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Morse index and symmetry breaking for an elliptic equation with negative exponent in expanding annuli

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Bifurcation of non-radial solutions from radial solutions of a semilinear elliptic equation with negative exponent in expanding annuli of $\mathbb{R}^2$ is studied. To obtain the main results, we use a blow-up argument via the Morse index of the regular entire solutions of the equation

$$\Delta u = \frac{\lambda}{u^2} \quad \text{in } \mathbb{R}^2.$$

The main results of this paper can be seen as applications of the results obtained recently for finite Morse index solutions of the equation

$$\Delta u = \frac{\lambda}{u^p} \quad \text{in } \mathbb{R}^N$$

with $N \geqslant 2$ and $p > 0$.

Keywords: expanding annuli; bifurcation; symmetry breaking; negative exponent; Morse index

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1. Introduction

We study symmetry breaking of radial solutions to the equation

$$\left.\begin{aligned} -\Delta v &= \frac{\lambda}{(1-v)^2} \quad \text{in } D_R, \\ 0 < v &< 1 \quad \text{in } D_R, \\ v &= 0 \quad \text{on } \partial D_R, \end{aligned}\right\} \tag{1.1}$$

where $D_R = \{x \in \mathbb{R}^2 : R < |x| < R+1\}$ with $R > 1$, with λ a positive parameter. We will obtain bifurcations of non-radial solutions from the radial solutions of (1.1).

Problem (1.1) arises from the use of electrostatic forces to provide actuation, which is a method of central importance in microelectromechanical systems and in nanoelectromechanical systems; see, for example [21, 25]. It has been studied by many authors in recent years; see, for example, [3–5, 7, 9–14, 19] and the references therein.

Let v be a solution to the problem

$$\left.\begin{aligned} -\Delta v &= \frac{\lambda}{(1-v)^2} \quad \text{in } \Omega, \\ 0 < v &< 1 \quad \text{in } \Omega, \\ v &= 0 \quad \text{on } \partial\Omega, \end{aligned}\right\} \tag{1.2}$$

where $\Omega \subset \mathbb{R}^2$, $\lambda > 0$. Denote by Q_v the bilinear form associated with v, i.e.

$$Q_v(\phi,\psi) = \int_\Omega \nabla\phi\cdot\nabla\psi \,\mathrm{d}x - \int_\Omega \frac{2\lambda}{(1-v)^3}\phi\psi\,\mathrm{d}x, \quad \phi,\psi \in H_0^1(\Omega).$$

The Morse index at (λ, v), denoted by $i(\lambda, v)$, is the maximal dimension of a subspace $X \subset H_0^1(\Omega)$ such that

$$Q_v(\psi,\psi) < 0 \quad \forall \psi \in X \setminus \{0\}.$$

This is equivalent to saying that $i(\lambda, v)$ is the number of negative eigenvalues of $-\Delta - (2\lambda/(1-v)^3)I$ computed with their multiplicity. If $\Omega = \{x \in \mathbb{R}^2 \colon a < |x| < b\}$ with $b > a > 0$ and v is a radial solution of (1.2), the radial Morse index at (λ, v), $i_{\mathrm{rad}}(\lambda, v)$, is the number of negative eigenvalues of the problem

$$\begin{gathered} \psi'' + \frac{1}{r}\psi' + \frac{2\lambda}{(1-v)^3}\psi = -\mu\psi, \quad r \in (a,b), \\ \psi(a) = \psi(b) = 0. \end{gathered}$$

For any fixed $R > 1$, variants of the arguments in the proof of theorem 1.1 of [6] imply that there exists a $\lambda_R^* \in (0,\infty)$ such that problem (1.1) admits no radial solutions for $\lambda > \lambda_R^*$, one and only one radial solution v_R^* for $\lambda = \lambda_R^*$, and exactly two radial solutions $0 < \underline{v}_R^\lambda < \bar{v}_R^\lambda$ for $0 < \lambda < \lambda_R^*$. It is well known that $\underline{v}_R^\lambda$ $(0 < \lambda < \lambda_R^*)$ and v_R^* are *stable*, i.e. $i(\lambda, \underline{v}_R^\lambda) = i_{\mathrm{rad}}(\lambda, \underline{v}_R^\lambda) = 0$ for $\lambda \in (0, \lambda_R^*]$. Moreover, v_R^* is the unique radial solution of (1.1) at which the first eigenvalue of the linearized problem is 0. Note that $\bar{v}_R^\lambda$ is radially non-degenerate for $\lambda \in (0, \lambda_R^*)$. To see this, we notice that if μ is an eigenvalue of the problem

$$\begin{gathered} -\phi''(r) - \frac{1}{r}\phi'(r) - \frac{2\lambda}{(1-\bar{v}_R^\lambda)^3}\phi = \mu\phi, \quad R < r < R+1, \\ \phi(R) = \phi(R+1) = 0, \end{gathered}$$

then μ is simple. By the bifurcation theory on the critical point of odd multiplicity [23, 24], if there is some $\mu = 0$, it is actually a secondary bifurcation point, i.e. there is a new radial solution branch of (1.1) coming from $(\lambda, \bar{v}_R^\lambda)$ (note that the nonlinearity of (1.1) is convex). This contradicts the fact that (1.1) has exactly two radial solutions for $\lambda \in (0, \lambda_R^*)$. It was also shown in [6] that the upper branch of radial solutions of (1.1) has non-radially symmetric bifurcations at infinitely many $\lambda_N \in (0, \lambda_R^*)$.

Arguments similar to those in the proof of theorem 1.1 of [6] imply that a similar result is also true for the problem

$$\left.\begin{aligned} -v'' = \frac{\lambda}{(1-v)^2} \quad &\text{in } (0,1), \\ 0 < v < 1 \quad &\text{in } (0,1), \\ v(0) = 0 = v(1), & \end{aligned}\right\} \tag{1.3}$$

i.e. there exists a $\lambda_0^* \in (0,\infty)$ such that problem (1.3) admits no solutions for $\lambda > \lambda_0^*$, one and only one solution v_0^* for $\lambda = \lambda_0^*$, and exactly two solutions $0 < \underline{v}_0^\lambda < \bar{v}_0^\lambda$ for $0 < \lambda < \lambda_0^*$. It is well known that $\underline{v}_0^\lambda$ $(0 < \lambda < \lambda_0^*)$ and v_0^* are *stable* and the Morse index of $\bar{v}_0^\lambda$ $(0 < \lambda < \lambda_0^*)$ is not less than 1. Moreover, v_0^* is the unique solution of (1.3) at which the first eigenvalue of the linearized problem is 0.

In this paper we are interested in studying the asymptotic behaviours of the radial solutions and bifurcation of non-radial solutions from a radial solution of (1.1) when R varies. To do this, we first need to know the asymptotic behaviour of λ_R^* as $R \to \infty$. The main results of this paper can then be expressed as the following theorems.

THEOREM 1.1. *We have*

$$\lim_{R\to+\infty} \lambda_R^* = \lambda_0^*. \tag{1.4}$$

Moreover, for $\lambda \in (0,\lambda_0^*)$, $\underline{v}_R^\lambda(R+t) \to \underline{v}_0^\lambda(t)$ *uniformly for* $t \in [0,1]$ *as* $R \to \infty$, *and* $v_R^*(R+t) \to v_0^*(t)$ *uniformly for* $t \in [0,1]$ *as* $R \to \infty$.

THEOREM 1.2. *For any fixed* $\lambda \in (0,\lambda_0^*)$ *there exist* $R_* \gg 1$ *and a sequence* $\{R_j\}$ *with* $R_j \geqslant R_*$ *and* $R_j \to \infty$ *as* $j \to \infty$ *such that the Morse index* $i(\lambda, \bar{v}_{R_j}^\lambda) \to \infty$ *as* $j \to \infty$.

THEOREM 1.3. *For any fixed* $\lambda \in (0,\lambda_0^*)$ *there exists a sequence* $\{R_k\}$ *with* $R_k \geqslant R_*$ *and* $R_k \to \infty$ *as* $k \to \infty$ *such that a non-radial bifurcation occurs at* $(\lambda, R_k, \bar{v}_{R_k}^\lambda)$.

Bifurcation and symmetry breaking of positive radial solutions of the equations with 'positive exponent' in annular domains have been studied by many authors; see, for example, [1, 8, 15–18] and references therein.

2. Preliminaries

In this section we obtain some results on the spectrum of the linearized operator associated with a radial solution v of the problem

$$\left.\begin{aligned} -\Delta v = \frac{\lambda}{(1-v)^2} \quad &\text{in } D, \\ 0 < v < 1 \quad &\text{in } D, \\ v = 0 \quad &\text{on } \partial D, \end{aligned}\right\} \tag{2.1}$$

where $D = \{x \in \mathbb{R}^2 \colon \alpha < |x| < \beta\}$ $(0 < \alpha < \beta < \infty)$, $\lambda > 0$. Note that v depends on λ. The linearized operator at v is defined as

$$L_v = -\Delta - \frac{2\lambda}{(1-v)^3} I.$$

We recall that a solution v to (2.1) is degenerate if the equation

$$\left.\begin{aligned} -\Delta z - \frac{2\lambda}{(1-v)^3} z = 0 \quad \text{in } D, \\ z = 0 \quad \text{on } \partial D, \end{aligned}\right\} \tag{2.2}$$

admits non-trivial solutions.

In order to study the spectrum of the operator L_v, as in [1] we introduce the operators $\tilde{L}\colon H^2(D) \cap H_0^1(D) \to L^2(D)$,

$$\tilde{L}_v := |x|^2 \left(-\Delta - \frac{2\lambda}{(1-v)^3} I \right), \quad x \in D, \tag{2.3}$$

and $\hat{L}\colon H^2(\alpha,\beta) \cap H_0^1(\alpha,\beta) \to L^2(\alpha,\beta)$,

$$\hat{L}_v(w) := r^2 \left(-w'' - \frac{1}{r} w' - \frac{2\lambda}{(1-v)^3} w \right), \quad r \in (\alpha,\beta). \tag{2.4}$$

Note that the eigenvalues of $\tilde{L}_v$ are given by

$$\tilde{\mu}_i = \inf_{W \subset H_0^1(D),\, \dim W = i} \; \max_{w \in W,\, w \neq 0} \frac{\int_D |\nabla w|^2 - \int_D (2\lambda/(1-v)^3) w^2}{\int_D |x|^{-2} w^2} \tag{2.5}$$

and the eigenvalues $\hat{\mu}_i$ of $\hat{L}_v$ are obtained in the same way just replacing the space $H_0^1(D)$ by $H_0^1(\alpha,\beta)$.

The proofs of the following lemmas can be obtained by arguments similar to those in the proofs of lemmas 2.1–2.3 in [8].

Lemma 2.1. *Let v be a radial solution to (2.1). Then the Morse index $i(\lambda, v)$ of v is equal to the number of negative eigenvalues $\tilde{\mu}_v$ of the operator $\tilde{L}_v$.*

Denoting by $\sigma(\cdot)$ the spectrum of a linear operator, we have the following lemma.

Lemma 2.2. *Let v be a radial solution to (2.1). Then*

$$\sigma(\tilde{L}_v) = \sigma(\hat{L}_v) + \sigma(-\Delta_{S^1}).$$

Let us denote by λ_k, $k = 0, 1, 2, \ldots$, the eigenvalues of the operator $-\Delta_{S^1} := -\partial^2/\partial\theta^2$ and by ϕ_k the corresponding eigenfunctions. It is known that $\lambda_k = k^2$. Let us denote by $\hat{\mu}_i$ the eigenvalues of $\hat{L}_v$ and by w_i the corresponding eigenfunctions with $\|w_i\|_\infty = 1$.

Lemma 2.3. *Let v be a radial solution to (2.1) that is non-degenerate in the space $H^1_{0,\text{rad}}(D)$. Then problem (2.2) has a non-trivial solution if and only if there exists $k \geqslant 1$ such that*

$$\hat{\mu}_1 + \lambda_k = 0. \tag{2.6}$$

Moreover, the solutions z to (2.2) can be written as

$$z(x) = w_1(|x|)\phi_k\left(\frac{x}{|x|}\right),$$

where $w_1(r)$ is the first positive eigenfunction of $\hat{L}_v$ and $\phi_k(\theta)$ is the eigenfunction of $-\partial^2/\partial\theta^2$ relative to the eigenvalue λ_k.

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Remark 2.4. It follows from the lemmas above that the Morse index of $\tilde{L}_v$ depends only on $\hat{\mu}_1$.

3. Asymptotic behaviours and Morse index of the radial solutions

In this section we consider the asymptotic behaviours and Morse index of the radial solutions of (1.1). Note that the domains D_R are expanding annuli when R varies. We will present the proof of theorems 1.1 and 1.2 in this section.

Lemma 3.1. *Let* $0 < a < \frac{1}{4}$ *and* $A_a = \{x\colon 1 - a < |x| < 1\} \subset \mathbb{R}^2$. *Then there is a* $0 < \lambda_a^* < \infty$ *such that for* $\lambda \in (0, \lambda_a^*)$ *the problem*

$$\left.\begin{aligned} -\Delta u &= \frac{\lambda}{(1-u)^2} && \text{in } A_a, \\ 0 < u &< 1 && \text{in } A_a, \\ u &= 0 && \text{on } \partial A_a, \end{aligned}\right\} \tag{3.1}$$

admits exactly two radial solutions $0 < \underline{u}_\lambda^a < \bar{u}_\lambda^a$ *with* $\max_{A_a} \bar{u}_\lambda^a \to 1$ *as* $\lambda \to 0^+$; *(3.1) admits exactly one radial solution* u_*^a *for* $\lambda = \lambda_a^*$; *(3.1) does not admit a solution for* $\lambda > \lambda_a^*$. *Moreover,* λ_a^* *is a decreasing function with respect to* a *and* $\lim_{a\to 0^+} \lambda_a^* = \infty$.

Proof. For each fixed $a \in (0, \frac{1}{4})$, it is known from [6] that

$$\lambda_a^* = \sup\{\lambda \in (0,\infty)\colon \text{(3.1) admits a minimal radial solution}\}$$

is bounded, and, for $\lambda \in (0, \lambda_a^*)$, (3.1) admits exactly two radial solutions $0 < \underline{u}_\lambda^a < \bar{u}_\lambda^a$ with $\max_{A_a} \bar{u}_\lambda^a \to 1$ as $\lambda \to 0^+$; (3.1) admits exactly one radial solution u_*^a for $\lambda = \lambda_a^*$; (3.1) does not admit a solution in $H_0^1(A_a)$ for $\lambda > \lambda_a^*$.

For any $a_1, a_2 \in (0, \frac{1}{4})$, $a_2 > a_1$, we see that $A_{a_1} \subset A_{a_2}$ and $\underline{u}_*^{a_2}$ is a supersolution to the problem

$$\left.\begin{aligned} -\Delta u &= \frac{\lambda_{a_2}^*}{(1-u)^2} && \text{in } A_{a_1}, \\ 0 < u &< 1 && \text{in } A_{a_1}, \\ u &= 0 && \text{on } \partial A_{a_1}. \end{aligned}\right\} \tag{3.2}$$

This implies that (3.1) with $a = a_1$ admits a minimal radial solution for $\lambda = \lambda_{a_2}^*$ (note that $u \equiv 0$ is a subsolution to (3.2)). Therefore, $\lambda_{a_2}^* \leqslant \lambda_{a_1}^*$. This shows that λ_a^* is a decreasing function with respect to a, and $\lim_{a\to 0^+} \lambda_a^*$ exists and can be $+\infty$.

To show that $\lim_{a\to 0^+} \lambda_a^* = +\infty$, we first notice that $\operatorname{meas}(A_a) \to 0$ as $a \to 0^+$. Thus, for any sufficiently small $a > 0$, we can choose a proper annulus A'_a that contains A_a with $\operatorname{meas}(A'_a) \to 0$ as $a \to 0^+$ such that the first eigenfunction φ_1^a with $\|\varphi_1^a\|_\infty = 1$ of the problem

$$\left.\begin{aligned} -\Delta\varphi &= \sigma\varphi && \text{in } A'_a, \\ \varphi &= 0 && \text{on } \partial A'_a, \end{aligned}\right\} \tag{3.3}$$

satisfies that $\frac{1}{2} < m_a \leqslant \varphi_1^a \leqslant 1$ on A_a. Indeed, for any sufficiently small $a > 0$, there is a constant $1 \leqslant M \leqslant 1/2a$, which is independent of a and will be determined later, such that $1 - aM \geqslant \frac{1}{2}$ (note that M can be chosen to be sufficiently large if a is sufficiently small). Set $A'_a = \{x\colon 1 - aM < |x| < 1 + aM\}$. Then $\varphi_1 := \varphi_1^a$ is a positive radial solution of (3.3) and

$$\left.\begin{aligned} -\varphi_1'' - \frac{1}{r}\varphi_1' = \sigma_1\varphi_1, \\ 1 - aM < r < 1 + aM, \\ \varphi_1(1 - aM) = \varphi_1(1 + aM) = 0, \end{aligned}\right\} \tag{3.4}$$

where $\sigma_1 := \sigma_1^a$ is the first eigenvalue of (3.3) corresponding to the first eigenfunction φ_1. Set $r = 1 + a\rho$ and $\tilde{\varphi}_1(\rho) = \varphi_1(r)$. Then we have that

$$\left.\begin{aligned} -\tilde{\varphi}_1'' - \frac{a}{1 + a\rho}\tilde{\varphi}_1' = \sigma_1 a^2\tilde{\varphi}_1, \\ -M < \rho < M, \\ \tilde{\varphi}_1(-M) = \tilde{\varphi}_1(M) = 0. \end{aligned}\right\} \tag{3.5}$$

By comparing σ_1 with the first eigenvalue of the eigenvalue problem on the ball $B_{aM}(x_0)$, we obtain

$$-\Delta\varphi = \sigma\varphi \quad \text{in } B_{aM}(x_0), \qquad \varphi = 0 \quad \text{on } \partial B_{aM}(x_0),$$

where $x_0 = (1, 0)$, and we find that there is a constant $C(M)$ such that

$$\sigma_1 a^2 \leqslant C(M),$$

since $B_{aM}(x_0) \subset A'_a$. We now can apply the L^q estimate and embedding theorems to conclude that there is a sequence of $a \to 0^+$, say a_n, such that $\tilde{\varphi}_1^{a_n} \to \psi$ on $C^1([-M, M])$ as $n \to \infty$, where $\psi(\rho)$ is the positive function satisfying

$$-\psi'' = \mu\psi \quad \text{on } (-M, M),\ \psi(-M) = \psi(M) = 0,$$

with $\|\psi\|_\infty = 1$ and some constant $\mu \in [0, C(M)]$. Clearly, $\mu = (\pi/2M)^2$ and $\psi(\rho) = \cos(\pi\rho/2M)$. By a compactness and uniqueness argument we have

$$\varphi_1^a(1 + a\rho) \to \cos\left(\frac{\pi\rho}{2M}\right) \quad \text{in } C^1([-M, M]) \quad \text{and} \quad a^2\sigma_1^a \to \left(\frac{\pi}{2M}\right)^2 \quad \text{as } a \to 0^+.$$

Therefore, for sufficiently small $a > 0$ and $1 - aM/3 < r < 1 + aM/3$, we have

$$\varphi_1^a(r) > \tfrac{1}{2} \quad \text{and} \quad \frac{\pi^2}{8M^2a^2} < \sigma_1^a < \frac{\pi^2}{2M^2a^2}. \tag{3.6}$$

Fix an $M \geqslant 6$. We have $A_a \subset A'_a$ and $\varphi_1^a > \frac{1}{2}$ on A_a. Clearly, $\frac{1}{3}\varphi_1^a$ is a supersolution to (3.1) with $\lambda_a = \frac{4}{27}\sigma_1^a m_a$, where $m_a := \inf_{A_a}\varphi_1^a > 1/2$. This implies that

$$\lambda_a \leqslant \lambda_a^*. \tag{3.7}$$

From (3.6) we have $\sigma_1^a \to +\infty$ as $a \to 0^+$. It follows that $\lambda_a \to +\infty$ as $a \to 0^+$, and hence $\lambda_a^* \to +\infty$ as $a \to 0^+$ by (3.7). □

LEMMA 3.2. *Let $D_R = \{x \in \mathbb{R}^2 \colon R < |x| < R+1\}$ with $R > 1$. There exists $\lambda_R^* > 0$ such that the problem*

$$\left.\begin{aligned} -\Delta v &= \frac{\lambda}{(1-v)^2} \quad \text{in } D_R, \\ 0 < v &< 1 \quad \text{in } D_R, \\ v &= 0 \quad \text{on } \partial D_R, \end{aligned}\right\} \tag{3.8}$$

admits no radial solutions for $\lambda > \lambda_R^*$, one and only one radial solution v_R^* for $\lambda = \lambda_R^*$ and exactly two radial solutions $0 < \underline{v}_R^\lambda < \bar{v}_R^\lambda$ for $0 < \lambda < \lambda_R^*$. Moreover, there exists $C > 0$ independent of R such that, for all R sufficiently large,

$$\lambda_R^* \geqslant C. \tag{3.9}$$

Proof. We only need to show (3.9). Making the transformation

$$u(\rho) = v(r), \qquad \rho = \frac{r}{R+1},$$

we see that u satisfies the problem

$$\left.\begin{aligned} -\Delta u &= \frac{\lambda(R+1)^2}{(1-u)^2} \quad \text{in } A_{1/(R+1)}, \\ 0 < u &< 1 \quad \text{in } A_{1/(R+1)}, \\ u &= 0 \quad \text{on } \partial A_{1/(R+1)}. \end{aligned}\right\} \tag{3.10}$$

It follows from lemma 3.1 that

$$\lambda_R^*(R+1)^2 = \lambda_{1/(R+1)}^*. \tag{3.11}$$

It is known from (3.7) and (3.6) that, for $R > 0$ sufficiently large, there exists $C > 0$ independent of R such that

$$\lambda_{1/(R+1)}^* \geqslant C(R+1)^2. \tag{3.12}$$

Both (3.11) and (3.12) imply (3.9). □

Let $R_* \gg 1$ be a large number. For each $R \geqslant R_*$, we now study the linearized problem at the solution $(\lambda, \bar{v}_R^\lambda)$ of (3.8):

$$\left.\begin{aligned} \Delta h + \frac{2\lambda}{(1-\bar{v}_R^\lambda)^3} h &= -\mu h \quad \text{in } D_R, \\ h &= 0 \quad \text{on } \partial D_R. \end{aligned}\right\} \tag{3.13}$$

It is known from Crandall and Rabinowitz [2] that the Morse index at $(\lambda, \bar{v}_R^\lambda)$, i.e. the number of negative eigenvalues of the problem (3.13), satisfies $i(\lambda, \bar{v}_R^\lambda) \geqslant 1$, and $i_{\text{rad}}(\lambda, \bar{v}_R^\lambda) \geqslant 1$ for $\lambda \in (0, \lambda_R^*)$, since $f(s) = 1/(1-s)^2$ is a convex function for $s \in (0,1)$.

Let $\Lambda^* = \inf_{R \geqslant R_*} \lambda_R^*$. It is known from (3.9) that $\Lambda^* \geqslant C$.

Proof of theorem 1.1. Let σ_R^1 be the first eigenvalue of the problem

$$\begin{aligned} -\Delta\phi &= \sigma_R^1 \phi \quad \text{in } D_R, \\ \phi &= 0 \quad \text{on } \partial D_R. \end{aligned}$$

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It is known from [18, lemma A.1] that $\sigma_R^1 \to \pi^2$ as $R \to \infty$. Arguments similar to those in the proof of theorem 3.1 of [6] imply that, for R sufficiently large,

$$\lambda_R^* \leqslant \tfrac{4}{27}\sigma_R^1 \leqslant \tfrac{8}{27}\pi^2.$$

It is also known from lemma 3.2 that $\lambda_R^* \geqslant C$ for some $C > 0$ independent of R. Therefore, there is an $R_{**} > R_*$ such that, for $R \geqslant R_{**}$,

$$C \leqslant \lambda_R^* \leqslant \tfrac{8}{27}\pi^2. \tag{3.14}$$

For any $\lambda \in (0, \inf_{R \geqslant R_{**}} \lambda_R^*]$, we know that $i(\lambda, \underline{v}_R) = 0$ for all $R \geqslant R_{**}$. Then we claim that there is a $C := C(\lambda) > 0$, which is independent of R, such that, for all $R \geqslant R_{**}$,

$$(1 - \underline{v}_R(x))^{-3/2} \leqslant C \operatorname{dist}^{-1}(x, \partial D_R) \quad \forall x \in D_R. \tag{3.15}$$

Note that we write $\underline{v}_R^\lambda$ as $\underline{v}_R$. Equation (3.15) implies that

$$1 - \underline{v}_R(x) \geqslant C \operatorname{dist}^{2/3}(x, \partial D_R) \quad \forall x \in D_R, \tag{3.16}$$

which provides a uniform positive lower bound of $1 - \underline{v}_R$ and means that $\underline{v}_R$ is a classical solution.

Assume that (3.15) fails. Then there exist sequences $\{R_k\}$, $\{\underline{v}_k\} \equiv \{\underline{v}_{R_k}\}$, $y_k \in D_{R_k}$, such that $\underline{v}_k$ solves the problem

$$\left.\begin{aligned} -\Delta v = \frac{\lambda}{(1-v)^2} \quad & \text{in } D_{R_k}, \\ 0 < v < 1 \quad & \text{in } D_{R_k}, \\ v = 0 \quad & \text{on } \partial D_{R_k}, \end{aligned}\right\} \tag{3.17}$$

and the functions

$$M_k(x) := (1 - \underline{v}_k(x))^{-3/2}, \quad k = 1, 2, \dots,$$

satisfy

$$M_k(y_k) > 2k \operatorname{dist}^{-1}(y_k, \partial D_{R_k}).$$

By a doubling lemma (see [22, lemma 5.1]), we see that there exists $x_k \in D_{R_k}$ such that

$$M_k(x_k) \geqslant M_k(y_k), \qquad M_k(x_k) > 2k \operatorname{dist}^{-1}(x_k, \partial D_{R_k})$$

and

$$M_k(z) \leqslant 2M_k(x_k), \qquad |z - x_k| \leqslant k M_k^{-1}(x_k).$$

Now we rescale $1 - \underline{v}_k$ by setting

$$w_k(y) := \tau_k^{-2/3}(1 - \underline{v}_k(x_k + \tau_k y)), \quad |y| \leqslant k \text{ with } \tau_k = M_k^{-1}(x_k).$$

The function w_k solves

$$\Delta w_k = \frac{\lambda}{w_k^3} w_k, \quad |y| \leqslant k.$$

Moreover,

$$w_k^{-3/2}(0) = \tau_k M_k(x_k) = 1$$

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and

$$\frac{\lambda}{w_k^3(y)} \leqslant 4\lambda, \quad |y| \leqslant k \text{ for all } k.$$

By the Harnack inequality, there is a constant C, independent of k, such that

$$w_k(y) \leqslant C, \quad |y| \leqslant k/2.$$

Note that since the Morse index of $i(\lambda, \underline{v}_k) = 0$ for all k, we see that $i(\lambda, w_k) = i(\lambda, \underline{v}_k) = 0$ for all k. By using elliptic L^q estimates and standard embeddings, we deduce that some subsequence of $\{w_k\}$ converges in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$ to a (classical) solution w of the equation

$$\Delta w = \frac{\lambda}{w^2} \quad \text{in } \mathbb{R}^2.$$

Moreover, $w^{-3/2}(0) = 1$ and $i(\lambda, w) = 0$ (see step 1 in the proof of theorem 1.2 below). On the other hand, it is known from [4, theorem 1.1] and [3, theorem 1.1] that the Morse index of any regular entire solution z of the equation

$$\Delta z = \frac{\lambda}{z^p} \quad \text{in } \mathbb{R}^N$$

is ∞, provided that $p > p_c(N)$ for some $p_c(N)$ given there. Therefore, we obtain that $i(\lambda, w) = \infty$ (note that $p_c(2) = 0$ if $N = 2$ and $2 > 0$ in our case here). This contradicts $i(\lambda, w) = 0$ obtained above and hence (3.15) and (3.16) hold.

We now use the blow-up argument again to get a lower bound for $1 - \underline{v}_R$. We have that there exists $\varepsilon := \varepsilon(\lambda) > 0$ independent of R such that, for all $R \geqslant R_{**}$,

$$1 - \underline{v}_R \geqslant \varepsilon \quad \text{in } D_R. \tag{3.18}$$

If, on the contrary, there is a sequence $\{R_k\}$ with $R_k \to \infty$ as $k \to \infty$ such that $1 - \underline{v}_{R_k}(x_k) \to 0$ as $k \to \infty$, where $\underline{v}_{R_k}(x_k) = \max_{D_{R_k}} \underline{v}_{R_k}$, then define $\delta_k := 1 - \underline{v}_{R_k}(x_k)$ (which tends to 0 as $k \to \infty$), where $\underline{v}_{R_k}(x_k) = \max_{D_{R_k}} \underline{v}_{R_k}$. Setting

$$y = \delta_k^{-3/2}(x - \eta_k), \qquad \tilde{v}_k(y) = \underline{v}_{R_k}(x),$$

where $\eta_k \in \partial D_{R_k}$ such that $\mathrm{dist}(x_k, \eta_k) = \mathrm{dist}(x_k, \partial D_{R_k})$, we see that $\tilde{v}_k$ satisfies the problem

$$\begin{aligned} -\Delta_y \tilde{v}_k &= \frac{\lambda \delta_k^3}{(1 - \tilde{v}_k)^2} \quad && \text{in } \Omega_k, \\ \tilde{v}_k &= 0 && \text{on } \partial\Omega_k, \end{aligned}$$

where $\Omega_k = \{y = \delta_k^{-3/2}(x - \eta_k) \colon x \in D_{R_k}\}$. Moreover,

$$1 - \tilde{v}_k = 1 - \underline{v}_{R_k} \geqslant \delta_k \quad \text{in } \Omega_k.$$

Therefore,

$$\frac{\lambda \delta_k^3}{(1 - \tilde{v}_k)^2} = \frac{\lambda \delta_k}{((1 - \tilde{v}_k)/\delta_k)^2} \leqslant \lambda \delta_k.$$

On the other hand, it follows from (3.15) that

$$\delta_k^{-3/2} \,\mathrm{dist}(x_k, \partial D_{R_k}) \leqslant C \quad \text{as } k \to \infty.$$

By using elliptic L^p estimates and standard embeddings, we deduce that some subsequence of $\{\tilde{v}_k\}$ converges in $C^1_{\mathrm{loc}}(\Gamma)$ to a (classical) solution $\tilde{v}$ of

$$\begin{aligned} -\Delta_y \tilde{v} &= 0 \quad \text{in } \Gamma, \\ \tilde{v} &= 0 \quad \text{on } \partial\Gamma, \end{aligned}$$

where Γ is a half-space of $\mathbb{R}^2$, which implies that $\tilde{v} \equiv 0$. This contradicts the fact that $\tilde{v}(\tilde{y}) = 1$ and $\operatorname{dist}(\tilde{y}, 0) \leqslant C$, where $\tilde{y} = \lim_{k\to\infty} \delta_k^{-3/2}(x_k - \eta_k)$. Thus, (3.18) holds.

We now show that

$$\underline{v}_R(R+t) \to \underline{v}_0(t) \quad \text{uniformly for } t \in [0,1] \text{ as } R \to \infty, \tag{3.19}$$

where $\underline{v}_0$ is the minimal solution of the problem

$$\left.\begin{aligned} -v_0'' &= \frac{\lambda}{(1-v_0)^2} \quad \text{in } (0,1), \\ 0 < v_0 &< 1 \quad \text{in } (0,1), \\ v_0(0) &= v_0(1) = 0. \end{aligned}\right\} \tag{3.20}$$

Set $\tilde{v}_R(t) = \underline{v}_R(t+R)$. Then, for all $R \geqslant R_{**}$,

$$\begin{aligned} \int_0^1 \frac{\lambda \tilde{v}_R}{(1-\tilde{v}_R)^2}\,\mathrm{d}t &= \int_R^{R+1} \frac{\lambda \underline{v}_R}{(1-\underline{v}_R)^2}\,\mathrm{d}r \\ &\leqslant \frac{1}{R}\int_R^{R+1} \frac{\lambda r \underline{v}_R}{(1-\underline{v}_R)^2}\,\mathrm{d}r \\ &\leqslant C, \end{aligned} \tag{3.21}$$

where $C > 0$ is independent of R (we use (3.18) here).

Multiplying (3.8) by $\underline{v}_R$ and integrating, we see from (3.21) that

$$\int_R^{R+1} r(\underline{v}_R')^2\,\mathrm{d}r = \int_R^{R+1} \frac{\lambda r \underline{v}_R}{(1-\underline{v}_R)^2}\,\mathrm{d}r \leqslant CR.$$

This implies that

$$R\biggl(\int_0^1 (\tilde{v}_R')^2\,\mathrm{d}t\biggr) \leqslant \int_R^{R+1} r(\underline{v}_R')^2\,\mathrm{d}r \leqslant CR,$$

and hence

$$\int_0^1 (\tilde{v}_R')^2\,\mathrm{d}t \leqslant C. \tag{3.22}$$

Observe that $\tilde{v}_R$ satisfies the problem

$$\begin{aligned} -\tilde{v}_R'' - \frac{1}{t+R}\tilde{v}_R' &= \frac{\lambda}{(1-\tilde{v}_R)^2} \quad \text{in } (0,1), \\ 0 < \tilde{v}_R &< 1 \quad \text{in } (0,1), \\ \tilde{v}_R(0) &= \tilde{v}_R(1) = 0. \end{aligned}$$

Therefore, since, by (3.22), $\tilde{v}_R$ is bounded in $H_0^1(0,1)$, it is also bounded in $C^1(0,1)$. Consequently, there is a sequence $\{R_k\}$ with $R_k \to \infty$ as $k \to \infty$ such that $\tilde{v}_{R_k} \to \hat{v}_0$ uniformly for $t \in [0,1]$ as $k \to \infty$, where $\hat{v}_0$ satisfies (3.20). Let us show that $\hat{v}_0 = \underline{v}_0$. We know that

$$i(\lambda, \underline{v}_{R_k}) = i_{\mathrm{rad}}(\lambda, \underline{v}_{R_k}) = i(\lambda, \tilde{v}_{R_k}) = i_{\mathrm{rad}}(\lambda, \tilde{v}_{R_k}) = 0 \quad \text{for all } R_k \geqslant R_{**}.$$

Then $i_{\mathrm{rad}}(\lambda, \hat{v}_0) = 0$ (note that $i(\lambda, \hat{v}_0) = i_{\mathrm{rad}}(\lambda, \hat{v}_0)$), and hence $\hat{v}_0 \equiv \underline{v}_0$. By a compact and uniqueness argument we have that $\underline{v}_R(R+t) \to \underline{v}_0(t)$ uniformly for $t \in [0,1]$ as $R \to \infty$. This also implies that $\min_{R \geqslant R_{**}} \lambda_R^* \leqslant \lambda_0^*$.

We now show (1.4). If, on the contrary, there is a sequence $\{R_k\}$ with $R_k \to \infty$ as $k \to \infty$ such that $\lambda^*_{R_k} \to \hat{\lambda} \neq \lambda_0^*$ as $k \to \infty$, then it follows from (3.14) that $0 < \hat{\lambda} < \infty$. Arguments similar to those in the proof of (3.19) imply that $v^*_{R_k}(R_k + t) \to \hat{v}(t)$ uniformly for $t \in [0,1]$ as $k \to \infty$, where $\hat{v}$ solves the ordinary differential equation

$$\begin{aligned} -\hat{v}'' &= \frac{\hat{\lambda}}{(1-\hat{v})^2} \quad \text{in } (0,1), \\ 0 < \hat{v} &< 1 \quad \text{in } (0,1), \\ \hat{v}(0) &= \hat{v}(1) = 0. \end{aligned}$$

On the other hand, we know that for all k the first eigenvalues $\mu_1(v^*_{R_k})$ of the problem

$$\begin{aligned} -\phi'' - \frac{1}{r}\phi' - \frac{2\lambda^*_{R_k}}{(1 - v^*_{R_k})^3}\phi &= \mu\phi \quad \text{in } (R_k, R_k + 1), \\ \phi(R_k) &= \phi(R_k + 1) = 0, \end{aligned}$$

equal 0 for all k. By the continuity of μ, we easily see that $0 = \mu_1(v^*_{R_k}) = \mu_1(\tilde{v}^*_{R_k}) \to \mu_1(\hat{v})$ as $k \to \infty$, where $\tilde{v}^*_{R_k}(t) = v^*_{R_k}(R+t)$, and $\mu_1(\hat{v})$ is the first eigenvalue of the problem

$$\begin{aligned} -\psi'' - \frac{2\hat{\lambda}}{(1-\hat{v})^3}\psi &= \mu\psi \quad \text{in } (0,1), \\ \psi(0) &= \psi(1) = 0. \end{aligned}$$

Therefore, $\hat{\lambda} = \lambda_0^*$ and $\hat{v} \equiv v_0^*$. This contradicts our assumption that $\hat{\lambda} \neq \lambda_0^*$, and hence (1.4) holds and for any $\lambda \in (0, \lambda_0^*)$, $\underline{v}_R(R+t) \to \underline{v}_0(t)$ uniformly for $t \in [0,1]$ as $R \to \infty$, $v_R^*(R+t) \to v_0^*(t)$ uniformly for $t \in [0,1]$ as $R \to \infty$. It follows from (3.11) that

$$\lim_{R \to \infty} \frac{\lambda^*_{1/(R+1)}}{(R+1)^2} = \lambda_0^*. \tag{3.23}$$

This also implies that

$$\lim_{a \to 0^+} a^2 \lambda_a^* = \lambda_0^*, \tag{3.24}$$

where λ_a^* is as in lemma 3.1. □

Proof of theorem 1.2. Since $\lambda \in (0, \lambda_0^*)$ and $\lim_{R\to\infty} \lambda_R^* = \lambda_0^*$, we can choose R_{**} (given above) such that $\lambda \in (0, \lambda_R^*)$, provided that $R > R_{**}$.

Suppose that the conclusion of theorem 1.2 does not hold; we see that there exists an integer $\theta \geqslant 1$ independent of R such that

$$i(\lambda, \bar{v}_R^{\lambda}) \leqslant \theta \quad \forall R \geqslant R_{**}. \tag{3.25}$$

For convenience, we write $\bar{v}_R^{\lambda}$ as $\bar{v}_R$ in the following.

The proof can be divided into several steps.

Step 1. We show that there is a $C > 0$ independent of R such that, for any $R \geqslant R_{**}$,

$$(1 - \bar{v}_R(x))^{-3/2} \leqslant C \operatorname{dist}^{-1}(x, \partial D_R) \quad \forall x \in D_R. \tag{3.26}$$

Equation (3.26) implies that

$$1 - \bar{v}_R(x) \geqslant C \operatorname{dist}^{2/3}(x, \partial D_R) \quad \forall x \in D_R, \tag{3.27}$$

which provides a uniform positive lower bound of $1 - \bar{v}_R$ and means that $\bar{v}_R$ is a classical solution of the problem

$$\left.\begin{aligned} -\Delta v &= \frac{\lambda}{(1-v)^2} \quad \text{in } D_R, \\ 0 < v &< 1 \quad \text{in } D_R, \\ v &= 0 \quad \text{on } \partial D_R. \end{aligned}\right\} \tag{3.28}$$

The proof of (3.26) is similar to that of (3.15) provided that (3.25) holds. Indeed, we use the blow-up argument as in the proof of (3.15) and (3.18). If we define $\tilde{M}_k(x) := (1 - \bar{v}_k(x))^{-3/2}$ and

$$\tilde{w}_k(y) := \tilde{\tau}_k^{-2/3}(1 - \bar{v}_k(x_k + \tilde{\tau}_k y)), \quad |y| \leqslant k \text{ with } \tilde{\tau}_k = \tilde{M}_k^{-1}(x_k),$$

we see that $\tilde{w}_k$ solves

$$\Delta \tilde{w}_k = \frac{\lambda}{\tilde{w}_k^2}, \quad |y| \leqslant k.$$

Moreover,

$$\tilde{w}_k^{-3/2}(0) = \tilde{\tau}_k \tilde{M}_k(x_k) = 1$$

and

$$\tilde{w}_k^{-3/2}(y) \leqslant 2, \quad |y| \leqslant k.$$

On the other hand, we see that for the first eigenvalue we have

$$\begin{aligned} \mu_1^{(k)} &= \inf_{\varphi \in H_0^1(D_{R_k}),\, \varphi \not\equiv 0} \frac{\int_{D_{R_k}} [|\nabla_x \varphi|^2 - (2\lambda/(1-\bar{v}_k)^3)\varphi^2] \,\mathrm{d}x}{\int_{D_{R_k}} \varphi^2 \,\mathrm{d}x} \\ &= \inf_{\psi \in H_0^1(\tilde{D}_{R_k}),\, \psi \not\equiv 0} \frac{\int_{\tilde{D}_{R_k}} [|\nabla_y \psi|^2 - (2\lambda/\tilde{w}_k^3)\psi^2] \,\mathrm{d}y}{\tilde{\tau}_k^2 \int_{\tilde{D}_{R_k}} \psi^2 \,\mathrm{d}y}, \end{aligned}$$

where $\tilde{D}_{R_k} = \{y\colon x_k + \tilde{\tau}_k y \in D_{R_k}\}$, $\psi(y) := \varphi(x)$ and $x = x_k + \tilde{\tau}_k y$. This implies that

$$\tilde{\mu}_1^{(k)} = \tilde{\tau}_k^2 \mu_1^{(k)},$$

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where

$$\tilde{\mu}_1^{(k)} = \inf_{\psi \in H_0^1(\tilde{D}_{R_k}),\, \psi \not\equiv 0} \frac{\int_{\tilde{D}_{R_k}} [|\nabla_y \psi|^2 - (2\lambda/\tilde{w}_k^3)\psi^2]\,\mathrm{d}y}{\int_{\tilde{D}_{R_k}} \psi^2\,\mathrm{d}y}.$$

By the fact that

$$\mu_j^{(k)} = \inf_{\mathcal{A} \subset H_0^1(D_{R_k}),\, \dim \mathcal{A} = j} \max_{\varphi \in \mathcal{A},\, \varphi \neq 0} \frac{\int_{D_{R_k}} [|\nabla_x \varphi|^2 - (2\lambda/(1-\bar{v}_k)^3)\varphi^2]\,\mathrm{d}x}{\int_{D_{R_k}} \varphi^2\,\mathrm{d}x},$$

we also see that

$$\tilde{\mu}_j^{(k)} = \tilde{\tau}_k^2 \mu_j^{(k)},$$

where

$$\tilde{\mu}_j^{(k)} = \inf_{\mathcal{B} \subset H_0^1(\tilde{D}_{R_k}),\, \dim \mathcal{B} = j} \max_{\psi \in \mathcal{B},\, \psi \neq 0} \frac{\int_{\tilde{D}_{R_k}} [|\nabla_y \psi|^2 - (2\lambda/\tilde{w}_k^3)\psi^2]\,\mathrm{d}y}{\int_{\tilde{D}_{R_k}} \psi^2\,\mathrm{d}y}.$$

(Note that $\tilde{\tau}_k^2 \to 0$ as $k \to \infty$.) This and (3.25) imply that

$$i(\lambda, \tilde{w}_k) \leqslant \theta. \tag{3.29}$$

Therefore, the maximal dimension of all subspaces X of $C_0^1(\tilde{D}_{R_k})$ such that

$$\int_{\tilde{D}_{R_k}} \left[|\nabla_y \psi|^2 - \frac{2\lambda}{\tilde{w}_k^3}\psi^2 \right] \mathrm{d}y < 0 \quad \forall \psi \in X \setminus \{0\}$$

is not bigger than θ.

On the other hand, arguments similar to those in the proof of (3.15) imply that $\{\tilde{w}_k\}$ converges in $C^1_{\mathrm{loc}}(\mathbb{R}^2)$ to a (classical) solution $\tilde{w}$ of the equation

$$\Delta \tilde{w} = \frac{\lambda}{\tilde{w}^2} \quad \text{in } \mathbb{R}^2.$$

Moreover, $\tilde{w}^{-3/2}(0) = 1$. It follows from (3.29) that the maximal dimension of all subspaces X of $C_0^1(\mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^2} \left[|\nabla_y \psi|^2 - \frac{2\lambda}{\tilde{w}^3}\psi^2 \right] \mathrm{d}y < 0 \quad \forall \psi \in X \setminus \{0\}$$

is not bigger than θ, and thus

$$i(\lambda, \tilde{w}) \leqslant \theta. \tag{3.30}$$

It is known from [4, theorem 1.1] and [3, theorem 1.1] that $i(\lambda, \tilde{w}) = \infty$ (note that $p_c(2) = 0$ if $N = 2$). This contradicts (3.30).

Using (3.27) and arguments similar to those in the proof of (3.18), we have that there exists $\bar{\varepsilon} := \bar{\varepsilon}(\lambda) > 0$ such that, for $R > R_{**}$,

$$1 - \bar{v}_R \geqslant \bar{\varepsilon} > 0 \quad \text{in } D_R.$$

Step 2. We show that $\bar{v}_R(R+t) \to \bar{v}_0^\lambda(t)$ uniformly for $t \in [0,1]$ as $R \to \infty$, where $\bar{v}_0^\lambda(t)$ is the solution on the upper solution branch of the problem

$$\left.\begin{aligned} -v_0'' &= \frac{\lambda}{(1-v_0)^2} \quad \text{in } (0,1),\\ 0 < v_0 &< 1 \quad \text{in } (0,1),\\ v_0(0) &= v_0(1) = 0. \end{aligned}\right\} \tag{3.31}$$

The proof of this step is similar to that of (3.19). To guarantee that the limit is $\bar{v}_0^\lambda$, we need to notice that $i_{\mathrm{rad}}(\lambda, \bar{v}_R) \geqslant 1$ for all $R \geqslant R_{**}$.

Step 3. We show that if $\hat{\mu}_1(R)$ is the first eigenvalue of the operator $\hat{L}_R$ defined by $\hat{L}_R \colon H^2(R, R+1) \cap H_0^1(R, R+1) \to L^2(R, R+1)$,

$$\hat{L}_R(w) = r^2\biggl(-w'' - \frac{1}{r}w' - \frac{2\lambda}{(1-\bar{v}_R)^3}w\biggr), \quad r \in (R, R+1),$$

then

$$\hat{\mu}_1(R) = \sigma_1 R^2 + o(R^2) \quad \text{as } R \to \infty, \tag{3.32}$$

where $\sigma_1 < 0$ is the smallest eigenvalue for the problem

$$\left.\begin{aligned} -w'' - \frac{2\lambda}{(1-\bar{v}_0^\lambda)^3}w &= \sigma w \quad \text{in } (0,1),\\ w(0) = w(1) &= 0, \end{aligned}\right\} \tag{3.33}$$

and $\bar{v}_0^\lambda$ is as in step 2.

The eigenvalue $\hat{\mu}_1(R)$ can be characterized as

$$\hat{\mu}_1(R) = \inf_{w \in H_0^1(R,R+1),\, w \not\equiv 0} \frac{\int_R^{R+1} r(w')^2\,\mathrm{d}r - \int_R^{R+1} 2\lambda r(1-\bar{v}_R)^{-3}w^2\,\mathrm{d}r}{\int_R^{R+1} r^{-1}w^2\,\mathrm{d}r}. \tag{3.34}$$

To estimate $\hat{\mu}_1(R)$, let us consider a function $\hat{\phi} \in C_0^\infty(0,1)$, $\hat{\phi} \geqslant 0$, and set $\phi(r) = \hat{\phi}(r-R)$. Then

$$\begin{aligned} \hat{\mu}_1(R) &\leqslant \frac{\int_R^{R+1} r(\phi')^2\,\mathrm{d}r - \int_R^{R+1} 2\lambda r(1-\bar{v}_R)^{-3}\phi^2\,\mathrm{d}r}{\int_R^{R+1} r^{-1}\phi^2\,\mathrm{d}r}\\ &= \frac{\int_0^1 (R+t)(\hat{\phi}')^2\,\mathrm{d}t - \int_0^1 2\lambda(R+t)(1-\tilde{v}_R)^{-3}\hat{\phi}^2\,\mathrm{d}t}{\int_0^1 (R+t)^{-1}\hat{\phi}^2\,\mathrm{d}t}\\ &\leqslant CR^2. \end{aligned} \tag{3.35}$$

In order to get the reverse inequality, let us denote by $w_{1,R}$ the eigenfunction associated with $\hat{\mu}_1(R)$ with $\|w_{1,R}\|_{L^\infty(D_R)} = 1$. Inserting $w_{1,R}$ into (3.34), we have, for a positive constant $C > 0$,

$$\hat{\mu}_1(R) = \frac{\int_R^{R+1} r(w_{1,R}')^2\,\mathrm{d}r - \int_R^{R+1} 2\lambda r(1-\bar{v}_R)^{-3} w_{1,R}^2\,\mathrm{d}r}{\int_R^{R+1} r^{-1} w_{1,R}^2\,\mathrm{d}r}$$
$$\geqslant \frac{-2\lambda K \int_R^{R+1} r w_{1,R}^2\,\mathrm{d}r}{\int_R^{R+1} r^{-1} w_{1,R}^2\,\mathrm{d}r}$$
$$\geqslant -CR^2 \tag{3.36}$$

since we know from step 1 that $(1-\bar{v}_R)^{-3} \leqslant K$.

Now we define $\tilde{w}_{1,R}(t) = w_{1,R}(t+R)$ in $(0,1)$, and see that $\tilde{w}_{1,R}$ satisfies the problem

$$\left.\begin{aligned} -\tilde{w}_{1,R}'' - \frac{1}{t+R}\tilde{w}_{1,R}' - \frac{2\lambda}{(1-\tilde{v}_R)^3}\tilde{w}_{1,R} = \hat{\mu}_1(R)\frac{\tilde{w}_{1,R}}{(t+R)^2} \quad \text{in } (0,1),\\ \tilde{w}_{1,R}(0) = \tilde{w}_{1,R}(1) = 0. \end{aligned}\right\} \tag{3.37}$$

It is known from (3.35) and (3.36) that $|\hat{\mu}_1(R)/R^2| \leqslant C$ for R sufficiently large. Since $\|\tilde{w}_{1,R}\|_\infty = 1$, the regularity argument as above implies that there is a sequence $\{R_k\}$ with $R_k \to \infty$ as $k \to \infty$ such that $\tilde{w}_{1,R_k} \to \xi_1 \geqslant 0$ uniformly in $[0,1]$ as $k \to \infty$, and ξ_1 with $\|\xi_1\|_{L^\infty(0,1)} = 1$ is a solution to the problem

$$\left.\begin{aligned} -\xi_1'' - \frac{2\lambda}{(1-\bar{v}_0^\lambda)^3}\xi_1 = \sigma_1\xi_1 \quad \text{in } (0,1),\\ \xi_1(0) = \xi_1(1) = 0, \end{aligned}\right\} \tag{3.38}$$

where $\bar{v}_0^\lambda$ is as in step 2 and $\sigma_1 = \lim_{k\to\infty}(\hat{\mu}_1(R_k)/R_k^2)$. The strong maximum principle implies that $\xi_1 > 0$ in $(0,1)$. Hence, σ_1 is the first eigenvalue of problem (3.33) and obviously $\sigma_1 < 0$; ξ_1 is the corresponding eigenfunction. The uniqueness of the first eigenvalue and corresponding eigenfunction of (3.38) implies that

$$\lim_{R\to\infty} \frac{\hat{\mu}_1(R)}{R^2} = \sigma < 0, \quad \tilde{w}_{1,R} \to \xi_1 \quad \text{uniformly in } [0,1] \text{ as } R \to \infty.$$

Then we have, with a change of variable, from (3.36),

$$\hat{\mu}_1(R) = \frac{\int_0^1 (t+R)(\tilde{w}_{1,R}')^2\,\mathrm{d}t - \int_0^1 2\lambda(t+R)(1-\tilde{v}_R)^{-3}\tilde{w}_{1,R}^2\,\mathrm{d}t}{\int_R^{R+1} (t+R)^{-1}\tilde{w}_{1,R}^2\,\mathrm{d}t},$$

and hence, passing to the limit, as $R \to \infty$,

$$\frac{\hat{\mu}_1(R)}{R^2} = \frac{\int_0^1 (\xi_1')^2\,\mathrm{d}t - 2\lambda\int_0^1 (1-\bar{v}_0^\lambda)^{-3}\xi_1^2\,\mathrm{d}t}{\int_0^1 \xi_1^2\,\mathrm{d}t} + o(1).$$

From this and (3.38) we obtain (3.32).

Step 4. We complete the proof of this theorem.

It is known from lemma 2.2 that $\tilde{\mu}(R,k) := \hat{\mu}_1(R) + k^2$, $k = 0,1,2,\dots$, are eigenvalues of the operator $\tilde{L}_{\bar{v}_R^\lambda}$. Note that $\bar{v}_R^\lambda$ is non-degenerate in the space $H_{0,\mathrm{rad}}^1(D_R)$.

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Then, for any large R, $\tilde{\mu}(R,k) < 0$ for $k = 0, 1, \dots, [\sqrt{-\hat{\mu}_1(R)}]$, where $[a]$ is the integer part of a. It follows from step 3 that there are infinitely many negative $\tilde{\mu}(R,k)$ as $R \to \infty$. Therefore, lemma 2.1 implies that

$$i(\lambda, \bar{v}_R) \to +\infty \quad \text{as } R \to \infty.$$

This contradicts (3.25) and completes the proof of this theorem. □

COROLLARY 3.3. *There exists a sequence $\{R_k\}$ with $R_k \geqslant R_{**}$ and $R_k \to +\infty$ as $k \to \infty$ such that, for each k, $\bar{v}^\lambda_{R_k}$ is degenerate.*

Proof. It is known from theorem 1.2 that there exists a sequence $\{R_j\}$ with $R_j \geqslant R_{**}$ and $R_j \to \infty$ such that $i(\lambda, \bar{v}^\lambda_{R_j}) \to \infty$ as $j \to \infty$. We can choose a subsequence of $\{R_j\}$ (still denoted by $\{R_j\}$) such that, for any j,

$$i(\lambda, \bar{v}^\lambda_{R_j}) < i(\lambda, \bar{v}^\lambda_{R_{j+1}}).$$

By the continuity of the eigenvalues of $L_{\bar{v}^\lambda_R}$ with respect to R, we easily see that for each j there exists $k = k(j)$ such that $R_k \in [R_j, R_{j+1}]$ and $\bar{v}^\lambda_{R_k}$ is degenerate. It is known from lemma 2.3 that R_k can be obtained from $\hat{\mu}_1(R_k) + \lambda_k = 0$. This implies our conclusion. □

4. Bifurcation results

In this section we obtain the non-radial bifurcation from the radial solution $\bar{v}_R$ by showing that there is a change in the Leray–Schauder degree of certain associated maps as R crosses R_k, where R_k is given in corollary 3.3. The main ideas are similar to those in [8].

Let $\alpha, \beta \in \mathbb{R}$ such that $0 < \alpha < \beta$, and let $D := \{x \in \mathbb{R}^2,\ \alpha < |x| < \beta\}$. For any $R \in (1, \infty)$, let

$$h_R \colon D_R \to D$$

be the diffeomorphism that maps D_R into D, i.e.

$$h_R(r,\theta) = (\alpha + (\beta - \alpha)(r - R), \theta).$$

This map h_R induces the map

$$h^*_R \colon C^0_0(\overline{D_R}) \to C^0_0(\bar{D})$$

defined by $h^*_R(v)(x) = v(h^{-1}_R(x))$ for $x \in D$.

Then our (1.1) in D_R becomes

$$\left.\begin{aligned} L_R w &= \frac{\lambda}{(1-w)^2} \quad \text{in } D, \\ 0 < w &< 1 \quad \text{in } D, \\ w &= 0 \quad \text{on } \partial D, \end{aligned}\right\} \tag{4.1}$$

where $L_R(w) = h^*_R(-\Delta((h^*_R)^{-1}(w)))$. Finally, we can rewrite (4.1) as $w = T_R(w)$, where $T_R(w) = L^{-1}_R(\lambda/(1-w)^2)$. It is easy to see that, for any $R > R_{**}$, T_R is well defined from the set $\mathcal{E} = \{w \in C^0_0(\bar{D}); 0 < w < 1 \text{ in } D\}$ into $C^0_0(\bar{D})$ and is a compact operator.

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If $\mathcal{F}$ is an open bounded set in $\mathcal{E}$ such that $I - T_R \neq 0$ on $\partial\mathcal{F}$, then the Leray–Schauder degree for the map $I - T_R$, i.e. $\deg(I - T_R, \mathcal{F}, 0)$ is well defined.

Applying [20, proposition 2, p. 243], we have that

$$\deg(I - T_R, \mathcal{F}, 0) = \deg(I - P_R, (h_R^*)^{-1}(\mathcal{F}), 0), \tag{4.2}$$

where $P_R(u) = (-\Delta)^{-1}(\lambda/(1-u)^2)$ and is defined in $\mathcal{E}_R := \{u \in C_0^0(\overline{D_R})\colon 0 < u < 1 \text{ in } D_R\}$.

The operator P_R is differentiable at $\bar{v}_R$, and, as we have seen before, the operator $I - P_R'(\bar{v}_R)$ is invertible if $R \neq R_k$ and, near R_k, $R_k > R_{**}$ is as in corollary 3.3. Hence, we have that, for any $R \neq R_k$ and near R_k,

$$\deg(I - P_R, (h_R^*)^{-1}(\mathcal{F}), 0) = \deg(I - P_R'(\bar{v}_R), (h_R^*)^{-1}(\mathcal{F}), 0) = (-1)^{i(\lambda, \bar{v}_R)} \tag{4.3}$$

if $(h_R^*)^{-1}(\mathcal{F})$ is a neighbourhood of $\bar{v}_R$ in $\mathcal{E}_R$ such that $\bar{v}_R$ is the only solution of $(I - P_R)(u) = 0$ in the closure of $(h_R^*)^{-1}(\mathcal{F})$.

Proof of theorem 1.3. We consider the set $\mathcal{H}_R$ of $C^0(\overline{D_R})$ given by

$$\mathcal{H}_R = \{u \in \mathcal{E}_R,\ u(x_1, x_2) = u(-x_1, x_2)\}.$$

It follows from [26, proposition 5.2] that for any i the eigenspace V_i of the operator $-\Delta_{S^1}$, spanned by the eigenfunctions $\phi_i(x)$ corresponding to the eigenvalue λ_i, which are invariant for x_1, i.e. $\phi_i(x_1, x_2) = \phi_i(-x_1, x_2)$, is one dimensional. Thus, the Morse index $i(\lambda, \bar{v}_R)$ in $\mathcal{H}_R$ grows by 1 when R crosses R_k. Then $i(\lambda, \bar{v}_{R_k+\varepsilon}) = i(\lambda, \bar{v}_{R_k-\varepsilon}) + 1$ if ε is small enough.

Then for any positive, small enough ε, letting $w_R = h_R^*(\bar{v}_R)$, we obtain from (4.2) and (4.3) that

$$\deg(I - T_{R_k-\varepsilon}, \mathcal{F}_{R_k-\varepsilon}, 0) = -\deg(I - T_{R_k+\varepsilon}, \mathcal{F}_{R_k+\varepsilon}, 0)$$

if $\mathcal{F}_R$ is a neighbourhood of w_R in $\mathcal{E}$ and the functions in $\mathcal{F}_R$ are invariant for x_1. This implies that there is a change in the degree at the point (λ, R_k, w_{R_k}) and then bifurcation must occur. Moreover, the bifurcating solutions are regular and non-radial, since for each R and λ, (1.1) admits exactly two radial solutions $\underline{v}_R$ and $\bar{v}_R$. This completes the proof. □

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