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Indirect solution of optimal control problems with state variable inequality constraints: finite difference approximation

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SUMMARY
In this paper, a method for the indirect solution of the optimal control problem (OCP) in the presence of pure state variable inequality constraints (SVICs) and mixed state-control inequality constraints (SCIC), without a need for a close initial guess is presented. In the proposed method, using the finite difference approximation (FDA), the pure SVICs are converted to SCIC. Here, the distance of the constraint function to the feasibility bounds of the constraint is computed in every situation and the control signal is chosen appropriately to facilitate the constraint stays safe. In this method, prior knowledge of the numbers and sequences of activation times is not required. So, it can be simply implemented in continuous boundary value problem (BVP) solvers. The proposed method simply applies the SVICs and since the constraint is directly applied on the control signal, it improves the convergence. On the other hand, because of the convergence problem in the indirect solution of OCP, the simple homotopy continuation method (HCM) is used to overcome the initial guess problem by deploying a secondary OCP for which the initial guess can be zero. The proposed approach is applied on a few comprehensive problems in the presence of different constraints. Simulations are compared with the direct solution of the OCP to confirm the accuracy and with the penalty function method and the sequential constraint-free OCP to confirm the convergence. The results indicate that the FDA method for handling the constraints along with the HCM is easy to apply with acceptable accuracy and convergence, even for highly nonlinear problems in robotic systems such as the constrained time optimal control of a two-link manipulator (TLM) and a three-link common industrial robot.

KEYWORDS: Optimal control; Indirect solution; Pontryagin maximum principle; State variable inequality constraint; Homotopy method; Finite difference approximation.

1. Introduction

In practice, OCP is surrounded by different parameters and usually one target is chosen as the minimization objective while the others are rewritten as constraints. Because of their inherent properties, numerical methods to solve the OCP are divided into three basic approaches: *dynamic programming*, *direct method* and *indirect method*.

Dynamic programming[1] uses the principle of optimality of sub-arcs in the continuous time case and leads to the Hamilton–Jacobi–Bellman equation. Although, it can be approximately computed, it suffers from the “Bellman’s Curse of Dimensionality” and it is not usually practical in high-dimensional state spaces.[2] As regards, the optimality condition extraction, some authors prefer to consider it as an indirect method.

Direct methods[3] are based on discretization of the dynamic variables (states, controls) leading to a parameter optimization problem. They provide sub-optimum solutions and less accurate results when compared to indirect methods. Then a nonlinear optimization, evolutionary or classical stochastic technique is applied to obtain the optimal values of the parameters. Various methods of converting

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the OCP into a parameter optimization problem have been developed over a long period of time by many researchers. The methods differ principally by the choice of the unknowns, the type of method used to integrate the state differential equations and the order of integration as reviewed by Sargent.[3] Different direct methods such as trust-region approach,[4] sequential quadratic programming,[5] interior-point approach,[6] and spectral methods[7] have been developed to solve OCP up to now. Despite convergence improvement, constraint simple application[8] and easy initial guess,[9] direct methods have a no confidence of optimality and they also suffer from numerical explosion with approximate solution.

Indirect methods use the Pontryagin's Maximum Principle (PMP) to convert the OCP to the BVP.[10] They consist of dynamics equations, a new set of equations called costate/adjoint equations and equations dealing with Boundary Conditions (BCs). The whole set of these equations yields a two-point BVP, which often numerically solved by shooting techniques, collocation and finite difference methods.[11] It shows fast numerical convergence and high accuracy in the local neighborhood of optimal solution, which makes them mainly attractive in many situations such as the low thrust trajectory design[12] and recently in the robotic fields for accurate trajectory planning problem.[13,14] Although, this approach has been used successfully in many applications, two major problems can be seen in the indirect solution of the OCP: Convergence and Constraint handling. Less of many efforts for resolving these intricacies are practical and easy-to-apply. In the following, these problems are explained more.

Convergence: As a major problem, the indirect methods need a very close initial guess to converge to the solution. Finding an appropriate initial guess is firstly solved by hybridizing direct methods for converging to the near optimal solution, and then indirect methods for adjusting of accuracy.[15] In many practical cases, implementation of this approach is so hard that many practitioners relinquish this approach. HCM as a second and proper resolution of OCP has been reviewed by Zhulin[16] and then used by some authors e.g., Steinboeck[17] and Hermant.[18] HCM has a long history where Le Verrier (1886) and others used for the numerical solution of equations. Principally, the outlook of HCM is to solve a main difficult problem by starting from the related simple problem with known solution. By related we mean that there must be found solutions series of the problem called zero-paths.

The HCM has also been applied on the BVP solution and the OCP.[18] In this technique, the problem is translated into the one-parameter chain of problems, which it is smoothly continued from the easy auxiliary OCP to the original hard one. It is self—contained and does not require any initial near—optimal trajectory from a direct solution.[19] In this method, the problem is translated into the one-parameter chain of problems called sliding homotopy parameter (namely c), which it is smoothly continued from an easy auxiliary OCP for $c = 0$ to the original hard one for $c = 1$. In each sequence the result at c_i is considered as the initial guess of $c_{i+1.}$

Constraints: Inequality constraints are differential laws that form the physical restrictions, classified to pure SVIC, mixed SCIC and pure Control Variable Inequality Constraint (CVIC). It is shown that using the constrained form of the PMP, CVICs can be simply applied by using of the saturation functions on the controls. Also in many cases, the SCIC can be applied by using the Lagrange multiplier method. Nevertheless, SVICs are essentially more complex than others, since they depend on the history of controls.[20] SVICs are also known as "continuous," "path," "point-wise," or "all-time" constraints in the literature. Application of SVIC in the OCP is a challenging task and many different methods have been presented to solve it. These schemes can be broadly classified into two classes: analytical and numerical.

SVICs were first analytically considered by Gamkrelidze,[21] by adjoining of time derivation of constraints in the OCP. This application is considered thereafter as an indirect adjoining method. On the other hand, the exterior penalty functions are introduced to apply SVICs by Kelley.[22] An approximately approaching interior penalty function that can be assumed a HCM, is proposed by Lasdon *et al.*[23] This technique is applied to design the constrained optimal trajectory of robotic systems by Moradi *et al.*[24] The exterior penalty functions usually violate the constraints. The interior penalty method is studied by Malisani *et al.*,[25] where in that method the dynamics of the system, the fitness function and the BCs are changed in order to transform the constrained OCP into the unconstrained OCP by using penalty function augmented to the main Lagrangian. Graichen *et al.*[26] proposed a method formerly based on the interior penalty function that uses the saturation function, where the medium variable should be infinite to touch the saturation bound corresponds to the boundary of the SVICs. This problem is known as the unboundedness of the auxiliary states and controls. A technique is used to restrict this effect using the additional penalty term. Therefore, the solution is still approximated and the penalty function is necessary and also this transformation technique actually increases the nonlinearity of the overall equations.

Valentine[27] converted the inequality SVIC into an equality SCIC by introducing a slack variable where developed consequently by Jacobson[28] and combined with HCM by Fabian.[29] Additional virtual control also proposed by Gerdts *et al.*[30] for linear SVIC-OCPs. This method applies some additional nonlinearity in cause of the variable changing and the additional states and controls, also it is required to solve the singular OCP.

The continuous assumption in the analytical methods is not generally applicable, due to discontinuities in the adjoint variables at junction-points where the inequalities become active or cease to be active. Therefore, accurate solutions cannot be achieved by continuous BVP solvers such as *bvp4c* command in the MATLAB® at all, unless to use the multipoint BVP which needs the computation of activation times for the constraints. Direct adjoint methods[31] generally needs to convert the two-point BVP into the multipoint BVP. In this technique, computing the switch time is required which it is hard to implement in many practical cases. However, these obstacles hold back practitioners from using of the indirect method.

On the other hand, the generality of continuous BVP solvers, motivates researchers to develop the approximate but simple methods to solve constrained OCPs.[30] A lot of numerical techniques for determining the approximate solution of OCP have proposed up to now, such as the multiple shooting methods,[32] discretization methods,[33] ε-convergent methods,[34] Sequential Constraint-free Optimal control Problem (SCOP),[35] non-smooth Newton methods,[33] feasible direction methods,[36] control parameterization methods[37] and smooth penalty approach.[38] Multiple shooting methods are very popular due to its simplicity and inherent convergence property, but it is required to have the prior knowledge of the number and sequence of the activation time of constraints. In SCOP, as a heuristic effort to deal with SVICs, knowing of the number of active constraints is not necessary. This method converts the original OCP to the interior constrained OCPs with applied constraints on the internal boundaries. Another similar method is proposed by Keulen *et al.*[39] in which the unconstrained OCP is solved first, and then the problem is split to consider the constraints. It is shown that this algorithm can be converged to the constrained case. However, in these methods, there is no confidence to reach the satisfactory accurate solution in the some finite iteration. Also because of increasing the interior BCs, these methods are hardly converged in the practical OCPs. Moreover, in these methods, the nonlinear constraints are almost cannot be considered.

As a major contribution of this paper, SVIC is converted to SCIC using the FDA. This method can be interpreted as an approximate indirect adjoining of constraints introduced by Gamkrelidze[21] which has not been considered before. The major problem of the indirect adjoint method is the difficulty in checking of the SVIC's activation and it becomes so simple using the FDA proposed here. In this method, intuitively, in every situation the distance of the constraint function to the feasibility bounds of the constraint is computed and the control signal is chosen to facilitate the constraint stays safe. The proposed method simply applies SVICs and since the constraint is directly applied on the control signal, it improves the convergence. In comparison with the other existing techniques such as the penalty function based methods or transformation based methods; the proposed method has a larger convergence rate. As well, the slack variable solution method needs to solve a singular OCP which in the FDA method leads to the regular OCP. In comparison with the multiple shooting solution methods and direct adjoining, the presented FDA method does not need the prior knowledge of the numbers and sequences of activation time. Therefore, its implementation is simple in the continuous BVP solvers. It should be noted that the method is more suitable to deal with the OCPs in the presence of the first-order SVICs. On the other hand as mentioned before, the original problem of the OCP has the convergence problem and usually cannot be solved by the zero initial condition. To overcome this problem, the simple or progressive HCM used before by Fabian,[29] Graichen *et al.*[26] and Malisani *et al.*,[25] is applied to improve the initial guess.

The paper is organized as follows: problem definition and the solution method based on the HCM are presented in Section 2. In Section 3, types of the OCP's constraints are introduced and the general form of converting the SVIC to SCIC based on the FDA idea is proposed. In Section 4, simulation studies are applied on a one-link manipulator and the obtained results are compared with the direct solution, penalty approach and the SCOP algorithm. Then in Section 5, several constrained OCPs are solved in order to show the efficiency and the ability of the presented approach. Finally, the conclusion is presented.

2. The Constrained OCP and the HCM

2.1. Problem definition

The formal statement of a constrained OCP is to find an admissible control history $\mathbf{u}$: $[t_0 t_f] \to \Omega \subseteq \mathbb{R}^r$ generating the corresponding state trajectory $\mathbf{x}$:$[t_0 t_f] \to \Gamma \subseteq \mathbb{R}^n$. This admissible control, minimizes the performance index

$$J = \phi\left(\mathbf{x}_f, \mathbf{b}, t_f\right) + \int_{t_0}^{t_f} L\left(\mathbf{x}, \mathbf{u}, \mathbf{b}, t\right) dt, \tag{1}$$

subject to the system equations as an Ordinary Differential Equation (ODE) and the constraints in the form of the State-Control Inequality Constraints (SCIC), the pure SVIC, the Integral Value Equality Constraints (IVEC) and the Parameter Interval Constraints (PIC) as follows:

$$\text{ODE} : \dot{\mathbf{x}} = \mathbf{f}\left(\mathbf{x}, \mathbf{u}, \mathbf{b}, t\right), \tag{2}$$

$$\text{SCIC} : \mathbf{h}\left(\mathbf{u}, \mathbf{x}, \mathbf{b}, t\right) \leq \mathbf{0}, \tag{3}$$

$$\text{SVIC} : \mathbf{g}\left(\mathbf{x}, t, \mathbf{b}\right) \leq \mathbf{0}, \tag{4}$$

$$\text{IVEC} : \int_{t_0}^{t_f} \mathbf{k}\left(\mathbf{x}, \mathbf{u}, \mathbf{b}, t\right) dt = \mathbf{k}_f \tag{5}$$

$$\text{PIC} : \mathbf{b}_{\min} \leq \mathbf{b} \leq \mathbf{b}_{\max}, \tag{6}$$

and the given initial and final conditions

$$\mathbf{x}(t_0, \mathbf{b}) = \mathbf{x}_0, \mathbf{x}(t_f, \mathbf{b}) = \mathbf{x}_f. \tag{7}$$

Here, $\mathbf{x} \in \mathbb{R}^n$ is the state vector, $\mathbf{u} \in \mathbb{R}^r$ is the control vector, Γ is the feasible state subspace of $\mathbb{R}^n$, $\mathbf{b} \in \mathbb{R}^m$ is the vector of the unknown parameters, t_0 and t_f are the start and final time, $\boldsymbol{x}_0$ and $\boldsymbol{x}_f$ are the predefined initial and final state, respectively. ϕ and L are scalar functions in which ϕ is the final state penalty term and L is the path dependent cost density. Since in this paper, SVIC is approximated by SCIC, the augmented Hamiltonian can be defined only for the ODE and SCIC as

$$H = L + \boldsymbol{\lambda}^T \mathbf{f} + \boldsymbol{\mu}^T \mathbf{h}, \tag{8}$$

where $\boldsymbol{\lambda}$ is the multiplier associated with ODE, called costate and $\boldsymbol{\mu}$ is the multiplier associated with the path constraints. Consequently, the PMP leads to the following conditions:

i. Costate ODE:

$$\dot{\boldsymbol{\lambda}} = -H_{\mathbf{x}} = -L_{\mathbf{x}} - \boldsymbol{\lambda}^T \mathbf{f}_{\mathbf{x}} - \boldsymbol{\mu}^T \mathbf{h}_{\mathbf{x}}, \tag{9}$$

ii. Minimum condition for Hamiltonian:

$$H\left(\mathbf{x}^*, \boldsymbol{\lambda}^*, \mathbf{u}^*\right) = \min\left\{H\left(\mathbf{x}, \boldsymbol{\lambda}, \mathbf{u}\right), \mathbf{h} \leq \mathbf{0}\right\}, \tag{10}$$

iii. Local minimum condition for the augmented Hamiltonian:

$$0 = H_{\mathbf{u}} = L_{\mathbf{u}} + \boldsymbol{\lambda}^T \mathbf{f}_{\mathbf{u}} + \boldsymbol{\mu}^T \mathbf{h}_{\mathbf{u}}, \tag{11}$$

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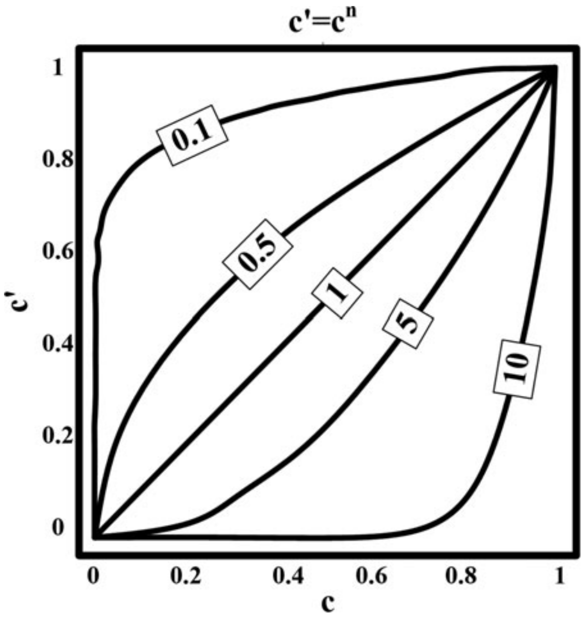

Fig. 1. Effect of the homotopy power.

iv. Sign of the multiplier μ and the complementary condition:

$$\begin{aligned} &\mathbf{h} = \mathbf{0} \Rightarrow \boldsymbol{\mu} \geq \mathbf{0}\,(\text{SCIC Active}) \\ &\mathbf{h} \leq \mathbf{0} \Rightarrow \boldsymbol{\mu} = \mathbf{0}\,(\text{SCIC is not Active}) \end{aligned} \tag{12}$$

v. Optimality with respect to the parameter:

$$\dot{\mathbf{m}} = -H_{\mathbf{b}},\ \mathbf{m}(0) = \mathbf{0},\ \mathbf{m}(t_f) = \phi_{\mathbf{b}}. \tag{13}$$

By using of the condition (iii), it is possible to find the μ when this constraint becomes activated. This form of the equations is generally known as the constrained form of PMP.

2.2. *Applying HCM into OCP*

In the HCM, a parameter such as c is defined, so that for $c = 1$ the original system is obtained, and for $c = 0$ an easy-to-solve problem is attained. A general homotopic problem can be presented as a function of unknowns $\mathbf{x}$ and a continuation parameter $c\in[0\ 1]$ as

$$X\,(\mathbf{x}, c) = 0, \tag{14}$$

where the X is the equation involved in the problem. The convex homotopy can be defined as

$$X = c^n X_1 + (1 - c^n)X_0, \tag{15}$$

where X is the equation when the homotopy parameter is c, n is the power of homotopy, X_0 and X_1 are the start and the target value of X, so that $X = X_0$ if $c = 0$ and $X = X_1$ if $c = 1$. Selection of n is an important job to have an appropriate gradually improvement of the nonlinearities. The effect of n which can be interpreted as the inertia of homotopy is shown in Fig. 1.

In the following, HCM is applied on the all parts of OCP including the system, performance index, BCs and the unknown parameters.

System: The system can be stated in the simple homotopy structure as

$$\mathbf{f}_o(\mathbf{x}, \mathbf{u}, \mathbf{b}, c) = c^{n_s}\mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{b}) + (1 - c^{n_s})\hat{\mathbf{f}}(\mathbf{x}, \mathbf{u}), \tag{16}$$

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where $\hat{\mathbf{f}}(\mathbf{x}, \mathbf{u})$ can be a simple system such as the linearized system, or the decoupled system, and n_s denotes for the homotopy power of the system.

Performance Index: The application of HCM on the performance index is possible whereas the continuation parameter can be considered as

$$\mathbf{J}_o(c) = c^{n_p}\left[\varphi(\mathbf{x}_f, \mathbf{b}) + \int_{t_0}^{t_f} L(\mathbf{x}, \mathbf{u}, \mathbf{b})\, dt\right] + (1 - c^{n_p})\left[\hat{\varphi}(\mathbf{x}_f, \mathbf{b}) + \int_{t_0}^{t_f} \hat{L}(\mathbf{x}, \mathbf{u}, \mathbf{b})\, dt\right], \quad (17)$$

where $\hat{\varphi}$ is the simplified final position penalty, $\hat{L}$ is the simplified performance index, n_p is the homotopy power of performance index.

Boundary Conditions: Intuitively, when the problem is concerned by the irregular BCs, the algorithm will diverge. In many cases, the solution is simple for some BCs and then sliding of BCs from the simple one to the original one is possible using the HCM. Homotopy for BC can be proposed as

$$c^{n_B}\mathbf{B}(\mathbf{x}, \mathbf{b}) + (1 - c^{n_B})\hat{\mathbf{B}}(\mathbf{x}, \mathbf{b}) = \mathbf{0}, \quad (18)$$

where $\mathbf{B}$ denotes the original BC, $\hat{\mathbf{B}}$ denotes the simple BC and n_B is the power of BC homotopy.

Unknown Parameters: Since there is no assurance to see the unknown parameters in the initial system, then the unknown parameters should be attended in the performance index at the initial solution ($c = 0$). The parameters can be included in the additional performance index in the form of

$$L_{\mathbf{b}} = (1 - c^{n_b}) \int_{t_0}^{t_f} \|\mathbf{b} - \mathbf{b}_0\|^2 dt, \quad (19)$$

where b_0 is the initial value for the parameters and n_b is a homotopy power for the parameters. This term will be augmented to the performance index for deriving of the parameters from b_0 towards their optimal values. This addition authorizes the initial system to not contain any parameter.

3. Constraints in the OCP and the FDA Method

In this section, different types of constraints including the SCIC, SVIC, IVEC, PIC in the OCP are considered and the proposed method for converting SVIC to SCIC based on the FDA idea is presented.

SCIC (3): Using the condition (10), it can be performed as

$$\min_{\delta \mathbf{u}} L(\mathbf{u} + \delta\mathbf{u}, \mathbf{x})\, \text{s.t.} h_j(\mathbf{u} + \delta\mathbf{u}, \mathbf{x}, \mathbf{b}, t + \Delta t) \le 0\,;\, j = 1 \ldots n_h, \quad (20)$$

where L is the performance index defined in Eq. (1) and $\delta\mathbf{u}$ is the vector of control deviation. Particularly, for an explicit SCIC as

$$h(u_i, \mathbf{x}) \le 0 \Rightarrow u_i \le \hat{h}_h(\mathbf{x}) \text{ or } \hat{h}_l(\mathbf{x}) \le u_i, \quad (21)$$

where h is the implicit SCIC, u_i is the control, $\hat{h}_h$ is the upper bound saturation function and $\hat{h}_l$ is the lower bound saturation function, the implementation of Eq. (10) becomes so simple and it can be applied by the saturation function as follows while the corresponding control variation can be computed as

$$\delta u_{h,i} = \hat{h}_h(\mathbf{x}) - u_i,\, \delta u_{l,i} = \hat{h}_l(\mathbf{x}) - u_i. \quad (22)$$

SVIC (4) (*Applied with the proposed FDA method*): It depends on the current and history of the control values. Although some methods are reviewed in the introduction, most of those methods increase stiffness of the problem and it cannot be simply solved even if the homotopy method is applied. These methods are rarely used in the realistic applications in cause of their inherited computational hardness in the form of the indirect solutions. Indeed, these methods need conditions related to state inequality, meanwhile this is hard to realize in the numerical computations. Then,

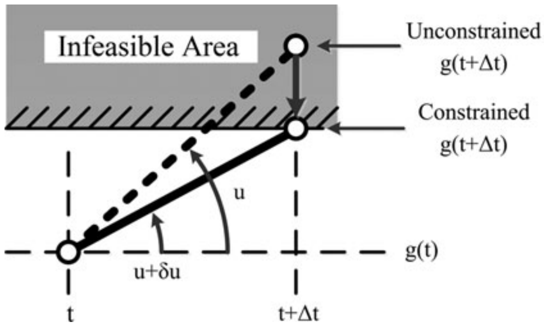

Fig. 2. Illustrative representation for the constraint handling.

both of the direct and indirect adjoin methods cannot simply be implemented in the usual manner. In the following, a method is proposed to apply the approximate indirect adjoin method with FDA state inequality check.

Suppose that the constraint g is satisfied at the time t and it is violated at the time $t+\Delta t$ as shown in the Fig. 2. The control signal $\mathbf{u}(t)$ must be modified as $\mathbf{u}(t)+\delta\mathbf{u}$ in order to eliminate the violation at the time $t+\Delta t$ and then the SVIC's bound touched by the trajectory.

Using the Taylor series expansion and the ODE (2), $\mathbf{x}(t+\Delta t)$ can be written as

$$\mathbf{x}(t+\Delta t) = \mathbf{x}(t) + \dot{\mathbf{x}}(t)\,\Delta t + O\left(\Delta t^2\right) \cong \mathbf{x}(t) + \dot{\mathbf{x}}(t)\,\Delta t = \mathbf{x}(t) + \mathbf{f}\left[\mathbf{x}(t), \mathbf{u}(t), \mathbf{b}\right]\Delta t. \quad (23)$$

Then by neglecting the higher order terms, an approximation of the SVIC (4) at the time $t+\Delta t$ can be stated as

$$\mathbf{g}\left(\mathbf{x}(t+\Delta t), t+\Delta t, \mathbf{b}\right) \leq \mathbf{0} \Rightarrow \mathbf{g}\left(\mathbf{x}(t) + \dot{\mathbf{x}}(t)\,\Delta t, t+\Delta t, \mathbf{b}\right) \leq \mathbf{0}. \quad (24)$$

By substituting the ODE (2) into Eq. (24), it turns to following SCIC

$$\mathbf{g}\left(\mathbf{x} + \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{b})\,\Delta t, t+\Delta t, \mathbf{b}\right) \leq \mathbf{0}. \quad (25)$$

Recall that constraining of the state $\mathbf{x}(t+\Delta t)$ needs manipulation of the control $\mathbf{u}(t)$.

IVEC (2): is considered by introducing of the new state for each constraint as follow

$$\dot{\mathbf{x}}_{n+1..n_{\mathbf{k}}} = \mathbf{k}(\mathbf{x}, \mathbf{u}, \mathbf{b}, t), \mathbf{x}_{n+1..n_{\mathbf{k}}}(t_0) = \mathbf{0}, \mathbf{x}_{n+1..n_{\mathbf{k}}}(t_f) = \mathbf{k}_f. \quad (26)$$

This alteration, calls to mind the presentation of fitness functional as a new state called "cost state" that it may be preferred to perform as

$$\dot{x}_{n+1} = L(\mathbf{x}, \mathbf{u}, \mathbf{b}, t), \mathbf{x}_{n+1}(t_0) = 0, \quad (27)$$

and then the Bolza form of performance index is expressed as Meyer form as follow

$$J = x_{n+1}(t_f). \quad (28)$$

PIC (6): It can be handled by transformations expressed in the finite or semi-finite forms as

$$\begin{aligned} &\text{finite}: \mathbf{b}_{\min} \leq \mathbf{b} \leq \mathbf{b}_{\max} \Rightarrow \mathbf{b} = \mathbf{b}_{\min} + (\mathbf{b}_{\max} - \mathbf{b}_{\min})\frac{1+\sin\mathbf{b}'}{2} \\ &\text{semifinite}: \begin{cases} \mathbf{b}_{\min} \leq \mathbf{b} \Rightarrow \mathbf{b} = \mathbf{b}_{\min} + \mathbf{b}'^2 \\ \mathbf{b} \leq \mathbf{b}_{\max} \Rightarrow \mathbf{b} = \mathbf{b}_{\max} - \mathbf{b}'^2 \end{cases} \end{aligned} \quad (29)$$

where $\mathbf{b}'$ is the new unconstraint parameter.

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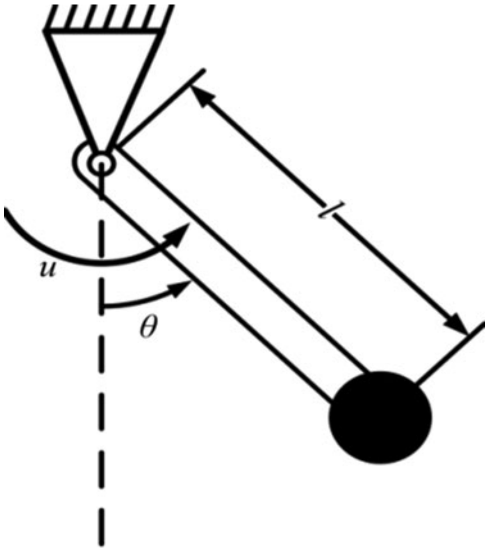

Fig. 3. The one-link manipulator.

4. Simulation Study for One-Link Manipulator

In this section, the proposed algorithm is applied on a one-link manipulator. At first, the optimality conditions are extracted by using the PMP. In Section 4.1, details of constraint application using penalty method, the SCOP and the proposed FDA are stated. In Section 4.2, formerly an effort-OCP for the one-link manipulator is solved and the FDA-based constrained solution is compared with the direct solution of the problem. Then, the obtained results are compared with each other to show the superiority of the FDA. The manipulator is presented in Fig. 3.

4.1. The OCP statement

Dynamic equations of this problem by defining the state vector as $x_1 = \theta$ and $x_2 = \dot{\theta}$, becomes

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{v}{\text{mL}^2}x_2 + \frac{1}{\text{mL}^2}u, \end{aligned} \tag{30}$$

where $m = 2$ kg, $l = 1$ m, $v = 6$ kg m^{2}s^{-1}. The effort-OCP is defined as

$$\min \frac{1}{2}\int_0^2 \left[(x_1 - 0.4)^2 + (u - 7.63)^2\right] dt \tag{31}$$

subject to

$$\begin{aligned} &\text{ODE}: \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{g}{l}\sin x_1 - \frac{v}{\text{mL}^2}x_2 + \frac{1}{\text{mL}^2}u \end{cases} \\ &\text{CIC}: |u| \le 10, \quad \text{SVIC}: x_2 \le 0.4, \end{aligned} \tag{32}$$

Timeinterval :$t \in [0, 2]$,

$$\text{BCs}: x_1(0) = 0,\ x_2(0) = 0,\ x_1(2) = 0.4,\ x_2(2) = 0.$$

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4.2. Constraint application methods

The penalty function method: It is possible to select an exponentially penalty function to turn constrained optimization to unconstrained one, proposed by Murphy[40] and followed by some others.[41–43] By introducing the costate vector as $[p_1 p_2]$, the Hamiltonian can be stated as

$$\begin{aligned} H = \exp\left(\frac{x_2 - 0.4}{1 - c + \varepsilon}\right) + 0.5\,(x_1 - 0.4)^2 + 0.5\,(u - 7.63)^2 + p_1 x_2 \\ + p_2(-9.8\sin x_1 - 3x_2 + 0.5u) + \mu_1(u - 10) - \mu_2(u + 10), \end{aligned} \tag{33}$$

where $\varepsilon > 0$ is the tolerance parameter to avoids the singularity of the problem in $c = 1$. Applying of the PMP leads to the following optimal control as

$$H_u = 0 \Rightarrow u = 7.63 - 0.5p_2 + \mu_1 - \mu_2. \tag{34}$$

Therefore, the overall equations can be stated as Eq. (30) and costate equations as

$$\begin{aligned} \dot{p}_1 &= -\frac{\partial H}{\partial x_1} = 0.4 + 9.8p_2\cos x_1 - x_1, \\ \dot{p}_2 &= -\frac{\partial H}{\partial x_1} = 3p_2 - p_1 - \frac{1}{1 - c + \varepsilon} exp\left(\frac{x_2 - 0.4}{1 - c + \varepsilon}\right). \end{aligned} \tag{35}$$

The FDA method: Using the FDA, the SVIC can be converted to the SCIC as

$$x_2 \le 0.4 \Rightarrow x_2 + \dot{x}_2\Delta t \le 0.4 \Rightarrow x_2 + \left(-9.8\sin x_1 - 3x_2 + \frac{u}{2}\right)\Delta t \le 0.4, \tag{36}$$

and so, the SCIC can be stated as

$$u \le \min\left[10, 2\left(\frac{0.4 - x_2}{\Delta t} + 9.8\sin x_1 + 3x_2\right)\right]. \tag{37}$$

Now, the Hamiltonian can be stated as

$$\begin{aligned} H = 0.5\,(x_1 - 0.4)^2 + 0.5\,(u - 7.63)^2 + p_1 x_2 + p_2\,(-9.8\sin x_1 - 3x_2 + 0.5u) \\ + \mu_1(u - 10) - \mu_2\,(u + 10) + c\mu_3\left[u - 2\left(\frac{0.4 - x_2}{\Delta t} + 9.8\sin x_1 + 3x_2\right)\right]. \end{aligned} \tag{38}$$

Applying of the PMP leads to following optimal control and costate equations as

$$H_u = 0 \Rightarrow u = 7.63 - 0.5p_2 - \mu_1 + \mu_2 - c\mu_3. \tag{39}$$

Therefore, the overall equations can be stated as Eq. (30) and

$$\begin{aligned} \dot{p}_1 &= -\frac{\partial H}{\partial x_1} = 9.8p_2\cos x_1 - x_1 + 19.6\mu_3\cos x_1 \times c, \\ \dot{p}_2 &= -\frac{\partial H}{\partial x_1} = 3p_2 - p_1 + 2\mu_3\left(3 - \frac{1}{\Delta t}\right) \times c. \end{aligned} \tag{40}$$

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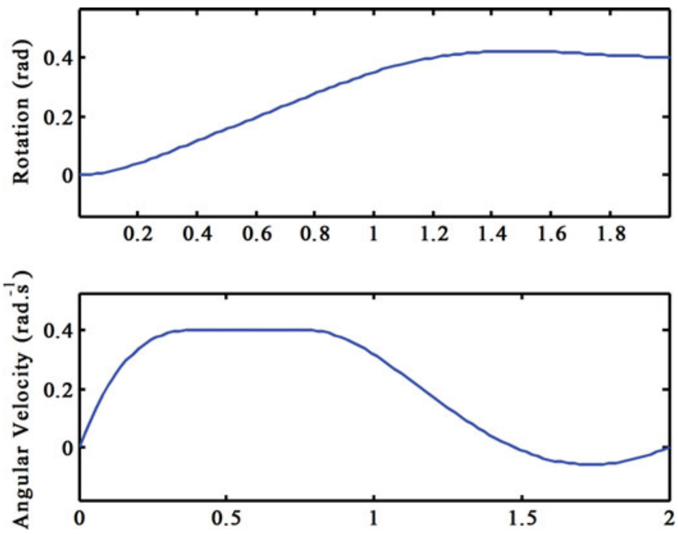

Fig. 4. States of the one-link manipulator.

μ_3 can be found when the SCIC becomes activated. We here consider that the constraints are not conflicted here and in any time only one of the constraints is active. It can be written,

$$\left.\begin{aligned} u &= 2\left(\tfrac{0.4-x_2}{\Delta t}+9.8\sin x_1+3x_2\right)\\ &= 7.63-0.5p_2\underbrace{-\mu_1+\mu_2}_{=0}-\mu_3 \end{aligned}\right\}\Rightarrow \mu_3 = 7.63-0.5p_2-2\left(\frac{0.4-x_2}{\Delta t}+9.8\sin x_1+3x_2\right). \tag{41}$$

The SCOP method: Conceptually, this simple method is based on addition of some internal BCs in the farthest points of trajectory that violate the constraint to force back the trajectory to its edge.[39] Therefore, this method needs not any additional equations to derive.

4.3. Accuracy analysis

The results of the indirect solution are shown in Figs. 4 and 5. The code I for this problem is given in the Appendix to plot the Fig. 4. In that code, you can see details of the FDA implementation such as multiplier computation, and also implementation of the progressive HCM. The algorithm is compared with the others indirect solution methods such as penalty method and SCOP and the results are shown in the Fig. 6. As it can be seen, the accuracy of the FDA is better than SCOP and penalty method. The penalty method shows an offset error and the SCOP shows the oscillating behavior.

To check the accuracy of the result, the proposed method is also compared with a direct solution of the OCP by the nonlinear programming method.[44] Figure 7 shows the negligible difference between the direct and the indirect solution.

5. Solving the Several OCPs

5.1. The Van der Pol oscillator OCP

As a first problem, the Van der Pol constrained OCP[31] is solved with the proposed method. The problem is minimization of the quadratic cost in the effort and the states of the motion in the unstable area of the nonlinear mass-spring-damper. This problem can be reformulated from the Lagrange

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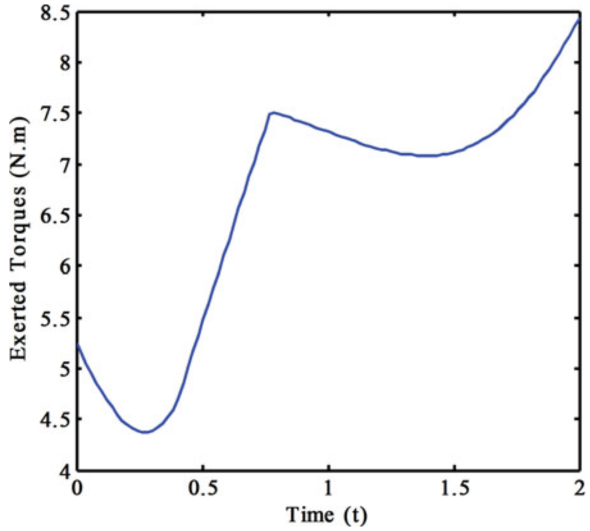

Fig. 5. Optimal control of the one-link manipulator.

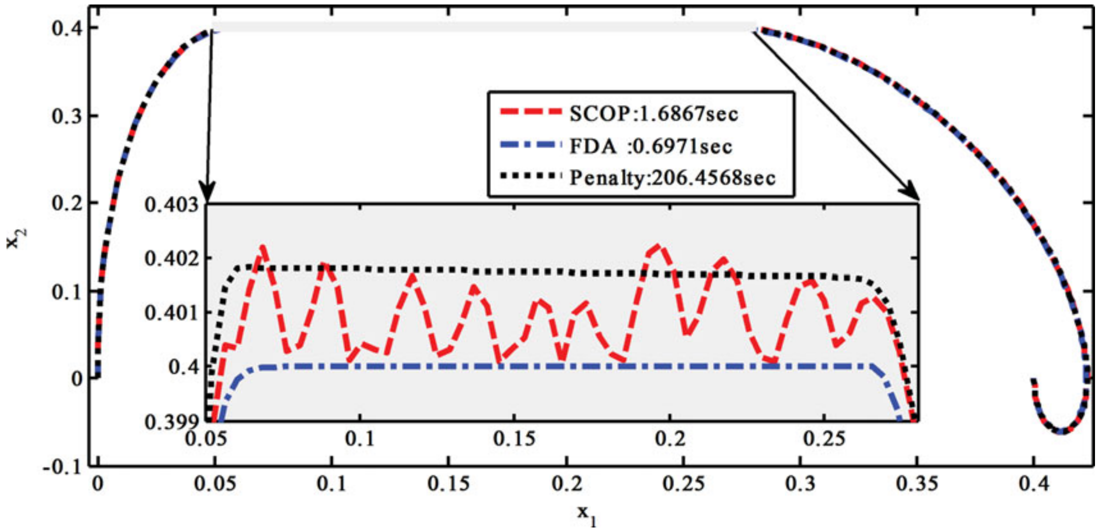

Fig. 6. The Optimal phase trajectory and comparison with the SCOP, the FDA and the penalty methods.

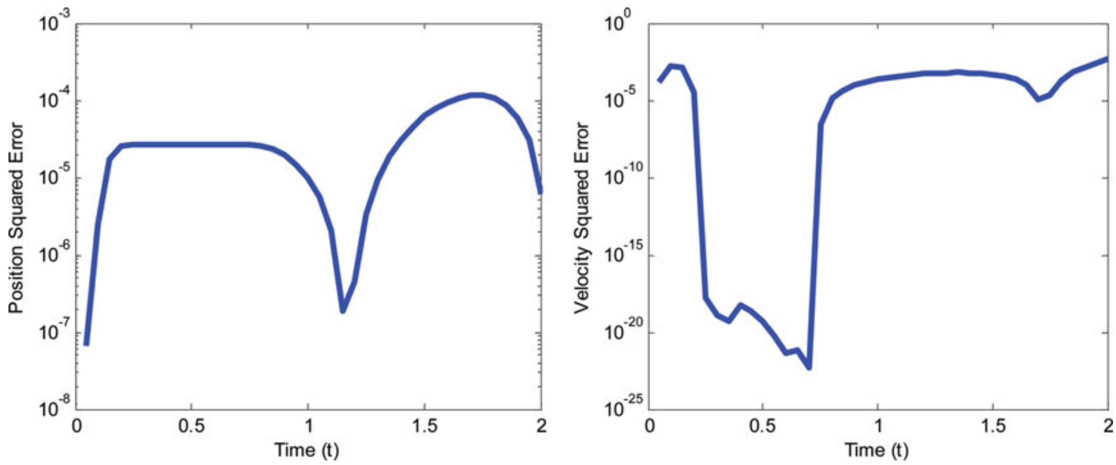

Fig. 7. The squared error between the direct and the indirect solutions.

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problem to the Meyer problem by defining of x_3 as a cost state. Therefore, it can be expressed as

$$\begin{aligned}
&\min \int_0^5 \left(x_1^2 + x_2^2 + u^2\right) dt = \min x_3\,(5) \\
&\text{subject to} \\
&\text{ODE}: \begin{cases} \text{Vanderpole}: \begin{cases} \dot{x}_1 = \left(1 - x_2^2\right) x_1 - x_2 + u \\ \dot{x}_2 = x_1 \end{cases} \\ \text{Cost} - \text{state}: \dot{x}_3 = x_1^2 + x_2^2 + u^2 \end{cases} \\
&\text{CIC}: -0.3 \le u \le 1.0, \\
&\text{SVIC}: x_1 \ge -0.4 \\
&\text{Time interval}: 0 \le t \le 5 \\
&\text{BCs}: x_1\,(0) = 0,\, x_2\,(0) = 1,\, x_3\,(0) = 0, \\
&x_1, x_2, x_3 @ t = 5 \text{ are free.}
\end{aligned} \tag{42}$$

Using the FDA method, the SVIC can be approximated by the following SCIC as

$$\begin{aligned}
&x_1 \ge -0.4 \Rightarrow x_1 + \dot{x}_1 \Delta t \ge -0.4 \Rightarrow x_1 + \left(\left(1 - x_2^2\right) x_1 - x_2 + u\right) \Delta t \ge -0.4, \\
&\Rightarrow -\frac{0.4 + x_1}{\Delta t} - \left(\left(1 - x_2^2\right) x_1 - x_2\right) \le u.
\end{aligned} \tag{43}$$

By defining the costate vector as $[p_1 p_2 p_3]^{\mathrm{T}}$, the Hamiltonian can be written as

$$\begin{aligned}
H = {} & p_1 \left[\left(1 - x_2^2\right) x_1 - x_2 + u\right] + p_2 x_1 + p_3 \left(x_1^2 + x_2^2 + u^2\right) \\
& + \mu_1 (u - 1) - \mu_2 (u + 0.3) + \mu_3 \left[-\frac{0.4 + x_1}{\Delta t} - \left(\left(1 - x_2^2\right) x_1 - x_2\right) - u\right].
\end{aligned} \tag{44}$$

Using PMP, the optimal control can be achieved as

$$H_u = p_1 + 2 p_3 u + \mu_1 - \mu_2 - \mu_3 = 0 \Rightarrow u = -\frac{p_1 + \mu_1 - \mu_2 - \mu_3}{2 p_3}, \tag{45}$$

and the costate equations becomes

$$\begin{aligned}
\dot{p}_1 &= -\frac{\partial H}{\partial x_1} = -p_1 \left(1 - x_2^2\right) - p_2 - 2 p_3 x_1 + \mu_3 \left(\frac{1}{\Delta t} + 1 - x_2^2\right), \\
\dot{p}_2 &= -\frac{\partial H}{\partial x_2} = 2 p_1 x_1 x_2 + p_1 - 2 p_3 x_2 - \mu_3 \left(2 x_2 x_1 + 1\right), \\
\dot{p}_3 &= -\frac{\partial H}{\partial x_3} = 0.
\end{aligned} \tag{46}$$

with the BCs

$$\mathbf{x}_1\,(0) = [\,0 \quad 1 \quad 0\,]^T,\ \mathbf{p}(5.0) = [\,0 \quad 0 \quad 1\,]^T. \tag{47}$$

In the case that SCIC becomes activated and at the same time upper and lower saturations of the control input are not, it is possible to find μ_3 as

$$\left.\begin{aligned} u = 2\left(x_2 - \tfrac{0.4 + x_2}{\Delta t} - \left(1 - x_2^2\right)\right) x_1 \\ = \tfrac{\mu_3 - p_1}{2 p_3} = \tfrac{\mu_3 - p_1}{2} \end{aligned}\right\} \Rightarrow \mu_3 = p_1 + 4_3 \left(x_2 - \frac{0.4 + x_2}{\Delta t} - \left(1 - x_2^2\right)\right) x_1. \tag{48}$$

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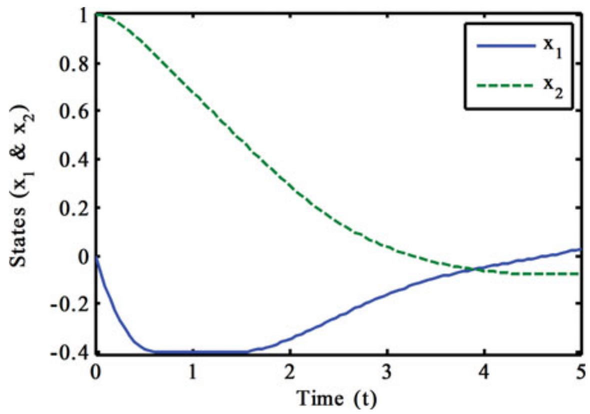

Fig. 8. The first and second states.

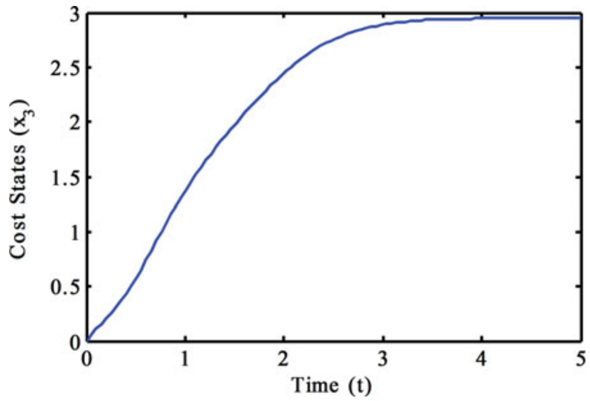

Fig. 9. The third state.

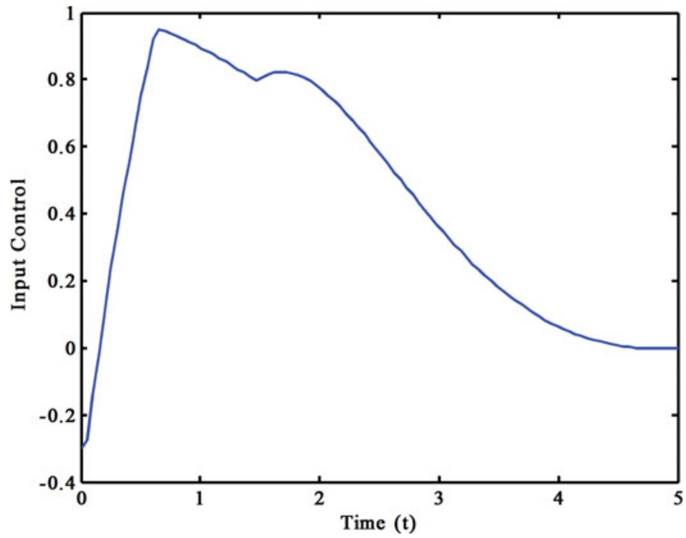

Fig. 10. Optimal control of the Van der Pol problem.

The states are shown in Figs. 8 and 9 and the value of optimal control is shown in Fig. 10. As it can be seen from Fig. 9 the minimum value of the fitness is 3.00. This value corresponds with the previously solved results.[31]

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Table I. Parameters of two-link manipulator[46].

Parameter	Value	Unit
Mass	$m_1 = 29.58, m_2 = 15$	kg
Payload mass	$m_p = 6$	kg
Length of links	$l_1 = 0.4, l_2 = 0.25$	m
Moment of inertia	$I_1 = 0.416739, I_2 = 0.205625$	kg.m^2
Torque limits	$\|u_1\| \leq 25, \|u_2\| \leq 9$	N.m

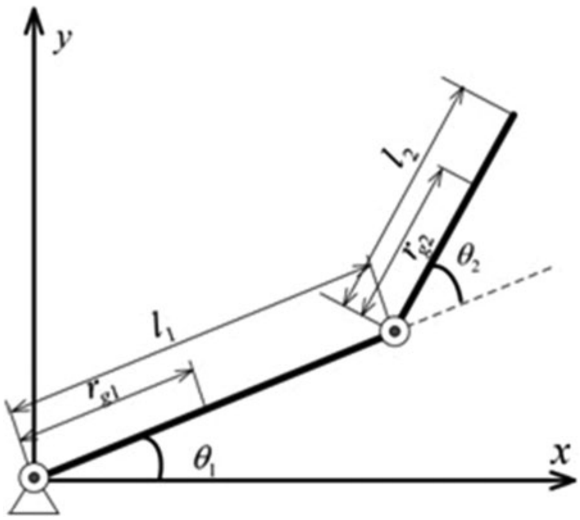

Fig. 11. Schematic view of the TLM.

5.2. The time-optimal maneuvering of the two-link manipulator

5.2.1. Dynamic equations. The velocity constrained time OCP (TOCP) of TLM is a hard OCP. In this section, the eligibility of HCM for a hard problem is shown. The ODE of the TLM is presented in Fig. 11:[45]

$$\begin{bmatrix} M_{11} & M_{12} \\ M_{12} & M_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{49}$$

where

$$\begin{aligned} M_{11} &= m_1 r_{g1}^2 + m_2 \left(l_1^2 + 2 l_1 r_{g2} \cos\theta_2 + r_{g2}^2 \right) + I_1 + I_2, \\ M_{12} &= m_2 \left(l_1 r_{g2}^2 \cos\theta_2 + r_{g2}^2 \right) + I_2,\ M_{22} = m_2 r_{g2}^2 + I_2, \\ C_1 &= -m_2 r_{g2} l_1 \sin\theta_2 \left(2\dot{\theta}_1 \dot{\theta}_2 + \dot{\theta}_2^2 \right),\ C_2 = -m_2 r_{g2} l_1 \dot{\theta}_1^2 \sin\theta_2, \\ G_1 &= \left(m_2 r_{g2} + m_2 l_1 \right) g \cos\theta_1 + m_2 r_{g1} g \cos(\theta_1 + \theta_2),\ G_2 = -m_2 r_{g2} g \cos(\theta_1 + \theta_2). \end{aligned} \tag{50}$$

For link i ($i = 1, 2$), m_i denotes the mass, l_i denotes the length, I_i denotes the mass moment of inertia, r_{gi} denotes the distance from the joint i to the center of mass of the link i. The parameters of the TLM are listed in Table I.

In TOCP, final time is unknown and by scaling of the time, it can be converted to the fixed final time problem where the final time is a parameter which should be computed. The normalized time τ

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can be defined as

$$\tau = \frac{t}{t_f}, \tau_0 = 0, \tau_f = 1. \tag{51}$$

So, differentiation operator will be changed as

$$\frac{\mathrm{d}}{\mathrm{d}t} \to \frac{1}{t_f}\frac{\mathrm{d}}{\mathrm{d}\tau}. \tag{52}$$

The TOCP can be considered with the initial position of the end-effector in XY plane at $t = 0$ is $P_0 = (x_0, y_0)$ and the final position at $t = t_f$ is $\mathrm{P}_f = (\mathrm{x}_f, \mathrm{y}_f)$ with zero velocities. So by solving the inverse kinematic equations, one can write the BCs as follows

$$\begin{aligned} &\theta_1(0) = \theta_{10}, \theta_2(0) = \theta_{20}, \theta_1(t_f) = \theta_{1f}, \theta_2(t_f) = \theta_{2f}, \\ &\dot{\theta}_1(0) = \dot{\theta}_2(0) = \dot{\theta}_1(t_f) = \dot{\theta}_2(t_f) = 0. \end{aligned} \tag{53}$$

The state vector is defined as

$$\mathbf{x} = [\theta_1 \quad \theta_2 \quad \dot{\theta}_1 \quad \dot{\theta}_2]^T, \tag{54}$$

and then the state space form of the Eq. (49), becomes

$$\begin{aligned} \dot{x}_1 &= t_f x_3, \\ \dot{x}_2 &= t_f x_4, \\ \dot{x}_3 &= t_f \frac{M_{22}(u_1 - C_1 - G_1) - M_{12}(u_2 - C_2 - G_2)}{M_{11}M_{22} - M_{12}^2} \\ \dot{x}_4 &= t_f \frac{M_{11}(u_2 - C_2 - G_2) - M_{12}(u_1 - C_1 - G_1)}{M_{11}M_{22} - M_{12}^2} \end{aligned} \tag{55}$$

In this equation, M_{ij}, C_i and G_i $(i, j = 1, 2)$ should be substituted from the Eq. (50).

5.2.2. Optimality conditions. The performance index selected as

$$J = ct_f^2 + (1-c)(t_f - t_{f0})^2 + \int_0^1 \left[(1 - c + \varepsilon)\left(u_1^2 + u_2^2\right)\right], \tag{56}$$

where ε is a very little positive number in order to avoid from the improperness of the control and t_{f0} is the initial final time that it can be selected so far the OCP can be solved. The homotopic parameter c, transforms effort-OCP in t_{f0} to TOCP. Since the final time must be positive $t_f > 0$, using Eq. (29) t_f^2 is considered instead of t_f.

The problem is divided into two sub-problems: the unconstrained and the constrained TOCP. Note that both above problems are constrained in control effort with emphasis on the existence of the SVIC.

5.2.3. Unconstrained TOCP. The costate vector is defined as $\boldsymbol{\lambda} = [x_5 \; x_6 \; x_7 \; x_8]^T$, and an additional state should be defined for handling the unknown final time as x_9. The unconstrained Hamiltonian is

$$H = (1 - c + \varepsilon)\left(u_1^2 + u_2^2\right) + t_f \boldsymbol{\lambda}^T \dot{\mathbf{x}} + \boldsymbol{\mu}_1^T \mathbf{h}_1, \tag{57}$$

where $\dot{\mathbf{x}}$ can be substituted from the Eq. (55), $\boldsymbol{\mu}_1 = [\mu_1 \; \mu_2 \; \mu_3 \; \mu_4]^T$ and constraint vector is $\mathbf{h}_1 = [u_1 - 25 \; -u_1 - 25 \; u_2 - 9 \; -u_2 - 9]^T$. By substituting of the Eq. (55) into the Eq. (57), and

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application of the PMP, according to the Eq. (9) the costate equations are obtained as follows

$$\dot{x}_{i+4} = -t_f \frac{\partial H'}{\partial x_i}, \quad i \in [1 \ldots 4], \tag{58}$$

where $H' = \frac{(C_1+G_1)(x_8 M_{12} - x_7 M_{22}) + (C_2+G_2)(x_7 M_{12} - x_8 M_{11})}{M_{11} M_{22} - M_{12}^2}$. The additional state equation using the Eq. (13) can be computed as

$$\dot{x}_9 = \frac{\partial H}{\partial t_f} = \sum_{i=1}^{4} x_{i+4} \dot{x}_i,$$
$$x_9(0) = 0, x_9(1) = -2ct_f - 2(1-c)(t_f - t_{f0}). \tag{59}$$

After that, the optimal control can be expressed as

$$\frac{\partial H}{\partial u_1} = \frac{\partial H}{\partial u_2} = 0. \tag{60}$$

So, by substituting the unconstrained Hamiltonian from Eq. (57) into Eq. (60), the optimal control laws can be expressed as

$$u_1^* = \frac{0.5}{1 - c + \varepsilon} \frac{x_8 M_{12} - x_7 M_{22}}{M_{11} M_{22} - M_{12}^2},$$
$$u_2^* = \frac{0.5}{1 - c + \varepsilon} \frac{x_7 M_{12} - x_8 M_{11}}{M_{11} M_{22} - M_{12}^2}. \tag{61}$$

The vector $\boldsymbol{\mu}$ can be found by Eq. (10). The result is equal by application of the saturation as Eq. (62). The limits are stated in Table I as follow

$$-\mathbf{u}_{\max} \leq \mathbf{u} \leq \mathbf{u}_{\max}. \tag{62}$$

Finally, by substituting Eq. (61) into Eqs. (55) and (58), nine ODEs will be obtained which with eight BCs given in Eq. (53) and two BCs given in Eq. (59), construct a two-point BVP. When the unconstrained TOCP of the TLM is solved using the HCM, the final time is found to be 1.08187 s which corresponds the results given in ref. [46]. The inputs of the joints are shown in (A)–(A) and (B). The angular position and velocity of the links are shown in Figs. 12(C)–(F). In all the figures, the faintly gray lines seen in the back of the bolded final trajectory are the homotopic sequential solutions devoted to the continuation from the effort-OCP to the TOCP.

5.2.4. Constrained TOCP. In this section, for the previous problem some inequality constraints are considered as

$$\left|\dot{\theta}_2\right| \leq 3. \tag{63}$$

These two SVICs should be expanded as

$$\begin{bmatrix} \dot{\theta}_2 - 3 & -3 - \dot{\theta}_2 \end{bmatrix}^T \leq \mathbf{0}. \tag{64}$$

Using FDA, SVICs can be expressed as

$$\begin{bmatrix} \dot{\theta}_2 - 3 \\ -\dot{\theta}_2 - 3 \end{bmatrix} \leq \mathbf{0} \Rightarrow \begin{bmatrix} \dot{\theta}_2 + \ddot{\theta}_2 \Delta t - 3 \\ -\dot{\theta}_2 - \ddot{\theta}_2 \Delta t - 3 \end{bmatrix} \Rightarrow \frac{-\dot{\theta}_2 - 3}{\Delta t} \leq \ddot{\theta}_2 \leq \frac{\dot{\theta}_2 - 3}{\Delta t}. \tag{65}$$

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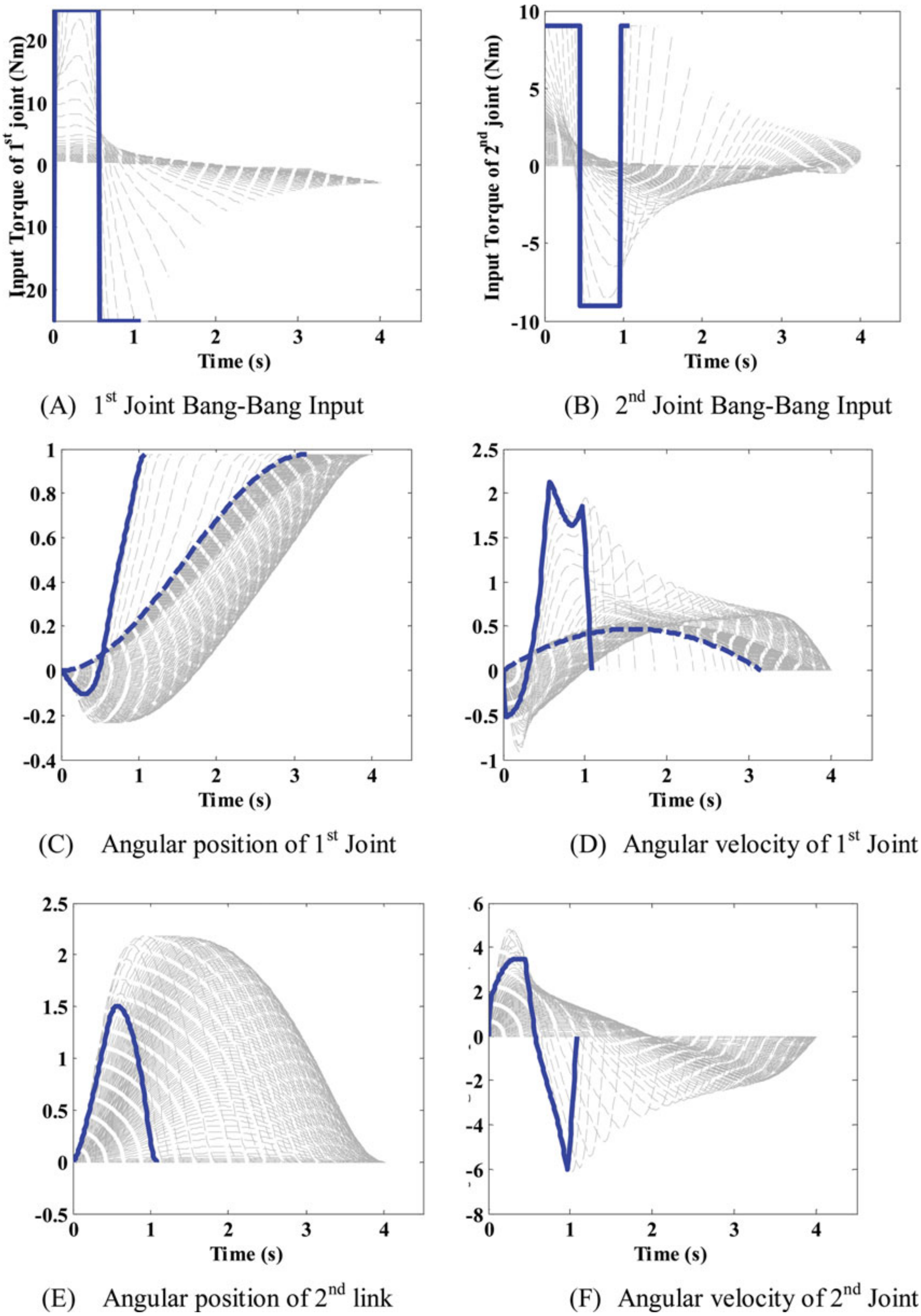

Fig. 12. Results of the unconstrained case. (A) First Joint Bang-Bang Input. (B) Second Joint Bang-Bang Input. (C) Angular position of first Joint. (D) Angular velocity of first Joint. (E) Angular position of second link. (F) Angular velocity of second Joint.

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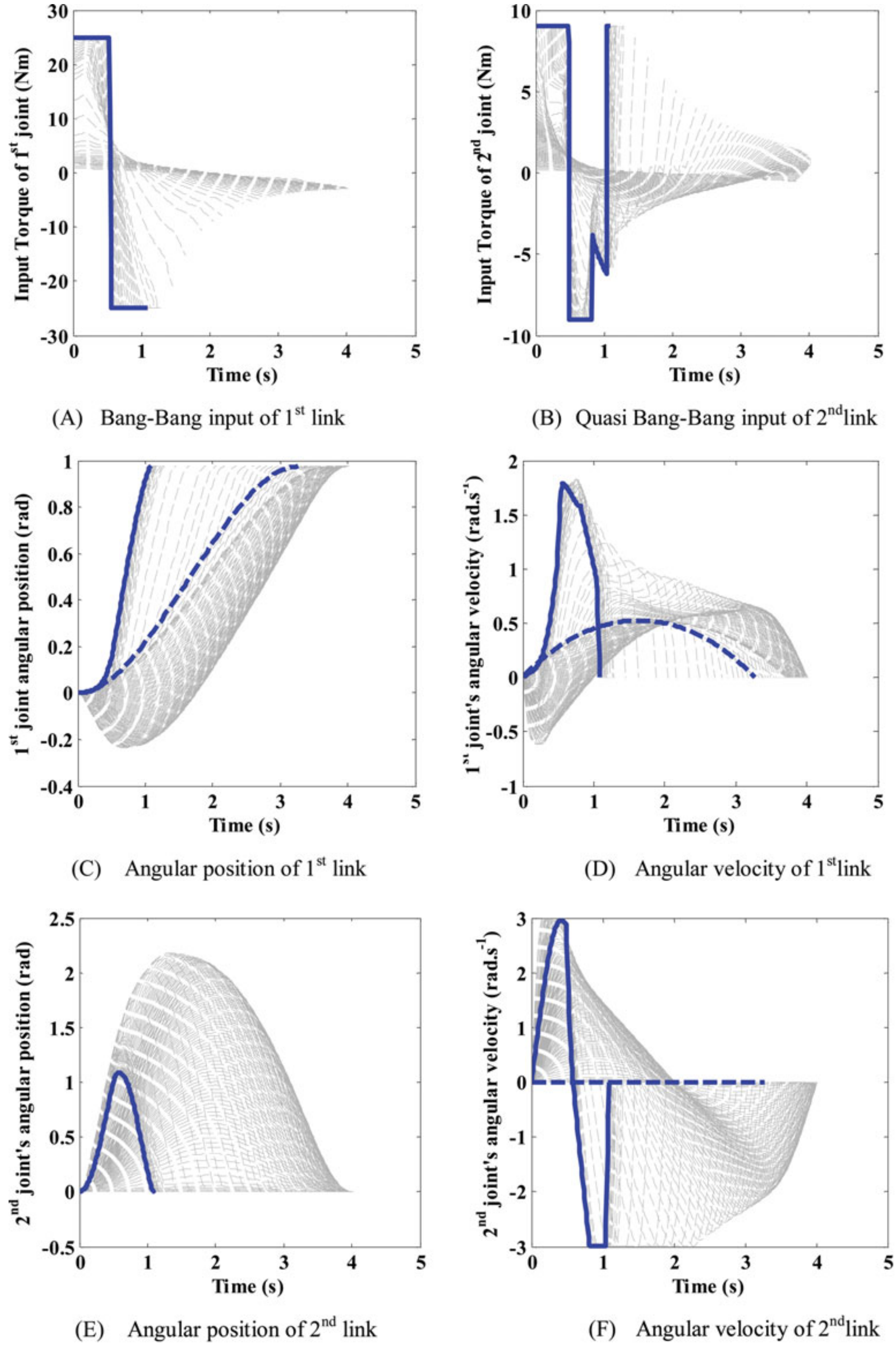

Fig. 13. Results for constrained case of TLM. (A) Bang-Bang input of first link. (B) Quasi Bang-Bang input of second link. (C) Angular position of first link. (D) Angular velocity of first link. (E) Angular position of second link. (F) Angular velocity of second link.

By substituting $\ddot{\theta}_2$ from Eq. (55) into Eq. (65), one can write

$$\frac{-\dot{\theta}_2 - 3}{\Delta t} \leq t_f \frac{M_{11}\left(u_2 - C_2 - G_2\right) - M_{12}\left(u_1 - C_1 - G_1\right)}{M_{11}M_{22} - M_{12}^2} \leq \frac{\dot{\theta}_2 - 3}{\Delta t}. \tag{66}$$

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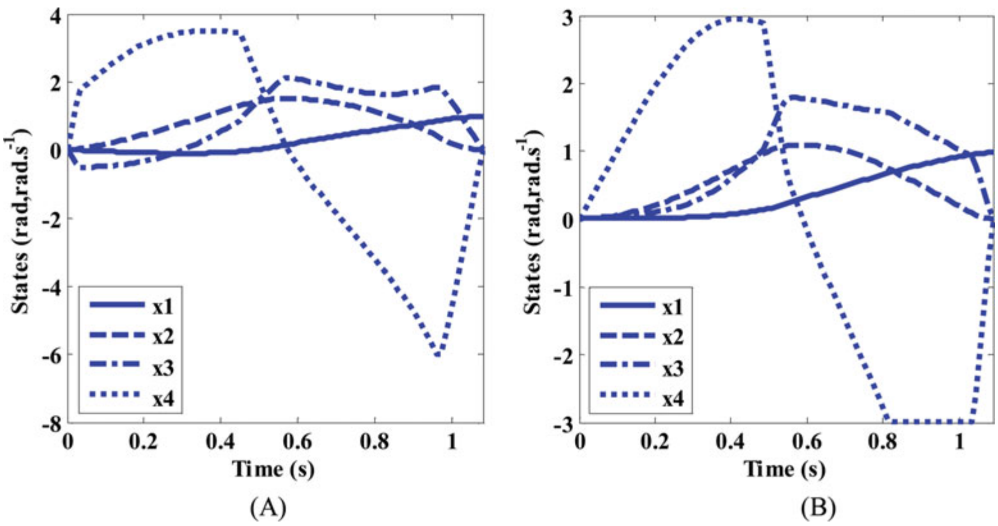

Fig. 14. States of the (A) unconstrained and (B) constrained case.

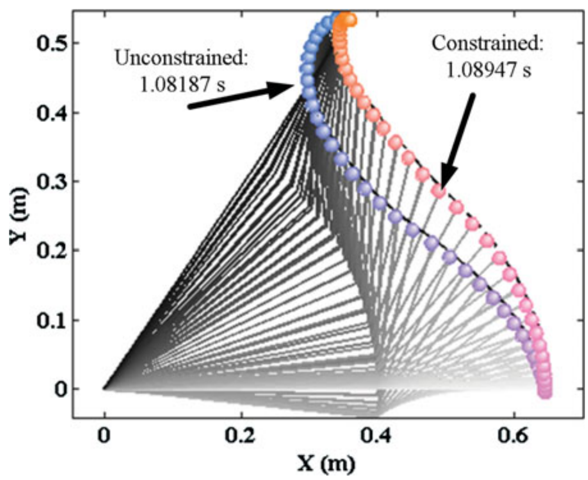

Fig. 15. Constrained and unconstrained time optimal trajectory.

which it can be rewritten as

$$\mathbf{h}_2 = \begin{bmatrix} M_{11}(u_2 - C_2 - G_2) - M_{12}(u_1 - C_1 - G_1) - \left(M_{11}M_{22} - M_{12}^2\right)\left(\frac{-\dot{\theta}_2 - 3}{t_f \Delta t}\right) \\ M_{11}(u_2 - C_2 - G_2) - M_{12}(u_1 - C_1 - G_1) - \left(M_{11}M_{22} - M_{12}^2\right)\left(\frac{\dot{\theta}_2 - 3}{t_f \Delta t}\right) \end{bmatrix} + \kappa \leq \mathbf{0} \quad (67)$$

The additional terms on the unconstrained Hamiltonian Eq. (57) can be expressed as

$$H_{\text{constrained}} = H + \boldsymbol{\mu}_2^T \mathbf{h}_2, \quad (68)$$

where $\boldsymbol{h}_2$ is stated in (67) and $\boldsymbol{\mu}_2 = [\,\mu_5\ \mu_6\,]^T$. Variation in the control signal can be found by using the quadratic programming as

$$\min_{\delta u_1, \delta u_2} \left(\delta u_1^2 + \delta u_2^2\right), \quad (69)$$

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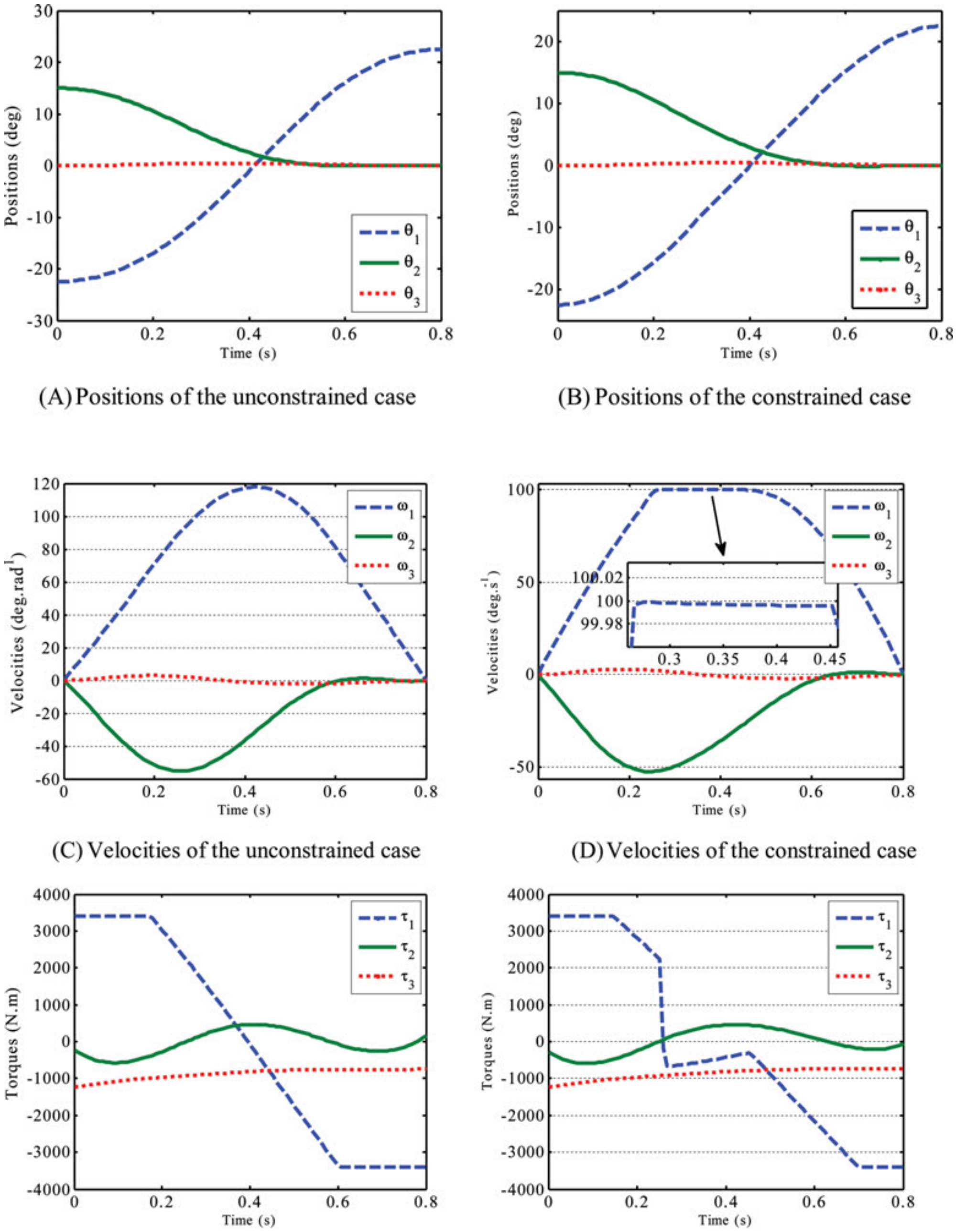

(A) Positions of the unconstrained case
(B) Positions of the constrained case
(C) Velocities of the unconstrained case
(D) Velocities of the constrained case
(E) Torques of the unconstrained case
(F) Torques of the constrained case

Fig. 16. The results for the ABB robot. (A) Positions of the unconstrained case. (B) Positions of the constrained case. (C) Velocities of the unconstrained case. (D) Velocities of the constrained case. (E) Torques of the unconstrained case. (F) Torques of the constrained case.

subject to Eqs. (67) and (62). This quadratic programming can be solved using the *quadprog* command in the MATLAB®.

Notice that κ is a homotopic factor for lightly approaching of the constraints to its rough boundaries. κ should be a large number in $c = 0$ and becomes $\kappa = 0$ when $c = 1$. The obtained optimal time is 1.089468 s that it is only 7.59 ms more than the unconstrained TOCP. The control inputs are shown in Figs. 13(A) and (B). The angular position and velocity of the 1st link and 2nd link are shown in from Figs. 13(C)–(F) respectively. In order to compare the constrained and unconstrained results, the states for these two cases are shown in Figs. 14(A) and (B) correspondingly. As it can be seen, the velocity of the 2nd link is bounded between the given constraints accurately. The stroboscopic plot of the TLM's motion for two cases is shown in the Fig. 15.

5.3. The ABB 6400 IRB 2.8 robot

Now, the proposed method is applied on the three major axes of the ABB robot. All the parameters are the same as used in ref. [24]. The vector of the rotational position, rotational velocity and the corresponding torques are denoted by $[\theta_1\theta_2\theta_3]^{\mathrm{T}}$, $[\omega_1\omega_2\omega_3]^{\mathrm{T}}$ and $[\tau_1\tau_2\tau_3]^{\mathrm{T}}$ respectively. The task is the effort optimal that moves the robot from the position $[-22.5°15°0]^{\mathrm{T}}$ to $[-22.5°00]^{\mathrm{T}}$ as a rest to rest motion in 0.8s. The velocities of the joints are bounded at $100°\ \mathrm{s}^{-1}$. Problem is solved for two cases, without the state constraints and with the state constraints. The results of the first case are shown in Figs. 16(A), (C) and (E). In the first joint's velocity it is seen that the trajectory violates the velocity limit. The results of the constrained problem solved by the proposed FDA method are shown in Figs. 16(B), (D) and (C). As it can be seen both the control constraint and the velocity constraint have been satisfied pretty well.

6. Conclusion

Indirect solution of OCP is an important method in the accurate trajectory planning and the open-loop nonlinear control. Two major problems dealing with the indirect methods, the convergence problem and the constraint handling are studied in this article from the practical view of point. In this paper, the HCM is extended to cover ODE, BCs and performance index. On the other hand, the FDA is handled to convert SVIC to SCIC where this conversion simplifies drastically the OCP solution. Using these conceptions, the MATLAB's® BVP solver (*bvp4c*) is equipped to adapt with the HCM and the constraint handling by FDA. Different simulations indicate that not only the proposed method can solve the easy OCPs accurately, but it also can easily solve the so nonlinear and complicated problems such as the constrained TOCP of the TLM and the ABB 640 robot that have not been solved before. As future works, the FDA can be considered in the trajectory planning of nonlinear and coupled systems. Also, it is interesting to generalize the method for velocity constrained robotic manipulators.

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Appendix

The inequality constraint application for the one link OCP presented above is included in the sequel.

```
function OneLink
clc
Nmax=800; nn=15;
tf=2;% Final time
options=bvpset('Nmax',Nmax,'AbsTol',1e-6,'RelTol',1e-6);
onelink_bc=@(xa,xb)([xa(1);xb(1)-0.4;xa(2);xb(2)]);
onelink_init=@(t)([-0.1*t^3+0.3*t^2; -0.6*t^2+1.2*t;-1;1]);
sol = bvpinit(linspace(0,tf,nn),onelink_init);
for con=[0 0.0415 0.071 0.12 0.194 0.3999 1] % homotopy zero path
 disp(con)
 sol= bvp4c(@(t,x)onelink_ode(t,x,con),onelink_bc,sol,options);
end
t=linspace(0,tf,100);
x = deval(sol,t);
for i=1:100
    [dxdt u(i)]=onelink_ode(t(i),x(:,i),1);
end
figure(1);  subplot(2,1,1);
plot(t,x(1,:),'-','Linewidth',2);axis equal;
ylabel('Rotation (rad)');
subplot(2,1,2);
plot(t,x(2,:),'-','Linewidth',2);axis equal;
ylabel('Angular Velocity (rad.s^-^1)');
figure(2)
plot(t,u,'-','Linewidth',2);
ylabel('Exerted Torques (N.m)');
xlabel('Time (t)');
end
function [dxdt u] =onelink_ode(~,x,con)
u=7.63-.5*x(4);
deltat=0.001; vmax=0.4; umax=10;
umaxi=min([2*((vmax-x(2))/deltat+9.8*sin(x(1))+3.*x(2)),umax]);
umini=-max([umaxi,umax]);
u=(u<umini)*umini+(u>=umini)*(u<=umaxi)*u+(u>umaxi)*umaxi;
if u==umaxi
  mu3=con*(7.63-.5*x(4)-u)*2/deltat;
else
  mu3=0;
end
dxdt=[ x(2);
       0.5*u-9.8*sin(x(1))-3*x(2);
      -x(1)+0.4+9.8*x(4)*cos(x(1))-9.8*mu3*cos(x(1));
      -x(3)+3*x(4)-3*mu3*(0.2-deltat)];
end
```

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