Proceedings of the Royal Society of Edinburgh, **137A**, 625–645, *2007*

Existence and multiplicity of periodic solutions for the Duffing equation with singularity

Jing Xia and **Zaihong Wang**
Department of Mathematics, Capital Normal University, Beijing 100037, People's Republic of China (zhwang@mail.cnu.edu.cn)

(MS received 25 July 2005; accepted 17 May 2006)

In this paper, we consider the existence and multiplicity of periodic solutions for the Duffing equation $x'' + g(x) = p(t)$ with a singularity. When the time map has oscillating properties, $g(x)$ possesses a singularity at the origin and tends to $+\infty$ as $x \to +\infty$ and other conditions hold. We obtain the existence of harmonic solutions and the multiplicity of subharmonic solutions of the given equation by using the phase-plane analysis methods and the generalized Poincaré–Birkhoff twist theorem.

1. Introduction

We are concerned with the existence and multiplicity of periodic solutions of the Duffing equation

$$x'' + g(x) = p(t), \tag{1.1}$$

where $g : \mathbb{R}^+ \to \mathbb{R}$ is locally Lipschitz continuous and has a singularity at the origin, $p(t)$ is continuous and has the least period 2π.

The existence and multiplicity of periodic solutions of equations with singularities have been investigated extensively because of their background in applied sciences (see [2, 3, 8–10, 12, 14, 16–18] and the references therein). For example, the Brillouin electron-beam-focusing problem [6, p. 264] is to find the existence of positive 2π-periodic solutions of

$$x'' + a(1 + \cos 2t)x = \frac{1}{x}, \quad a > 0,$$

satisfying periodic boundary conditions

$$x(0) = x(\pi) > 0, \qquad x'(0) = x'(\pi) = 0.$$

Lazer and Solimini [10] first studied the existence of periodic solutions of (1.1) with a singularity. Assume that $g : (-\infty, 0) \cup (0, +\infty) \to \mathbb{R}$ is continuous and satisfies

$$g(x)x > 0, \quad x \neq 0,$$

and

$$\lim_{|x|\to+\infty} g(x) = 0, \quad \lim_{x\to 0^+} g(x) = +\infty, \quad \lim_{x\to 0^-} g(x) = -\infty.$$

https://doi.org/10.1017/S0308210505000879 Published online by Cambridge University Press

They proved that (1.1) has at least one 2π-periodic solution if and only if

$$\int_0^{2\pi} p(s)\,\mathrm{d}s \neq 0.$$

When $g(x)$ is sublinear at infinity and has a singularity at the origin, Fonda *et al*. [8] proved the existence of infinitely many subharmonic solutions for (1.1) by using the critical-point theory. The superlinear case on a bounded interval was also studied in [8] by using the generalized Poincaré–Birkhoff fixed-point theorem.

Del Pino *et al*. [3] studied the existence of periodic solutions of

$$x'' + f(t, x) = 0, \tag{1.2}$$

where $f : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ is continuous and 2π-periodic in t. Assume that there exist positive constants c, c', δ and $r \geqslant 1$ such that

$$\frac{c}{x^r} \leqslant -f(t, x) \leqslant \frac{c'}{x^r}, \tag{1.3}$$

where $t \in [0, 2\pi]$, $0 < x < \delta$. Furthermore, there is an integer $n \geqslant 0$ such that, for $t \in [0, 2\pi]$,

$$\frac{n^2}{4} < \lim_{x\to+\infty} \inf \frac{f(t, x)}{x} \leqslant \lim_{x\to+\infty} \sup \frac{f(t, x)}{x} < \frac{(n + 1)^2}{4}. \tag{1.4}$$

They proved that (1.2) has at least one periodic solution under conditions (1.3) and (1.4).

The multiplicity of periodic solutions of the superlinear equation (1.2) with a singularity was studied in [2]. Assume that

$$-\infty \leqslant \lim_{s\to0^+} \sup sf(t, s) < 0$$

and

$$\lim_{s\to+\infty} \frac{f(t, s)}{s} = +\infty.$$

It was proved in [2] that there exists an integer $n_0 > 0$ such that, for any $n \geqslant n_0$, (1.2) has two 2π-periodic solutions $x_n^+(t)$ and $x_n^-(t)$.

The generalization of [3] was done for equation of the type of (1.1) in [16]. Assume that $g(x)$ satisfies

$$\lim_{x\to0^+} g(x) = -\infty, \tag{g_0}$$

and

$$\frac{n^2}{4} < \lim_{x\to+\infty} \inf \frac{g(x)}{x} \leqslant \lim_{x\to+\infty} \sup \frac{g(x)}{x} < \frac{(n + 1)^2}{4},$$

and the primitive function $G(x)$ of $g(x)$ satisfies

$$\lim_{x\to0^+} G(x) = +\infty, \quad G(x) = \int_1^x g(s)\,\mathrm{d}s. \tag{G_0}$$

It was proved in [16] that (1.1) has at least one 2π-periodic solution.

https://doi.org/10.1017/S0308210505000879 Published online by Cambridge University Press

Here we will deal with the existence and multiplicity of periodic solutions for (1.1) with a singularity under generalized conditions. Assume that $g(x)$ satisfies

$$\lim_{x\to+\infty} g(x) = +\infty, \tag{g_1}$$

and that $G(x)$ satisfies a similar condition to that in [1]:

$$\begin{aligned}&\text{for all } c_1 > 0, \text{ there exists } c_2 > 0 \text{ such that if } x \gg 1,\ y \gg 1\\ &\qquad\text{and } |\sqrt{G(x)} - \sqrt{G(y)}| < c_1, \text{ then } |x-y| < c_2.\end{aligned} \tag{G_1}$$

One can check that the condition (G_1) is satisfied if $g(x)$ satisfies semilinear condition, as follows:

$$0 < \liminf_{x\to+\infty} \frac{g(x)}{x} \leqslant \limsup_{x\to+\infty} \frac{g(x)}{x} < +\infty.$$

It is well known that time map plays an important role in studying the existence and multiplicity of periodic solutions of (1.1). We now introduce the time map. Consider the autonomous system

$$x'' + g(x) = 0,$$

or its equivalent system

$$x' = y, \quad y' = -g(x). \tag{1.5}$$

The first integral of (1.5) is the curve

$$\varGamma_c : \tfrac{1}{2}y^2 + G(x) = c,$$

where c is an arbitrary constant.

From (g_0), (G_0) and (g_1) we know that, for c sufficiently large, $\varGamma_c$ is a closed curve. Let $(x(t), y(t))$ be any solution of (1.5) whose orbit is $\varGamma_c$. Clearly, this solution is periodic. Let $\tau(c)$ denote the least positive period of this solution. It is not hard to check that

$$\tau(c) = \sqrt{2}\int_{h(c)}^{d(c)} \frac{\mathrm{d}x}{\sqrt{c - G(x)}}, \tag{1.6}$$

where $0 < h(c) < d(c)$, $G(h(c)) = G(d(c)) = c$, $\lim_{c\to+\infty} h(c) = 0$, $\lim_{c\to+\infty} d(c) = +\infty$. We note that there is little difference between this time map $\tau(c)$ and the time map in [4].

Assume that $\tau(c)$ satisfies the following conditions, as in [4, 7, 13].

(τ_0) There exist a constant $\sigma > 0$, an integer $m > 0$ and two sequences $\{a_k\}$ and $\{b_k\}$, with $a_k \to +\infty$ and $b_k \to +\infty$ as $k \to +\infty$ such that

$$\tau(a_k) < \frac{2\pi}{m} - \sigma, \qquad \tau(b_k) > \frac{2\pi}{m} + \sigma.$$

(τ_1) There exist two positive integers m and n with $(m, n) = 1$, a positive constant $\beta > 0$ and two sequences $\{a_k\}$ and $\{b_k\}$, with $a_k \to +\infty$ and $b_k \to +\infty$ as $k \to +\infty$ such that

$$\tau(a_k) < \frac{2n\pi}{m} - \beta, \qquad \tau(b_k) > \frac{2n\pi}{m} + \beta.$$

https://doi.org/10.1017/S0308210505000879 Published online by Cambridge University Press

We obtain the following theorems.

THEOREM 1.1. *Assume that conditions* (g_0), (g_1), (G_0), (G_1) *and* (τ_0) *hold. Then (1.1) has infinitely many* 2π*-periodic solutions.*

THEOREM 1.2. *Assume that conditions* (g_0), (g_1), (G_0), (G_1) *and* (τ_1) *hold. Then (1.1) has at least one harmonic solution and infinitely many* n*-order subharmonic solutions.*

2. Several lemmas

In this paper, we will use the generalized Poincaré–Birkhoff twist theorem to study the existence and multiplicity of periodic solutions of (1.1). We now introduce this theorem.

Let D denote an annular region in the (x, y)-plane. The boundary of D consists of two simple closed curves: the inner boundary curve C_1 and the outer boundary curve C_2. We denote by D_1 the simple connected open set bounded by C_1.

Consider an area-preserving homeomorphism $T : D \subset \mathbb{R}^2 \to \mathbb{R}^2$. Suppose that $T(D) \subset \mathbb{R}^2 - \{O\}$, where O is the origin. Let (r, θ) be the polar coordinate of (x, y). That is, $x = r\cos\theta$, $y = r\sin\theta$. Assume that T is given by

$$r^* = f(r, \theta), \quad \theta^* = \theta + g(r, \theta),$$

where f and g are continuous in (r, θ), and 2π-periodic in θ.

THEOREM 2.1 (generalized Poincaré–Birkhoff twist theorem [5]). *Besides the above-mentioned assumptions, we assume that*

(1) C_1 *is star-shaped about the origin,*

(2) $\mathrm{O} \in T(D_1)$,

(3) $g(r, \theta) > 0 (< 0)$, $(r\cos\theta, r\sin\theta) \in C_1$; $g(r, \theta) < 0 (> 0)$, $(r\cos\theta, r\sin\theta) \in C_2$.

Then T *has at least two fixed points in* D.

In order to use the phase-plane analysis methods conveniently, we consider the equation

$$x'' + g(x) = p(t), \tag{2.1}$$

where $g : (-1, +\infty) \to \mathbb{R}$ is continuous and exhibits a singularity at -1. In fact, we can take a transformation $x = 1 + u$ to achieve this aim. Under this transformation, (2.1) becomes $u'' + g(1 + u) = p(t)$. Set

$$\hat{g}(u) = g(1 + u), \qquad \hat{G}(u) = \int_0^u \hat{g}(s)\,\mathrm{d}s.$$

We then have

$$\lim_{u\to -1^+} \hat{g}(u) = -\infty \quad \text{and} \quad \lim_{u\to -1^+} \hat{G}(u) = +\infty.$$

https://doi.org/10.1017/S0308210505000879 Published online by Cambridge University Press

For simplicity, from now on, we assume that g satisfies the following conditions:

$$\lim_{x\to -1^+} g(x) = -\infty, \tag{g_0'}$$

and

$$\lim_{x\to -1^+} G(x) = +\infty, \quad G(x) = \int_0^x g(s)\,\mathrm{d}s. \tag{G_0'}$$

In this case, $h(c)$ and $d(c)$ in (1.6) satisfy

$$\lim_{c\to +\infty} h(c) = -1, \quad \lim_{c\to +\infty} d(c) = +\infty.$$

We will prove theorems 1.1 and 1.2 under conditions (g_0') and (G_0') instead of conditions (g_0) and (G_0).

Consider the system equivalent to (2.1):

$$x' = y, \quad y' = -g(x) + p(t). \tag{2.2}$$

Let $(x(t), y(t)) = (x(t; x_0, y_0), y(t; x_0, y_0))$ be the solution of (2.2) through the initial point

$$x(0; x_0, y_0) = x_0, \qquad y(0; x_0, y_0) = y_0.$$

LEMMA 2.2. *Assume that conditions (G_0') and (g_1) hold. Then every solution of system (2.2) exists uniquely on the whole t-axis.*

Proof. Define a potential function

$$V(x, y) = \tfrac{1}{2}y^2 + G(x). \tag{2.3}$$

It follows from (G_0') and (g_1) that we can find a constant $M > 0$ such that $G(x) + M \geqslant 0$. Set

$$V(t) = V(x(t), y(t)).$$

Then we have

$$\begin{aligned} V'(t) &= y(t)y'(t) + g(x(t))x'(t) \\ &= y(t)(-g(x(t)) + p(t)) + g(x(t))y(t) \\ &= y(t)p(t) \\ &\leqslant \tfrac{1}{2}(y^2(t) + p^2(t)) \\ &\leqslant \tfrac{1}{2}(y^2(t) + M') \\ &\leqslant \tfrac{1}{2}y^2(t) + G(x(t)) + M'' \\ &= V(t) + M'', \end{aligned}$$

where $M' = \max\{p^2(t) \mid t \in R\}$, $M'' = M + \frac{1}{2}M'$. Hence,

$$V(t) \leqslant V(t_0)e^{\tau} + M''e^{\tau}, \quad t \in [t_0, t_0 + \tau).$$

Therefore, there is no blow-up in any finite interval for every solution $(x(t), y(t))$ of system (2.2). This shows that every solution of system (2.2) is defined uniquely on the whole t-axis. This completes the proof of lemma 2.2. □

https://doi.org/10.1017/S0308210505000879 Published online by Cambridge University Press

According to lemma 2.2, the Poincaré map $P : (-1, +\infty) \times \mathbb{R} \to \mathbb{R}^2$ is well defined by

$$P : (x_0, y_0) \mapsto (x_1, y_1) = (x(2\pi; x_0, y_0), y(2\pi; x_0, y_0)).$$

Obviously, the fixed points of the Poincaré map P correspond to 2π-periodic solutions of system (2.2).

To depict the position of orbit $(x(t), y(t))$ for $t \in [0, 2\pi]$, we introduce a function $l : (-1, +\infty) \times \mathbb{R} \to \mathbb{R}^+$,

$$l(x, y) = x^2 + y^2 + \frac{1}{(1 + x)^2}.$$

Lemma 2.3. *Assume that* (G_0') *and* (g_1) *hold. Then, for any* $r > 0$*, there exists* $\varrho > 0$ *sufficiently large that, for* $l(x_0, y_0) \geqslant \varrho^2$,

$$l(x(t), y(t)) \geqslant r^2, \quad t \in [0, 2\pi],$$

where $(x(t), y(t))$ *is the solution of system (2.2) through the initial point* (x_0, y_0).

Proof. Define $V(x, y)$ as in (2.3). From (G_0') and (g_1) we know that there is a constant $M > 0$ such that $G(x) + M \geqslant 0$ and

$$\lim_{l(x,y)\to+\infty} (V(x, y) + M) = +\infty. \tag{2.4}$$

Write

$$W(t) = V(x(t), y(t)) + M.$$

Then

$$W'(t) = y(t)p(t) \geqslant -2\sqrt{W(t)}L, \quad L = \max_{t\in[0,2\pi]} |p(t)|,$$

and thus

$$\sqrt{W(t)} \geqslant \sqrt{W(0)} - Lt. \tag{2.5}$$

For each $r > 0$, set

$$m(r) = \max_{l(x,y)\leqslant r^2} (V(x, y) + M).$$

It follows from (2.4) that there exists a $\varrho > 0$ sufficiently large that, if $l(x, y) \geqslant \varrho^2$, then

$$V(x, y) + M > (\sqrt{m(r)} + 2\pi L)^2.$$

Assume that $l(x_0, y_0) \geqslant \varrho^2$. From (2.5) we get

$$\begin{aligned}\sqrt{V(x(t), y(t)) + M} &> \sqrt{V(x_0, y_0) + M} - Lt \\ &> \sqrt{m(r)} + 2L\pi - Lt \\ &\geqslant \sqrt{m(r)}, \quad t \in [0, 2\pi].\end{aligned}$$

Consequently, we have

$$l(x(t), y(t)) \geqslant r^2, \quad t \in [0, 2\pi].$$

□

https://doi.org/10.1017/S0308210505000879 Published online by Cambridge University Press

According to lemma 2.3, if $l(x_0, y_0)$ is sufficiently large, then $x^2(t) + y^2(t) > 0$, $t \in [0, 2\pi]$. Therefore, we can take the polar coordinate transformation $x = r\cos\theta$, $y = r\sin\theta$. Thus, system (2.2) becomes

$$\left.\begin{aligned} \frac{\mathrm{d}r}{\mathrm{d}t} &= r\sin\theta\cos\theta - g(r\cos\theta)\sin\theta + p(t)\sin\theta, \\ \frac{\mathrm{d}\theta}{\mathrm{d}t} &= -\sin^2\theta - \frac{1}{r}g(r\cos\theta)\cos\theta + \frac{1}{r}p(t)\cos\theta. \end{aligned}\right\} \tag{2.6}$$

Let $(r(t), \theta(t)) = (r(t; r_0, \theta_0), \theta(t; r_0, \theta_0))$ denote the solution of (2.6) satisfying

$$r(0) = r_0, \quad \theta(0) = \theta_0.$$

Then we can rewrite the Poincaré map P as follows:

$$P : (r_0, \theta_0) \to (r_1, \theta_1) = (r(2\pi; r_0, \theta_0), \theta(2\pi; r_0, \theta_0)),$$

with $r_0 \cos\theta_0 = x_0 > -1$, $r_0 \sin\theta_0 = y_0$.

Lemma 2.4. *Assume that (g_0') and (g_1) hold. Then there exists a $l_0 > 0$ such that, for $l(x, y) \geqslant l_0$, $\theta'(t) < 0$, $t \in [0, 2\pi]$.*

Proof. It follows from (g_0') and (g_1) that there exist positive constants c_1, c_2, $c_2 < 1$, such that

$$g(x) - p(t) > 0, \quad x \in (c_1, +\infty), \quad t \in [0, 2\pi]; \tag{2.7}$$

$$g(x) - p(t) < 0, \quad x \in (-1, -c_2), \quad t \in [0, 2\pi]. \tag{2.8}$$

If $x(t) > c_1$ or $-1 < x(t) < -c_2$, $t \in [0, 2\pi]$, then we conclude from (2.6)–(2.8) that

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} < -\sin^2\theta \leqslant 0. \tag{2.9}$$

If $-c_2 \leqslant x(t) \leqslant c_1$, $t \in [0, 2\pi]$, then there exists a constant $l_0 > 0$ such that

$$|\sin\theta(t)| \geqslant \tfrac{1}{2} \quad \text{and} \quad \frac{|g(x(t)) - p(t)|}{r} \leqslant \frac{1}{8}$$

for $l(x_0, y_0) \geqslant l_0$. Consequently,

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} \leqslant -\frac{1}{4} + \frac{1}{8} < 0, \quad t \in [0, 2\pi]. \tag{2.10}$$

From (2.9) and (2.10) we get the conclusion of lemma 2.4. □

Consider the autonomous system

$$x' = y, \quad y' = -g(x). \tag{2.11}$$

Lemma 2.5. *Assume that (g_0'), (G_0') and (g_1) hold. There then exists a $c_0 > 0$ such that, for $c \geqslant c_0$, $\Gamma_c : \frac{1}{2}y^2 + G(x) = c$ is a star-shaped closed curve about the origin* O.

Proof. From (G_0') and (g_1) we have

$$\lim_{x\to -1} G(x) = +\infty \quad \text{and} \quad \lim_{x\to +\infty} G(x) = +\infty.$$

https://doi.org/10.1017/S0308210505000879 Published online by Cambridge University Press

It is easy to show that there exists a $c_0' > 0$ such that Γ_c is a closed curve for $c \geqslant c_0'$. In fact, it follows from (g_0') and (g_1) that, for sufficiently large c, Γ_c intersects with the x-axis at exactly two points. On the other hand, Γ_c is symmetric about the x-axis. Hence, Γ_c is a closed curve for sufficiently large c. Under the polar coordinate transformation

$$x = \rho\cos\varphi, \qquad y = \rho\sin\varphi,$$

system (2.11) becomes

$$\left.\begin{aligned} \dot\rho &= \rho\sin\varphi\cos\varphi - g(\rho\cos\varphi)\sin\varphi, \\ \dot\varphi &= -\sin^2\varphi - \frac{1}{\rho}g(\rho\cos\varphi)\cos\varphi. \end{aligned}\right\} \tag{2.12}$$

Let

$$(\rho(t), \varphi(t)) = (\rho(t;\rho_0,\varphi_0), \varphi(t;\rho_0,\varphi_0))$$

be the solution of (2.12) through (ρ_0, φ_0). Using the same method as in the proof of lemma 2.4, we can prove that there exists $c_0'' > 0$ such that $\dot\varphi(t) < 0$ for $(\rho_0,\varphi_0) \in \Gamma_c$, $c \geqslant c_0''$. Therefore, $\varphi(t)$ is strictly decreasing. Set $c_0 = \max\{c_0', c_0''\}$. We then obtain the conclusion of lemma 2.5. □

Lemma 2.6. *Assume that (g_0') and (G_0') hold. Let $t_1 < t_2 < t_3 < +\infty$, $x(t_1) = 0$, $y(t_1) < 0$; $x(t_2) < 0$, $y(t_2) = 0$; $x(t_3) = 0$, $y(t_3) > 0$ and $x(t) \leqslant 0$, $t \in [t_1, t_3]$, where $(x(t), y(t))$ is any solution of system (2.2). Then $t_3 - t_1 = o(1)$, as $l(x_0, y_0) \to +\infty$.*

Proof. We now deal with solutions of system (2.2). First we estimate $t_2 - t_1$. Assume that $|p(t)| \leqslant L$. Set

$$W(x, y) = \tfrac{1}{2}y^2 + G(x) + Lx, \qquad W(t) = W(x(t), y(t)).$$

Thus, we get

$$W'(t) = y(t)y'(t) + g(x(t))x'(t) + Lx'(t) = (L + p(t))y(t) \leqslant 0.$$

Therefore, $W(t)$ is decreasing in the interval $[t_1, t_2]$. For $t \in [t_1, t_2]$, we have

$$\tfrac{1}{2}y^2(t) + G(x(t)) + Lx(t) \geqslant \tfrac{1}{2}y^2(t_2) + G(x(t_2)) + Lx(t_2).$$

Therefore,

$$y^2(t) \geqslant 2(G(x(t_2)) - G(x(t))) + 2L(x(t_2) - x(t)).$$

Thus,

$$-x'(t) = -y(t) \geqslant \sqrt{2(G(x(t_2)) - G(x(t))) + 2L(x(t_2) - x(t))}.$$

Accordingly,

$$\frac{-x'(t)}{\sqrt{2(G(x(t_2)) - G(x(t))) + 2L(x(t_2) - x(t))}} \geqslant 1.$$

Integrating both sides of the above inequality in the interval $[t_1, t_2]$, we get

$$t_2 - t_1 \leqslant \int_s^0 \frac{\mathrm{d}x}{\sqrt{2(G(s) - G(x)) + 2L(s - x)}},$$

https://doi.org/10.1017/S0308210505000879 Published online by Cambridge University Press

where $s = x(t_2)$. By lemma 2.3, $s \to -1^+$ as $l(x_0, y_0) \to +\infty$. We claim that

$$\lim_{s\to -1^+} \int_s^0 \frac{\mathrm{d}x}{\sqrt{2(G(s)-G(x)) + 2L(s-x)}} = 0.$$

Let $\eta > 0$ be a sufficiently small constant. Write

$$I_1 = \int_s^{s+\eta} \frac{\mathrm{d}x}{\sqrt{2(G(s)-G(x)) + 2L(s-x)}},$$
$$I_2 = \int_{s+\eta}^0 \frac{\mathrm{d}x}{\sqrt{2(G(s)-G(x)) + 2L(s-x)}}.$$

If $x \in (s, s+\eta)$, then

$$G(s) - G(x) = g(\xi)(s-x) = -g(\xi)(x-s), \quad \xi \in (s,x) \subset (s, s+\eta).$$

Set $\mu(s,\eta) = \inf\{-g(x) \mid x \in (s, s+\eta)\}$. Then

$$G(s) - G(x) \geqslant \mu(s,\eta)(x-s).$$

We know from (g_0') that $\mu(s,\eta) \to +\infty$ as $s \to -1^+$ and $\eta \to 0^+$. Hence,

$$I_1 \leqslant \int_s^{s+\eta} \frac{\mathrm{d}x}{\sqrt{2[\mu(s,\eta)-L](x-s)}} = \frac{\sqrt{2\eta}}{\sqrt{\mu(s,\eta)-L}}.$$

Obviously, $I_1 \to 0$ for $s \to -1^+$, $\eta \to 0^+$. It follows from (G_0') that

$$\lim_{s\to -1^+} (G(s) - G(x)) = +\infty \quad \text{for all } x \in (s+\eta, 0),$$

which implies that $I_2 \to 0$ as $s \to -1^+$. Thus, we have finished the proof of the claim. Set

$$\bar{W}(x,y) = \tfrac{1}{2}y^2 + G(x) - Lx.$$

Analogously to the estimate of $t_2 - t_1$, we may obtain $t_3 - t_2 = o(1)$ as $l(x_0, y_0) \to +\infty$. Consequently, the conclusion of lemma 2.6 holds. □

Remark 2.7. In particular, if (g_0') and (G_0') hold, then we have

$$\lim_{c\to +\infty} \int_{h(c)}^0 \frac{\mathrm{d}x}{\sqrt{c - G(x)}} = 0.$$

We can get this conclusion by putting $p(t) = 0$ in lemma 2.6. Furthermore, if (g_0) and (G_0) hold, then we have

$$\lim_{c\to +\infty} \int_{h(c)}^1 \frac{\mathrm{d}x}{\sqrt{c - G(x)}} = 0.$$

Similar conclusions can also be found in [1].

https://doi.org/10.1017/S0308210505000879 Published online by Cambridge University Press

LEMMA 2.8. *Assume that* (g_1) *holds. If* $1 \leqslant x, y \leqslant c$ *and* $|x-y| \leqslant E$ *for a fixed positive constant* E*, then*

$$\lim_{c\to+\infty}\int_x^y \frac{\mathrm{d}s}{\sqrt{G(c)-G(s)}} = 0.$$

Proof. Without loss of generality, we assume that $1 \leqslant x \leqslant y \leqslant c$. From (g_1) we know that there exists a constant x_0 (where $x_0 \geqslant 1+E$) such that $g(x) \geqslant 0$, for $x \geqslant x_0 - E$. Obviously, the conclusion holds provided that $1 \leqslant x \leqslant y \leqslant x_0$. In what follows, we assume that $y \geqslant x_0$. Thus, $x \geqslant x_0 - E$. We shall proceed in two steps.

STEP 1 ($y \geqslant \frac{1}{2}c$). For $x \leqslant s \leqslant y$, we have

$$\begin{aligned} G(c)-G(s) &= \int_s^c g(u)\,\mathrm{d}u \geqslant \min_{s\leqslant\xi\leqslant c} g(\xi)(c-s) \\ &\geqslant \min_{x\leqslant\xi\leqslant c} g(\xi)(c-s) \\ &\geqslant \min_{y-E\leqslant\xi\leqslant c} g(\xi)(c-s) \\ &\geqslant \min_{\frac{1}{2}c-E\leqslant\xi\leqslant c} g(\xi)(c-s) \\ &= \mu(c)(c-s), \end{aligned}$$

where $\mu(c) = \min\{g(\xi) : \frac{1}{2}c - E \leqslant \xi \leqslant c\}$. From (g_1) we know that $\mu(c) \to +\infty (c \to +\infty)$. As a result, we have

$$\begin{aligned} 0 &\leqslant \int_x^y \frac{\mathrm{d}s}{\sqrt{G(c)-G(s)}} \leqslant \int_x^y \frac{\mathrm{d}s}{\sqrt{\mu(c)(c-s)}} \\ &\leqslant \frac{1}{\sqrt{\mu(c)}}\int_{y-E}^y \frac{\mathrm{d}s}{\sqrt{c-s}} \leqslant \frac{1}{\sqrt{\mu(c)}}\int_{c-E}^c \frac{\mathrm{d}s}{\sqrt{c-s}} \\ &= \frac{2\sqrt{E}}{\sqrt{\mu(c)}}. \end{aligned}$$

Since

$$\lim_{c\to+\infty} \frac{2\sqrt{E}}{\sqrt{\mu(c)}} = 0,$$

we obtain

$$\lim_{c\to+\infty}\int_x^y \frac{\mathrm{d}s}{\sqrt{G(c)-G(s)}} = 0.$$

STEP 2 ($y \leqslant \frac{1}{2}c$). For $x \leqslant s \leqslant y$, we have

$$G(c)-G(s) \geqslant G(c) - G(\tfrac{1}{2}c).$$

Thus,

$$0 \leqslant \int_x^y \frac{\mathrm{d}s}{\sqrt{G(c)-G(s)}} \leqslant \int_x^y \frac{\mathrm{d}s}{\sqrt{G(c)-G(\frac{1}{2}c)}} \leqslant \frac{E}{\sqrt{G(c)-G(\frac{1}{2}c)}},$$

https://doi.org/10.1017/S0308210505000879 Published online by Cambridge University Press

which implies that

$$\lim_{c\to+\infty}\int_x^y \frac{\mathrm{d}s}{\sqrt{G(c)-G(s)}} = 0.$$

From steps 1 and 2 we obtain the conclusion of lemma 2.8. □

Denote by $\tau(r_0,\theta_0)$ the time for the solution $(x(t),y(t))$ of (2.2) to make one turn around the origin.

LEMMA 2.9. *Assume that conditions (g_0'), (g_1), (G_0') and (G_1) hold. Let τ_1 be a positive constant. Then, for any $\varepsilon>0$, there exists a constant $c(\varepsilon,\tau_1)>0$ such that, if $c\geqslant c(\varepsilon,\tau_1)$ and $\tau(c)\leqslant\tau_1$, then*

$$|\tau(r_0,\theta_0)-\tau(c)|<\varepsilon,\quad (r_0\cos\theta_0, r_0\sin\theta_0)\in\Gamma_c.$$

Proof. From (G_0') and (g_1) we know that there is a constant $M>0$ such that $2G(x)+M>0$. Set

$$u(t)=\sqrt{y^2(t)+2G(x(t))+M}.$$

Then

$$|u'(t)| = \frac{|y(t)p(t)|}{\sqrt{y^2(t)+2G(x(t))+M}} \leqslant |p(t)|.$$

In what follows, we will first estimate $\tau(r_0,\theta_0)$ with $(r_0\cos\theta_0, r_0\sin\theta_0)=(x_0,y_0)\in\Gamma_c$ and $\tau(c)\leqslant\tau_1$ for sufficiently large c under the additional assumption that $\tau(r_0,\theta_0)\leqslant 2\tau_1$. For $t,s\in[0,2\tau_1]$, we have

$$|u(t)-u(s)|\leqslant e,\quad e=\int_0^{2\tau_1}|p(s)|\,\mathrm{d}s. \tag{2.13}$$

Since $u(0)=\sqrt{y_0^2+2G(x_0)+M}=\sqrt{2c+M}$, we have

$$\sqrt{2c+M}-e\leqslant u(t)\leqslant\sqrt{2c+M}+e,\quad t\in[0,2\tau_1].$$

Consequently,

$$\sqrt{2G(d)+M}-e\leqslant u(t)\leqslant\sqrt{2G(d)+M}+e,$$

where $d=d(c)$. According to (g_1), if x is sufficiently large, then $\sqrt{2G(x)+M}$ is increasing and tends to $+\infty$. Hence, there exist constants $a>d>b>1$ such that

$$\sqrt{2G(a)+M}=\sqrt{2G(d)+M}+e,\qquad \sqrt{2G(b)+M}=\sqrt{2G(d)+M}-e, \tag{2.14}$$

and

$$\sqrt{2G(b)+M}\leqslant u(t)\leqslant\sqrt{2G(a)+M}. \tag{2.15}$$

From (2.14) we get

$$\sqrt{2G(a)+M}-\sqrt{2G(b)+M}=2e.$$

As a result,

$$\frac{G(a)-G(b)}{\sqrt{2G(a)+M}+\sqrt{2G(b)+M}}=e.$$

https://doi.org/10.1017/S0308210505000879 Published online by Cambridge University Press

Accordingly,

$$\frac{(\sqrt{G(a)}-\sqrt{G(b)})(\sqrt{G(a)}+\sqrt{G(b)})}{\sqrt{2G(a)+M}+\sqrt{2G(b)+M}}=e.$$

Therefore,

$$|\sqrt{G(a)}-\sqrt{G(b)}|<2e.$$

It follows from (G_1) that there exists a constant $\varsigma>0$ such that

$$|a-b|<\varsigma. \tag{2.16}$$

From (2.15) we have

$$2G(b)\leqslant y^2(t)+2G(x(t))\leqslant 2G(a)$$

or

$$2(G(b)-G(x(t)))\leqslant y^2(t)\leqslant 2(G(a)-G(x(t))). \tag{2.17}$$

Now, we proceed in two steps. We will always assume that $\tau(c)\leqslant\tau_1$ for sufficiently large c.

STEP 1 ($(x_0,y_0)=(0,\sqrt{2c})$). Let $0=t_0<t_1<t_2$, satisfying

$$x(t_1)>0,\quad y(t_1)=0;\quad x(t)\geqslant 0,\quad y(t)\geqslant 0,\quad t\in[t_0,t_1];$$
$$x(t_2)=0,\quad y(t_2)<0;\quad x(t)\geqslant 0,\quad y(t)\leqslant 0,\quad t\in[t_1,t_2].$$

First we estimate t_1-t_0. Let $t_b\in(t_0,t_1)$, satisfying $x(t_b)=b$, $0\leqslant x(t)\leqslant b$, $t\in[0,t_b]$. It follows from (2.17) that

$$\sqrt{2(G(b)-G(x(t)))}\leqslant\dot{x}(t)\leqslant\sqrt{2(G(a)-G(x(t)))}.$$

Hence,

$$\frac{\dot{x}(t)}{\sqrt{2(G(a)-G(x(t)))}}\leqslant 1\leqslant\frac{\dot{x}(t)}{\sqrt{2(G(b)-G(x(t)))}}.$$

Integrating the above inequality over the interval $[t_0,t_b]$, we derive

$$\int_0^b\frac{\mathrm{d}x}{\sqrt{2(G(a)-G(x))}}\leqslant t_b-t_0\leqslant\int_0^b\frac{\mathrm{d}x}{\sqrt{2(G(b)-G(x))}}.$$

By the equality

$$\int_0^b\frac{\mathrm{d}x}{\sqrt{2(G(a)-G(x))}}=\int_0^a\frac{\mathrm{d}x}{\sqrt{2(G(a)-G(x))}}+\int_a^b\frac{\mathrm{d}x}{\sqrt{2(G(a)-G(x))}},$$

using lemma 2.8 and (2.16), we obtain

$$\int_0^b\frac{\mathrm{d}x}{\sqrt{2(G(a)-G(x))}}=\int_0^a\frac{\mathrm{d}x}{\sqrt{2(G(a)-G(x))}}+o(1),\quad c\to+\infty.$$

Thus,

$$\int_0^a\frac{\mathrm{d}x}{\sqrt{2(G(a)-G(x))}}+o(1)\leqslant t_b-t_0\leqslant\int_0^b\frac{\mathrm{d}x}{\sqrt{2(G(b)-G(x))}}. \tag{2.18}$$

Using [1, lemma 5.4], we have

$$\int_0^b \frac{\mathrm{d}x}{\sqrt{2(G(b)-G(x))}} = \int_0^a \frac{\mathrm{d}x}{\sqrt{2(G(a)-G(x))}} + o(1)$$
$$= \int_0^d \frac{\mathrm{d}x}{\sqrt{2(G(d)-G(x))}} + o(1)$$

as $c \to +\infty$. From (2.18), we get

$$t_b - t_0 = \int_0^d \frac{\mathrm{d}x}{\sqrt{2(G(d)-G(x))}} + o(1). \tag{2.19}$$

Next, we estimate $t_1 - t_b$. Write $\delta(b) = \inf\{g(x) : x \geqslant b\}$. Since $\dot{x}(t_1) = y(t_1) = 0$, for $t \in (t_b, t_1)$, we have

$$\dot{x}(t) = \dot{x}(t) - \dot{x}(t_1) = \int_{t_1}^t \ddot{x}(s)\,\mathrm{d}s = \int_{t_1}^t \dot{y}(s)\,\mathrm{d}s = -\int_{t_1}^t g(x(s))\,\mathrm{d}s + \int_{t_1}^t p(s)\,\mathrm{d}s.$$

Therefore,

$$\dot{x}(t) \geqslant \int_t^{t_1} \delta(b)\,\mathrm{d}s - \int_t^{t_1} p(s)\,\mathrm{d}s \geqslant (t_1 - t)\delta(b) - e.$$

Consequently,

$$\int_{t_b}^{t_1} \dot{x}(s)\,\mathrm{d}s \geqslant \int_{t_b}^{t_1} (t_1 - s)\delta(b)\,\mathrm{d}s - 2e\tau_1.$$

Thus,

$$\tfrac{1}{2}(t_1 - t_b)^2\delta(b) - 2e\tau_1 \leqslant x(t_1) - x(t_b) \leqslant a - b < \varsigma,$$

which yields

$$(t_1 - t_b)^2 \leqslant \frac{2\varsigma + 4e\tau_1}{\delta(b)}.$$

From (g_1) we know that $\delta(b) \to +\infty$, $b \to +\infty$. Therefore,

$$t_1 - t_b = o(1), \quad b \to +\infty. \tag{2.20}$$

It follows from (2.19), (2.20) and remark 2.7 that

$$t_1 - t_0 = (t_1 - t_b) + (t_b - t_0) = \int_0^d \frac{\mathrm{d}x}{\sqrt{2(G(d)-G(x))}} + o(1) = \tfrac{1}{2}\tau(c) + o(1). \tag{2.21}$$

Similarly,

$$t_2 - t_1 = \tfrac{1}{2}\tau(c) + o(1). \tag{2.22}$$

From (2.21), (2.22) and lemma 2.6 we have

$$\tau(r_0, \theta_0) = \sqrt{2}\int_0^{d(c)} \frac{\mathrm{d}x}{\sqrt{c - G(x)}} + o(1) = \tau(c) + o(1), \quad c \to +\infty.$$

Thus, we have proved that, for any $\varepsilon > 0$, there exists $c(\varepsilon, \tau_1) > 0$ such that, if $c \geqslant c(\varepsilon, \tau_1)$ and $\tau(c) \leqslant \tau_1$, then $|\tau(r_0, \theta_0) - \tau(c)| < \varepsilon$.

https://doi.org/10.1017/S0308210505000879 Published online by Cambridge University Press

Step 2 ($(x_0, y_0) \in \Gamma_c$, $(x_0, y_0) \neq (0, \sqrt{2c})$). Without loss of generality, we assume that $x_0 > 0$, $y_0 > 0$. The other cases can be treated similarly. Using the same methods as in step 1, we can prove that the required time for solution $(x(t), y(t))$ ($(x_0, y_0) \in \Gamma_c$) to pass through the regions $\{(x, y) : x \geqslant x_0, y \geqslant 0\}$ and $\{(x, y) : 0 \leqslant x \leqslant x_0, y \geqslant 0\}$ is $\frac{1}{2}\tau(c) + o(1)$ for $c \to +\infty$. Therefore, we only need to prove that the time Δt for solution $(x(t), y(t))$ to pass through the region

$$\mathcal{D} = \left\{(x, y) \in R^2 \;\middle|\; x \geqslant x_0,\ y \geqslant \frac{y_0}{x_0}x,\ G(b) \leqslant \tfrac{1}{2}y^2 + G(x) \leqslant G(a)\right\}$$

or

$$\mathcal{D} = \left\{(x, y) \in R^2 \;\middle|\; x \leqslant x_0,\ 0 \leqslant y \leqslant \frac{y_0}{x_0}x,\ G(b) \leqslant \tfrac{1}{2}y^2 + G(x) \leqslant G(a)\right\}.$$

satisfies $\Delta t = o(1)$ for $c \to +\infty$. Assume that $x_0 < b$. The other case can be shown by using the conclusion in step 1. We will estimate the time Δt in two cases.

Case 1 ($G(x_0) \geqslant \frac{1}{2}G(d)$). Assume that the ray line $y = y_0 x/x_0$ intersects with $x = x_0$, $\frac{1}{2}y^2 + G(x) = G(b)$ and $\frac{1}{2}y^2 + G(x) = G(a)$ at points (x_0, y_0), (x_-, y_-) and (x_+, y_+), respectively. We then have

$$\tfrac{1}{2}y_-^2 + G(x_-) = G(b), \qquad \tfrac{1}{2}y_+^2 + G(x_+) = G(a). \tag{2.23}$$

Recalling (2.14), we know that

$$G(b) = G(d) - e\sqrt{2G(d) + M} + \tfrac{1}{2}e^2, \tag{2.24}$$

$$G(a) - G(b) = 2e\sqrt{2G(d) + M}. \tag{2.25}$$

On the basis of (2.23) and (2.24), we get

$$\begin{aligned} G(x_-) &= G(b) - \tfrac{1}{2}y_-^2 \geqslant G(b) - \tfrac{1}{2}y_0^2 \\ &= G(b) + G(x_0) - G(d) \geqslant G(b) - \tfrac{1}{2}G(d) \\ &= \tfrac{1}{2}G(d) - e\sqrt{2G(d) + M} + \tfrac{1}{2}e^2. \end{aligned}$$

Since

$$G(x_+) - G(x_-) = (G(a) - G(b)) - \tfrac{1}{2}(y_+^2 - y_-^2),$$

and $y_+ \geqslant y_-$, we have

$$G(x_+) - G(x_-) \leqslant G(a) - G(b).$$

It follows from (2.25) that

$$G(x_+) - G(x_-) \leqslant 2e\sqrt{2G(d) + M}.$$

Accordingly,

$$\begin{aligned} \sqrt{2G(x_+)} - \sqrt{2G(x_-)} &= \frac{2(G(x_+) - G(x_-))}{\sqrt{2G(x_+)} + \sqrt{2G(x_-)}} \leqslant \frac{G(x_+) - G(x_-)}{\sqrt{2G(x_-)}} \\ &\leqslant \frac{2e\sqrt{2G(d) + M}}{\sqrt{G(d) - 2e\sqrt{2G(d) + M} + e^2}}. \end{aligned}$$

https://doi.org/10.1017/S0308210505000879 Published online by Cambridge University Press

Since

$$\lim_{d\to+\infty}\frac{2e\sqrt{2G(d)+M}}{\sqrt{G(d)-2e\sqrt{2G(d)+M}+e^2}}=2\sqrt{2}e,$$

there exists a constant $c_1>0$ such that

$$|\sqrt{G(x_+)}-\sqrt{G(x_-)}|<c_1.$$

From condition (G_1), there exists $c_2>0$ such that

$$|x_+-x_-|<c_2. \tag{2.26}$$

In the case when $(x(t),y(t))\in\mathcal{D}$, then $\frac{1}{2}y^2(t)+G(x(t))\geqslant G(b)$. Thus,

$$\sqrt{2(G(b)-G(x(t)))}\leqslant y(t)=\dot{x}(t).$$

If $x_-\leqslant x(t)\leqslant x_+\leqslant b$, then we have

$$\Delta t\leqslant\int_{x_-}^{x_+}\frac{\mathrm{d}s}{\sqrt{2(G(b)-G(s))}}.$$

It follows from (2.26) and lemma 2.8 that $\Delta t<\varepsilon$ when b is sufficiently large. If $x_+\geqslant b$, we divide the interval $[x_-,x_+]$ into two intervals: $[x_-,b]$ and $[b,x_+]$. In $[x_-,b]$, using the fact that $|b-x_-|<c_2$ and lemma 2.8, we know that the time Δt_1 for solution $(x(t),y(t))$ to go through field $\{(x,y)\in R^2\mid x_-\leqslant x\leqslant b, y\geqslant 0\}$ satisfies $\Delta t_1<\frac{1}{2}\varepsilon$ for sufficiently large b. On the other hand, using the same method as in estimating t_b-t_1 in step 1, we know that the time Δt_2 taken for solution $(x(t),y(t))$ to go through the field $\{(x,y)\in R^2\mid x_-\leqslant x\leqslant b, y\geqslant 0\}$ satisfies $\Delta t_2<\frac{1}{2}\varepsilon$ for sufficiently large b. Consequently, we know that the time Δt taken for solution $(x(t),y(t))$ to go through region $\{(x,y)\in R^2\mid x_-\leqslant x\leqslant x_+, y\geqslant 0\}$ satisfies $\Delta t<\varepsilon$.

CASE 2 $(G(x_0)\leqslant\frac{1}{2}G(d))$. Since

$$y_0^2+2G(x_0)=2G(d),$$

we have

$$y_0^2\geqslant G(d).$$

Hence,

$$\frac{x_0}{y_0}\leqslant\frac{d}{\sqrt{G(d)}}\leqslant\int_0^d\frac{2\mathrm{d}s}{\sqrt{G(d)-G(s)}}=\sqrt{2}\tau(c).$$

Recalling (2.23), we get

$$\tfrac{1}{2}y_-^2=G(b)-G(x_-)\geqslant G(b)-G(x_0)\geqslant G(b)-\tfrac{1}{2}G(d).$$

Thus,

$$|y_+-y_-|\leqslant\sqrt{|y_+^2-y_-^2|}\leqslant\sqrt{2(G(a)-G(b))}$$

and

$$\frac{|y_+-y_-|}{y_-}\leqslant\frac{\sqrt{2(G(a)-G(b))}}{\sqrt{2G(b)-G(d)}}.$$

https://doi.org/10.1017/S0308210505000879 Published online by Cambridge University Press

We conclude from (2.24) and (2.25) that

$$\lim_{d\to\infty}\frac{\sqrt{2(G(a)-G(b))}}{\sqrt{2G(b)-G(d)}}=\lim_{d\to\infty}\frac{\sqrt{4e\sqrt{2G(d)+M}}}{\sqrt{2G(d)-2e\sqrt{2G(d)+M}+e^2-G(d)}}$$
$$=\lim_{d\to\infty}\frac{\sqrt{4e\sqrt{2G(d)+M}}}{\sqrt{G(d)-2e\sqrt{2G(d)+M}+e^2}}=0.$$

Consequently,

$$\int_{x_-}^{x_+}\frac{\mathrm{d}x}{y}\leqslant\frac{|x_+-x_-|}{y_-}=\frac{|(x_0y_+/y_0)-(x_0y_-/y_0)|}{y_-}$$
$$=\frac{|y_+-y_-|}{y_-}\frac{x_0}{y_0}$$
$$\leqslant\frac{|y_+-y_-|}{y_-}\sqrt{2}\tau(c),$$

which implies that

$$\int_{x_-}^{x_+}\frac{\mathrm{d}x}{y}<\varepsilon$$

for sufficiently large d.

Combining cases 1 and 2, we obtain

$$|\tau(r_0,\theta_0)-\tau(c)|<\varepsilon,\quad (r_0\cos\theta_0,r_0\sin\theta_0)\in\Gamma_c.$$

So far, we have proved the conclusion of lemma 2.9 under additional assumption $\tau(r_0,\theta_0)\leqslant 2\tau_1$ with $(r_0\cos\theta_0,r_0\sin\theta_0)\in\Gamma_c$ and $\tau(c)\leqslant\tau_1$ for c sufficiently large.

Finally, we shall show that $\tau(r_0,\theta_0)\leqslant 2\tau_1$ holds for $(r_0\cos\theta_0,r_0\sin\theta_0)\in\Gamma_c$ with $\tau(c)\leqslant\tau_1$ and sufficiently large c. Suppose that there exist arbitrarily large c and some point (r_0,θ_0) such that $\tau(r_0,\theta_0)>2\tau_1$ for $(r_0\cos\theta_0,r_0\sin\theta_0)\in\Gamma_c$ and $\tau(c)\leqslant\tau_1$. We also see that (2.17) holds during the time interval $[0,2\tau_1]$. Applying (2.17), lemma 2.6 and the conclusion in steps 1 and 2, we can derive $\tau(r_0,\theta_0)<2\tau_1$ for sufficiently large c. This is a contradiction. □

Denote by $\tau_j(r_0,\theta_0)$ the time taken for the solution $(r(t),\theta(t))$ of system (2.6) to make j clockwise turns around the origin.

Lemma 2.10. *Assume that (g_0'), (g_1), (G_0') and (G_1) hold. Let τ_2 be a positive constant. Then, for any $\varepsilon>0$, there exists $C(\varepsilon,\tau_2)>0$ such that, if $c\geqslant C(\varepsilon,\tau_2)$ and $\tau(c)\leqslant\tau_2$, then*

$$|\tau_j(r_0,\theta_0)-j\tau(c)|<j\varepsilon.$$

Proof. Using the similar methods to those in the proof of lemma 2.9, we see that there exists a positive constant e' such that, for $t,s\in[0,(j+1)\tau_2]$,

$$|u(t)-u(s)|\leqslant e',\quad e'=\int_0^{(j+1)\tau_2}|p(s)|\,\mathrm{d}s,$$

where $u(\cdot)$ is given in the proof of lemma 2.9. Let $(x(t), y(t))$ be the solution of (2.2) through $(x_0, y_0) \in \Gamma_c$. There then exist positive constants ς', a' and b' (with a', b' depending on c) satisfying $|a' - b'| \leqslant \varsigma'$ such that, for $t \in [0, (j+1)\tau_2]$, $(x(t), y(t))$ lies between

$$\Gamma_A : \tfrac{1}{2}y^2 + G(x) = G(a')$$

and

$$\Gamma_B : \tfrac{1}{2}y^2 + G(x) = G(b').$$

Following lemma 2.9 and its proof, we can estimate one by one the required time for the solution $(x(t), y(t))$ to make j turns around the origin. From [1, lemma 5.4] we know that the conclusion of lemma 2.10 holds. □

3. Proof of the theorems

Proof of theorem 1.1. Consider the Poincaré map $P : (r_0, \theta_0) \mapsto (r_1, \theta_1)$,

$$r_1 = r(2\pi; r_0, \theta_0), \quad \theta_1 = \Theta(r_0, \theta_0) + \theta_0,$$

where $\Theta(r_0, \theta_0) = \theta(2\pi; r_0, \theta_0) - \theta_0 + 2m\pi$. The map P is an area-preserving homeomorphism. Let $0 < \varepsilon < \sigma$. It follows from lemma 2.10 and condition (τ_0) that

$$\tau_m(r_0, \theta_0) < m\tau(a_k) + m\varepsilon < 2\pi - m(\sigma - \varepsilon), \quad (r_0 \cos\theta_0, r_0 \sin\theta_0) \in \Gamma_{a_k}, \tag{3.1}$$

$$\tau_m(r_0, \theta_0) > m\tau(b_k) - m\varepsilon > 2\pi + m(\sigma - \varepsilon), \quad (r_0 \cos\theta_0, r_0 \sin\theta_0) \in \Gamma_{b_k}. \tag{3.2}$$

Then, from (3.1), (3.2) we have

$$\theta(2\pi; r_0, \theta_0) - \theta_0 < -2m\pi, \quad (r_0 \cos\theta_0, r_0 \sin\theta_0) \in \Gamma_{a_k},$$
$$\theta(2\pi; r_0, \theta_0) - \theta_0 > -2m\pi, \quad (r_0 \cos\theta_0, r_0 \sin\theta_0) \in \Gamma_{b_k}.$$

Thus, we have proved that the twist condition (3) of the generalized Poincaré–Birkhoff twist theorem (theorem 2.1) is satisfied for the Poincaré map P. Moreover, condition (1) of theorem 2.1 holds by lemma 2.5. Finally, from lemma 2.3 we know that, if $l(x_0, y_0)$ is sufficiently large, then $r(2\pi; r_0, \theta_0) > 0$. Therefore, $\mathrm{O} \in P(D_1)$, where D_1 is an open region with boundary Γ_{a_k} (k is sufficiently large). Thus, condition (2) of theorem 2.1 is satisfied. Hence, the map P has at least two fixed points in D (D is the annulus bounded by Γ_{a_k} and Γ_{b_k}). These two fixed points correspond to two 2π-periodic solutions of system (2.2). For brevity, we assume that $a_k < b_k < a_{k+1}$, $k \in N$. Otherwise, we could take two subsequences of $\{a_k\}$ and $\{b_k\}$ satisfying this inequality. Then for every sufficiently large k, the map P has at least two fixed points in annulus bounded by Γ_{a_k} and Γ_{b_k}. Therefore, the map P has infinitely many fixed points. Consequently, equation (1.1) has infinitely many 2π-periodic solutions. □

Proof of theorem 1.2. Let P^i be the ith iterate of the Poincaré map P for any positive integer i. Namely, $P^1 = P$, $P^2 = P \circ P$ and so on. Thus, P^n can be expressed in the form $P^n : (r_0, \theta_0) \mapsto (r^*, \theta^*)$,

$$r^* = r(2n\pi; r_0, \theta_0), \quad \theta^* = \Theta(r_0, \theta_0) + \theta_0,$$

where $\Theta(r_0,\theta_0)=\theta(2n\pi;r_0,\theta_0)-\theta_0+2m\pi$. Since $(m,n)=1$, we can get

$$\min_{j\in N, i=1,2,\dots,n-1}\left|\frac{2n\pi}{m}-\frac{2i\pi}{j}\right|>0.$$

For simplicity, we assume that

$$0<\beta\leqslant\min_{j\in N, i=1,2,\dots,n-1}\left|\frac{2n\pi}{m}-\frac{2i\pi}{j}\right|.$$

Set $\varepsilon_0<\frac{1}{4}\beta$, $\tilde{\beta}=\frac{1}{4}\beta$. Since $\tau(c)$ is continuous for sufficiently large c, we can find two sequences $\{\tilde{a}_k\}$, $\{\tilde{b}_k\}$ with

$$\tilde{a}_k,\tilde{b}_k>C\left(\varepsilon_0,\frac{2n\pi}{m}+\frac{\beta}{2}\right),\quad \tilde{a}_k<\tilde{b}_k$$

and $\lim_{k\to+\infty}\tilde{a}_k=\lim_{k\to+\infty}\tilde{b}_k=+\infty$ such that

$$\tau(\tilde{a}_k)=\frac{2n\pi}{m}-\tilde{\beta},\qquad \tau(\tilde{b}_k)=\frac{2n\pi}{m}+\tilde{\beta},$$

and

$$\frac{2n\pi}{m}-\tilde{\beta}<\tau(c)<\frac{2n\pi}{m}+\tilde{\beta},\quad c\in(\tilde{a}_k,\tilde{b}_k).$$

For $c\in[\tilde{a}_k,\tilde{b}_k]$, we derive from lemma 2.10 and the choice of $\tilde{\beta}$ that, for all $j\in N$, $i=1,2,\dots,n-1$,

$$|\theta(2i\pi;r_0,\theta_0)-\theta_0+2j\pi|>0,\quad (r_0\cos\theta_0,r_0\sin\theta_0)\in\Gamma_c.$$

In fact, suppose that there exist some $j\in N$, $1\leqslant i\leqslant n-1$, and $r_0,\theta_0\in R$ such that

$$|\theta(2i\pi;r_0,\theta_0)-\theta_0+2j\pi|=0,\quad (r_0\cos\theta_0,r_0\sin\theta_0)\in\Gamma_c.$$

This is to say that $(r(t),\theta(t))$ makes j turns around the origin during time $2i\pi$. On the other hand, from lemma 2.10 we have that, for sufficiently large c,

$$|\tau_j(r_0,\theta_0)-j\tau(c)|<j\varepsilon_0,\quad (r_0\cos\theta_0,r_0\sin\theta_0)\in\Gamma_c.$$

Hence,

$$\left|\frac{2i\pi}{j}-\tau(c)\right|<\varepsilon_0.$$

Furthermore,

$$\left|\frac{2i\pi}{j}-\frac{2n\pi}{m}\right|<\left|\frac{2i\pi}{j}-\tau(c)\right|+\left|\tau(c)-\frac{2n\pi}{m}\right|<\varepsilon_0+\tilde{\beta}<\tfrac{1}{2}\beta,$$

which is contrary to the choice of β. Therefore, from the Poincaré–Bohl theorem [11] we know that the map $P(=P^1)$ has at least one fixed point in the region bounded by $\Gamma_{\tilde{a}_k}$ and thus the system (2.2) has at least one harmonic solution. Meanwhile, P^i has no fixed point on the annulus $A_k=\{(x,y)\in\Gamma_c\mid \tilde{a}_k<c<\tilde{b}_k\}$ for $i=1,2,\dots,n-1$. On the other hand, from lemma 2.10 we have

$$\tau_m(r_0,\theta_0)<m\tau(\tilde{a}_k)+m\varepsilon_0=2n\pi-m(\tilde{\beta}-\varepsilon_0),\quad (r_0\cos\theta_0,r_0\sin\theta_0)\in\Gamma_{\tilde{a}_k},\quad (3.3)$$

$$\tau_m(r_0,\theta_0)>m\tau(\tilde{b}_k)-m\varepsilon_0=2n\pi+m(\tilde{\beta}-\varepsilon_0),\quad (r_0\cos\theta_0,r_0\sin\theta_0)\in\Gamma_{\tilde{b}_k}.\quad (3.4)$$

According to (3.3) and (3.4), we know that

$$(2n\pi; r_0, \theta_0) - \theta_0 < -2m\pi, \quad (r_0 \cos\theta_0, r_0 \sin\theta_0) \in \varGamma_{\tilde{a}_k};$$
$$\theta(2n\pi; r_0, \theta_0) - \theta_0 > -2m\pi, \quad (r_0 \cos\theta_0, r_0 \sin\theta_0) \in \varGamma_{\tilde{b}_k}.$$

Thus, P^n is twisting on the annulus A_k. Moreover, $\mathrm{O} \in P^n(B_k)$ as k is sufficiently large, where B_k is the region surrounded by $\varGamma_{\tilde{b}_k}$. We recall, by lemma 2.5, that $\varGamma_{\tilde{a}_k}$ is star-shaped with respect to the origin O. Therefore, all assumptions of the generalized Poincaré–Birkhoff twist theorem are satisfied for the map P^n. Hence, P^n has at least two fixed points on the annulus A_k. However, P^i, $1 \leqslant i \leqslant n-1$, has no fixed point on the annulus A_k. Thus, the fixed points of the map P^n on A_k correspond to n-order subharmonic solutions of system (2.2). Consequently, (1.1) has infinitely many n-order subharmonic solutions. □

Corollary 3.1. *Assume that* (g_0), (g_1), (G_0) *and* (G_1) *hold and*

$$\Delta\tau = \limsup_{c\to+\infty} \tau(c) - \liminf_{c\to+\infty} \tau(c) > 0.$$

Then (1.1) has at least one harmonic solution and there exists an integer $l_0 > 0$ *such that (1.1) has infinitely many* l*-order subharmonic solutions for* $l \geqslant l_0$.

Proof. The proof of [7, theorem 2.3] and theorem 1.2 yields the conclusion of corollary 3.1. □

Remark 3.2. Let $\gamma > 0$ be an arbitrary fixed constant. Write

$$\tau_\gamma(c) = \sqrt{2} \int_\gamma^{d(c)} \frac{\mathrm{d}x}{\sqrt{c - G(x)}}.$$

From remark 2.7 and lemma 2.6 we know that, if (g_0), (G_0) hold and $\tau_\gamma(c)$ satisfies (τ_0) or (τ_1), then $\tau(c)$ also satisfies (τ_0) or (τ_1) (σ or β may be different). Thus, theorems 1.1 and 1.2 still hold, provided that $\tau_\gamma(c)$ satisfies (τ_0) or (τ_1).

4. An example

In this section, we construct an example to show an application of our conclusions.

Assume that $h : R^+ \to R$ is continuous and satisfies the following conditions:

(1) $h(x) \geqslant \sqrt{x} - x$;

(2) there is a constant $M > 0$ such that

$$|H(x)| = \left| \int_1^x h(s)\,\mathrm{d}s \right| \leqslant M;$$

(3) $h(x_n) = \sqrt{x_n} - x_n$ for $x_n = \mathrm{e}^{2n\pi}$, $n \in N \cup \{0\}$.

Define $g : R^+ \to R$ as follows:

$$g(x) = \begin{cases} \dfrac{\ln x}{x} + 1, & 0 < x \leqslant 1, \\ 2x - x\cos\ln x + h(x), & x > 1. \end{cases}$$

https://doi.org/10.1017/S0308210505000879 Published online by Cambridge University Press

Obviously,

$$\lim_{x\to 0^+} g(x) = -\infty, \qquad \lim_{x\to +\infty} g(x) = +\infty.$$

Thus, $g(x)$ satisfies (g_0) and (g_1). Furthermore,

$$G(x) = \begin{cases} \frac{1}{2}\ln^2 x + x - 1, & 0 < x \leqslant 1, \\ x^2 - \frac{1}{5}x^2(\sin\ln x + 2\cos\ln x) - \frac{3}{5} + H(x), & x > 1, \end{cases}$$

where

$$H(x) = \int_1^x h(s)\,\mathrm{d}s.$$

It is easy to see that

$$\lim_{x\to 0^+} G(x) = +\infty.$$

Therefore, condition (G_0) is satisfied. Since $\hat{g}(x) = 2x - x\cos\ln x$ satisfies the semilinear condition for large positive x, as far as the function $\hat{g}(x)$ is concerned, condition (G_1) is satisfied. Using [15, lemma 2.3], we know that the function $g(x)$ also satisfies (G_1). By a simple calculation, we obtain

$$G_* = \liminf_{x\to+\infty} \frac{2G(x)}{x^2} = 2\left(1 - \frac{\sqrt{5}}{5}\right), \qquad G^* = \limsup_{x\to+\infty} \frac{2G(x)}{x^2} = 2\left(1 + \frac{\sqrt{5}}{5}\right).$$

It follows from remark 2.7 and [7] that

$$\left[\frac{\pi}{\sqrt{G^*}}, \frac{\pi}{\sqrt{G_*}}\right] \subset [\tau_*, \tau^*],$$

where

$$\tau_* = \liminf_{c\to+\infty} \tau(c), \qquad \tau^* = \limsup_{c\to+\infty} \tau(c).$$

Therefore,

$$\Delta\tau = \limsup_{c\to+\infty} \tau(c) - \liminf_{c\to+\infty} \tau(c) = \tau^* - \tau_* > 0.$$

According to corollary 3.1, we know that (1.1) has at least one harmonic solution and there exists $l_0 > 0$ such that, for $l \geqslant l_0$, (1.1) has infinitely many l-order subharmonic solutions.

Acknowledgments

This research was supported by the National Natural Science Foundation of China, Grant no. 10471099, the Scientific Research Foundation for the Returned Overseas Chinese Scholars, Beijing Personnel Bureau, State Education Ministry, Beijing Education Committee Grant no. KM200410028003 and the key project of NSFB-FBEC under Contract no. KZ200310028010. The authors are grateful to the referee for many valuable suggestions to make the paper more readable.

References

1 A. Capietto, J. Mawhin and F. Zanolin. A continuation theorem for periodic boundary value problems with oscillatory nonlinearities. *Nonlin. Diff. Eqns Applic.* **2** (1995), 133–163.

2 M. del Pino and R. Manásevich. Infinitely many T-periodic solutions for a problem arising in nonlinear elasticity. *J. Diff. Eqns* **103** (1993), 260–277.

3 M. del Pino, R. Manásevich and A. Montero. T-periodic solutions for some second order differential equations with singularities. *Proc. R. Soc. Edinb.* A **120** (1992), 231–243.

4 T. Ding. An infinite class of periodic solutions of periodically perturbed Duffing equations at resonance. *Proc. Am. Math. Soc.* **86** (1982), 47–54.

5 W. Ding. A generalization of the Poincaré–Birkhoff theorem. *Proc. Am. Math. Soc.* **88** (1983), 341–346.

6 T. Ding. *Applications of qualitative methods of ordinary differential equations* (Beijing: Higher Education Press, 2004).

7 T. Ding, R. Iannacci and F. Zanolin. Existence and multiplicity results for periodic solutions of semilinear Duffing equations. *J. Diff. Eqns* **2** (1993), 364–409.

8 A. Fonda, R. Manásevich and F. Zanolin. Subharmonic solutions for some second-order differential equations with singularities. *SIAM J. Math. Analysis* **24** (1993), 1294–1311.

9 P. Habets and L. Sanchez. Periodic solutions of some Liénard equations with singularities. *Proc. Am. Math. Soc.* **109** (1990), 1035–1044.

10 A. C. Lazer and S. Solimini. On periodic solutions of nonlinear differential equations with singularities. *Proc. Am. Math. Soc.* **88** (1987), 109–114.

11 N. G. Lloyd. *Degree theory* (Cambridge University Press, 1978).

12 P. Omari and W. Ye. Necessary and sufficient conditions for the existence of periodic solutions of second order differential equations with singular nonlinearities. *Diff. Integ. Eqns* **8** (1995), 1843–1858.

13 D. Qian. Time-maps and Duffing equations crossing resonance points. *Sci. China* A **23** (1993), 471–479.

14 S. Solimini. On forced dynamical systems with a singularity of repulsive type. *Nonlin. Analysis* **14** (1990), 489–500.

15 Z. Wang. Multiplicity of periodic solutions for Duffing equation crossing resonance points. *Chin. Ann. Math.* A **18** (1997), 421–432.

16 Z. Wang. Periodic solutions of the second order differential equations with singularity. *Nonlin. Analysis* **58** (2004), 319–331.

17 P. Yan and M. Zhang. Higher order non-resonance for differential equations with singularities. *Math. Meth. Appl. Sci.* **26** (2003), 1067–1074.

18 M. Zhang. Periodic solutions of Liénard equations with singular forces of repulsive type. *J. Math. Analysis Applic.* **203** (1996), 254–269.

(*Issued 8 June 2007*)