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A note on non-degenerate umbilics and the path formulation for bifurcation problems

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Path formulation can be used to classify and structure efficiently multiparameter bifurcation problems around fundamental singularities: the cores. The non-degenerate umbilic singularities are the generic cores for four situations in corank 2: the general or gradient problems and the $\mathbb{Z}_2$-equivariant (general or gradient) problems. Those categories determine an interesting 'Russian doll' type of structure in the universal unfoldings of the umbilic singularities.

One advantage of our approach is that we can handle one, two or more parameters using the same framework (even considering some special parameter structure, for instance, some internal hierarchy). We classify the generic bifurcations that occur in those cases with one or two parameters.

1. Introduction

The theory of parametrized contact-equivalence of Golubitsky and Schaeffer [10] has been very successful for the understanding and classification of the qualitative local behaviour of bifurcation diagrams and their perturbations. By bifurcation diagrams we mean the zero-set of parametrized equations of the type $f(x, \lambda) = 0$, where x represents the state space and λ the bifurcation parameter(s). Both are finite dimensional or we assume that a reduction of Lyapunov–Schmidt-type is applicable. For the local behaviour, we mean that we consider germs near the origin. Two bifurcation germs f, g are $\mathcal{K}_\lambda$-*equivalent* if there exist (orientation-preserving) changes of coordinates (T, X, L) around the origin such that

$$g(x, \lambda) = T(x, \lambda)f(X(x, \lambda), L(\lambda)). \tag{1.1}$$

Clearly, (T, X, L) induces a local diffeomorphism between the zero-sets of f and g and preserves the special role of the bifurcation parameters (cf. [10, 11]). In [2, 6, 7, 13], an alternative point of view has been developed: the *path formulation*. In this paper we show how path formulation can be used to efficiently classify and structure multi-parameter bifurcation problems. It organizes $\mathcal{K}_\lambda$-equivalence by distinguishing the singular behaviour due to the core of the bifurcation germ from the effects of the parameters. The *core* of a bifurcation problem $f(x, \lambda) = 0$ is the germ $f_0(x) = f(x, 0)$ obtained by setting the parameters equal to zero. It represents the singular behaviour independently of the way the parameter(s) enter. In corank 1, many results appear in our references (see [6] for a comprehensive account). Here, we are interested in corank-2 problems. In particular, the non-degenerate umbilic singularities are the generic cores in four situations: the general or gradient problems and the $\mathbb{Z}_2$-equivariant (general or gradient) problems where $\mathbb{Z}_2$ acts on the second component of $\mathbb{R}^2$ via $\kappa(x, y) = (x, -y)$. The non-degenerate umbilic singularities $f_0 : (\mathbb{R}^2, 0) \to \mathbb{R}^2$ are defined by $f_0(x, y) = (x^2 + \varepsilon y^2, 2\varepsilon xy)$, where $\varepsilon^2 = 1$. When $\varepsilon = 1$, we have the *elliptic umbilic*; when $\varepsilon = -1$, the *hyperbolic umbilic*. In this paper, we discuss aspects of the path formulation for bifurcation problems based on f_0. More precisely, to a bifurcation problem $f : (\mathbb{R}^2 \times \mathbb{R}^k, 0) \to \mathbb{R}^2$ with k bifurcation parameters and core f_0, we associate a *path* $\bar{\alpha} : (\mathbb{R}^k, 0) \to \mathbb{R}^a$ in the parameter space $\mathbb{R}^a$ of the universal unfolding F_a of f_0 in the *relevant* category ($\mathcal{K}$, gradient or $\mathcal{K}^{\mathbb{Z}_2}$) such that f and the pull-back $\bar{\alpha}^* F_a$ are $\mathcal{K}_\lambda$-equivalent (see § 1.1 for details). Then the description of such bifurcation problems and their deformations can be broadly understood as deformations of paths via changes of coordinates respecting the discriminant of the projection of $F_a^{-1}(0)$ onto the parameter space $\mathbb{R}^a$. The exact details will be explained later.

The $\mathcal{K}$-universal unfolding of f_0 in the general corank-2 category is

$$F_0 : (\mathbb{R}^2 \times \mathbb{R}^4, 0) \to \mathbb{R}^2$$

(of codimension 4), defined as

$$F_0(x, y, \alpha_1, \alpha_2, \beta, \gamma) = \begin{pmatrix} x^2 + \varepsilon y^2 + \alpha_1 x + \alpha_2 + \gamma y \\ 2\varepsilon xy - \alpha_1 \varepsilon y + \beta - \gamma x \end{pmatrix}. \tag{1.2}$$

The universal unfoldings of f_0 in the other categories are imbedded into F_0. When $\gamma = 0$, equation (1.2) is the universal unfolding in the gradient case (cf. § 1.2) and when $\gamma = \beta = 0$, it is so in the $\mathbb{Z}_2$-equivariant cases (both $\mathcal{K}^{\mathbb{Z}_2}$ and gradient). Note that when $\varepsilon = -1$ and $\alpha_2 = \beta = \gamma = 0$, equation (1.2) is also the universal unfolding for the generic $\mathbb{D}_3$-equivariant core ($\mathbb{D}_3$ acts as the group of isometry of the equilateral triangle). The universal unfolding of the umbilic singularities therefore have an interesting 'Russian doll' type of structure of universal unfoldings in all those categories.

In this paper we classify the generic bifurcations with one or two parameters that occur in those cases using the path formulation for bifurcation problems. Some results are known with one bifurcation parameter. The generic corank-2 case has been studied in [10] and some of the gradient cases in [16]. Later, in [2], a comprehensive theory was developed, but the examples concentrated on the equivariant gradient case. There is an extensive classification of $\mathbb{Z}_2$-equivariant bifurcation germs in [5]. One advantage of our approach is that we can handle one, two or more parameter situations using the same framework. We can even consider some special parameter structure (for instance, some internal hierarchy (see [6,7])). In § 3 we illustrate this by classifying the generic one-parameter bifurcation germs of corank 2 in our four categories (recovering and extending previously known results) and get as new results the generic *two*-parameter bifurcation germs of corank 2. We finish this introduction with more details on the principles of path equivalence and their applications to the classification of gradient bifurcation problems.

1.1. Principles of path equivalence

Given a bifurcation germ f, we construct the path $\bar{\alpha}$ representing it by considering f as an unfolding with parameters λ of the core f_0 in the relevant category. Let F_a be the universal unfolding of f_0 in such a category. The theory of unfolding then means that $f(x,\lambda) = T(x,\lambda)F_a(X(x,\lambda),\bar{\alpha}(\lambda))$ for some $\bar{\alpha}$. This means that f and the pull back $\bar{\alpha}^*F_a$ are $\mathcal{K}_\lambda$-equivalent with equivalence (T,X,I). The qualitative study of the zero set of bifurcation problems with the same core is obtained through the study of their associated paths, in particular, their position with respect to the discriminant variety $\varDelta^{F_a}$ associated with F_a. More precisely, let

$$\pi_{F_a} : (F_a^{-1}(0), 0) \to \mathbb{R}^a$$

be the restriction of the natural projection $\pi : (\mathbb{R}^{n+a}, 0) \to \mathbb{R}^a$. Then

$$\varDelta^{F_a} = \pi_{F_a}(\varSigma_{F_a}),$$

where $\varSigma_{F_a}$ is the local bifurcation set of F_a. Clearly, $\varDelta^{F_a}$ monitors when, and 'how', a path $\bar{\alpha}$ induces a crossing of $\varSigma_{F_a}$, that is, when there is a local change in behaviour of the zero-set. In § 2 we define those varieties in more detail. We prefer to choose $\varDelta^{F_a}$ as the *real* slice of the discriminant of the complexification of F_a. This means that we can complexify the situation and use the power of singularity theory in the complex realm. For finite-codimension problems, we do not loose anything.

The idea of the path formulation goes back at least to Arnold [1] and was the original starting point of the work [9], where the very fruitful $\mathcal{K}_\lambda$-equivalence approach had finally been developed because the technicalities of the path formulation could not easily be overcome at the time. The ideas behind the path formulation were resurrected in [12, 13] for the usual contact-equivalence and in [2] for (symmetric) gradient problems. It followed recent progresses in singularity theory allowing the handling of variety-preserving contact-equivalence. Since then, an algebraic formulation has been derived in [8], which shows that the main features of the path formulation occur naturally in the algebra of $\mathcal{K}_\lambda$-theory via the concept of liftable vector fields (cf. [4]). Fix a universal unfolding of F_a of f_0 in the appropriate category. We say that the two paths $\bar{\alpha}, \bar{\beta} : (\mathbb{R}^k, 0) \to (\mathbb{R}^a, 0)$ are *path equivalent* if

$$\bar{\alpha}(\lambda) = H(\lambda, \bar{\beta}(L(\lambda))), \tag{1.3}$$

where

$$L : (\mathbb{R}^k, 0) \to (\mathbb{R}^k, 0)$$

is an orientation-preserving diffeomorphism and

$$H : (\mathbb{R}^{k+a}, 0) \to (\mathbb{R}^a, 0)$$

is a λ-parametrized family of local diffeomorphism on a $\mathbb{R}^a$ path connected to the identity that preserves the discriminant $\varDelta^{F_a}$ of F_a in the sense that $H(\lambda, \varDelta^{F_a}) \subset \varDelta^{F_a}$ for $\lambda \in (\mathbb{R}^k, 0)$. More precisely, we choose $\varDelta^{F_a}$ as the *real* slice of the discriminant of the complexification of F_a (cf. [7]). For a fixed F_a, the set of path equivalences $\mathcal{K}_{\Delta}^{F_a}$ form a geometric subgroup of $\mathcal{K}$ in the sense of Damon, hence the usual theory and calculations of singularity apply. Note that we cannot, in general, simplify H in (1.3) as a λ-parametrized matrix as with the usual $\mathcal{K}$-equivalence, and an explicit description of the diffeomorphisms H is, in general, impossible. Nevertheless, the tangent spaces of paths can be determined explicitly at lower order or with the help of computer algebra packages. In particular, the extended tangent space of a path $\bar{\alpha}$ is given by the $\mathcal{E}_\lambda$-module

$$\langle \bar{\alpha}_\lambda \rangle_{\mathcal{E}_\lambda} + \bar{\alpha}^*(\mathrm{Derlog}(\Delta^{F_a}))_{\mathcal{E}_\lambda}, \tag{1.4}$$

where $\mathrm{Derlog}(\Delta^{F_a})$ is the $\mathcal{E}_\alpha$-module of vector fields tangent to the discriminant Δ^{F_a} (cf. § 2). We denote by $\mathcal{E}_z$ the ring of smooth germs $f : (\mathbb{R}^n, 0) \to \mathbb{R}$ with variable z and by $\boldsymbol{\mathcal{E}}_z$ the $\mathcal{E}_z$-module of smooth germs $f : (\mathbb{R}^n, 0) \to \mathbb{R}^m$ when m is clear from the context. We denote by $\mathcal{O}_z$ ($\boldsymbol{\mathcal{O}}_z$) the same rings (modules) of analytic germs. Let R be a local ring. We denote by $\langle m_1 \cdots m_k \rangle_R$ the R-module generated by the m_i. Note that (1.3) is the definition of contact equivalence of sections over Δ^{F_a}. In general, we need to use the subgroup of diffeomorphisms liftable over $F_a^{-1}(0)$ (cf. [7, 8]). In our contexts, both groups indeed coincide.

1.2. Variational bifurcation

In [2], we derived a theory for (equivariant) gradient bifurcation problems. Let $g : (\mathbb{R}^n \times \mathbb{R}^k, 0) \to \mathbb{R}$ be a germ. We say that $\nabla_x g(x, \lambda) = 0$ is a *gradient bifurcation problem* when $\nabla_x g(0, 0) = 0$ and $\nabla_x^2 g(0, 0) = 0$. Its *potential* is g. Gradient bifurcation problems form an $\mathcal{E}_\lambda$-submodule of $\boldsymbol{\mathcal{E}}_{(x,\lambda)}$, denoted by

$$\boldsymbol{\mathcal{E}}_\nabla = \{\nabla_x g(x, \lambda) \mid g \in \mathcal{E}_{(x,\lambda)}\}.$$

Contact equivalence is also an equivalence relation on $\boldsymbol{\mathcal{E}}_\nabla$. But, for an arbitrary contact equivalence, $(T, X, L) \cdot \nabla_x f$ is not necessarily in $\boldsymbol{\mathcal{E}}_\nabla$. Therefore, some modification of the usual techniques is necessary in order to describe the contact classes and their perturbations inside $\boldsymbol{\mathcal{E}}_\nabla$.

A natural framework for such problems is right equivalence for potentials with some special consideration for the parameter λ: $f, g \in \mathcal{E}_{(x,\lambda)}$ are equivalent if there exists a change of coordinates (X, L) such that

$$f(x, \lambda) = g(X(x, \lambda), L(\lambda)). \tag{1.5}$$

Although this theory has an elegant simplicity, it turns out to be inadequate. Clearly, if f, g satisfy (1.5), then $\nabla_x f$ is contact equivalent to $\nabla_x g$, but the converse is not true in general. There are two distinct difficulties involving different levels of complexity. The first obstruction is linked with the difference between contact equivalence ($\mathcal{K}$) of gradients and right equivalence ($\mathcal{R}$) of potentials and does not involve the distinguished parameter. In particular, $\mathcal{R}$-equivalence can introduce *moduli* (parameters that cannot be scaled away by a smooth change of coordinates) that are irrelevant in the context of $\mathcal{K}$-equivalence. Examples are the $T_{p,q,r}$ singularities, $x^p + y^q + z^r + mxyz$ with integer exponent such that $1/p + 1/q + 1/r < 1$. The parameter m is an $\mathcal{R}$-moduli, but can be scaled away in a $\mathcal{K}$-equivalence of the gradients. A second difficulty is more fundamental: the change of coordinates (X, L) in (1.5) is too restrictive with the distinguished parameter λ. Singularities of infinite codimension arise immediately. For example, the potential $f_1(x,\lambda) = \frac{1}{4}x^4 + \frac{1}{2}\lambda x^2$ of the pitchfork is of infinite $\mathcal{R}_\lambda$-codimension, but $\nabla_x f_1$ is of $\mathcal{K}_\lambda$-codimension 2. A similar, more immediate, fact is that $x^4+x^3+\lambda x$ is $\mathcal{K}_\lambda$-contact equivalent to $x^3+\lambda x$, but, at the potential level, $\frac{1}{5}x^5 + \frac{1}{4}x^4 + \frac{1}{2}\lambda x^2$ is *not* $\mathcal{R}_\lambda$-equivalent to $\frac{1}{4}x^4 + \frac{1}{2}\lambda x^2$. The first obstruction is more a nuisance than a difficulty: it introduces unnecessary distinctions in the classification. The second is fundamental as it precludes finite codimension. In some cases, we can use left–right equivalence ($\mathcal{A}_\lambda$) with parameters, but this is again not satisfactory. The $\mathcal{A}_\lambda$-codimension increases much more rapidly than the $\mathcal{K}_\lambda$-codimension and, in most cases, the $\mathcal{A}_\lambda$-codimension is still infinite.

Another approach suggested in [9] is to use the concept of paths in the parameter space. It has been extended and applied to some one-parameter bifurcation problems in $\mathbb{R}^2$ by Zuppa [16]. In those two works, the path formulation is used with the $\mathcal{R}$-universal unfolding of the core f_0. But this still does not solve the problem of the appearance of unwanted moduli in the classification of the cores. For these reasons, we took a hybrid approach in [2]. We developed a theory based on the 'gradient' part of the tangent spaces used in the classical approach. Although there is not an *a priori* group of change of coordinates, the principal results go through. In the present context, the two theories coincide and the path formulation goes through applied to the $\mathcal{R}$-unfolding of the umbilics when $\gamma = 0$.

2. Cores and Derlogs

We denote by $\nabla_x G_0$ the restriction of F_0 to the gradient situation, $\gamma = 0$, of potential

$$G_0(x,y,\alpha_1,\alpha_2,\beta) = \tfrac{1}{3}x^3 + \varepsilon xy^2 + \tfrac{1}{2}\alpha_1(x^2 - \varepsilon y^2) + \alpha_2 x + \beta y.$$

It represents the $\mathcal{R}$-universal unfolding of f_0 of codimension 3. Because G_0 is quasi-homogeneous, $\nabla_x G_0$ is also the gradient universal unfolding of $\nabla_x f_0$ (cf. [2]). With the $\mathbb{Z}_2$-symmetry, $F_0^{\mathbb{Z}_2}$ is the restriction of F_0 when $\beta = \gamma = 0$,

$$F_0^{\mathbb{Z}_2}(x,y,\alpha_1,\alpha_2) = \begin{pmatrix} x^2 + \varepsilon y^2 + \alpha_1 x + \alpha_2 \\ 2\varepsilon xy - \varepsilon\alpha_1 y \end{pmatrix}. \tag{2.1}$$

Note that $F_0^{\mathbb{Z}_2}$ is also the gradient of the $\mathbb{Z}_2$-invariant potential

$$\tfrac{1}{3}x^3 + \varepsilon xy^2 + \tfrac{1}{2}\alpha_1(x^2 - \varepsilon y^2) + \alpha_2 x.$$

Therefore, in both $\mathbb{Z}_2$-equivariant cases, the problem is of codimension 2. This equality means that the gradient and dissipative theories for the non-degenerate umbilics are equal with a $\mathbb{Z}_2$-symmetry. We recall what is needed is the next section.

2.1. $\mathbb{Z}_2$-equivariant problems

The ring of $\mathbb{Z}_2$-invariant germs is generated by x and $v = y^2$ and the module of equivariant germs is freely generated over the ring of invariant germs by $(1,0)$ and $(0,y)$. A $\mathbb{Z}_2$-equivariant map F on $\mathbb{R}^2$, with parameter a, has components $P(x,v,a)$ and $yQ(x,v,a)$. It is a gradient if and only if $2P_v \equiv Q_x$ and if G is a $\mathbb{Z}_2$-invariant potential, then $F(z,a) = \nabla_z G(z,a) = (G_x, 2yG_v)$, so $P = G_x$ and $Q = 2G_v$. The solution set $F(z,a)=0$ consists of two pieces, distinguished by the symmetry of the solution.

(1) $\mathrm{Fix}(\mathbb{Z}_2)$ of equation $P(x,0,a)=0$. The eigenvalues of $F_z(x,0,a)$ are $P_x(x,0,a)$ and $Q(x,0,a)$. The local bifurcation varieties are

$$\mathcal{B}_x = \{(x,0,a) \mid P(x,0,a)=P_x(x,0,a)=0\}$$

and

$$\mathcal{P}_\kappa = \{(x,0,a) \mid P(x,0,a)=Q(x,0,a)=0\}.$$

(2) $\mathrm{Fix}(\mathbf{1})$ of equation $P(x,v,a)=Q(x,v,a)=0$. The eigenvalues of $F_z(z,a)$ satisfy $\operatorname{tr} F_z(z,a) = P_x + 2vQ_v$ and $\det F_z(z,a) = 2v(P_xQ_v - Q_xP_v)$. The local bifurcation variety $\mathcal{B}_\kappa$ satisfies

$$P(x,v,a)=Q(x,v,a)=P_x(x,v,a)Q_v(x,v,a)-Q_x(x,v,a)P_v(x,v,a)=0.$$

There are possible Hopf bifurcation points near the roots of

$$P(x,v,a)=Q(x,v,a)=P_x(x,v,a)+2vQ_v(x,v,a)=0$$

satisfying $P_xQ_v - Q_xP_v > 0$, but those points are not invariant of the contact equivalence. When f is not a gradient, we can ascertain their existence using continuity arguments along branches of solutions where the determinant does not change sign but the trace does (cf. [11, p. 429]).

Proposition 2.1. *The non-degenerate umbilics are the generic cores in the $\mathbb{Z}_2$-equivariant general and gradient cases with the same universal unfolding.*

Proof. For all cases, the calculation is classic. We use the usual techniques for (equivariant) contact-equivalence. The generators for the $\mathcal{K}^{\mathbb{Z}_2}$-tangent space are $(p,0)$, $(vq,0)$, $(0,p)$, $(0,q)$, (vp_v, vq_v) and (p_x,q_x). The first and last are generated over $\mathcal{M}_{(x,v)}$ to eliminate the higher-order terms. □

2.2. Discriminants

The local bifurcation set of F_0 is

$$\Sigma_{F_0} = \{(x,y,\alpha,\beta,\gamma) \mid F_0(x,y,\alpha,\beta,\gamma)=0 \text{ and } d_{(x,y)}F_0(x,y,\alpha,\beta,\gamma) \text{ is singular}\}.$$

It is a conical set of dimension 3 with singularity at the origin constituted generically of fold points. The discriminant of F_0 is the variety of reduced equation $h=0$, where $h(\alpha,\beta,\gamma)$ is equal to

$$\begin{aligned}(\alpha_1^2 - 4\alpha_2)(3\alpha_1^2+4\alpha_2)^3 &+ 32\varepsilon\beta^2(9\alpha_1^4 - 48\alpha_1^2\alpha_2 + 16\alpha_2^2)\\ &- 256\beta^4 - 1024(\beta^2+3\varepsilon\alpha_2^2)\alpha_1\beta\gamma\\ &\quad - 4(144\alpha_1^2\beta^2 + 384\alpha_2\beta^2 + 27\varepsilon\alpha_1^6 - 144\varepsilon\alpha_1^2\alpha_2^2 + 128\varepsilon\alpha_2^3)\gamma^2\\ &\qquad + 18(9\alpha_1^4 - 16\alpha_2^2 + 16\varepsilon\beta^2)\gamma^4 - 108\varepsilon\alpha_1^2\gamma^6 + 27\gamma^8.\end{aligned}$$

The local bifurcation set Σ_{G_0} of G_0 is the set of critical points of G_0. It is a conical set of dimension 2, also with singularity at the origin and constituted generically of fold points. The discriminant of G_0 is the variety of equation

$$\begin{aligned} h^{\nabla}(\alpha,\beta) &= (\alpha_1^2-4\alpha_2)(3\alpha_1^2+4\alpha_2)^3+32\varepsilon\beta^2(9\alpha_1^4-48\alpha_1^2\alpha_2+16\alpha_2^2)-256\beta^4 \\ &= 0. \end{aligned}$$

The discriminant of $F_0^{\mathbb{Z}_2}$ is formed of the projections of the local bifurcation varieties $\mathcal{P}_\kappa$ and $\mathcal{B}_x$, generically formed of pitchforks or folds, respectively. The equation of the (reducible but principal) discriminant is

$$\begin{aligned} h^{\mathbb{Z}_2}(\alpha) &= (3\alpha_1^2+4\alpha_2)(\alpha_1^2-4\alpha_2) \\ &= 0. \end{aligned}$$

2.3. Derlogs and liftable vector fields

For the equivalence of the singularity theories for finite-codimension bifurcation germs and their associated paths, we actually need the notion of vector fields liftable via the projections π_{F_a}. Without loss of generality, we can consider analytic germs when dealing with finite-codimension problems. A vector field germ $\xi:(\mathbb{C}^a,0)\to\mathbb{C}^a$ is *liftable* over π_{F_a} if there exists a vector field germ $\eta:(\mathbb{C}^{2+a},0)\to\mathbb{C}^2$ and a matrix map germ $T:(\mathbb{C}^{2+a},0)\to \mathrm{M}(2,\mathbb{C})$ in the right category such that

$$(F_a)_z(z,\alpha)\eta(z,\alpha)+(F_a)_\alpha(z,\alpha)\xi(\alpha)=T(z,\alpha)F_a(z,\alpha). \tag{2.2}$$

This definition is geometric in the sense that ξ lifts to vector fields (η,ξ) tangent to $F_a^{-1}(0)$ at its smooth points. In our problems, the liftable vector fields are exactly the vector fields tangent to the discriminant. Let $\mathcal{I}(\Delta^{F_a})$ denote the ideal of germs vanishing on Δ^{F_a}. Define

$$\mathrm{Derlog}(\Delta^{F_a})=\{\xi\in\boldsymbol{\mathcal{O}}_a \mid \xi(\mathcal{I}(\Delta^{F_a}))\subset\mathcal{I}(\Delta^{F_a})\}.$$

It extends to the coherent sheaf of vector fields tangent to Δ^{F_a} because it can also be defined as the kernel of an epimorphism of coherent modules (cf. [7] and references therein). The discriminant Δ^{F_a} is a *free* (or *Saito*) *divisor* if $\mathrm{Derlog}(\Delta^{F_a})$ is a locally free $\mathcal{O}_a$-module (of rank a). In order to calculate the generators of $\mathrm{Derlog}(\Delta^{F_a})$, we use the following result.

THEOREM 2.2 (cf. [15]).

(a) *If the vector fields* $\{\xi_i\}_{i=1}^a$ *are in* $\mathrm{Derlog}(\Delta^{F_a})$ *and the determinant* $|\xi_1\cdots\xi_a|$ *is a reduced defining equation for* Δ^{F_a}, *then those vector fields generate freely* $\mathrm{Derlog}(\Delta^{F_a})$.

(b) *If the vector fields* $\{\xi_i\}_{i=1}^a$ *form a Lie algebra and* $|\xi_1\cdots\xi_a|=0$ *is a reduced defining equation for a hypersurface* Δ *of* $\mathbb{C}^a$, *then they generate freely* $\mathrm{Derlog}(\Delta)$.

We therefore obtain the following result.

THEOREM 2.3.

(a) *The Derlog of* $h = 0$ *is freely generated over* $\mathcal{O}_{(\alpha,\beta,\gamma)}$ *by the nilpotent basis*

$$\begin{aligned}
\xi_1 &= (\alpha_1, 2\alpha_2, 2\beta, \gamma),\\
\xi_2 &= (3\varepsilon\gamma, -2\varepsilon\beta, -2\alpha_2, 3\alpha_1),\\
\xi_3 &= (8\alpha_2, -8\alpha_1\alpha_2 + 3\alpha_1^3 - 3\varepsilon\alpha_1\gamma^2 - 8\varepsilon\beta\gamma,\\
&\qquad\qquad 8\alpha_2\gamma + 8\alpha_1\beta + 3\varepsilon\gamma^3 - 3\alpha_1^2\gamma, -8\beta),\\
\xi_4 &= (-8\varepsilon\beta, 3\gamma^3 - 3\varepsilon\alpha_1^2\gamma - 8\varepsilon\alpha_1\beta - 8\varepsilon\alpha_2\gamma,\\
&\qquad\qquad 8\alpha_1\alpha_2 + 8\varepsilon\beta\gamma + 3\alpha_1^3 - 3\varepsilon\alpha_1\gamma^2, 8\alpha_2).
\end{aligned}$$

A nilpotent basis consists of an Euler field (like ξ_1*) and a basis of the annihilator of* h*. All the elements in* Derlog$(h = 0)$ *lift. Note that another basis of* Derlog$(h = 0)$ *is to be found in [13].*

(b) *The Derlog of* $h^\nabla = 0$ *is generated by the following:*

$$\begin{aligned}
\phi_1 &= (\alpha_1, 2\alpha_2, 2\beta),\\
\phi_2 &= (8\alpha_2^2 - 8\varepsilon\beta^2, -8\alpha_1\alpha_2^2 + 3\alpha_1^3\alpha_2 - 8\varepsilon\alpha_1\beta^2, 16\alpha_1\alpha_2\beta + 3\alpha_1^3\beta),\\
\phi_3 &= (24\alpha_1\alpha_2, -16\varepsilon\beta^2 - 24\alpha_1^2\alpha_2 + 9\alpha_1^4, -16\alpha_2\beta + 24\alpha_1^2\beta),\\
\phi_4 &= (24\varepsilon\alpha_1\beta, -16\varepsilon\alpha_2\beta + 24\varepsilon\alpha_1^2\beta, -(4\alpha_2 + 3\alpha_1^2)^2).
\end{aligned}$$

Note that Derlog$(h^\nabla = 0)$ *is not freely generated over* $\mathcal{O}_{(\alpha,\beta)}$*, but it is a Lie sub-algebra of* Derlog$(h = 0)$*. Moreover,* ϕ_1 *is an Euler field and* ϕ_2*,* ϕ_3*,* ϕ_4 *are in the annihilator of* h *and all the elements in* Derlog$(h^\nabla = 0)$ *lift.*

(c) *The Derlog of* $h^{\mathbb{Z}_2} = 0$ *is freely generated over* $\mathcal{O}_\alpha$ *by the nilpotent basis*

$$\zeta_1 = (\alpha_1, 2\alpha_2), \qquad \zeta_2 = (8\alpha_2 + 2\alpha_1^2, 3\alpha_1^3 - 4\alpha_1\alpha_2),$$

and all the elements in Derlog$(h^{\mathbb{Z}_2} = 0)$ *lift.*

Proof. (a), (c) These are classical calculations, as they follow equivalently from either Saito's criteria or because the Lie Algebra structure, with respect to the usual bracket of vector fields in $\mathbb{R}^4$, of Derlog$(h = 0)$ is $[\xi_1, \xi_2] = 0$, $[\xi_1, \xi_3] = \xi_3$, $[\xi_1, \xi_4] = \xi_4$, $[\xi_2, \xi_3] = -\xi_4$, $[\xi_2, \xi_4] = -\varepsilon\xi_3$ and $[\xi_3, \xi_4] = 16(\alpha_1^2 - \varepsilon\gamma^2)\xi_2$. Note that the first vector field is the Euler field, $\xi_1(h) = 8h$, and the others satisfy $\xi_i(h) = 0$, $i = 2, 3, 4$. Similarly, $\zeta_1(h^{\mathbb{Z}_2}) = 4h^{\mathbb{Z}_2}$ and $\zeta_2(h^{\mathbb{Z}_2}) = 0$.

We can explicitly calculate the lifts using (2.2). We find that ξ_1 lifts to (x, y, ξ_1), ξ_2 to $(y, \varepsilon x, -\xi_2)$, ξ_3 to

$$(8x^2 + 2\alpha_1 x + 2\gamma y + 4\alpha_2 - 3\alpha_1^2 + 3\varepsilon\gamma^2, 8xy - 2\varepsilon\gamma x - 2\alpha_1 y + 4\varepsilon\beta, \xi_3)$$

and ξ_4 to

$$(8xy - 2\varepsilon\gamma x + 2\alpha_1 y + 4\varepsilon\beta, 8y^2 + 6\varepsilon\alpha_1 x + 2\varepsilon\gamma y + 4\varepsilon\alpha_2 + 3\varepsilon\alpha_1^2 - 3\gamma^2, \xi_4).$$

Setting $\beta = \gamma = 0$, we have $\zeta_1 = \xi_1$ and $\zeta_2 = \xi_3 + 2\alpha_1\xi_1$ implying that they also lift.

(b) First, note that, setting $\gamma = 0$, $\phi_2 = \alpha_2\xi_3 + \beta\xi_4$, $\phi_3 = 8\beta\xi_2 + 3\alpha_1\xi_3$ and $\phi_4 = 8\alpha_2\xi_2 - 3\alpha_1\xi_4$. These relations imply that the ϕ_i also lift. To calculate the Lie algebra structure of the ϕ_i, note that

$$[p\mu, q\nu] = pq[\mu,\nu] + p\mu(q)\nu - q\nu(p)\mu.$$

So

$$\begin{aligned}
[\phi_1,\phi_2] &= 3\phi_2,\\
[\phi_1,\phi_3] &= 2\phi_3,\\
[\phi_1,\phi_4] &= 2\phi_4,\\
[\phi_2,\phi_3] &= (16\alpha_1\alpha_2 - 3\alpha_1^3)\phi_3 + 24(\alpha_2 - \alpha_1^2)\phi_2,\\
[\phi_2,\phi_4] &= 24\varepsilon\beta\phi_2 - 8\varepsilon\alpha_1\beta\phi_3 - (3\alpha_1^3 + 8\alpha_1\alpha_2)\phi_4
\end{aligned}$$

and

$$[\phi_3,\phi_4] = -16\varepsilon\beta\phi_3 + 16\alpha_2\phi_4.$$

Note that

$$|\phi_1\phi_2\phi_3| = 64\beta h^{\nabla},$$

which means that they are a free basis for Derlog$(\beta h^{\nabla} = 0)$ because they form a Lie sub-algebra (cf. [3]). Finally, we would like to show that any $\zeta \in$ Derlog$(h^{\nabla} = 0)$ can be decomposed into a linear combination of the ϕ. The claim will hold if we can decompose it into a sum $\zeta = \zeta_1 + p\phi_4$ for some function p, where ζ_1 is in Derlog$(\beta h^{\nabla} = 0)$. Note that the third component of ϕ_4 when restricted to $\beta = 0$ is $(4\alpha_2 + 3\alpha_1^2)^2$. Moreover, $h^{\nabla}_{\alpha_1}$, $h^{\nabla}_{\alpha_2}$ and h^{∇} are with factor $(3\alpha_1^2 + 4\alpha_2)^2$ modulo β^2. And so, the third component of ζ has the same factor which can be eliminated using $p\phi_4$. □

3. Generic bifurcation problems

3.1. One bifurcation parameter

Without the gradient conditions, corank-2 problems with one bifurcation parameter are of high codimension; the simplest one is codimension 3 (the so-called 'hill-top' bifurcation, [10]), then we jump to topological codimension 5 and higher (cf. [9,10]). Some results for gradient bifurcation problems are available from Zuppa [16]. The coefficients δ, δ_1 are ± 1. In (3.3) m is modal and satisfies $m^2 \neq \varepsilon$. The parameters $\hat{\beta}_i$ are the unfolding parameters.

Theorem 3.1.

(a) *The generic bifurcation problems of corank* 2 *with one bifurcation parameter are of codimension 3 with universal unfolding*

$$F_1(x,y,\lambda,\hat{\beta}) = \begin{pmatrix} x^2 + \varepsilon y^2 + \delta\lambda + \hat{\beta}_1 x + \hat{\beta}_3 y \\ 2\varepsilon xy - \varepsilon\hat{\beta}_1 y - \hat{\beta}_3 x + \hat{\beta}_2 \end{pmatrix} \tag{3.1}$$

or

$$F_2(x,y,\lambda,\hat{\beta}) = \begin{pmatrix} x^2 + y^2 + \hat{\beta}_1 x + \hat{\beta}_3 y + \hat{\beta}_2 \\ 2\varepsilon xy + \delta\lambda - \varepsilon\hat{\beta}_1 y - \hat{\beta}_3 x \end{pmatrix}. \tag{3.2}$$

When $\varepsilon = -1$, *only* F_1 *is necessary.*

(b) *The generic gradient bifurcation problem of corank* 2 *with one bifurcation parameter is of topological codimension* 2 *with universal unfolding*

$$F(x, y, \lambda, \hat{\beta}) = \begin{pmatrix} x^2 + \varepsilon y^2 + \delta_1 \lambda x + \delta\lambda + \hat{\beta}_1 \\ 2\varepsilon xy - \varepsilon\delta_1 \lambda y + m\lambda + \hat{\beta}_2 \end{pmatrix}. \tag{3.3}$$

(c) *The generic (gradient)* $\mathbb{Z}_2$*-equivariant bifurcation problem of corank* 2 *with one bifurcation parameter is of codimension* 1 *with universal unfolding*

$$F(x, y, \lambda, \hat{\beta}) = \begin{pmatrix} x^2 + \varepsilon y^2 + \hat{\beta} x + \delta\lambda \\ 2\varepsilon xy - \varepsilon\hat{\beta} y \end{pmatrix}. \tag{3.4}$$

Proof. (a) We are going to prove that the generic path in the general corank-2 case is $(0, \delta\lambda, 0, 0)$ of universal unfolding $(\beta_1, \delta\lambda, \beta_2, \beta_3)$. First note that the path $\bar{\alpha}(\lambda) = (a\lambda, b\lambda, c\lambda, d\lambda)$ is of codimension 3 and the quadratic terms in λ are in its unipotent tangent space if $\varepsilon c^2 - b^2 \neq 0$. We can use Nakayama's lemma on the terms of lower order of the generators of Derlog(Δ^{F_a}):

$$\begin{aligned} \bar{\alpha}_\lambda &= (a, b, c, d), \\ \xi_1 &= (a\lambda, 2b\lambda, 2c\lambda, d\lambda), \\ \xi_2 &= (3\varepsilon d\lambda, -2\varepsilon c\lambda, -2b\lambda, 3a\lambda), \\ \xi_3 &= (b\lambda, 0, 0, -c\lambda), \\ \xi_4 &= (-\varepsilon c\lambda, 0, 0, b\lambda) \end{aligned}$$

modulo $\mathcal{M}_\lambda^2$. The condition is always satisfied for either b or c non zero if $\varepsilon = -1$ and for $b \neq \pm c$ if $\varepsilon = 1$. This corresponds to the conditions (H1) and (H3) of [10, p. 403]. In those cases we can change coordinates to set-up $b = \delta = \pm 1$ and $c = 0$.

(b) We proceed like in (a). Let $\bar{\alpha}(\lambda) = (a\lambda, b\lambda, c\lambda)$. The tangent space is generated by

$$(a, b, c), \qquad (0, b\lambda, c\lambda), \qquad (3\varepsilon ac\lambda^2, -2\varepsilon bc\lambda^2, -2b^2\lambda^2),$$
$$(3ab\lambda^2, -2\varepsilon c^2\lambda^2, -2bc\lambda^2), \qquad ((b^2 - \varepsilon c^2)\lambda^2, 0, 0).$$

With the same conditions as in (a), the quadratic terms in λ^2 are contained in the unipotent tangent space. Via rescaling we find the final result as c cannot be now eliminated and so becomes a modal parameter.

(c) We proceed in a similar way with $\bar{\alpha}(\lambda) = (a\lambda, b\lambda)$ and the generators (a, b), $(0, b\lambda)$ and $(b\lambda, 0)$ modulo $\mathcal{M}_\lambda^2$. When $b \neq 0$ the quadratic terms can be removed from the unipotent tangent space as well as a. □

3.2. Two-bifurcation parameter

In [14] one can find a classification of corank 1 two-parameter bifurcation germs. No results in corank 2 are previously available. We denote the bifurcation parameters by $\Lambda = (\lambda, \mu)$. As previously, the coefficients δ_i are ± 1. In (b) m is modal and must be non-zero. The parameters $\hat{\beta}_i$ are the unfolding parameters.

THEOREM 3.2.

(a) *The generic bifurcation problems of corank* 2 *with two bifurcation parameters are of codimension* 2 *with universal unfolding*

$$F(x,y,\Lambda,\hat\beta) = \begin{pmatrix} x^2+\varepsilon y^2+(\delta_3\lambda+\hat\beta_1)\,x+\hat\beta_2\,y+\delta_1\lambda \\ 2\varepsilon xy-\varepsilon(\delta_3\lambda+\hat\beta_1)\,y-\hat\beta_2\,x+\delta_2\mu \end{pmatrix}. \tag{3.5}$$

(b) *The generic gradient bifurcation problem of corank* 2 *with two bifurcation parameters is of topological codimension* 1 *with universal unfolding,*

$$F(x,y,\Lambda,\hat\beta) = \begin{pmatrix} x^2+\varepsilon y^2+\delta_1\lambda x+\delta_2\mu \\ 2\varepsilon xy-\varepsilon\delta_1\lambda y+\delta_3\lambda+m\mu+\hat\beta \end{pmatrix}. \tag{3.6}$$

(c) *The generic (gradient)* $\mathbb{Z}_2$*-equivariant bifurcation problem of corank* 2 *with two bifurcation parameters is of codimension* 0 *with universal unfolding*

$$f(x,y,\Lambda) = \begin{pmatrix} x^2+\varepsilon y^2+\delta_2\mu x+\delta_1\lambda \\ 2\varepsilon xy-\varepsilon\delta_2\mu y \end{pmatrix}. \tag{3.7}$$

Proof. The proofs follow the pattern of theorem 3.1. In this case the tangent spaces are submodules of $\boldsymbol{\mathcal{O}}_{(\lambda,\mu)}$ over $\mathcal{O}_{(\lambda,\mu)}$ and we consider the generators $\bar\alpha_\lambda$ and $\bar\alpha_\mu$. □

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References

1 V. I. Arnold. Wavefront evolution and equivariant Morse lemma. *Commun. Pure Appl. Math.* **29** (1976), 557–582.

2 T. J. Bridges and J. E. Furter. *Singularity theory and equivariant symplectic maps.* Springer Lectures Notes, vol. 1558 (Springer, 1993).

3 J. W. Bruce and D. L. Fidal. Generic functions on bifurcation varieties. I. (Unpublished notes.)

4 J. W. Bruce, A. du Plessis and L. C. Wilson. Discriminants and liftable vector fields. *J. Alg. Geom.* **3** (1994), 725–753.

5 G. Dangelmayer and D. Armbruster. Classification of $\mathbb{Z}_2$-equivariant imperfect bifurcations with corank 2. *Proc. Lond. Math. Soc.* **46** (1983), 517–546.

6 J. E. Furter. Geometric path formulation for bifurcation problems. *J. Nat. Geom.* **12** (1997), 1–100.

7 J. E. Furter and A. M. Sitta. Geometric theory of bifurcation problems: path formulation. (Preprint, Brunel University, 2004.)

8 J. E. Furter, A. M. Sitta and I. Stewart. Algebraic path formulation for equivariant bifurcation problems. *Math. Proc. Camb. Phil. Soc.* **124** (1998), 275–304.

9 M. Golubitsky and D. G. Schaeffer. A theory for imperfect bifurcation via singularity theory. *Commun. Pure Appl. Math.* **32** (1979), 21–98.

10 M. Golubitsky and D. G. Schaeffer. *Singularities and groups in bifurcation theory. I.* Applied Mathematical Sciences, vol. 51 (Springer, 1985).

11 M. Golubitsky, I. Stewart and D. G. Schaeffer. *Singularities and groups in bifurcation theory. II.* Applied Mathematical Sciences, vol. 69 (Springer, 1988).

12 D. Mond and J. Montaldi. Deformations of maps on complete intersections, Damon's $\mathcal{K}_V$-equivalence and bifurcations. In *Singularities* (ed. J. P. Brasselet). LMS Lecture Notes, vol. 201, pp. 263–284 (Cambridge University Press, 1994).

13 J. Montaldi. The path formulation of bifurcation theory. In *Dynamics, bifurcation and symmetry: new trends and new tools* (ed. P. Chossat), pp. 259–278. Nato ASI Series, vol. 437 (Dordrecht: Kluwer, 1994).

14 M. Peters. Multiparameter bifurcation problems. PhD thesis, University of Warwick (1991).

15 K. Saito. Theory of logarithmic differential forms and logarithmic vectorfields. *J. Fac. Sci. Univ. Tokyo* 1A **27** (1980), 265–291.

16 C. Zuppa. Bifurcations imparfaites de type potentiel et D-équivalences de déformations des fonctions. *Duke Math. J.* **51** (1984), 729–763.

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