Combinatorics, Probability and Computing (2014) 23, 116–134.
doi:10.1017/S0963548313000461

A Marstrand Theorem for Subsets of Integers

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Received 29 June 2012; revised 23 August 2013; first published online 25 October 2013

We propose a counting dimension for subsets of $\mathbb{Z}$ and prove that, under certain conditions on $E, F \subset \mathbb{Z}$, for Lebesgue almost every $\lambda \in \mathbb{R}$ the counting dimension of $E + \lfloor \lambda F \rfloor$ is at least the minimum between 1 and the sum of the counting dimensions of E and F. Furthermore, if the sum of the counting dimensions of E and F is larger than 1, then $E + \lfloor \lambda F \rfloor$ has positive upper Banach density for Lebesgue almost every $\lambda \in \mathbb{R}$. The result has direct consequences when E, F are arithmetic sets, *e.g.*, the integer values of a polynomial with integer coefficients.

2010 *Mathematics subject classification*: Primary 05A99
Secondary 37E15

1. Introduction

The purpose of this paper is to prove a Marstrand-type theorem for a class of subsets of the integers.

The well-known theorem of Marstrand [10] on geometric measure theory states the following. If $K \subset \mathbb{R}^2$ is a Borel set then, for almost every direction, its projection to $\mathbb{R}$ in the respective direction has Hausdorff dimension equal to the minimum between one and the Hausdorff dimension of K; if in addition K has Hausdorff dimension greater than one, then almost every such projection has positive Lebesgue measure. When $K = K_1 \times K_2$ for $K_1, K_2 \subset \mathbb{R}$, the projections are affine images of the *arithmetic sum*

$$K_1 + \lambda K_2 = \{x + \lambda y : x \in K_1, y \in K_2\},$$

and Marstrand's theorem states that $K_1 + \lambda K_2$ has the aforementioned properties for Lebesgue-a.e. $\lambda \in \mathbb{R}$.[1]

[1] Almost every $\lambda \in \mathbb{R}$ with respect to Lebesgue measure.

The investigation of such *arithmetic sums* is an active area of mathematics, specially because of its applications in various fields, *e.g.*, diophantine approximations and dynamical bifurcations.

Given $E \subset \mathbb{Z}$, let $d^*(E)$ denote its *upper Banach density*

$$d^*(E) = \limsup_{|I|\to\infty} \frac{|E \cap I|}{|I|},$$

where I runs over all intervals of $\mathbb{Z}$. A remarkable result in additive combinatorics is Szemerédi's theorem [13]: if $d^*(E) > 0$, then E contains arbitrarily long arithmetic progressions. One can interpret this result by saying that density represents the correct notion of largeness needed to preserve finite configurations of $\mathbb{Z}$.

Szemerédi's theorem does not apply to subsets of zero upper Banach density. Many of these sets are of interest, and they may also contain combinatorially rich patterns. For example, sets formed by the integer values of a polynomial with integer coefficients have a special interest in ergodic theory and its connections with combinatorics [2, 3]. Another example is the prime numbers: they have zero density (by the prime number theorem), and yet there are arbitrarily long arithmetic progressions of primes [6].

A set $E \subset \mathbb{Z}$ of zero upper Banach density occupies portions in intervals of $\mathbb{Z}$ that grow sublinearly as the lengths of the intervals grow, but there still may exist some sublinear growth speed, *e.g.*, the number of perfect squares on $(0, n]$ is about $n^{0.5}$. This exponent represents, in some sense, a dimension of $\{n^2 : n \in \mathbb{Z}\}$ inside $\mathbb{Z}$. In this article, we propose a *counting dimension*

$$D(E) = \limsup_{|I|\to\infty} \frac{\log |E \cap I|}{\log |I|},$$

where I runs over all intervals of $\mathbb{Z}$. This definition captures the growth rate of $|E \cap I|$, and it allows us to compare largeness between sets of zero upper Banach density.

Until now, a theory of *fractal sets* in $\mathbb{Z}$ has not been developed, and this is the motivation for this article. We address the following question: To what extent do fractal sets in $\mathbb{Z}$ satisfy a Marstrand-like theorem? By a fractal set we mean a set $E \subset \mathbb{Z}$ with $D(E) < 1$. For example, if $p \in \mathbb{Z}[x]$ has degree $d > 0$, then $E = \{p(n)\}_{n\in\mathbb{N}}$ has counting dimension $\frac{1}{d}$ (see Section 3.1). The first main result of this paper is that such *polynomial sets* satisfy a Marstrand-like theorem.

Theorem 1.1. *Let $p_i \in \mathbb{Z}[x]$ with degree $d_i > 0$, and let $E_i = \{p_i(n)\}_{n\in\mathbb{Z}}$. Then*

$$D(E_0 + \lfloor\lambda_1 E_1\rfloor + \cdots + \lfloor\lambda_k E_k\rfloor) \geqslant \min\left\{1, \frac{1}{d_0} + \frac{1}{d_1} + \cdots + \frac{1}{d_k}\right\}$$

for Lebesgue-a.e. $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$. If

$$\sum_{i=0}^{k} \frac{1}{d_i} > 1,$$

then $E_0 + \lfloor\lambda_1 E_1\rfloor + \cdots + \lfloor\lambda_k E_k\rfloor$ has positive upper Banach density for Lebesgue-a.e. $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$.

Theorem 1.1 is the consequence of a more general result, which is the main result of this paper. It identifies sufficient conditions for a Marstrand theorem on $\mathbb{Z}$ to hold.

Theorem 1.2. *Let $E, F \subset \mathbb{Z}$ be regular compatible sets. Then*

$$D(E + \lfloor \lambda F \rfloor) \geqslant \min\{1, D(E) + D(F)\}$$

for Lebesgue-a.e. $\lambda \in \mathbb{R}$. If $D(E) + D(F) > 1$, then $E + \lfloor \lambda F \rfloor$ has positive upper Banach density for Lebesgue-a.e. $\lambda \in \mathbb{R}$.

Let us briefly explain the notions of regularity and compatibility. The counting dimension says that $|E \cap I|$, along a sequence of intervals I, grows like $|I|^{D(E)}$, with a small error on the exponent $D(E)$. We say that E is *regular* when there is no error at all, *i.e.*, when the cardinality of $|E \cap I|$ is, up to a multiplicative constant, of the order of $|I|^{D(E)}$. See Section 4.1 for the definition.

Now let E, F be two regular sets. There are intervals I such that $|E \cap I|$ has order $|I|^{D(E)}$ and intervals J such that $|F \cap J|$ has order $|J|^{D(F)}$. In general, $|I|$ and $|J|$ are incomparable. We say that E and F are *compatible* if $|I|$ and $|J|$ are asymptotic. See Section 4.1 for the definition.

The quantities $d^*(E)$ and $D(E)$ are similar to the Lebesgue measure and box dimension on $\mathbb{R}$. It is because of this association that we call Theorem 1.2 a Marstrand theorem for subsets of integers. Most results of this paper were motivated by known facts in geometric measure theory. We will try to refer to these facts.

The notions of regularity and compatibility are both satisfied by many arithmetic subsets of $\mathbb{Z}$, *e.g.*, the integer values of a polynomial with integer coefficients and, more generally, by *universal* sets: those sets that exhibit the expected growth rate along intervals of arbitrary length (see Definition 4.4). For these sets, Theorem 1.2 can be inductively applied to give the result below.

Theorem 1.3. *Let $E_0, \ldots, E_k$ be universal subsets of $\mathbb{Z}$. Then*

$$D(E_0 + \lfloor \lambda_1 E_1 \rfloor + \cdots + \lfloor \lambda_k E_k \rfloor) \geqslant \min\{1, D(E_0) + D(E_1) + \cdots + D(E_k)\}$$

for Lebesgue-a.e. $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$. If

$$\sum_{i=0}^{k} D(E_i) > 1,$$

then $E_0 + \lfloor \lambda_1 E_1 \rfloor + \cdots + \lfloor \lambda_k E_k \rfloor$ has positive upper Banach density for Lebesgue-a.e. $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$.

Integer values of a polynomial with integer coefficients are universal sets, thus Theorem 1.1 follows from Theorem 1.3.

The proof of Theorem 1.2 is based on the ideas developed in [8, 9]. The cardinality of a regular subset of $\mathbb{Z}$ along an increasing sequence of intervals exhibits an exponential behaviour ruled out by its counting dimension. If this holds for two regular subsets $E, F \subset \mathbb{Z}$, the compatibility assumption allows to estimate the cardinality of $E + \lfloor \lambda F \rfloor$

along the respective arithmetic sums of intervals. A double-counting argument estimates the size of the 'bad' parameters for which such cardinality is small.

The paper is organized as follows. In Section 2 we provide basic notations and definitions. In Section 3 we discuss some examples, including the sets given by integer values of a polynomial with integer coefficients. In Section 4 we introduce the notions of regularity and compatibility. In Section 4.3 we construct a counterexample to Theorem 1.2 when the sets are not compatible (thus regularity and compatibility are not only sufficient but also necessary conditions for the validity of Theorem 1.2). In Section 4.4 we construct a counterexample to Theorem 1.2 when the space of parameters is $\mathbb{Z}$ (thus $\mathbb{R}$ is the correct space of parameters). In Section 5 we prove Theorems 1.2 and 1.3. We also collect some final remarks and questions in Section 6.

2. Preliminaries

2.1. General notation

Given a set X, $|X|$ denotes the cardinality of X. $\mathbb{Z}$ denotes the set of integers and $\mathbb{N}$ the set of positive integers.

Definition 2.1. Let $f, g : \mathbb{Z}$ or $\mathbb{N} \to \mathbb{R}$. We write $f \lesssim g$ if there exists $C > 0$ such that

$$|f(n)| \leqslant C|g(n)|, \quad \text{for all } n \in \mathbb{Z} \text{ or } \mathbb{N}.$$

If $f \lesssim g$ and $g \lesssim f$, we write $f \sim g$. We write $f \approx g$ if

$$\lim_{|n| \to \infty} \frac{f(n)}{g(n)} = 1.$$

Given $x \in \mathbb{R}$, $\lfloor x \rfloor$ is the integer part of x. For $k \geqslant 1$, m_k is the Lebesgue measure of $\mathbb{R}^k$. Let $m = m_1$. The letter I denotes an interval of $\mathbb{Z}$, *e.g.*, $I = (M, N] = \{M + 1, \ldots, N\}$. The *length* of I is $|I| = N - M$.

Given $E \subset \mathbb{Z}$ and $\lambda \in \mathbb{R}$, let $\lambda E = \{\lambda n : n \in E\} \subset \mathbb{R}$ and $\lfloor \lambda E \rfloor = \{\lfloor \lambda n \rfloor : n \in E\} \subset \mathbb{Z}$.

2.2. Counting dimension

Definition 2.2. The *upper Banach density* of $E \subset \mathbb{Z}$ is

$$d^*(E) = \limsup_{|I| \to \infty} \frac{|E \cap I|}{|I|},$$

where I runs over all intervals of $\mathbb{Z}$.

Definition 2.3. The *counting dimension*, or simply *dimension*, of $E \subset \mathbb{Z}$ is

$$D(E) = \limsup_{|I| \to \infty} \frac{\log |E \cap I|}{\log |I|},$$

where I runs over all intervals of $\mathbb{Z}$.

The above definition is similar to the box dimension on $\mathbb{R}$. Similar definitions appeared in [1, 5]. Note that $D(E) \in [0, 1]$, and if $d^*(E) > 0$ then $D(E) = 1$.

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Here is an alternative definition of $D(E)$ that is similar in spirit to the Hausdorff dimension on $\mathbb{R}$. Let α be a non-negative real number.

Definition 2.4. The *counting α-measure* of $E \subset \mathbb{Z}$ is

$$H_\alpha(E) = \limsup_{|I|\to\infty} \frac{|E \cap I|}{|I|^\alpha},$$

where I runs over all intervals of $\mathbb{Z}$.

Clearly, $H_\alpha(E) \in [0,\infty]$. For a fixed $E \subset \mathbb{Z}$, the numbers $H_\alpha(E)$ are decreasing in α. Furthermore,

$$\alpha < D(E) \implies H_\alpha(E) = \infty \quad \text{and} \quad \alpha > D(E) \implies H_\alpha(E) = 0,$$

and thus there is a unique $\alpha \geqslant 0$ such that

$$H_\beta(E) = \begin{cases} \infty & \text{if } 0 \leqslant \beta < \alpha, \\ 0 & \text{if } \beta > \alpha. \end{cases}$$

Thus $D(E) = \alpha$, *i.e.*, $D(E)$ is the parameter α where $H_\alpha(E)$ decreases from infinity to zero.

Here is an analogue to Frostman's lemma (see Theorem 8.8 of [11]). If $\beta > D(E)$, then

$$|E \cap I| \lesssim |I|^\beta, \tag{2.1}$$

where I runs over all intervals of $\mathbb{Z}$. Conversely, if (2.1) holds, then $D(E) \leqslant \beta$.

Below we collect some basic properties of D and H_α. All proofs are direct.

(i) If $E \subset F$, then $D(E) \leqslant D(F)$.
(ii) $D(E \cup F) = \max\{D(E), D(F)\}$.
(iii) If $\lambda \geqslant 1$, then

$$H_\alpha(\lfloor \lambda E \rfloor) = \lambda^{-\alpha} H_\alpha(E). \tag{2.2}$$

Remark 2.5. We have $\lfloor -x \rfloor = -\lfloor x \rfloor$ or $\lfloor -x \rfloor = -\lfloor x \rfloor - 1$, so $D(\lfloor -\lambda E \rfloor) = D(\lfloor \lambda E \rfloor)$. Also, $0 < H_\alpha(\lfloor -\lambda E \rfloor) < \infty$ if and only if $0 < H_\alpha(\lfloor \lambda E \rfloor) < \infty$. So we assume from now on that $\lambda > 0$.

3. Examples

Example 1. Let $\alpha \in (0,1]$, and let $E_\alpha = \{\lfloor n^{1/\alpha} \rfloor : n \in \mathbb{N}\}$. We claim that $H_\alpha(E_\alpha) = 1$. Because[2] $(x+y)^\alpha \leqslant x^\alpha + y^\alpha$ for $x, y \geqslant 0$, we have

$$\frac{|E_\alpha \cap (M,N]|}{(N-M)^\alpha} \leqslant \frac{(N+1)^\alpha - (M+1)^\alpha}{(N-M)^\alpha} \leqslant 1,$$

[2] For each $t \geqslant 0$, the function $x \in [0,t] \mapsto x^\alpha + (t-x)^\alpha$ is concave, so it attains its minima at $x = 0$ and $x = t$. Thus $x^\alpha + (t-x)^\alpha \geqslant t^\alpha$ for any $x \in [0,t]$.

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and thus $H_\alpha(E_\alpha) \leqslant 1$. On the other hand,

$$\frac{|E_\alpha \cap (0,N]|}{N^\alpha} \geqslant \frac{N^\alpha - 1}{N^\alpha},$$

so $H_\alpha(E_\alpha) \geqslant 1$.

Example 2. The prime numbers have dimension one. This follows from the prime number theorem:

$$\lim_{n\to\infty} \frac{\log|\{1 \leqslant p \leqslant n : p \text{ is prime}\}|}{\log n} = \lim_{n\to\infty} \frac{\log n - \log\log n}{\log n} = 1.$$

3.1. Polynomial sets of $\mathbb{Z}$

Definition 3.1. A *polynomial set of* $\mathbb{Z}$ is a set $E = \{p(n) : n \in \mathbb{N}\}$, where $p(x) \in \mathbb{Z}[x]$ is non-constant.

These are the sets we consider in Theorem 1.1. Let us calculate their counting dimension. Call $E, F \subset \mathbb{Z}$ *asymptotic* if

$$E = \{\cdots < a_{-1} < a_0 < a_1 < \cdots\}, \quad F = \{\cdots < b_{-1} < b_0 < b_1 < \cdots\}$$

and there exists $i \geqslant 0$ such that

$$a_{n-i} \leqslant b_n \leqslant a_{n+i}, \quad \text{for all } n \in \mathbb{Z}. \tag{3.1}$$

Lemma 3.2. *If E, F are asymptotic and $\alpha > 0$, then $H_\alpha(E) = H_\alpha(F)$. In particular, $D(E) = D(F)$.*

Proof. Let $I = (M, N]$, and let $E \cap I = \{a_{m+1}, a_{m+2}, \ldots, a_n\}$. By (3.1),

$$\{b_{m+i+1}, \ldots, b_{n-i}\} \subset F \cap I \subset \{b_{m-i}, \ldots, b_{n+i+1}\},$$

and thus $|E \cap I| \approx |F \cap I|$, and so $H_\alpha(E) = H_\alpha(F)$. □

Let $E = \{p(n) : n \in \mathbb{N}\}$, where p has degree d. Assume that p has leading coefficient $a \geqslant 1$. Thus there exists $i \geqslant 0$ such that $a(n-i)^d < p(n) < a(n+i)^d$ for sufficiently large $n \in \mathbb{Z}$, so $E, aE_{\frac{1}{d}}$ are asymptotic, where $E_{\frac{1}{d}}$ is defined as in Example 1. By equality (2.2) and Lemma 3.2, it follows that $D(E) = \frac{1}{d}$ and that $H_{\frac{1}{d}}(E) = a^{-\frac{1}{d}}$.

3.2. Cantor sets in $\mathbb{Z}$

The classical ternary Cantor set of $\mathbb{R}$ is the set of real numbers on $[0,1]$ with only zeros and twos on the expansion in base 3. In analogy to this, define $E \subset \mathbb{Z}$ as

$$E = \left\{ \sum_{i=0}^{n} a_i 3^i : n \in \mathbb{N} \text{ and } a_i \in \{0,2\} \right\}. \tag{3.2}$$

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Fisher proved in [4] that $H_{\frac{\log 2}{\log 3}}(E) > 0$. In Lemma 3.4 below, we will prove that $H_{\frac{\log 2}{\log 3}}(E) < \infty$. In particular,

$$D(E) = \frac{\log 2}{\log 3}.$$

The renormalization of $E \cap (0, 3^n)$ via the map $x \mapsto \frac{x}{3^n}$ is a subset of $(0, 1)$ equal to the set of left endpoints of the remaining intervals of the nth step of the construction of the classical ternary Cantor set of $\mathbb{R}$, *i.e.*, if $K = \bigcup_{n\in E}[n, n+1]$, then $\frac{K\cap[0,3^n]}{3^n}$ is the nth step of the construction of the ternary Cantor set of $\mathbb{R}$.

More generally, let us define a class of Cantor sets in $\mathbb{Z}$. Fix $a \in \mathbb{N}$ and a binary matrix $A = (a_{ij})_{0\leqslant i,j\leqslant a-1}$. For $n \geqslant 0$, let

$$\Sigma_n(A) = \{(d_0 d_1 \cdots d_n) : a_{d_{i-1}d_i} = 1,\ 1 \leqslant i \leqslant n\},$$

and let $\Sigma^*(A) = \bigcup_{n\geqslant 0} \Sigma_n(A)$.

Definition 3.3. The *integer Cantor set* $E_A \subset \mathbb{Z}$ induced by A is

$$E_A = \{d_0 a^0 + \cdots + d_n a^n : (d_0 d_1 \cdots d_n) \in \Sigma^*(A)\}.$$

Here is our motivation for Definition 3.3. Dynamically defined topologically mixing Cantor sets of the real line are homeomorphic to subshifts of finite type (see, *e.g.*, [8]). After truncating the numbers, this is exactly what Definition 3.3 does.

Remember that the *Perron–Frobenius eigenvalue* of A is its largest eigenvalue $\lambda_+(A)$. It has multiplicity one and maximizes the absolute value of the eigenvalues of A. Furthermore, there exists $c = c(A) > 0$ such that

$$c^{-1}\lambda_+(A)^n \leqslant |\Sigma_n(A)| \leqslant c\lambda_+(A)^n, \quad \text{for all } n \geqslant 0. \tag{3.3}$$

See, *e.g.*, [7]. The dimension of E_A depends explicitly on a and on $\lambda_+(A)$.

Lemma 3.4. *If A is a binary $a \times a$ matrix, then*

$$D(E_A) = \frac{\log \lambda_+(A)}{\log a} \quad \textit{and} \quad 0 < H_{\frac{\log \lambda_+(A)}{\log a}}(E_A) < \infty.$$

Proof. Let $I = (M, N]$. We may assume that $M + 1, N \in E_A$, say

$$\begin{cases} M+1 &= x_0 a^0 + \cdots + x_n a^n, \\ N &= y_0 a^0 + \cdots + y_n a^n, \end{cases}$$

where $y_n > x_n$ (if $x_n = y_n$, then we can consider the translation of I by $-x_n a^n$). If $y_n \geqslant x_n + 2$, then

$$\begin{cases} M+1 &\leqslant (x_n+1)a^n \\ N &\geqslant (x_n+2)a^n \end{cases} \implies |I| \geqslant a^n.$$

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Because $I \subset (0, a^{n+1})$, we get that

$$\frac{|E_A \cap I|}{|I|^{\frac{\log \lambda_+(A)}{\log a}}} \leqslant \frac{|\Sigma_n(A)|}{a^{\frac{n \log \lambda_+(A)}{\log a}}} \leqslant \frac{c\lambda_+(A)^n}{\lambda_+(A)^n} = c. \tag{3.4}$$

If $y_n = x_n + 1$, let $i, j \in \{0, 1, \ldots, n-1\}$ such that

(i) $x_i < a - 1$ and $x_{i+1} = \cdots = x_{n-1} = a - 1$,

(ii) $y_j > 0$ and $y_{j+1} = \cdots = y_{n-1} = 0$.

Thus

$$\left\{ \begin{array}{lcl} M + 1 & \leqslant & (x_n + 1)a^n - a^i \\ N & \geqslant & (x_n + 1)a^n + a^j \end{array} \right. \implies |I| \geqslant a^i + a^j \geqslant a^{\max\{i,j\}}.$$

If $\sum_{l=0}^{n} z_l a^l \in I$, then necessarily $z_n \in \{x_n, x_n + 1\}$. In the first case $z_{i+1} = \cdots = z_{n-1} = a - 1$, and in the second case $z_{j+1} = \cdots = z_{n-1} = 0$. Thus

$$|E_A \cap I| \leqslant |\Sigma_i(A)| + |\Sigma_j(A)| \leqslant 2c\lambda_+(A)^{\max\{i,j\}},$$

and so

$$\frac{|E_A \cap I|}{|I|^{\frac{\log \lambda_+(A)}{\log a}}} \leqslant \frac{2c\lambda_+(A)^{\max\{i,j\}}}{a^{\frac{\max\{i,j\} \log \lambda_+(A)}{\log a}}} = 2c. \tag{3.5}$$

Estimates (3.4) and (3.5) give that

$$H_{\frac{\log \lambda_+(A)}{\log a}}(E_A) < \infty.$$

Furthermore,

$$\frac{|E_A \cap (0, a^n]|}{a^{\frac{n \log \lambda_+(A)}{\log a}}} \geqslant \frac{c^{-1}\lambda_+(A)^{n-1}}{\lambda_+(A)^n} = c^{-1}\lambda_+(A)^{-1},$$

and thus

$$H_{\frac{\log \lambda_+(A)}{\log a}}(E_A) > 0. \qquad \square$$

Here is a direct application of Lemma 3.4. If $X \subset \{0, \ldots, a-1\}$ and $A = (a_{ij})$ with $a_{ij} = 1$ if and only if $i, j \in X$, then

$$D(E_A) = \frac{\log |X|}{\log a}.$$

If $E, F \subset \mathbb{Z}$ with $D(E) + D(F) > 1$, then it is not true in general that $d^*(E + F) > 0$. This happens if the elements of $E + F$ have many representations as the sum of one element of E and other of F. This resonance phenomenon might decrease the dimension of $E + F$. For example, if $E = E_A$ and $F = E_B$, where $A = (a_{ij})_{0 \leqslant i,j \leqslant 11}$, $B = (b_{ij})_{0 \leqslant i,j \leqslant 11}$ are defined by

$$a_{ij} = 1 \iff 0 \leqslant i, j \leqslant 3 \quad \text{and} \quad b_{ij} = 1 \iff 4 \leqslant i, j \leqslant 7,$$

then

$$D(E) + D(F) = \frac{2 \log 4}{\log 12},$$

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while $E+F=E_C$ for $C=(c_{ij})_{0\leqslant i,j\leqslant 11}$ defined by

$$c_{ij}=1 \iff 4\leqslant i,j\leqslant 10.$$

$E+F$ has counting dimension equal to $\frac{\log 7}{\log 12}$, and thus $d^*(E+F)=0$. What Theorem 1.2 gives is that resonance is avoided if we change the scales of the sets.

3.3. Generalized IP-sets

This class of sets was suggested to us by Simon Griffiths and Rob Morris. Let $(k_n)_{n\geqslant 1}$, $(d_n)_{n\geqslant 1}$ be sequences of positive integers with $k_n\geqslant 2$.

Definition 3.5. The *generalized IP-set* associated to $(k_n)_{n\geqslant 1}$, $(d_n)_{n\geqslant 1}$ is the set

$$E=\left\{\sum_{i=1}^{n} x_i d_i : n\in\mathbb{N} \text{ and } 0\leqslant x_i<k_i\right\}.$$

We always assume that $d_n>\sum_{i=1}^{n-1}k_i d_i$. Thus the map $\sum_{i=1}^{n}x_i d_i\mapsto(x_1,\ldots,x_n)$ is a bijection from E to $\{(x_1,\ldots,x_n): n\in\mathbb{N}, x_n>0 \text{ and } 0\leqslant x_i<k_i\}$. Also, if this former set is colexicographically ordered,[3] then the map is order-preserving.

Lemma 3.6. *Let E be the generalized IP-set associated to* $(k_n)_{n\geqslant 1}$, $(d_n)_{n\geqslant 1}$, *and let* $p_n=k_1\cdots k_n$. *Then*

$$D(E)=\limsup_{n\to\infty}\frac{\log p_n}{\log k_n d_n}. \tag{3.6}$$

Proof. First, note that if $I=(0,\sum_{i=1}^{n}(k_i-1)d_i]$, then $|E\cap I|=p_n$ and $|I|\leqslant k_n d_n$. Thus

$$D(E)\geqslant\limsup_{n\to\infty}\frac{\log p_n}{\log k_n d_n}.$$

Now let $I=(M,N]$, say

$$\begin{cases} M+1 &= x_1d_1+\cdots+x_nd_n,\\ N &= y_1d_1+\cdots+y_nd_n.\end{cases}$$

As in Lemma 3.4, we can assume that $y_n>x_n$. Let $y_n=x_n+k$, where $0<k<k_n$. We divide the analysis into three cases.

Case 1: $M=0$. We have $|E\cap I|\leqslant(k+1)p_{n-1}$ and $|I|\geqslant kd_n$. Because the map

$$k\in\mathbb{N}\mapsto\frac{\log(k+1)p_{n-1}}{\log kd_n}$$

is increasing, we get that

$$\frac{\log|E\cap I|}{\log|I|}\leqslant\frac{\log(k_n+1)p_{n-1}}{\log k_nd_n}=(1+o(1))\frac{\log k_np_{n-1}}{\log k_nd_n}=(1+o(1))\frac{\log p_n}{\log k_nd_n}.$$

[3] The sequence $(x_1,\ldots,x_n)$ is smaller than $(y_1,\ldots,y_m)$ if $n<m$ or if there exists $i\in\{1,\ldots,n\}$ such that $x_i<y_i$ and $x_j=y_j$ for $j=i+1,\ldots,n$.

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Case 2: $k \geqslant 2$. We have $|E \cap I| \leqslant (k+1)p_{n-1}$ and $|I| \geqslant (k-1)d_n$, so we can proceed as in Case 1.

Case 3: $k = 1$. Let

$$P = \sum_{i=1}^{n-1}(k_i - 1)d_i + x_n d_n,$$

and let $I_1 = (M, P]$ and $I_2 = (P, N]$. Thus $E \cap I_2 = P + E \cap (0, N - P]$, so $|E \cap I_2|, |I_2|$ can be estimated as in Case 1. Similarly, we estimate $|E \cap I_1|, |I_1|$ (just consider a reflected version of E and apply Case 1). Note that either $|I_1| \to +\infty$ or $|I_2| \to +\infty$.

Thus

$$D(E) \leqslant \limsup_{n\to\infty} \frac{\log p_n}{\log k_n d_n}. \qquad \square$$

4. Regularity and compatibility

4.1. Regular sets

Definition 4.1. We call $E \subset \mathbb{Z}$ *regular* or *α-set* if $D(E) = \alpha$ and $0 < H_\alpha(E) < \infty$.

By Lemmas 3.2 and 3.4, polynomial sets and Cantor sets are regular. A general set is not regular, but it does contain many regular subsets.

Proposition 4.2. *Let $E \subset \mathbb{Z}$ and $0 \leqslant \alpha \leqslant 1$. If $H_\alpha(E) > 0$, then there is a regular subset $E' \subset E$ such that $D(E') = \alpha$. In particular, if $0 \leqslant \alpha < D(E)$ then there exists $E' \subset E$ regular such that $D(E') = \alpha$.* $\square$

Proof. This is an analogue of Theorem 8.19 of [11], and the idea of the proof is similar in spirit: we apply a dyadic argument to decrease $H_\alpha(E)$ in a controlled way. Given an interval $I \subset \mathbb{Z}$ and a subset $F \subset \mathbb{Z}$, define

$$s_F(I) \doteq \sup_{\substack{J \subset I \\ J \text{ interval}}} \frac{|F \cap J|}{|J|^\alpha}.$$

If $F = \{a_1, a_2, \ldots, a_k\} \subset I$, the operation of alternately discarding the elements of F,

$$F = \{a_1, a_2, \ldots, a_k\} \rightsquigarrow F' = \{a_1, a_3, a_5, \ldots, a_{2\lceil \frac{k-1}{2} \rceil - 1}, a_k\},$$

decreases $s_F(I)$ to approximately $\frac{s_F(I)}{2}$. More specifically, if $s_F(I) > 2$, then

$$\frac{1}{2} < s_{F'}(I) < s_F(I) - \frac{1}{2}.$$

Indeed:

- For every interval $J \subset I$, we have

$$\frac{|F' \cap J|}{|J|^\alpha} \leqslant \frac{1}{2} \cdot \frac{|F \cap J| + 1}{|J|^\alpha} \leqslant \frac{s_F(I)}{2} + \frac{1}{2} < s_F(I) - \frac{1}{2}.$$

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- If J maximizes $s_F(I)$, then

$$\frac{|F'\cap J|}{|J|^\alpha} \geqslant \frac{1}{2}\cdot\frac{|F\cap J|-1}{|J|^\alpha} > 1-\frac{1}{2|J|^\alpha} \geqslant \frac{1}{2}.$$

After a finite number of these operations, we get $F' \subset F$ with $\frac{1}{2} < s_{F'}(I) \leqslant 2$.

If $H_\alpha(E) < \infty$, there is nothing to do. Assume that $H_\alpha(E) = \infty$. We inductively construct a sequence $F_1 \subset F_2 \subset \cdots$ of finite subsets of E such that $F_n \subset I_n = (a_n, b_n]$ with $|I_n| \to +\infty$ and

(i) $\frac{1}{2} < s_{F_n}(I_n) \leqslant 3$,

(ii) there is an interval $J_n \subset I_n$ such that $|J_n| \geqslant n$ and

$$\frac{|F_n\cap J_n|}{|J_n|^\alpha} > \frac{1}{2}.$$

Once these properties hold, $E' = \bigcup_{n\geqslant 1} F_n$ will satisfy the required conditions.

Take any $a \in E$ and $I_1 = \{a\}$. Assume I_n, F_n, J_n are defined and satisfy (i) and (ii). Because $H_\alpha(E) = \infty$, there exists an interval J_{n+1} disjoint from $(a_n - |I_n|^{\frac{1}{\alpha}}, b_n + |I_n|^{\frac{1}{\alpha}}]$ such that

$$\frac{|E\cap J_{n+1}|}{|J_{n+1}|^\alpha} \geqslant (n+1)^{1-\alpha}.$$

Thus we can restrict J_{n+1} to a smaller interval of size at least $n+1$, also denoted by J_{n+1}, such that

$$s_E(J_{n+1}) = \frac{|E\cap J_{n+1}|}{|J_{n+1}|^\alpha}. \tag{4.1}$$

Consider $F'_{n+1} = E \cap J_{n+1}$ and apply the dyadic operation to F'_{n+1} until

$$\frac{1}{2} < s_{F'_{n+1}}(J_{n+1}) = \frac{|F'_{n+1}\cap J_{n+1}|}{|J_{n+1}|^\alpha} \leqslant 2. \tag{4.2}$$

Let $I_{n+1} = I_n \cup K_n \cup J_{n+1}$ be the convex hull of I_n and J_{n+1}, and let $F_{n+1} = F_n \cup F'_{n+1}$. Condition (ii) is satisfied because of (4.2). To prove (i), let I be a sub-interval of I_{n+1}. We have three cases.

- $I \subset I_n \cup K_n$. By (i),

$$\frac{|F_{n+1}\cap I|}{|I|^\alpha} \leqslant \frac{|F_n\cap(I\cap I_n)|}{|I\cap I_n|^\alpha} \leqslant 3.$$

- $I \subset K_n \cup J_{n+1}$. By (4.2),

$$\frac{|F_{n+1}\cap I|}{|I|^\alpha} \leqslant \frac{|F'_{n+1}\cap(I\cap J_{n+1})|}{|I\cap J_{n+1}|^\alpha} \leqslant 2.$$

- $I \supset K_n$. Because $|K_n| \geqslant |I_n|^{\frac{1}{\alpha}}$,

$$\begin{aligned}\frac{|F_{n+1}\cap I|}{|I|^\alpha} &= \frac{|F_n\cap(I\cap I_n)| + |F'_{n+1}\cap(I\cap J_{n+1})|}{(|I\cap I_n| + |K_n| + |I\cap J_{n+1}|)^\alpha}\\ &\leqslant \frac{|F_n\cap(I\cap I_n)|}{(|I\cap I_n| + |K_n|)^\alpha} + \frac{|F'_{n+1}\cap(I\cap J_{n+1})|}{(|I\cap J_{n+1}|)^\alpha}\end{aligned}$$

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$$\leqslant \frac{|I_n|}{|K_n|^\alpha} + s_{F'_{n+1}}(J_{n+1})$$
$$\leqslant 3.$$

This proves (i) and completes the inductive step. □

4.2. Compatible sets

Definition 4.3. We refer to two regular subsets $E, F \subset \mathbb{Z}$ as *compatible* if there are sequences $(I_n)_{n\geqslant 1}$, $(J_n)_{n\geqslant 1}$ of intervals with increasing lengths such that

(i) $|I_n| \sim |J_n|$,
(ii) $|E \cap I_n| \gtrsim |I_n|^{D(E)}$ and $|F \cap J_n| \gtrsim |J_n|^{D(F)}$.

In other words, two subsets are compatible if they have intervals of comparable lengths such that their cardinality on these intervals have the correct asymptotic.

Definition 4.4. We call a regular subset $E \subset \mathbb{Z}$ *universal* if there is a sequence $(I_n)_{n\geqslant 1}$ of intervals such that $|I_n| \sim n$ and $|E \cap I_n| \gtrsim |I_n|^{D(E)}$.

Each E_α is universal, as well as each polynomial sets (see Section 3.1). If E is universal and F is regular, then E, F are compatible. In particular, any two polynomial sets are compatible.

4.3. A counterexample to Theorem 1.2 for regular non-compatible sets

Let us show that compatibility is a necessary assumption for Theorem 1.2: we construct regular sets $E, F \subset \mathbb{Z}$ such that $D(E) + D(F) > 1$ and $E + \lfloor \lambda F \rfloor$ has zero upper Banach density for every $\lambda \in \mathbb{R}$. The idea is to construct E and F such that the intervals $I, J \subset \mathbb{Z}$ on which

$$\frac{|E \cap I|}{|I|^{D(E)}} \quad \text{and} \quad \frac{|F \cap J|}{|J|^{D(F)}}$$

are bounded away from zero have incomparable lengths.

Let $\alpha \in \left(\frac{1}{2}, 1\right)$, and let

$$E = \bigcup_{i \text{ odd}} (E_i \cap I_i), \quad F = \bigcup_{i \text{ even}} (E_i \cap I_i)$$

such that the following conditions hold:

(i) $E_i = \lfloor \mu_i \lfloor {\mu_i}^{-1} E_0 \rfloor \rfloor$, where $E_0 \subset \mathbb{N}$ with $H_\alpha(E_0) = \frac{1}{2}$,
(ii) $\mu_i > b_{i-1}^{2/(1-\alpha)}$ for all $i \geqslant 1$,
(iii) $(I_i)_{i\geqslant 1}$ is a disjoint sequence of intervals of increasing lengths such that

$$\lim_{i\to\infty} \frac{|E_i \cap I_i|}{|I_i|^\alpha} = \frac{1}{2}.$$

It is clear that we can inductively construct $(\mu_i)_{i\geqslant 1}, (E_i)_{i\geqslant 1}, (I_i)_{i\geqslant 1}$ such that $I_n = (a_n, b_n]$ and $0 < b_i < a_{i+1}$. Let us explain conditions (i)–(iii). Condition (i) gives an α-regular set such

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that the gap between consecutive elements is at least (of the order of) μ_i. Condition (ii) implies that E, F are incompatible, and also that the left endpoint of I_i is much larger than the right endpoint of I_{i-1}. Condition (iii) implies that E, F are α-sets.

Lemma 4.5. *Let $E, F \subset \mathbb{Z}$ with $D(E), D(F) < 1$, $A, B \subset \mathbb{Z}$ finite and $E' = E \cup A$, $F' = F \cup B$. Then $d^*(E' + \lfloor \lambda F' \rfloor) = d^*(E + \lfloor \lambda F \rfloor)$ for all $\lambda \in \mathbb{R}$.*

Proof. $E' + \lfloor \lambda F' \rfloor = (E + \lfloor \lambda F \rfloor) \cup (A + \lfloor \lambda F \rfloor) \cup (E + \lfloor \lambda B \rfloor) \cup (A + \lfloor \lambda B \rfloor)$, and each of the sets $A + \lfloor \lambda F \rfloor$, $E + \lfloor \lambda B \rfloor$, $A + \lfloor \lambda B \rfloor$ has dimension smaller than one. □

If we fix $\lambda > 0$, then for i, j large enough we have:
(iv) $\lfloor \lambda I_i \rfloor \cap \lfloor \lambda I_j \rfloor = \emptyset$,
(v) $b_i > \max\{4\lambda^{-1}, 4\lambda\}^{\frac{1-\alpha}{1+\alpha}}$.
By Lemma 4.5, we can delete the intervals I_i for small i without changing $d^*(E + \lfloor \lambda F \rfloor)$. Thus we can assume (iv) and (v) for all i, j. Under these assumptions,

$$(I_i + \lfloor \lambda I_j \rfloor) \cap (I_k + \lfloor \lambda I_l \rfloor) \neq \emptyset \iff i = k \text{ and } j, l < i \quad \text{or} \quad j = l \text{ and } i, k < j.$$

This follows from (ii): if $(I_i + \lfloor \lambda I_j \rfloor) \cap (I_k + \lfloor \lambda I_l \rfloor) \neq \emptyset$, then $\max\{i, j\} = \max\{k, l\}$. Here is another consequence of (ii): if $a, a' \in I_i$ with $a \neq a'$ and $j, j' < i$, then $(a + \lfloor \lambda I_j \rfloor) \cap (a' + \lfloor \lambda I_{j'} \rfloor) = \emptyset$. Indeed, $|a - a'| \geqslant \mu_i \gg \lambda b_{i-1}$. Thus

$$(E + \lfloor \lambda F \rfloor) \cap (a_i, a_{i+1}] = \begin{cases} \displaystyle\bigsqcup_{\substack{a \in I_i \\ j = 2, 4, \ldots, i-1}} (a + \lfloor \lambda I_j \rfloor) & \text{if } i \text{ is odd}, \\ \displaystyle\bigsqcup_{\substack{b \in I_i \\ j = 1, 3, \ldots, i-1}} (I_j + \lfloor \lambda b \rfloor) & \text{if } i \text{ is even}. \end{cases}$$

Here is a consequence of (v): if i is odd, $a, a' \in I_i$ with $a \neq a'$ and $j, k < i$ are even, then the gap between $a + \lfloor \lambda I_j \rfloor$ and $a' + \lfloor \lambda I_k \rfloor$ has length at least $\frac{|a - a'|}{2}$; if i is even, $b, b' \in I_i$ with $b \neq b'$ and $j, k < i$ are odd, then the gap between $I_j + \lfloor \lambda b \rfloor$ and $I_k + \lfloor \lambda b' \rfloor$ has length at least $\frac{|b - b'|}{4\lambda}$.

Now we prove that $d^*(E + \lfloor \lambda F \rfloor) = 0$. Let $I = (M, N] \subset \mathbb{Z}$ with $M + 1, N \in E + \lfloor \lambda F \rfloor$, say

$$\begin{cases} M + 1 = a' + \lfloor \lambda b' \rfloor \in I_k + \lfloor \lambda I_l \rfloor, \\ N = a + \lfloor \lambda b \rfloor \in I_i + \lfloor \lambda I_j \rfloor. \end{cases}$$

We have $\max\{i, j, k, l\} = i$ or j. Without loss of generality,[4] assume that $i > j$. We have two cases.

- $a = a'$. Here $r + \lfloor \lambda s \rfloor \in (E + \lfloor \lambda F \rfloor) \cap I$ if and only if $r = a$ and $s \in F \cap [b', b]$, so

$$\frac{|(E + \lfloor \lambda F \rfloor) \cap I|}{|I|} \leqslant \frac{|b - b' + 1|^\alpha}{|\lfloor \lambda b \rfloor - \lfloor \lambda b' \rfloor|} \sim \frac{|b - b'|^\alpha}{\lambda |b - b'|} = \frac{1}{\lambda |b - b'|^{1-\alpha}}. \tag{4.3}$$

[4] The reverse case is symmetric, because $E + \lfloor \lambda F \rfloor$ is basically $\lfloor \lambda (F + \lfloor \lambda^{-1} E \rfloor) \rfloor$ and, with this interpretation, the roles of I_i and I_j are interchanged.

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- $a > a'$. If $r + \lfloor \lambda s \rfloor \in (E + \lfloor \lambda F \rfloor) \cap I$, then $r \in E \cap [a', a]$ and $s \in I_2 \cup I_4 \cup \cdots \cup I_{i-1}$, so

$$\frac{|(E + \lfloor \lambda F \rfloor) \cap I|}{|I|} \leqslant \frac{b_{i-1} |E \cap [a', a]|}{\frac{|a-a'|}{2}} \sim \frac{b_{i-1} |a - a'|^{\alpha}}{|a - a'|} \leqslant \frac{b_{i-1}}{\mu_i{}^{1-\alpha}} \leqslant \frac{1}{b_{i-1}}, \tag{4.4}$$

where in the last inequality we used (ii).

By (4.3) and (4.4), it follows that $d^*(E + \lfloor \lambda F \rfloor) = 0$.

4.4. Another counterexample to Theorem 1.2

We now prove that the set of parameters in Theorem 1.2 cannot be $\mathbb{Z}$: we construct $E \subset \mathbb{Z}$ regular such that $D(E) = D(E + \lambda E)$ for all $\lambda \in \mathbb{Z}$.

Given $\alpha \in \left(\frac{1}{2}, 1\right)$ and $c \in \mathbb{Z}$, let $E(\alpha, c)$ be the generalized IP-set associated to the sequences $k_n = c2^n$ and $d_n = \lfloor 2^{n^2/2\alpha} \rfloor$. If n is large enough, then

$$\sum_{i=1}^{n-1} k_i d_i \leqslant c \sum_{i=1}^{n-1} 2^{\frac{i^2}{2\alpha} + i} \leqslant c \sum_{j=1}^{\frac{(n-1)^2}{2\alpha} + n - 1} 2^j < c 2^{\frac{(n-1)^2}{2\alpha} + n} < d_n.$$

By Lemma 3.6, $E(\alpha, c)$ has counting dimension α. Let $E = E(\alpha, 1)$. Thus

$$E + \lambda E = E(\alpha, \lambda + 1)$$

has counting dimension α. If E is regular, we are done. If not, we apply Proposition 4.2 to get a regular subset of E yet with counting dimension greater than $\frac{1}{2}$.

5. Proofs

Fix $E, F \subset \mathbb{Z}$ regular and compatible. Throughout the proof, we fix a compact interval $\Lambda \subset (0, +\infty)$. Given distinct points $z = (a, b)$ and $z' = (a', b')$ of $E \times F$, let

$$\Lambda_{z,z'} = \{\lambda \in \Lambda : a + \lfloor \lambda b \rfloor = a' + \lfloor \lambda b' \rfloor\}.$$

Clearly, $\Lambda_{z,z'}$ is empty if $b = b'$.

Lemma 5.1. *If $z = (a, b), z' = (a', b')$ are distinct points of $\mathbb{Z}^2$ and $\Lambda_{z,z'} \neq \emptyset$, then:*
(a) $m(\Lambda_{z,z'}) \lesssim |b - b'|^{-1}$,
(b) $|b - b'| \min \Lambda - 1 \leqslant |a - a'| \leqslant |b - b'| \max \Lambda + 1$.

Proof. Assume that $b > b'$, and let $\lambda \in \Lambda_{z,z'}$, say $a + \lfloor \lambda b \rfloor = n = a' + \lfloor \lambda b' \rfloor$. Thus

$$\begin{cases} n - a \leqslant \lambda b < n - a + 1 \\ n - a' \leqslant \lambda b' < n - a' + 1 \end{cases} \implies a' - a - 1 < \lambda(b - b') < a' - a + 1,$$

so

$$\Lambda_{z,z'} \subset \left(\frac{a' - a - 1}{b - b'}, \frac{a' - a + 1}{b - b'} \right),$$

which proves (a). Further,

$$\frac{a' - a - 1}{b - b'} \leqslant \min \Lambda_{z,z'} \leqslant \max \Lambda \implies a' - a \leqslant (b - b') \max \Lambda + 1$$

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and

$$\frac{a'-a+1}{b-b'} \geqslant \max \Lambda_{z,z'} \geqslant \min \Lambda \implies a'-a \geqslant \min(b-b')\Lambda - 1. \qquad \square$$

Lemma 5.1 expresses the crucial property of transversality that is present in most results related to Marstrand's theorem. By (b), if $\Lambda_{z,z'} \neq \emptyset$ then $|a-a'| \sim |b-b'|$.

Let $(I_n)_{n\geqslant 1}$ and $(J_n)_{n\geqslant 1}$ be sequences of intervals satisfying Definition 4.3. For each pair $(n,\lambda) \in \mathbb{N} \times \Lambda$, let

$$N_n(\lambda) = \{((a,b),(a',b')) \in ((E \cap I_n) \times (F \cap J_n))^2 : a + \lfloor \lambda b \rfloor = a' + \lfloor \lambda b' \rfloor\},$$

and let $\Delta_n = \int_\Lambda |N_n(\lambda)| dm(\lambda)$. By a double-counting argument,

$$\Delta_n = \sum_{z,z' \in (E\cap I_n)\times(F\cap J_n)} m(\Lambda_{z,z'}). \tag{5.1}$$

Lemma 5.2. *Denote $D(E) = \alpha$ and $D(F) = \beta$.*

(a) *If $\alpha + \beta < 1$, then $\Delta_n \lesssim |I_n|^{\alpha+\beta}$.*

(b) *If $\alpha + \beta > 1$, then $\Delta_n \lesssim |I_n|^{2\alpha+2\beta-1}$.*

Proof. By (5.1),

$$\begin{aligned}
\Delta_n &= \sum_{z,z' \in (E\cap I_n)\times(F\cap J_n)} m(\Lambda_{z,z'}) \\
&= \sum_{\substack{a\in E\cap I_n \\ b \in F\cap J_n}} \sum_{s=1}^{\log|I_n|} \sum_{\substack{a'\in E\cap I_n \\ |a-a'|\sim e^s}} \sum_{\substack{b'\in F\cap J_n \\ |b-b'|\sim e^s}} m(\Lambda_{z,z'}) \\
&\lesssim \sum_{\substack{a\in E\cap I_n \\ b \in F\cap J_n}} \sum_{s=1}^{\log|I_n|} e^{-s}(e^s)^\alpha (e^s)^\beta \\
&= \sum_{\substack{a\in E\cap I_n \\ b \in F\cap J_n}} \sum_{s=1}^{\log|I_n|} (e^s)^{\alpha+\beta-1} \\
&\lesssim |I_n|^{\alpha+\beta} \sum_{s=1}^{\log|I_n|} (e^{\alpha+\beta-1})^s,
\end{aligned}$$

and thus

$$\Delta_n \lesssim \begin{cases} |I_n|^{\alpha+\beta}|I_n|^{\alpha+\beta-1} = |I_n|^{2\alpha+2\beta-1} & \text{if } \alpha+\beta > 1, \\ |I_n|^{\alpha+\beta} & \text{if } \alpha+\beta < 1. \end{cases} \qquad \square$$

5.1. Proof of Theorem 1.2

We divide the proof into three parts.

(1) $\alpha + \beta < 1$. Fix $\varepsilon > 0$ and $n \geqslant 1$. By Lemma 5.2, the set of parameters $\lambda \in \Lambda$ such that

$$|N_n(\lambda)| \lesssim \frac{|I_n|^{\alpha+\beta}}{\varepsilon} \tag{5.2}$$

has Lebesgue measure at least $m(\Lambda) - \varepsilon$. We will prove that

$$\frac{|(E + \lfloor \lambda F \rfloor) \cap (I_n + \lfloor \lambda J_n \rfloor)|}{|I_n + \lfloor \lambda J_n \rfloor|^{\alpha+\beta}} \gtrsim \varepsilon \tag{5.3}$$

for every $\lambda \in \Lambda$ satisfying (5.2). For $(m, n, \lambda) \in \mathbb{Z} \times \mathbb{Z} \times \Lambda$, let

$$s(m, n, \lambda) = |\{(a, b) \in (E \cap I_n) \times (F \cap J_n) : a + \lfloor \lambda b \rfloor = m\}|.$$

Thus

$$\sum_{m \in \mathbb{Z}} s(m, n, \lambda) = |E \cap I_n||F \cap J_n| \sim |I_n|^{\alpha+\beta} \tag{5.4}$$

and

$$\sum_{m \in \mathbb{Z}} s(m, n, \lambda)^2 = |N_n(\lambda)| \lesssim \frac{|I_n|^{\alpha+\beta}}{\varepsilon}. \tag{5.5}$$

The numerator in (5.3) is at least the cardinality of the set $S(n, \lambda) = \{m \in \mathbb{Z} : s(m, n, \lambda) > 0\}$, because $(E + \lfloor \lambda F \rfloor) \cap (I_n + \lfloor \lambda J_n \rfloor)$ contains $S(n, \lambda)$. By the Cauchy–Schwarz inequality and (5.4), (5.5), we have

$$|S(n, \lambda)| \geqslant \frac{\left(\sum_{m \in \mathbb{Z}} s(m, n, \lambda)\right)^2}{\sum_{m \in \mathbb{Z}} s(m, n, \lambda)^2} \gtrsim \frac{(|I_n|^{\alpha+\beta})^2}{\frac{|I_n|^{\alpha+\beta}}{\varepsilon}} = \varepsilon |I_n|^{\alpha+\beta}.$$

Because $|I_n + \lfloor \lambda J_n \rfloor| \sim |I_n|$, we get that

$$\frac{|(E + \lfloor \lambda F \rfloor) \cap (I_n + \lfloor \lambda J_n \rfloor)|}{|I_n + \lfloor \lambda J_n \rfloor|^{\alpha+\beta}} \gtrsim \frac{|S(n, \lambda)|}{|I_n|^{\alpha+\beta}} \gtrsim \varepsilon,$$

establishing (5.3).

For each $n \geqslant 1$, let $G_\varepsilon^n = \{\lambda \in \Lambda : (5.3) \text{ holds}\}$. Then $m\left(\Lambda \backslash G_\varepsilon^n\right) \leqslant \varepsilon$, and the same holds for

$$G_\varepsilon = \bigcap_{n \geqslant 1} \bigcup_{l=n}^{\infty} G_\varepsilon^l.$$

For each $\lambda \in G_\varepsilon$, $H_{\alpha+\beta}(E + \lfloor \lambda F \rfloor) > 0$, thus $D(E + \lfloor \lambda F \rfloor) \geqslant \alpha + \beta$. Because

$$G = \bigcup_{n \geqslant 1} G_{1/n} \subset \Lambda$$

has Lebesgue measure $m(\Lambda)$, part (1) is complete.

(2) $\alpha + \beta > 1$. For a fixed $\varepsilon > 0$, Lemma 5.2 implies that the set of parameters $\lambda \in \Lambda$ such that

$$|N_n(\lambda)| \lesssim \frac{|I_n|^{2\alpha+2\beta-1}}{\varepsilon}$$

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has Lebesgue measure at least $m(\Lambda)-\varepsilon$. In this case,

$$|S(n,\lambda)| \geqslant \frac{\left(\sum_{m\in\mathbb{Z}} s(m,n,\lambda)\right)^2}{\sum_{m\in\mathbb{Z}} s(m,n,\lambda)^2} \gtrsim \frac{(|I_n|^{\alpha+\beta})^2}{\frac{|I_n|^{2\alpha+2\beta-1}}{\varepsilon}} = \varepsilon|I_n|,$$

and thus

$$\frac{|(E+\lfloor\lambda F\rfloor)\cap(I_n+\lfloor\lambda J_n\rfloor)|}{|I_n+\lfloor\lambda J_n\rfloor|} \gtrsim \frac{|S(n,\lambda)|}{|I_n|} \gtrsim \varepsilon.$$

The Borel–Cantelli argument is analogous to part (1).

(3) $\alpha+\beta=1$. Let $n\geqslant 1$. Because E is regular, there exists $E_n\subset E$, regular and compatible with F, such that $D(E)-\frac{1}{n}<D(E_n)<D(E)$. Thus $1-\frac{1}{n}<D(E_n)+D(F)<1$. By part (1), there is a set Λ_n of full Lebesgue measure such that $D(E_n+\lfloor\lambda F\rfloor)\geqslant 1-\frac{1}{n}$ for all $\lambda\in\Lambda_n$. The set $\Lambda=\bigcap_{n\geqslant 1}\Lambda_n$ has full Lebesgue measure and $D(E+\lfloor\lambda F\rfloor)\geqslant 1$ for all $\lambda\in\Lambda$.

5.2. Proof of Theorem 1.3

We divide the proof into two parts.

(1) $\sum_{i=0}^{k} D(E_i)\leqslant 1$. By Theorem 1.2,

$$D(E_0+\lfloor\lambda_1 E_1\rfloor)\geqslant D(E_0)+D(E_1),\quad m\text{-a.e. } \lambda_1\in\mathbb{R}.$$

To each of these parameters, apply Proposition 4.2 to obtain a regular subset $F_{\lambda_1}\subset E_0+\lfloor\lambda_1 E_1\rfloor$ such that $D(F_{\lambda_1})=D(E_0)+D(E_1)$. Because E_2 is universal, we can apply Theorem 1.2 again and get that

$$D(F_{\lambda_1}+\lfloor\lambda_2 E_2\rfloor)\geqslant D(E_0)+D(E_1)+D(E_2),\quad m\text{-a.e. } \lambda_2\in\mathbb{R}.$$

By Fubini's theorem,

$$D(E_0+\lfloor\lambda_1 E_1\rfloor+\lfloor\lambda_2 E_2\rfloor)\geqslant D(E_0)+D(E_1)+D(E_2),\quad m_2\text{-a.e. } (\lambda_1,\lambda_2)\in\mathbb{R}^2.$$

Iterating the above arguments, it follows that

$$D(E_0+\lfloor\lambda_1 E_1\rfloor+\cdots+\lfloor\lambda_k E_k\rfloor)\geqslant D(E_0)+\cdots+D(E_k),\quad m_k\text{-a.e. } (\lambda_1,\ldots,\lambda_k)\in\mathbb{R}^k.$$

(2) $\sum_{i=0}^{k} D(E_i)>1$. Without loss of generality, we can assume that

$$D(E_0)+\cdots+D(E_{k-1})\leqslant 1<D(E_0)+\cdots+D(E_{k-1})+D(E_k).$$

By part (1),

$$D(E_0+\lfloor\lambda_1 E_1\rfloor+\cdots+\lfloor\lambda_{k-1}E_{k-1}\rfloor)\geqslant D(E_0)+\cdots+D(E_{k-1})$$

for m_{k-1}-a.e. $(\lambda_1,\ldots,\lambda_{k-1})\in\mathbb{R}^{k-1}$. To each of these $(k-1)$-tuples, let

$$F_{(\lambda_1,\ldots,\lambda_{k-1})}\subset E_0+\cdots+\lfloor\lambda_{k-1}E_{k-1}\rfloor$$

regular with

$$D\big(F_{(\lambda_1,\ldots,\lambda_{k-1})}\big)=D(E_0)+\cdots+D(E_{k-1}).$$

https://doi.org/10.1017/S0963548313000461 Published online by Cambridge University Press

Because $D(F_{(\lambda_1,\dots,\lambda_{k-1})}) + D(E_k) > 1$, Theorem 1.2 gives

$$d^*\left(F_{(\lambda_1,\dots,\lambda_{k-1})} + \lfloor \lambda_k E_k \rfloor\right) > 0, \quad \text{for } m\text{-a.e. } \lambda_k \in \mathbb{R}.$$

By Fubini's theorem, the proof is complete.

6. Concluding remarks

We think there is a more specific way of defining the counting dimension that encodes the conditions of regularity and compatibility. A natural candidate would be a prototype of a Hausdorff dimension, where one looks to all covers, properly renormalized in the unit interval, and takes a lim inf. An alternative definition appeared in [12]. It would be natural to prove Marstrand-type results in this context.

Another interesting question is to consider arithmetic sums $E + \lambda F$, where $\lambda \in \mathbb{Z}$. These are genuine arithmetic sums and, as we saw in Section 4.4, their dimension may not increase. We think very strong conditions on the sets E, F are needed to prove analogous results about $E + \lambda F$ for $\lambda \in \mathbb{Z}$.

We also think the results obtained here work for subsets of $\mathbb{Z}^k$. Given $E \subset \mathbb{Z}^k$, the *upper Banach density* of E is equal to

$$d^*(E) = \limsup_{|I_1|,\dots,|I_k| \to \infty} \frac{|E \cap (I_1 \times \cdots \times I_k)|}{|I_1 \times \cdots \times I_k|},$$

where $I_1, \dots, I_k$ run over all intervals of $\mathbb{Z}$; the *counting dimension* of E is

$$D(E) = \limsup_{|I_1|,\dots,|I_k| \to \infty} \frac{\log |E \cap (I_1 \times \cdots \times I_k)|}{\log |I_1 \times \cdots \times I_k|},$$

where $I_1, \dots, I_k$ run over all intervals of $\mathbb{Z}$; and, for $\alpha \geqslant 0$, the *counting α-measure* of E is

$$H_\alpha(E) = \limsup_{|I_1|,\dots,|I_k| \to \infty} \frac{|E \cap (I_1 \times \cdots \times I_k)|}{|I_1 \times \cdots \times I_k|^\alpha},$$

where $I_1, \dots, I_k$ run over all intervals of $\mathbb{Z}$. These quantities satisfy similar properties to those in Section 2.2. The notion of regularity is defined in an analogous manner. For compatibility, we take into account the geometry of $\mathbb{Z}^k$. Two regular subsets $E, F \subset \mathbb{Z}^k$ are compatible if there exist sequences of boxes $R_n = I_1^n \times \cdots \times I_k^n$ and $S_n = J_1^n \times \cdots \times J_k^n$ such that

(i) $|I_i^n| \sim |J_i^n|$ for every $i = 1, 2, \dots, k$, and
(ii) $|E \cap R_n| \gtrsim |R_n|^{D(E)}$ and $|F \cap S_n| \gtrsim |S_n|^{D(F)}$.

We think the theory developed in this article can be extended to prove that if $E, F \subset \mathbb{Z}^k$ are two regular compatible subsets, then

$$D(E + \lfloor \lambda F \rfloor) \geqslant \min\{1, D(E) + D(F)\}$$

for Lebesgue-a.e. $\lambda \in \mathbb{R}$. If $D(E) + D(F) > 1$, then $E + \lfloor \lambda F \rfloor$ has positive upper Banach density for Lebesgue-a.e. $\lambda \in \mathbb{R}$.

Acknowledgements

The authors are grateful to IMPA for the excellent atmosphere during the preparation of this manuscript, to Simon Griffiths and Rob Morris for the conversations that gave rise to the counterexample of Section 4.4, and to the referee for many useful and detailed suggestions. Y.L. is also grateful to Enrique Pujals for his constant encouragement. During the preparation of this manuscript, Y.L. was a doctoral student at IMPA. This research was possible due to the support of the J. Palis 2010 Balzan Prize for Mathematics, CNPq-Brazil and Faperj-Brazil. Y.L. is supported by the Brin Fellowship.

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