BANACH SPACES

D. McLAUGHLIN, R. POLIQUIN, J. VANDERWERFF AND V. ZIZLER

ABSTRACT In this paper we study approximations of convex functions by twice Gâteaux differentiable convex functions We prove that convex functions (respectively norms) can be approximated by twice Gâteaux differentiable convex functions (respectively norms) in separable Banach spaces which have the Radon-Nikodym property and admit twice Gâteaux differentiable bump functions New characterizations of spaces isomorphic to Hilbert spaces are shown Locally uniformly rotund norms that are limits of C^k-smooth norms are constructed in separable spaces which admit C^k-smooth norms

1. **Introduction.** It is known that the existence of a twice Fréchet differentiable bump function on a Banach space X has a profound impact on the structure of X and thus is a very restrictive condition on X. For example the space $(\sum_{n=1}^{\infty} l_4^n)_2$ has a norm with modulus of smoothness of power type 2 yet admits no twice Fréchet differentiable bump function; see *e.g.* [DGZ$_2$, Chapter V]. It is also known that there is a norm on l_2 which cannot be approximated uniformly on bounded sets by functions with uniformly continuous second derivatives ([V$_1$]). However, it seems to be unknown whether every norm on l_2 can be approximated uniformly on bounded sets by twice Fréchet differentiable convex functions.

In Section 2, it is shown that the situation is different in the case of second-order Gâteaux differentiability. Motivated by a recent paper of Borwein and Noll ([BN]), we show that if a separable Banach space admits a norm with modulus of smoothness of power type 2, then convex functions (respectively norms) can be approximated by twice Gâteaux differentiable convex functions (respectively norms). Thus, such approximations are valid, for example, in $(\sum_{n=1}^{\infty} l_4^n)_2$. Moreover, using techniques of [BN] and [DGZ$_1$], it is proven that a space with the Radon-Nikodym property (RNP) admits a norm with modulus of smoothness of power type 2 provided it admits a continuous twice Gâteaux differentiable bump function; see [Bou] for properties of RNP spaces. As an application of this, the isomorphic characterizations of Hilbert spaces in [DGZ$_1$] and [F] are improved.

The third section shows that a separable Banach space X admits a locally uniformly rotund norm which is a limit of C^k-smooth norms provided X admits a C^k-smooth norm.

This work was supported in part by the Natural Sciences and Engineering Research Council of Canada under grant OGP41983 for the second author and grant OGP7926 for the fourth author

Received by the editors September 30, 1991

AMS subject classification Primary 46B20, secondary 46B22, 46C05

All Banach spaces (in short spaces) considered here are over the real field. We will say a function $\phi: X \longrightarrow \mathbb{R}$ is *twice Gâteaux differentiable at* $x \in X$ provided that $\phi'(y)$ exists for y in a neighborhood of x, and that the limit

$$\phi''(x)(h,k) = \lim_{t \downarrow 0} \frac{1}{t}\big(\phi'(x+tk) - \phi'(x)\big)(h)$$

exists for each $h, k \in X$, and that $\phi''(\cdot,\cdot)$ is a continuous symmetric bilinear form. A function $f: X \longrightarrow \mathbb{R}$ has a *second-order directional Taylor expansion at* x_0 if

$$f(x_0 + th) = f(x_0) + t\langle y^*, h\rangle + (t^2/2)\langle Th, h\rangle + o(t^2) \quad (t \longrightarrow 0)$$

where $T: X \longrightarrow X^*$ is a bounded linear operator and $y^*: X \longrightarrow \mathbb{R}$ is continuous and linear; *cf.* [BN].

Recall that a C^k-smooth function is a real-valued function which is continuously k-times Fréchet differentiable. A norm $\|\cdot\|$ is *locally uniformly rotund* (LUR) if $\|x - x_n\| \longrightarrow 0$, whenever $2\|x\|^2 + 2\|x_n\|^2 - \|x + x_n\|^2 \longrightarrow 0$. A norm is *uniformly rotund* (UR) if $\|x_n - y_n\| \longrightarrow 0$, whenever $\{x_n\}$ is bounded and $2\|x_n\|^2 + 2\|y_n\|^2 - \|x_n + y_n\|^2 \longrightarrow 0$. We will use the notation $B_X = \{x : \|x\| \leq 1\}$, $S_X = \{x : \|x\| = 1\}$, $B_r = \{x : \|x\| \leq r\}$ and $B(x_0, \epsilon) = \{x : \|x - x_0\| \leq \epsilon\}$.

The *modulus of smoothness* $\rho_X(\tau)$ of $(X, \|\cdot\|)$ is defined for $\tau > 0$ by

$$\rho_X(\tau) = \sup\Big\{\frac{1}{2}(\|x+y\| + \|x-y\| - 2) : \|x\| = 1, \|y\| \leq \tau\Big\};$$

$\rho_X(\tau)$ is of *power type p*, for $1 < p \leq 2$, if there exists a $C > 0$ such that $\rho_X(\tau) \leq C\tau^p$. In particular, such a norm is *uniformly smooth*, that is $\lim_{\tau\downarrow 0} \rho(\tau)/\tau = 0$. Recall that X admits a uniformly smooth norm if and only if X is super-reflexive and admits a UR norm ([En, p. 287]); in addition, the UR norms are dense among all norms on X (see *e.g.* [B, Exercise 1, p. 211]). From the proof of [FWZ, Lemma 2.4] it is easy to see that a norm $\|\cdot\|$ on X with modulus of smoothness of power type 2 has Lipschitz derivative on its sphere. Moreover, a direct computation then shows that $\|\cdot\|^2$ has Lipschitz derivative on all of X; the details are in [DGZ$_2$, Chapter V]. Finally, a norm $\|\cdot\|$ is *Lipschitz smooth* at $x \neq 0$ if there exists a $C > 0$ so that $\|x+h\| + \|x-h\| - 2\|x\| \leq C\|h\|^2$ for all $h \in X$ (*cf.* [FWZ, Lemma 2.4]).

2. **Second-order Gâteaux differentiability and approximation.** The following theorem summarizes our main results.

THEOREM 2.1. *For a separable Banach space X, the following are equivalent.*

(a) X has the RNP *and admits a continuous twice Gâteaux differentiable bump function.*

(b) X has the RNP *and admits a continuous bump function with second-order directional Taylor expansion at each point.*

(c) X admits a norm with modulus of smoothness of power type 2.

https://doi.org/10.4153/CJM-1993-032-9 Published online by Cambridge University Press

(d) *Every norm on X is a limit of* UR *norms which are twice Gâteaux differentiable on $X \setminus \{0\}$ and have moduli of smoothness of power type 2.*

(e) *Every convex function which is bounded on bounded subsets of X can be approximated uniformly on bounded sets by twice Gâteaux differentiable convex functions whose first derivatives are also Lipschitz.*

REMARK. Notice that (a) does not follow trivially from (b). Indeed, there are bump functions that have second-order Taylor expansions and yet do not possess a second-order Gâteaux derivative; consider the function $t^3 \cos(1/t)$ (*cf.* [BN, Section 3, Remark 2]). However, this cannot occur for convex functions as was demonstrated in [BN, Theorem 3.1]. Also, it is relatively easy to obtain any of the first three conditions of the above theorem from either of the last two. The main effort will involve showing that (b) implies (c) and that (d) and (e) can be obtained from (c).

Some preliminary definitions and results will be given before proving Theorem 2.1.

DEFINITION. For $1 < p \leq 2$ a real-valued function ϕ defined on a Banach space X has *directional modulus of smoothness of power type p* at $x \in X$ if for each $h \in S_X$ there exist $C > 0$ and $\delta > 0$ such that

$$|\phi(x+th) + \phi(x-th) - 2\phi(x)| \leq Ct^p \text{ whenever } t \in [0,\delta]. \tag{2.1}$$

If this happens for all $x \in X$, we say that ϕ has *pointwise directional modulus of smoothness of power type p*. If the constant C in (2.1) does not depend on h, we say that ϕ has *pointwise modulus of smoothness of power type p at x*.

The function ϕ is said to be *directionally Lipschitz* at x_0 if there exists a $\delta > 0$ such that given $h \in S_X$ there is a $C_h > 0$ for which $|\phi(x_0 + th) - \phi(x_0)| \leq C_h|t|$ whenever $|t| \leq \delta$.

FACT 2.2. *Suppose ϕ is continuous and pointwise directionally Lipschitz at x_0. If ϕ has directional modulus of smoothness of power type p at x_0, then so does $\psi = \phi^{-2}$ provided $\phi(x_0) \neq 0$.*

PROOF. Let $f(x) = \phi^{-1}(x)$. Choose $\delta > 0$ and $K > 0$ so that $|f(x)| \leq K$ whenever $\|x - x_0\| \leq \delta$. Fix $h \in S_X$, then for $0 \leq t \leq \delta$, we have:

$$\begin{aligned}
&|f(x_0+th) + f(x_0-th) - 2f(x_0)| \\
&\qquad \leq K^3|\phi(x_0+th)\phi(x_0) + \phi(x_0-th)\phi(x_0) - 2\phi(x_0+th)\phi(x_0-th)| \\
&\qquad = K^3|\phi(x_0+th)[2\phi(x_0) - \phi(x_0+th) - \phi(x_0-th)] \\
&\qquad\qquad + [\phi(x_0-th) - \phi(x_0+th)]\,[\phi(x_0) - \phi(x_0+th)]| \\
&\qquad = O(t^p).
\end{aligned}$$

By the above inequality and the fact that f is directionally Lipschitz at x_0, for $0 \leq t \leq \delta$ we obtain:

$$\begin{aligned}
&|f^2(x_0+th) + f^2(x_0-th) - 2f^2(x_0)| \\
&\qquad = |[f(x_0+th) + f(x_0-th) - 2f(x_0)]\,[f(x_0+th) + f(x_0)] \\
&\qquad\qquad + [f(x_0-th) - f(x_0)]\,[f(x_0-th) - f(x_0+th)]| \\
&\qquad = O(t^p).
\end{aligned}$$

■

https://doi.org/10.4153/CJM-1993-032-9 Published online by Cambridge University Press

The following proposition shows that the bump function in [DGZ$_1$, Theorem III.1] need only have a pointwise directional modulus of smoothness.

PROPOSITION 2.3. *Assume that X has the* RNP. *If X admits a continuous pointwise directionally Lipschitz bump function ϕ with pointwise directional modulus of smoothness of power type p, then X admits an equivalent norm with modulus of smoothness of power type p.*

PROOF. We essentially follow the proof of [DGZ$_1$, Theorem III.1].

First define $\psi: X \longrightarrow \mathbb{R}\cup\{\infty\}$ by $\psi(x) = \phi^{-2}(x)$ if $\phi(x) \neq 0$ and $\psi(x) = \infty$ if $\phi(x) = 0$. Let ψ^* be the Fenchel conjugate function of ψ *i.e.* for $y \in X^*$

$$\psi^*(y) = \sup\{\langle y, x\rangle - \psi(x) : x \in X\}.$$

Because $\psi(x) = \infty$ outside a bounded set, the function ψ^* is finite, convex and w^*-lower semicontinuous on X^*. Because X has RNP, the function ψ^* is Fréchet differentiable at each point of a norm dense G_δ subset Ω of X^* (*cf.* [C]) with derivative in X (x^* is in the subdifferential of ψ at x if and only if x is in the subdifferential of ψ^* at x^*; see [ET, Corollary 5.2, p. 22]). Let $\tilde{\psi}$ denote the Fenchel conjugate of ψ^* on X. It is shown in the proof of [DGZ$_1$, Theorem III.1] that if $y_0 \in \Omega$ and $x_0 = (\psi^*)'(y_0)$, then $\big(x_0, \tilde{\psi}(x_0)\big)$ is a strongly exposed point of the epigraph of $\tilde{\psi}$ (exposed by $(y_0, -1)$). Because of strong exposedness, the point $\big(x_0, \tilde{\psi}(x_0)\big)$ actually belongs to the epigraph of ψ and this means that $\psi(x_0) = \tilde{\psi}(x_0) < \infty$. By Fact 2.2, ψ has directional modulus of smoothness of power type p at x_0. Because $\tilde{\psi}$ is convex, majorized by ψ and agrees with ψ at x_0, it is straightforward to verify $\tilde{\psi}$ has directional modulus of smoothness of power type p at x_0.

We can now use the argument in [BN, Proposition 2.2] to show that $\tilde{\psi}$ has modulus of smoothness of power type p at x_0. Indeed, choose $\delta > 0$ so that $\tilde{\psi}$ is bounded and continuous on $B(x_0, \delta)$. Define F_n by

$$F_n = \{h \in B_X : \tilde{\psi}(x_0 + th) + \tilde{\psi}(x_0 - th) - 2\tilde{\psi}(x_0) \leq n\|th\|^p \text{ for } 0 < t \leq \delta\}.$$

Now F_n is closed; moreover $\cup_{n=1}^{\infty} F_n = B_X$ because $\tilde{\psi}$ is bounded on $B(x_0, \delta)$ and has directional modulus of smoothness of power type p at x_0. According to the Baire Category Theorem, there is a neighborhood V of a point h_0, in the interior of B_X, and n an integer, such that

$$\tilde{\psi}(x_0 + th) + \tilde{\psi}(x_0 - th) - 2\tilde{\psi}(x_0) \leq n\|th\|^p \text{ for all } h \in V, \quad 0 < t \leq \delta.$$

Consider the cone generated by taking the convex hull of $-h_0$ and V. This cone contains B_r for some $r > 0$. For some $k \geq n$, by convexity one has,

$$\tilde{\psi}(x_0 + th) + \tilde{\psi}(x_0 - th) - 2\tilde{\psi}(x_0) \leq k\|th\|^p \text{ for all } \|h\| \leq r, \quad 0 < t \leq \delta. \tag{2.2}$$

Because $\tilde{\psi}$ is convex, from (2.2) it follows that $\tilde{\psi}'(x_0)$ exists and equals y_0 and that

$$|\tilde{\psi}(x_0 + h) - \tilde{\psi}(x_0) - \langle y_0, h\rangle| \leq n\|h\|^p \text{ for } \|h\| \leq \delta r.$$

https://doi.org/10.4153/CJM-1993-032-9 Published online by Cambridge University Press

To complete the proof of Proposition 2.3, one need only reproduce (word for word) the proof of [DGZ_1, Theorem III.1] starting form equation (4). The argument shows that using the Baire Category Theorem we can produce an equivalent norm on X^* which has modulus of rotundity of power type $(1 - \mathrm{p}^{-1})^{-1}$. By duality X admits a norm with modulus of smoothness of power type p; see [B, Lemma 3, p. 208]. ∎

REMARK. It is immediate that any function with a second-order directional Taylor expansion at x_0 is directionally Lipschitz at x_0 and has directional modulus of smoothness of power type 2 at x_0. In particular, Proposition 2.3 (with $\mathrm{p} = 2$) is valid for RNP spaces admitting continuous twice Gâteaux differentiable bump functions.

We now develop some results concerning the approximation of convex functions. In what follows $f \,\square\, g$ denotes the *infimal convolution* of the convex functions f and g on a Banach space X. In other words, $f \,\square\, g(x) = \inf\{f(y) + g(x - y) : y \in X\}$.

LEMMA 2.4. *Suppose X is a Banach space and let f be a convex function on X which is bounded on bounded sets. If $\{g_k\}$ is a sequence of convex functions such that $g_k(0) \le \frac{1}{k}$ and $g_k(x) \ge k\|x\| - \frac{1}{k}$ for all $x \in X$, then $f \,\square\, g_k \to f$ uniformly on bounded subsets of X.*

PROOF. Let $r > 0$ and suppose that f has Lipschitz constant K on B_{r+1}. For $x_0 \in B_r$ fixed and for each k we can choose y_k so that $f \,\square\, g_k(x_0) \ge f(y_k) + g_k(x_0 - y_k) - \frac{1}{k}$. For any $k \ge K + 1$ with $k \ge 3$ we have

$$\begin{aligned} f(x_0) + \frac{1}{k} &\ge f(x_0) + g_k(0) \ge f \,\square\, g_k(x_0) \\ &\ge f(y_k) + g_k(x_0 - y_k) - \frac{1}{k} \\ &\ge f(y_k) + k\|x_0 - y_k\| - \frac{2}{k}. \end{aligned} \tag{2.3}$$

Let $\Lambda_0 \in \partial f(x_0)$, then $\|\Lambda_0\|^* \le K$ since f has Lipschitz constant K on B_{r+1}. Because $f(y_k) - f(x_0) \ge \Lambda_0(y_k) - \Lambda_0(x_0)$, we have

$$f(x_0) - f(y_k) \le \|\Lambda_0\|^* \|y_k - x_0\| \le K\|y_k - x_0\|.$$

Thus it follows from (2.3) that

$$K\|y_k - x_0\| + \frac{3}{k} \ge k\|x_0 - y_k\|.$$

In other words,

$$\|x_0 - y_k\| \le \frac{3}{k(k - K)}.$$

In particular, $y_k \in B_{r+1}$ and so $|f(y_k) - f(x_0)| \le K\|y_k - x_0\|$. From this we obtain

$$\begin{aligned} f(y_k) + k\|x_0 - y_k\| - \frac{2}{k} &\ge f(x_0) - K\|x_0 - y_k\| + k\|x_0 - y_k\| - \frac{2}{k} \\ &\ge f(x_0) - \frac{2}{k}. \end{aligned} \tag{2.4}$$

Clearly the lemma follows from (2.3) and (2.4). ∎

In the following proposition, part (a) generalizes [BN, Theorem 5.2(1)] while part (b) is well-known (see *e.g.* [B]) and is given here for the reader's convenience.

https://doi.org/10.4153/CJM-1993-032-9 Published online by Cambridge University Press

PROPOSITION 2.5. *Let X be a Banach space which has a norm with modulus of smoothness of power type 2.*

(a) Any convex function f which is bounded on bounded subset of X can be approximated uniformly on bounded sets by convex functions with Lipschitz derivatives.

(b) Every norm on X can be approximated by norms with moduli of smoothness of power type 2.

PROOF. Let $\|\cdot\|$ have modulus of smoothness of power type 2. Then $\|\cdot\|^2$ has Lipschitz derivative on all of X; hence so does g_k where $g_k(x) = k^4\|x\|^2$. Easily $g_k(x) \geq k\|x\| - \frac{1}{k}$ for all k and $g_k(0) = 0$, therefore $f \square g_k \longrightarrow f$ uniformly on bounded sets by Lemma 2.4.

To see that $f_k = f \square g_k$ has Lipschitz derivative for each k we use the Mean Value Theorem to choose $C_k > 0$ such that

$$g_k(x+h) + g_k(x-h) - 2g_k(x) \leq C_k\|h\|^2 \tag{2.5}$$

for all $x, h \in X$; *cf.* [FWZ, Lemma 2.4]. Fix an arbitrary $x_0 \in X$. Since X is reflexive we choose y_k so that $f_k(x_0) = f(y_k) + g_k(x_0 - y_k)$. Then, using (2.5), for any $h \in X$ we have

$$\begin{aligned} f_k(x_0+h) + f_k(x_0-h) - 2f_k(x_0) &\leq f(y_k) + g_k(x_0+h-y_k) + f(y_k) + g_k(x_0-h-y_k) \\ &\quad - 2\big(f(y_k) + g_k(x_0-y_k)\big) \\ &= g_k(x_0-y_k+h) + g_k(x_0-y_k-h) - 2g_k(x_0-y_k) \\ &\leq C_k\|h\|^2. \end{aligned}$$

Since C_k does not depend on x_0, it follows from the proof of [FWZ, Lemma 2.4] (see [DGZ$_2$, Chapter V]) that f_k' is Lipschitz. This proves (a).

To see (b), for a given norm $|\cdot|$ let $f = |\cdot|^2$. Then by (a) the norms $|\cdot|_k = (f \square g_k)^{\frac{1}{2}}$ have moduli of smoothness of power type 2 and converge to $|\cdot|$ uniformly on bounded sets. ∎

To obtain approximating functions which are twice Gâteaux differentiable we need a lemma whose proof is almost identical to the proof of [FWZ, Theorem 3.1].

LEMMA 2.6. *Let X be a separable Banach space and let $\epsilon > 0$ and $r > 0$ be given.*

(a) If f is a convex function whose first derivative is Lipschitz, then there is a convex function g such that $|g(x) - f(x)| < \epsilon$ for all $x \in B_r$ and g is twice Gâteaux differentiable with Lipschitz first derivative.

(b) If $\|\cdot\|$ is a norm with modulus of smoothness of power type 2, then there is a norm $\|\cdot\|_1$ such that $(1-\epsilon)\|x\| \leq \|x\|_1 \leq (1+\epsilon)\|x\|$ for all x and $\|\cdot\|_1$ is twice Gâteaux differentiable on $X \setminus \{0\}$ and has modulus of smoothness of power type 2.

PROOF. To begin the proof we fix $\epsilon > 0$ and $r > 0$. Let $C \in \mathbb{R}$ be such that f is Lipschitz with constant C on B_{r+1}, and select a set $\{h_i\}_{i=0}^{\infty}$ dense in S_X. Next, fix a C^∞-smooth function $\phi_0\colon \mathbb{R} \longrightarrow \mathbb{R}$ such that ϕ_0 is nonnegative and even, vanishes outside

$[\frac{-\epsilon}{2C}, \frac{\epsilon}{2C}]$ and satisfies $\int_{\mathbb{R}} \phi_0 = 1$. Setting $f_0 = f$ and $\phi_n = 2^n \phi_0(2^n t)$ for $t \in \mathbb{R}$, $n \geq 1$, we define a sequence of functions $\{f_n: X \longrightarrow \mathbb{R}\}_{n=1}^{\infty}$ by

$$f_n(x) = \int_{\mathbb{R}^{n+1}} f_0\Big(x - \sum_{i=0}^{n} t_i h_i\Big) \prod_{i=0}^{n} \phi_i(t_i)\, dt_0 \cdots dt_n.$$

As in the proof of [FWZ, Theorem 3.1] there is a function $g: X \longrightarrow \mathbb{R}$ such that $f_n \longrightarrow g$ uniformly on bounded sets and g is twice Gâteaux differentiable with Lipschitz first derivative.

Moreover, for $x \in B_r$ we have

$$\begin{aligned} |f(x) - g(x)| &= \lim_n \Big| \int_{\mathbb{R}^{n+1}} \Big[f_0(x) - f_0\Big(x - \sum_{i=0}^{n} t_i h_i\Big)\Big] \prod_{i=0}^{n} \phi_i(t_i)\, dt_0 \cdots dt_n \Big| \\ &\leq \int_{|t_i| \leq \epsilon/(2^{i+1}C)} C \Big\| \sum_{i=0}^{n} t_i h_i \Big\| \prod_{i=0}^{n} \phi_i(t_i)\, dt_0 \cdots dt_n \\ &\leq C \frac{\epsilon}{C} = \epsilon. \end{aligned}$$

In case (b) where the function f is a norm we set $f_0(x) = \|x\|^2$. It follows that the function g as obtained above is convex and even. By (a) choose g so that $g(x) \in [\|x\|^2 - \epsilon, \|x\|^2 + \epsilon]$ whenever $\|x\| \leq 5$. If we set $B = \{x \in X : g(x) \leq 16\}$ then as in [FWZ] the Minkowski functional of B is an equivalent norm $|\cdot|$ which is twice Gâteaux differentiable on $X \setminus \{0\}$ and has modulus of smoothness of power type 2. Let $\|\cdot\|_1 = 4|\cdot|$. Now $\|x\|_1 = 4$ if and only if $g(x) = 16$ which implies $16 - \epsilon \leq \|x\|^2 \leq 16 + \epsilon$. Therefore, $(1-\epsilon)\|x\| \leq \|x\|_1 \leq (1+\epsilon)\|x\|$ for all $x \in X$. ∎

PROOF OF THEOREM 2.1. It is obvious that (a) ⇒ (b), while (b) ⇒ (c) follows from Proposition 2.3. Next it is shown that (c) ⇒ (d).

STEP 1. If X admits a norm with modulus of smoothness of power type 2, then every UR norm is a limit of UR norms with moduli of smoothness of power type 2.

Let $|\cdot|$ be UR and let $\epsilon > 0$. By Proposition 2.5(b) choose norms $|\cdot|_n$ with moduli of smoothness of power type 2 so that $(1-\epsilon)|x| \leq |x|_n \leq |x|$ and $|\cdot|_n \longrightarrow |\cdot|$. Choose $C_n \geq 2$ so that $|x+h|_n^2 + |x-h|_n^2 - 2|x|_n^2 \leq C_n |h|_n^2$ for all $x, h \in X$ and define $|||\cdot|||$ by

$$|||x||| = \Big(|x|_1^2 + \epsilon \sum_{n=1}^{\infty} \frac{2^{-n}}{C_n} |x|_n^2 \Big)^{\frac{1}{2}}.$$

Easily $|||x+h|||^2 + |||x-h|||^2 - 2|||x|||^2 \leq (C_1 + 1)|||h|||^2$ for all $x, h \in X$ and $(1-\epsilon)|x| \leq |||x||| \leq (1+\epsilon)|x|$. To see that $|||\cdot|||$ is UR, suppose that

$$2|||x_n|||^2 + 2|||y_n|||^2 - |||x_n + y_n|||^2 \longrightarrow 0.$$

From this it follows, for each k, that

$$2|x_n|_k^2 + 2|y_n|_k^2 - |x_n + y_n|_k^2 \longrightarrow 0.$$

Thus,

$$2|x_n|^2 + 2|y_n|^2 - |x_n + y_n|^2 \longrightarrow 0.$$

Therefore $|x_n - y_n| \longrightarrow 0$ which implies that $|||x_n - y_n||| \longrightarrow 0$. That is, $|||\cdot|||$ is UR.

STEP 2. If the initial norm in Lemma 2.6(b) is UR, then so is the norm in the conclusion.

Let $|\cdot|$ be UR with modulus of smoothness of power type 2. By uniform rotundity, for a fixed $r > 0$, given $\delta > 0$ there exists $\epsilon > 0$ so that $2|x|^2 + 2|y|^2 \geq |x+y|^2 + 4\epsilon$ whenever $|x| \leq r+1, |y| \leq r+1$ and $|x-y| \geq \delta$. Hence using $|\cdot|$ to construct the functions f_n as in the proof of Lemma 2.6, for $|x-y| \geq \delta$ and $|x| \leq r$, $|y| \leq r$, we have

$$\begin{aligned} f_n\Big(\frac{x+y}{2}\Big) &= \int_{\mathbb{R}^{n+1}} \Big|\frac{x+y}{2} - \sum_{\iota=0}^{n} t_\iota h_\iota\Big|^2 \prod_{\iota=0}^{n} \phi(t_\iota)\, dt_0 \ldots dt_n \\ &= \int_{\mathbb{R}^{n+1}} \Big|\frac{x - \sum_{\iota=0}^{n} t_\iota h_\iota}{2} + \frac{y - \sum_{\iota=0}^{n} t_\iota h_\iota}{2}\Big|^2 \prod_{\iota=0}^{n} \phi(t_\iota)\, dt_0 \ldots dt_n \\ &\leq \int_{\mathbb{R}^{n+1}} \Big(\frac{1}{2}\Big|x - \sum_{\iota=0}^{n} t_\iota h_\iota\Big|^2 + \frac{1}{2}\Big|y - \sum_{\iota=0}^{n} t_\iota h_\iota\Big|^2 - \epsilon\Big) \prod_{\iota=0}^{n} \phi(t_\iota)\, dt_0 \ldots dt_n \\ &= \frac{1}{2} f_n(x) + \frac{1}{2} f_n(y) - \epsilon. \end{aligned}$$

Now $f_n \longrightarrow f$ for some f, therefore

$$f\Big(\frac{x+y}{2}\Big) \leq \frac{1}{2} f(x) + \frac{1}{2} f(y) - \epsilon.$$

Let $B = \{x : f(x) \leq M\}$ be the unit ball of some norm $\|\cdot\|$ we will show that $\|\cdot\|$ is UR. Now $B \subset \{x : |x| \leq r\}$ for some $r > 0$. Given $\delta > 0$, there exists an $\epsilon > 0$ so that

$$f\Big(\frac{x+y}{2}\Big) \leq \frac{1}{2} f(x) + \frac{1}{2} f(y) - \epsilon \leq M - \epsilon$$

whenever $\|x - y\| \geq \delta$ and $x, y \in B$. Since f is convex and bounded on bounded sets, it is certainly uniformly continuous on B. Thus there is an $\eta > 0$ such that $\|u - v\| \leq \eta$ and $u, v \in B$ imply $|f(u) - f(v)| \leq \epsilon$. Hence

$$f\Big((1+\eta)\Big(\frac{x+y}{2}\Big)\Big) \leq f\Big(\frac{x+y}{2}\Big) + \epsilon \leq M.$$

This implies

$$\Big\|\frac{x+y}{2}\Big\| \leq \frac{1}{1+\eta}$$

whenever $x, y \in B$ and $\|x - y\| \geq \delta$. Thus $\|\cdot\|$ is UR. This finishes Step 2.

Finally, since the UR norms are dense among all norms in X, Step 1, Step 2 and Lemma 2.6(b) show that (c) $\Rightarrow$ (d).

One obtains (d) $\Rightarrow$ (e) immediately from Proposition 2.5(a) and Lemma 2.6(a). To see that (e) $\Rightarrow$ (a), notice that (e) easily implies that X admits a continuous twice Gâteaux differentiable bump function with Lipschitz Fréchet derivative and therefore is superreflexive; see [FWZ, Theorem 3.2]. It is also easy to directly construct a norm with modulus of smoothness of power type 2 using (e) and the Implicit Function Theorem. ∎

REMARK. (a) If f is globally Lipschitz with Lipschitz constant K and g_k is as in Lemma 2.4, then arguing as in inequalities (2.3) and (2.4) for any $x_0 \in X$ we have

$$f(x_0) + \frac{1}{k} \geq f \,\square\, g(x_0) \geq f(x_0) + (k - K)\|x_0 - y_k\| - \frac{2}{k}.$$

Thus the approximation is uniform on all of X. Moreover in Lemma 2.6(a) the approximation is uniform on all of X provided the given function f is globally Lipschitz. Therefore the approximation in Theorem 2.1(e) will be uniform on all of X provided the initial function is globally Lipschitz.

(b) Variants of Proposition 2.5 are also valid, for example, in spaces which admit uniformly smooth norms or norms whose derivatives are α-Hölder on the sphere and for C^1-smoothness in reflexive spaces.

(c) Given f a convex function bounded on bounded sets of X, we construct in Theorem 2.1(e), Lemma 2.4, and Proposition 2.5 a sequence of convex functions $\{f_k\}$ converging uniformly to f on bounded sets (in Theorem 2.1(e) the f_k's are twice Gâteaux differentiable and in Proposition 2.5 they have Lipschitz derivative). It is easy to show that uniform convergence on bounded sets implies *Mosco*-convergence of the sequence $\{f_k\}$. Recall that $\{f_k\}$ Mosco-converges to f if for every $x \in X$ we have

$$\begin{aligned} &\forall x_k \longrightarrow x \text{ (weakly)}, \quad f(x) \leq \liminf f_k(x_k) \\ &\exists x_k \longrightarrow x \text{ (in norm)}, \quad f(x) \geq \limsup f_k(x_k) \end{aligned} \tag{2.6}$$

In our case, to establish that f_k Mosco-converges to f we need only verify (2.6). To this end let $\{x_k\}$ be a sequence weakly converging to x. Because the sequence is norm bounded, for a fixed $\epsilon > 0$ we have that $f_k(x_k) \geq f(x_k) - \epsilon$ for all large k. Thus

$$\liminf f_k(x_k) \geq \liminf f(x_k);$$

moreover, because the function f is weakly lower semicontinuous, it follows that

$$\liminf f_k(x_k) \geq f(x).$$

Since the spaces X we are dealing with are reflexive, the fact that the sequence $\{f_k\}$ Mosco-converges to f has many interesting and valuable properties, for example

$$f_k^* \text{ Mosco-converges to } f^*, \text{ and}$$
$$\text{if } x_k \in \operatorname{argmin} f_k \text{ with } x_k \longrightarrow x \text{ then } x \in \operatorname{argmin} f,$$

where h^* is the conjugate of h (see [ET]) and $\operatorname{argmin} h$ is the set of minimizers of h. For a complete survey on Mosco-convergence we refer the reader to [A].

We conclude this section with some isomorphic characterizations of Hilbert spaces.

COROLLARY 2.7. *Assume either X or X^* has the* RNP. *If both X and X^* have continuous pointwise directionally Lipschitz bump functions with pointwise directional moduli of smoothness of power type 2, then X is isomorphic to a Hilbert space.*

PROOF. From Proposition 2.3 it follows that that X is super-reflexive. Therefore we are in a situation where both X and X^* satisfy the hypothesis in Proposition 2.3. Applying Proposition 2.3 to both X and X^*, we conclude that both X and X^* have equivalent norms with moduli of smoothness of power type 2. If Y is a separable subspace of X, then Y^* admits a twice Gâteaux differentiable norm by [FWZ, Theorem 3.1]. Thus [FWZ, Theorems 2.7 and 2.8] show that Y is isomorphic to a Hilbert space. It follows that X is isomorphic to a Hilbert space because Hilbert spaces are separably determined—this can be shown directly or one can use Kwapien's theorem ([K]). ∎

It has recently been shown that a Banach space which admits a continuous twice Gâteaux differentiable bump function is an Asplund space, therefore the assumption that X or X^* has the RNP is redundant in Corollary 2.7 in the case that one of the bump functions is twice Gâteaux differentiable; the details are in [V_2].

Recall that in [F], Fabian defined an LD-*space* to be a Banach space on which every continuous convex function has a dense set of Lipschitz smooth points. Proposition 2.3 can be used to obtain the following improvement of [F, Theorem 3.3].

COROLLARY 2.8. *If X (X^*) admits a continuous bump function with pointwise directional modulus of smoothness of power type 2 and X^* (X) is an* LD-*space, then X is isomorphic to a Hilbert space.*

PROOF. First the nonparenthetical assertion: because X^* is LD, X has the RNP (see [Bou, Theorem 5.2.12]). According to Proposition 2.3 X admits a norm with modulus of smoothness of power type 2. Now let Y be a closed separable subspace of X. Invoking [FWZ, Theorem 3.1] yields a norm $|||\cdot|||$ which is twice Gâteaux differentiable on $Y\setminus\{0\}$. The proof is completed exactly as the proof of [F, Theorem 3.3].

For the parenthetical assertion: X^* has the RNP since X is an LD-space. Thus by Proposition 2.3 X is reflexive and the nonparenthetical assertion applies. ∎

3. **Approximating** LUR **norms by C^k-smooth norms in separable spaces.** The following remark illustrates the power of combining higher order smoothness with rotundity.

REMARK. Suppose X admits an LUR norm $\|\cdot\|$ that is Lipschitz smooth at each point of Ω a dense G_δ subset of X. Then X admits a norm with modulus of smoothness of power type 2.

To prove this, set $F_n = \{x : \|x+h\| + \|x-h\| - 2\|x\| \le n\|h\|^2 \text{ for all } h \in X\}$. Then F_n is closed and $\Omega \subset \cup F_n$. By the Baire Category Theorem, for some n_0, F_{n_0} has nonempty interior, say, $B(x_0, 2\epsilon) \subset F_{n_0}$ for some $\epsilon > 0$ and $x_0 \in X$. Therefore, $\|\cdot\|'$ is Lipschitz on $B(x_0,\epsilon)$. We now proceed as in the proof of [FWZ, Theorem 3.3]. Let $H = \{h \in X : \|x_0\|'(h) = 0\}$. Since $\|\cdot\|$ is LUR, there is a $\delta > 0$ such that for $h \in H$ and

$\|h\| > \epsilon$, we have $\|x_0+h\| \geq \|x_0\|+\delta$. For $h \in H$, let $\phi(h) = \|x_0+h\|+\|x_0-h\|-2\|x_0\|$. Set $Q = \{h \in H : \phi(h) \leq \frac{\delta}{2}\}$. Let q be the Minkowski functional of Q. The Implicit Function Theorem asserts that as an equivalent norm on H, q has Lipschitz derivative on its sphere. Thus there is a norm with modulus of smoothness of power type 2 on X.

Notice that the above remark shows that the norm constructed in the next proposition cannot, in general, be Lipschitz smooth (in particular twice Gâteaux differentiable by [BN, Proposition 2.2]) at each point of a dense G_δ set. See [PWZ, Proposition 2] for a construction on $c_0(\Gamma)$ which is similar to the following.

PROPOSITION 3.1. *Let X be a separable Banach space which admits a norm that is C^k-smooth on $X \setminus \{0\}$ for some $k \in \mathbb{N} \cup \{\infty\}$. Then there is an* LUR *norm on X which is C^1-smooth on $X \setminus \{0\}$ and is limit of norms which are C^k-smooth on $X \setminus \{0\}$.*

PROOF. Let the norm $\|\cdot\|$ be C^k-smooth $X \setminus \{0\}$ and $\{h_n\}_{n=1}^{\infty}$ be dense in S_X. Choose $f_n \in S_{X^*}$ such that $f_n(h_n) = 1$ and define the projections P_n by $P_n x = f_n(x)h_n$. For $m = 1, 2, \dots$ let ϕ_m be even, convex and C^∞-smooth functions on $\mathbb{R}$ such that $\phi_m(t) = 0$ if $|t| \leq \frac{1}{m}$ and $\phi_m(t) > 0$ if $|t| > \frac{1}{m}$; suppose also that $\phi_m(2) \leq \frac{1}{2}$ for all m. Now set

$$\theta_{n,m}(x) = \phi_m(\|x\|) + \phi_m(\|x - P_n x\|).$$

Observe that $\theta_{n,m}$ is C^k-smooth, even, convex and uniformly continuous on bounded subsets of X. If $V_{n,m} = \{x : \theta_{n,m}(x) \leq 1\}$, then $V_{n,m}$ is the unit ball of an equivalent norm $\|\cdot\|_{n,m}$. Because $\theta_{n,m}(x) \leq 1$ whenever $\|x\| \leq 1$, one has $\|\cdot\|_{n,m} \leq \|\cdot\|$. Moreover, $\theta_{n,m}(0) = 0$ and $\theta_{n,m}(x) = 1$ whenever $\|x\|_{n,m} = 1$; thus the convexity of $\theta_{n,m}$ implies $\theta'_{n,m}(x)(x) \geq 1$ whenever $\|x\|_{n,m} = 1$. According to the Implicit Function Theorem, $\|\cdot\|_{n,m}$ is C^k-smooth on $X \setminus \{0\}$.

Consider the norm $|||\cdot|||$ defined by

$$|||x||| = \Big(\|x\|^2 + \sum_{n,m} \frac{1}{2^{n+m}}\|x\|_{n,m}^2 + \sum_{n=1}^{\infty} \frac{1}{2^n} f_n^2(x)\Big)^{\frac{1}{2}}.$$

Notice that $|||\cdot|||^2$ is C^1-smooth because the sum of the derivatives of the terms in its definition converge uniformly on bounded sets. In addition, the norms

$$|||x|||_J = \Big(\|x\|^2 + \sum_{n,m \leq J} \frac{1}{2^{n+m}}\|x\|_{n,m}^2 + \sum_{n=1}^{J} \frac{1}{2^n} f_n^2(x)\Big)^{\frac{1}{2}}$$

are C^k-smooth on $X \setminus \{0\}$ and $|||\cdot|||_J \to |||\cdot|||$.

We will show that the norm $|||\cdot|||$ is LUR. For this, suppose that $|||x||| = 1$ and

$$2|||x|||^2 + 2|||x_\iota|||^2 - |||x + x_\iota|||^2 \to 0. \tag{3.1}$$

We now show, for every n, that $\|x_\iota - P_n x_\iota\| \to \|x - P_n x\|$. To do this, we first assume that $\limsup_\iota \|x_\iota - P_n x_\iota\| > \|x - P_n x\|$ for some n. Thus there is a subsequence $\{x_{\iota_j}\}$ such that $\|x_{\iota_j} - P_n x_{\iota_j}\| \geq \|x - P_n x\| + \delta$ for some $\delta > 0$ and for all j. Now fix m so that $\frac{1}{m} \leq \frac{\delta}{2}$. Because $\|x\|_{n,m} \leq \|x\| \leq |||x||| = 1$, we choose $\alpha \geq 1$ so that

$$\phi_m(\alpha\|x\|) + \phi_m(\alpha\|x - P_n x\|) = 1.$$

The definition of $|||\cdot|||$ and (3.1) imply $||x_\iota|| \longrightarrow ||x||$. Since ϕ_m is continuous and nondecreasing on $(0,\infty)$, we have

$$\liminf_j \phi_m(\alpha||x_{\iota_j}||) + \phi_m(\alpha||x_{\iota_j} - P_n x_{\iota_j}||) \geq \phi_m(\alpha||x||) + \phi_m(\alpha||x - P_n x|| + \delta).$$

Because ϕ_m is a convex function, it follows that

$$\phi_m\Big(\frac{2}{m}\Big) - \phi_m(0) \leq \phi_m\Big(\alpha||x - P_n x|| + \frac{2}{m}\Big) - \phi_m(\alpha||x - P_n x||).$$

Let $\lambda = \phi_m(\frac{2}{m}) - \phi_m(0) = \phi_m(\frac{2}{m}) > 0$. Because ϕ_m is nondecreasing on $(0,\infty)$ and $\delta \geq \frac{2}{m}$, the above inequality implies

$$\begin{aligned}\phi_m(\alpha||x||) + \phi_m(\alpha||x - P_n x|| + \delta) &\geq \phi_m(\alpha||x||) + \phi_m(\alpha||x - P_n x||) + \\ &\quad + \phi_m\Big(\alpha||x - P_n x|| + \frac{2}{m}\Big) - \phi_m(\alpha||x - P_n x||)\\ &\geq \phi_m(\alpha||x||) + \phi_m(\alpha||x - P_n x||) + \lambda\\ &= 1 + \lambda.\end{aligned}$$

Hence for some $j_0 \in \mathbb{N}$, $\phi_m(\alpha||x_{\iota_j}||) + \phi_m(\alpha||x_{\iota_j} - P_m x_{\iota_j}||) \geq 1 + \frac{\lambda}{2}$ for $j \geq j_0$. Since ϕ_m is uniformly continuous on bounded sets, there is an $\epsilon > 0$ so that

$$\phi_m\big((1-\epsilon)\alpha||x_{\iota_j}||\big) + \phi_m\big((1-\epsilon)\alpha||x_{\iota_j} - P_m x_{\iota_j}||\big) \geq 1 \text{ for all } j \geq j_0.$$

Hence $\liminf_j (1-\epsilon)||x_{\iota_j}||_{n,m} \geq ||x||_{n,m}$. However this leads to a contradiction since (3.1) and the definition of $|||\cdot|||$ imply that $||x_\iota||_{n,m} \longrightarrow ||x||_{n,m}$. Therefore, $\limsup_\iota ||x_\iota - P_n x_\iota|| \leq ||x - P_n x||$ for each n.

Similarly we see that $\liminf_\iota ||x_\iota - P_n x_\iota|| \geq ||x - P_n x||$ for each n. Therefore,

(3.2) $$||x_\iota - P_n x_\iota|| \longrightarrow ||x - P_n x|| \text{ for all } n.$$

We now show that $|||\cdot|||$ is LUR. Let $\epsilon > 0$ and recall that $P_n v = f_n(v)h_n$ where $||f_n||^* = ||h_n|| = f_n(h_n) = 1$. Since $\{h_n\}_{n=1}^\infty$ is dense in S_X, we choose and fix n such that

(3.3) $$||x - P_n x|| < \epsilon.$$

According to (3.2) and (3.3) there is an i_0 such that

(3.4) $$||x_\iota - P_n x_\iota|| < \epsilon \text{ for all } i \geq i_0.$$

Because of (3.1) and the definition of $|||\cdot|||$ it follows that $\lim_\iota f_n(x_\iota) = f_n(x)$. Thus replacing i_0 by a larger number if necessary we also have:

(3.5) $$|f_n(x_\iota) - f_n(x)| < \epsilon \text{ for all } i \geq i_0.$$

Finally, for $i \geq i_0$ (3.3), (3.4) and (3.5) imply

$$\begin{aligned}||x - x_\iota|| &\leq ||x_\iota - P_n x_\iota|| + ||P_n x_\iota - P_n x|| + ||P_n x - x||\\ &= ||x_\iota - P_n x_\iota|| + \big|\big|\big(f_n(x_\iota) - f_n(x)\big)h_n\big|\big| + ||P_n x - x||\\ &< 3\epsilon.\end{aligned}$$

Since $|||\cdot|||$ is equivalent to $||\cdot||$, $|||x - x_\iota||| \longrightarrow 0$. Therefore $|||\cdot|||$ is LUR. ■

It is not known if any norm on a space admitting a C^k-smooth norm can be approximated by C^k-smooth norms. However, approximations of LUR norms as in Proposition 3.1 are particularly useful for structural reasons: for example, they can be used to construct smooth homeomorphic maps of spaces into c_0 or l_2;see [DGZ$_2$, Chapter V].

https://doi.org/10.4153/CJM-1993-032-9 Published online by Cambridge University Press

PROPOSITION 3 2 *If X is a separable Banach space with a norm whose k th Frechet derivative is uniformly continuous on its sphere for some $k \in \mathbb{N}$, then X admits an* LUR *norm which has uniformly continuous k-th Frechet derivative on its sphere*

PROOF Essentially the same proof as in Proposition 3 1 works As before for $\|x\|_{n\,m} = 1$, we have $\theta'_{n\,m}(x)(x) \geq 1$, thus the Implicit Function Theorem asserts that $\|\ \|_{n\,m}$ has uniformly continuous k-th derivative since $\theta_{n\,m}$ has uniformly continuous k-th derivative Now define the norm $|||\ |||$ by

$$|||x||| = \Big(\|x\|^2 + \sum_{n\,m} \frac{1}{C_{n\,m}2^{n+m}}\|x\|^2_{n\,m} + \sum_{n\ 1}^{\infty} \frac{1}{2^n} f_n^2(x)\Big)^{1}$$

where $C_{n\,m} \geq 1$ is chosen so that the k-th derivative of $\frac{1}{C_{n,m}}\|\ \|^2_{n\,m}$ has norm ≤ 1 if $x \neq 0$ The rules for differentiating an infinite sum show that $|||\ |||$ has uniformly continuous k-th derivative on its sphere ■

Note that even in l_2 there is an LUR norm which is not a limit of functions with uniformly continuous second derivatives (see [V_1]) However, it seems to be unknown whether there is a UR norm which is a limit of norms with uniformly continuous k th derivative on the sphere under the hypothesis of Proposition 3 2

REMARK Suppose X does not contain a subspace isomorphic to $c_0(\mathbb{N})$ If X admits a norm whose k-th derivative is locally uniformly continuous on $X \setminus \{0\}$, then X admits a norm with uniformly continuous k-th derivative on its sphere

This remark was not included in the paper [FWZ] but follows easily from the results therein by [FWZ, Theorem 3 3(i)], X is super-reflexive Therefore there exists a strongly exposed point on B_X The proof of [FWZ, Theorem 3 3(ii)] then shows that the conclusion of the remark is valid

From the above remark and Proposition 3 2 we obtain

COROLLARY 3 3 *Suppose that X is separable and admits a norm which is C^{k+1} smooth on $X \setminus \{0\}$ for some $k \geq 1$ If X does not contain a subspace isomorphic to $c_0(\mathbb{N})$ then X admits an* LUR *norm which has uniformly continuous k th derivative on its sphere*

REFERENCES

[A] H Attouch *Variational convergence for functions and operators* Pitman New York 1984

[B] B Beauzamy *Introduction to Banach spaces and their geometry* North Holland Mathematics Studies **68** North Holland 1985

[BN] J M Borwein and D Noll *Second order differentiability of convex functions A D Alexandrov s Theorem in Hilbert space* preprint

[Bou] R D Bourgain *Geometric aspects of convex sets with the Radon Nikodym property* Lecture Notes in Mathematics **993** Springer Verlag 1983

[C] J B Collier *The dual of a space with the Radon Nikodym property* Pacific J Math **64**(1976) 103 106

[DGZ$_1$] R Deville G Godefroy and V Zizler *Smooth bump functions and geometry of Banach spaces* Mathematika to appear

[DGZ$_2$] ——— *Renormings and smoothness in Banach spaces* Longman Monographs in Pure and Applied Mathematics to appear

https://doi.org/10.4153/CJM-1993-032-9 Published online by Cambridge University Press

[ET] I Ekeland and R Temam, *Convex Analysis and Variational Problems*, Studies in Mathematics and its Applications, **1**, North-Holland, 1976

[En] P Enflo, *Banach spaces which can be given an equivalent uniformly convex norm*, Israel J Math **13** (1972), 281–288

[F] M Fabian, *Lipschitz smooth points of convex functions and isomorphic characterizations of Hilbert spaces*, Proc London Math Soc **51**(1985), 113–126

[FWZ] M Fabian, J H M Whitfield and V Zizler, *Norms with locally Lipschitzian derivatives*, Israel J Math **44**(1983), 262–276

[K] S Kwapien, *Isomorphic characterizations of inner product space by orthogonal series with vector valued coefficients*, Studia Math **44**(1972), 583–595

[LA-LI] J M Lasry and P L Lions, *A remark on regularization in Hilbert spaces*, Israel J Math **55**(1986), 257–266

[PWZ] J Pechanec, J H M Whitfield and V Zizler, *Norms locally dependent of finitely many coordinates*, An Acad Brasil Ciênc **53**(1981), 415–417

[V_1] J Vanderwerff, *Smooth approximations in Banach spaces*, Proc Amer Math Soc **115**(1992), 113–120

[V_2]——, *Second-order Gâteaux differentiability and an isomorphic characterization of Hilbert spaces*, Quart J Math Oxford, to appear

Department of Mathematics
University of Alberta
Edmonton, Alberta
T6G 2G1