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ON LARGE-DEVIATION PROBABILITIES FOR THE EMPIRICAL DISTRIBUTION OF BRANCHING RANDOM WALKS WITH HEAVY TAILS

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Abstract

Given a branching random walk $(Z_n)_{n\geq 0}$ on $\mathbb{R}$, let $Z_n(A)$ be the number of particles located in interval A at generation n. It is well known that under some mild conditions, $Z_n(\sqrt{n}A)/Z_n(\mathbb{R})$ converges almost surely to $\nu(A)$ as $n\to\infty$, where ν is the standard Gaussian measure. We investigate its large-deviation probabilities under the condition that the step size or offspring law has a heavy tail, i.e. a decay rate of $\mathbb{P}(Z_n(\sqrt{n}A)/Z_n(\mathbb{R})>p)$ as $n\to\infty$, where $p\in(\nu(A),1)$. Our results complete those in Chen and He (2019) and Louidor and Perkins (2015).

Keywords: Step size; offspring law; Schröder constant; Cramér's theorem

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1. Introduction and main results

1.1. Introduction

We consider a branching random walk (BRW) model, which is governed by a probability distribution $\{p_k\}_{k\geq 0}$ on the natural numbers (called the offspring distribution) and a real-valued random variable X (called the step size or displacement). This model is defined as follows. At time 0, there is one particle located at the origin. The particle dies and produces offspring according to the offspring distribution $\{p_k\}_{k\geq 0}$. Afterwards, the offspring particles move independently according to the law of X. This forms a process Z_1. For any point process Z_n, $n\geq 2$, we define it by the iteration $Z_n=\sum_{x\in Z_{n-1}}\tilde{Z}_1^x$, where $\tilde{Z}_1^x$ has the same distribution as $Z_1(\cdot - S_x)$, and $\{\tilde{Z}_1^x : x\in Z_{n-1}\}$ (conditioned on Z_{n-1}) are independent. Here and later, for a point process (also for a point measure) ξ, $x\in\xi$ means x is an atom of ξ, and S_x is the position of x (i.e. $\xi=\sum_{x\in\xi}\delta_{S_x}$).

We are interested here in the large-deviation probabilities of the corresponding empirical distribution, which is defined as

$$\bar{Z}_n(A):=\frac{Z_n(A)}{Z_n(\mathbb{R})}\qquad \text{for a measurable set } A\subset\mathbb{R}.$$

According to [7, Theorem 6]: if $\mathbb{E}[X]=0$, $\mathbb{E}\big[X^2\big]=1$, $\mathbb{E}[Z_1(\mathbb{R})\log Z_1(\mathbb{R})]<\infty$, and $\mathbb{E}[Z_1(\mathbb{R})]>1$, then, for any Borel-measurable set $A\subset\mathbb{R}$, $\bar{Z}_n(\sqrt{n}A)\to\nu(A)$ $\mathbb{P}$-a.s. (almost

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surely) on non-extinction, where ν is the standard Gaussian measure. So, it is natural to study the decay rate of $\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p)$ as $n \to \infty$, where $p \in (\nu(A), 1)$.

In fact, this question was considered in [32] under the assumption that $p_0 = p_1 = 0$ and $\mathbb{P}(X = 1) = \mathbb{P}(X = -1)$. Later, [13] further investigated the problem for unbounded displacements; see also [33]. However, they always assumed that the offspring law and step size have an exponential moment (or that the step size has a stretched exponential moment). So, we shall deal here with the case that the offspring law or step size has a heavy tail (i.e. $\mathbb{E}[e^{\theta Z_1(\mathbb{R})}] = +\infty$ or $\mathbb{E}[e^{\theta X}] = +\infty$ for any $\theta > 0$). We will see that the strategy for studying this problem and the answers obtained will be very different from theirs.

We also mention here that the BRW model has been extensively studied in recent decades due to its connection to many fields, such as Gaussian multiplicative chaos, random walks in random environments, random polymers, random algorithms, discrete Gaussian free field, etc; see [1, 9, 27, 30, 31] and references therein, and refer to [38] for a more detailed overview. The large-deviation probabilities (LDP) for BRW and branching Brownian motion (BBM) on the real line have attracted the attention of many researchers. For example, [14, 22, 26] considered the LDP and the moderate deviation of BRW's maximum (for BBM's maximum, see [11, 17–19]); [36] studied the lower deviation of BBM's local mass. See also [1] for the upper deviation of BBM's level sets. Some other related works include [6, 10, 37].

The a.s. behaviour of $\bar{Z}_n(\sqrt{n}A)$ has been considered by many researchers, e.g. [2, 7, 25, 28]. Moreover, that the a.s. convergence rate of $\bar{Z}_n(\sqrt{n}A) - \nu(A)$ tends to zero has also been well studied recently: [12] considered the branching Wiener process; [23] generalized Chen's results to the BRW, but a kind of Cramér's condition is needed for the step size; [24] studied the case when the step size of the BRW is lattice. These results show that $\sqrt{n}(\bar{Z}_n(\sqrt{n}A) - \nu(A))$ converges almost surely to a non-degenerate limit.

1.2. Main results

Before giving our results, we first introduce some notation. Let $\mathcal{A}$ be the algebra generated by $\{(-\infty, x], x \in \mathbb{R}\}$. For a non-empty set $A \in \mathcal{A}$ and $p \in (\nu(A), 1)$, define $I_A(p) = \inf\{|x| : \nu(A - x) \geq p\}$, $J_A(p) = \inf\left\{r : \sup_{x\in\mathbb{R}} \nu\left((A - x)/\sqrt{1-r}\right) \geq p,\ r \in [0, 1)\right\}$. Let $|Z_n| := Z_n(\mathbb{R})$, $m := \mathbb{E}[|Z_1|]$, and $b := \min\{k \geq 0 : p_k > 0\} \leq B := \sup\{k \geq 0 : p_k > 0\} \leq +\infty$. Recall that $\{p_k\}_{k\geq 0}$ is the offspring law, and X is the step size. In the remainder of this work we always need the following assumptions.

Assumption 1.

(i) $p_0 = 0$, $p_1 < 1$.

(ii) *X is symmetric and* $\mathbb{E}\left[X^2\right] = 1$.

(iii) *A is a no-empty set in* $\mathcal{A}$.

Remark 1. In Assumption 1(i), $p_0 = 0$ is made for convenience. If not, we can condition on non-extinction to obtain analogous results. Assumption 1(ii) is not essential either, but simplifies the proof. Assumption 1(iii) is crucial to our main results, and if $A \notin \mathcal{A}$, then the situation would be very different (see [32, Proposition 1.3]).

Now we are ready to state our main results. The first theorem concerns the case that the offspring law has a Pareto tail. Let $\Lambda(a) := \sup_{t\in\mathbb{R}}\{at - \log \mathbb{E}[e^{tX}]\}$ be the rate function in Cramér's theorem; see [15, Section 2.2].

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Theorem 1. *Take* $p \in (\nu(A), 1)$ *such that* $I_A(\cdot)$ *is continuous at* p *and* $I_A(p) < \infty$. *Suppose* $\mathbb{P}(|Z_1| > x) = \Theta(1)x^{-\beta}$ *as* $x \to +\infty$ *for some constant* $\beta > 1$. *Assume* $\mathbb{E}\left[e^{\theta X}\right] < \infty$ *for some* $\theta > 0$.

(i) *If* $0 < \beta - 1 < -\frac{\log p_1}{\log m}$, *then*

$$-\inf_{h>0} \Phi_1(h) \le \liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}\big(\bar{Z}_n(\sqrt{n}A) \ge p\big)$$
$$\le \limsup_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}\big(\bar{Z}_n(\sqrt{n}A) \ge p\big) \le -\sup_{h>0} \Phi_1'(h),$$

where $\Phi_1(h) := h(\beta - 1)\log m + \Lambda\big(\frac{I_A(p)}{h}\big)h$, $\Phi_1'(h) := (h(\beta - 1)\log m) \wedge \big(\Lambda\big(\frac{I_A(p)}{h}\big)h\big)$, *and we make the convention that* $-\log p_1 = +\infty$ *if* $p_1 = 0$.

(ii) *If* $\beta - 1 \ge -\frac{\log p_1}{\log m}$, *then*

$$-\inf_{h>0} \Phi_2(h) \le \liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \ge p)$$
$$\le \limsup_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \ge p) \le -\sup_{h>0} \Phi_2'(h),$$

where $\Phi_2(h) := -h\log p_1 + \Lambda\big(\frac{I_A(p)}{h}\big)h$, $\Phi_2'(h) := (-h\log p_1) \wedge \big(\Lambda\big(\frac{I_A(p)}{h}\big)h\big)$.

Remark 2. If $p_1 > 0$, $-\log p_1/\log m$ is the so-called Schröder constant, which determines the asymptotic behaviour of the harmonic moments of $|Z_n|$ (see Lemma 2). Furthermore, we will see that the asymptotic behaviour of $\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \ge p)$ mainly depends on the harmonic moments.

Remark 3. If $\beta \in (0, 1)$, [3] showed that $\bar{Z}_n(\sqrt{n}(-\infty, y])$ converges in distribution to a Bernoulli random variable.

The next theorem considers the case that the offspring law has a Weibull tail. As we can see in the following, the decay scales are the same as when that offspring law has an exponential moment. However, the Böttcher constant appears in the rate function.

Theorem 2. *Take* $p \in (\nu(A), 1)$ *such that* $I_A(\cdot)$ *is continuous at* p *and* $I_A(p) < \infty$. *Suppose* $\mathbb{P}(|Z_1| > x) \sim l_1 e^{-lx^\beta}$ *as* $x \to \infty$ *for some constants* $\beta \in (0, 1)$ *and* $l_1, l \in (0, +\infty)$. *Assume* $p_1 = 0$.

(i) *If* $\operatorname{ess\,sup} X = L \in (0, +\infty)$, *then*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \log\big[-\log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \ge p)\big] = \frac{I_A(p)}{L} \frac{\rho\beta}{\beta + \rho - \beta\rho} \log m, \quad (1)$$

where ρ *is the so-called Böttcher constant such that* $b = m^\rho$.

(ii) *If* $\mathbb{P}(X > x) = \Theta(1)e^{-\lambda x^\alpha}$ *as* $x \to \infty$ *for some constants* $\alpha \in (0, \infty)$ *and* $\lambda > 0$, *then*

$$\lim_{n\to\infty} \frac{(\log n)^{(\alpha-1)\vee 0}}{n^{\alpha/2}} \log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \ge p) = -\lambda I_A(p)^\alpha \left(\frac{2\beta\rho\log m}{\alpha(\beta + \rho - \beta\rho)}\right)^{(\alpha-1)\vee 0}. \quad (2)$$

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Remark 4. For the Schröder case (i.e. $p_1 > 0$), we can check that, combining the proof of [13, Theorem 1.1] and Lemma 5, we can generalize [13, Theorem 1.1] to the case that $|Z_1|$ has a Weibull tail.

As we can see in the above, when the step size has an exponential moment, the decay rate is very sensitive to the tail of the offspring law. However, in the next theorem we will see that when the step size has a Pareto tail, the offspring law seems to have little effect on the decay rate.

Theorem 3. *Take $p \in (\nu(A), 1)$ such that $I_A(\cdot)$ is continuous at p and $I_A(p) < \infty$. Suppose $\mathbb{E}\big[|Z_1|^\beta\big] < \infty$ for some $\beta > 1$ and $b < B$. Assume $\mathbb{P}(X > x) \sim \kappa x^{-\alpha}$ as $x \to \infty$ for some constants $\kappa > 0$ and $\alpha > 2$. Then*

$$\lim_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) = -\frac{\alpha}{2}.$$

Remark 5. If $b = B$, the above result is still true provided A is unbounded and $p \in \big(\nu(A), \nu(A) + b^{-1}(1 - \nu(A))\big)$.

The above theorems all assume that $I_A(p) < \infty$, and we have seen that in this case the law of step size plays an important role in the decay rate. However, in the following we will see that if $I_A(p) = \infty$, the decay rate mainly depends on the offspring law.

Theorem 4. *Suppose $I_A(p) = \infty$ and $J_A(\cdot)$ is continuous at p for $p \in (\nu(A), 1)$.*

(i) If $\mathbb{P}(|Z_1| > x) = \Theta(1)x^{-\beta}$ as $x \to +\infty$ for some constant $\beta > 1$, then

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) = \begin{cases} -J_A(p)(\beta - 1)\log m, & 0 < \beta - 1 < -\frac{\log p_1}{\log m}; \\ J_A(p)\log p_1, & \beta - 1 \geq -\frac{\log p_1}{\log m}. \end{cases}$$

(ii) Assume $\mathbb{P}(|Z_1| > x) \sim l_1 \mathrm{e}^{-lx^\beta}$ as $x \to \infty$ for some constants $\beta \in (0, 1)$ and $l_1, l \in (0, +\infty)$.

(a) *If $p_1 > 0$, then* $\lim_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) = J_A(p)\log p_1$.

(b) *If $p_1 = 0$, then*

$$\lim_{n\to\infty} \frac{1}{n} \log\big[-\log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p)\big] = J_A(p)\frac{\rho\beta}{\beta + \rho - \beta\rho}\log m.$$

Remark 6. In fact, Theorems 3 and 4 can also be generalized to the case that the step size X is in the domain of attraction of an α-stable law with $\alpha \in (0, 2]$. The results and proofs are similar; the main changes are to replace ν with the α-stable law and replace $\sqrt{n}$ with $\inf\{x : \mathbb{P}(|X| > x) < n^{-1}\}$ (which is a regular variation sequence with index $1/\alpha$).

Although, the proofs of the above four theorems differ from case to case, the basic strategy behind each of them is the same. If $I_A(p) < \infty$, we let x be the number realizing the infimum in the definition of $I_A(p)$, otherwise we let x realize the supremum in the definition of $J_A(p)$. The lower bound aims at achieving the event $\{\bar{Z}_n(\sqrt{n}A) \geq p)\}$ in the most effortless way. For Theorems 1, 2, and 4, our strategy is to let one particle reach around $x\sqrt{n}$ at some intermediate generation t_n, and then force its children to dominate the population size at time $t_n + 1$ (since in these theorems the offspring law has a heavy tail, this can be done with relatively high probability). Finally, optimizing for t_n yields the desired lower bound. For Theorem 3 the step size has a heavy tail, so particles can reach a high position in a short time. Therefore, in order to achieve $\{\bar{Z}_n(\sqrt{n}A) \geq p)\}$, we can let one particle reach around $x\sqrt{n}$ in the first generation and other particles stay around the origin.

For the upper bound all the theorems use a similar idea, which is borrowed from [32]. Our main tasks are to generalize their [32, Lemma 2.4] in the heavy-tail case and to study the asymptotic behaviours of the harmonic moments and stretched exponential moments of $|Z_n|$.

The rest of this paper is organised as follows. In Section 2 we present some preliminary results that are frequently used in our proofs. We consider the offspring law with a Pareto tail in Section 3, and a Weibull tail in Section 4. Section 5 is devoted to studying the case that the step size has a Pareto tail. The last section considers the case that $I_A(p) = +\infty$. We always use $C, C', C_0, C_1, \ldots$ and $c_1, c_2, \ldots$ to denote positive constants. As usual, we denote by $C(\epsilon, M)$ (or $C_{\epsilon,M}$) a positive constant depending only on ϵ and M. And, by convention, $f(x) = \Theta(1)g(x)$ as $x \to +\infty$ means there exist constants $C \geq C' > 0$ such that $C' \leq |f(x)/g(x)| \leq C$ for all $x > 1$, and $f(x) \sim g(x)$ means $\lim_{x\to\infty} f(x)/g(x) = 1$.

2. Preliminaries

In the following, we mainly concentrate on the offspring law $|Z_1|$ (or step size X) having one of the two typical heavy tails: Weibull or Pareto. For convenience, we write $|Z_1| \sim \text{Pareto}(\beta)$ if $\mathbb{P}(|Z_1| > x) = \Theta(1)x^{-\beta}$ as $x \to +\infty$ for some constant $\beta > 1$, and $|Z_1| \sim \text{Weibull}(\beta)$ if $\mathbb{P}(|Z_1| > x) \sim l_1 e^{-lx^\beta}$ as $x \to \infty$ for some constants $\beta \in (0, 1)$ and $l_1, l \in (0, +\infty)$. We denote by ν_n the distribution of $S_n := \sum_{i=1}^n X_i$, where $\{X_i\}_{i\geq 1}$ are independent and identically distributed (i.i.d.) copies of the step size X. We write $W_n := |Z_n|/m^n$. From [16, Theorems 1 and 3] and [29, Theorem 1], we have the following uniform bounds for W_n.

Lemma 1.

(i) If $|Z_1| \sim \text{Pareto}(\beta)$ *for some* $\beta > 1$ *then there exist constants* $0 < c_1 < c_2 < \infty$ *such that* $c_1 x^{-\beta} \leq \mathbb{P}(W_n > x) \leq c_2 x^{-\beta}$ *for all* $x > 1$ *and* $n \geq 1$*. Hence, for* $\alpha \in [1, \beta)$*,* $\sup_{n\geq 1} \mathbb{E}[W_n^\alpha] < \infty$.

(ii) If $|Z_1| \sim \text{Weibull}(\beta)$ *for some* $\beta \in (0, 1)$ *then, for every* $\epsilon \in (0, 1)$*, there exist constants* $c_\epsilon > c'_\epsilon > 0$ *depending only on* ϵ *such that* $c'_\epsilon e^{-l((m+\epsilon)x)^\beta} \leq \mathbb{P}(W_n > x) \leq c_\epsilon e^{-l((m-\epsilon)x)^\beta}$ *for all* $x > 0$ *and* $n \geq 1$*. Hence, for* $\alpha \in (0, \beta)$ *and* $\theta > 0$*,* $\sup_{n\geq 1} \mathbb{E}\left[e^{\theta W_n^\alpha}\right] < \infty$.

(ii) If $\mathbb{E}[|Z_1|^\beta] < \infty$ *for some* $\beta > 1$ *then*

$$\mathbb{E}\Big[\sup_{n\geq 1} W_n^\beta\Big] < \infty. \tag{3}$$

The following lemma gives the asymptotic behaviour of harmonic moments; see [35, Theorem 1].

Lemma 2. *If* $\mathbb{E}[|Z_1|] < \infty$ *and* $r > 0$ *then* $\lim_{n\to\infty} \mathbb{E}\left[|Z_n|^{-r}\right] A_n(r) = C_0$*, where* $C_0 \in (0, +\infty)$ *is a constant depending only on* $\{p_k\}_{k\geq 0}$ *and r, and*

$$A_n(r) = \begin{cases} p_1^{-n}, & r > -\dfrac{\log p_1}{\log m}; \\ np_1^{-n}, & r = -\dfrac{\log p_1}{\log m}; \\ m^{nr}, & r < -\dfrac{\log p_1}{\log m}. \end{cases}$$

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The next lemma considers the asymptotic behaviour of stretched exponential moments, and we will see that it is related to the LDP of $\bar{Z}_n(\sqrt{n}A)$ when $|Z_1|$ has a Weibull tail. Recall that ρ is the Böttcher constant such that $b=m^{\rho}$.

Lemma 3. *Assume that* $\mathbb{E}[|Z_1|\log|Z_1|]<\infty$. *For any* $\beta\in(0,1)$, $l\in(0,\infty)$,

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{E}\left[\mathrm{e}^{-l|Z_n|^{\beta}}\right]=p_1, \qquad \text{if } p_1>0;$$

$$\lim_{n\to\infty}\frac{1}{n}\log\left[-\log\mathbb{E}\left[\mathrm{e}^{-l|Z_n|^{\beta}}\right]\right]=\frac{\beta\rho}{\beta+\rho-\beta\rho}\log m, \quad \text{if } p_1=0.$$

Proof. We first consider the case $p_1=0$. Let d be the greatest common divisor of $\{j-k:j\neq k,\ p_jp_k>0\}$. According the proof of [20, Theorem 6], for any fixed $\varepsilon\in(0,1)$ there exist constants $c_3\geq c_4>0$ such that, for any $b^n\leq k\leq\lfloor m^{(1-\varepsilon)n}\rfloor$, $k=b^n \pmod d$, and $n\geq 1$,

$$\exp\left[-c_3\left(\frac{k}{m^n}\right)^{-\frac{\rho}{1-\rho}}\right]\leq m^n\mathbb{P}(|Z_n|=k)\leq\exp\left[-c_4\left(\frac{k}{m^n}\right)^{-\frac{\rho}{1-\rho}}\right]. \tag{4}$$

Hence, for the upper bound, we have

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{-l|Z_n|^{\beta}}\right]&\leq\mathbb{E}\left[\mathrm{e}^{-l|Z_n|^{\beta}}\mathbf{1}_{\{|Z_n|\leq m^{(1-\varepsilon)n}\}}\right]+\mathrm{e}^{-lm^{\beta(1-\varepsilon)n}}\\&\leq m^{-n}\sum_{\substack{b^n\leq k\leq\lfloor m^{(1-\varepsilon)n}\rfloor\\ k=b^n(\text{mod } d)}}\exp(-lk^{\beta})\exp\left[-c_4\left(\frac{k}{m^n}\right)^{-\frac{\rho}{1-\rho}}\right]+\mathrm{e}^{-lm^{\beta(1-\varepsilon)n}}.\end{aligned} \tag{5}$$

Note that there exists a positive constant T depending only on l, c_4, ρ, and β such that, for n large enough,

$$\min_{b^n\leq x\leq\lfloor m^{(1-\varepsilon)n}\rfloor}\left\{lx^{\beta}+c_4\left(\frac{x}{m^n}\right)^{-\frac{\rho}{1-\rho}}\right\}\geq Tm^{\frac{\beta\rho n}{\beta+\rho-\beta\rho}}.$$

As a consequence, if $1-\varepsilon>\frac{\rho}{\beta+\rho-\beta\rho}$ then

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{-l|Z_n|^{\beta}}\right]&\leq m^{\varepsilon n}\exp\left(-Tm^{\frac{\beta\rho n}{\beta+\rho-\beta\rho}}\right)+\mathrm{e}^{-lm^{\beta(1-\varepsilon)n}}\\&\leq 2m^{\varepsilon n}\exp\left(-(T\wedge l)m^{\frac{\beta\rho n}{\beta+\rho-\beta\rho}}\right),\end{aligned}$$

which yields

$$\liminf_{n\to\infty}\frac{1}{n}\log\left[-\log\mathbb{E}\left[\mathrm{e}^{-l|Z_n|^{\beta}}\right]\right]\geq\frac{\beta\rho}{\beta+\rho-\beta\rho}\log m.$$

For the lower bound, similarly, using the left-hand side of (4) we have, for n large enough,

$$\begin{aligned}\mathbb{E}\left[\mathrm{e}^{-l|Z_n|^{\beta}}\right]&\geq m^{-n}\sum_{\substack{b^n\leq k\leq\lfloor m^{(1-\varepsilon)n}\rfloor\\ k=b^n(\text{mod } d)}}\exp(-lk^{\beta})\exp\left[-c_3\left(\frac{k}{m^n}\right)^{-\frac{\rho}{1-\rho}}\right]\\&\geq m^{-n}\exp\left(-T'm^{\frac{\beta\rho n}{\beta+\rho-\beta\rho}}\right),\end{aligned}$$

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where $T'>0$ depends on l, c_3, ρ, β, and d. Taking limits yields

$$\limsup_{n\to\infty}\frac{1}{n}\log\left[-\log\mathbb{E}\left[\mathrm{e}^{-l|Z_n|^{\beta}}\right]\right]\leq\frac{\beta\rho}{\beta+\rho-\beta\rho}\log m.$$

For $p_1>0$, by [21, Lemma 13] there exists a universal constant $c>0$ such that, for any $k,n\geq 1$, $\mathbb{P}(|Z_n|=k)\leq cp_1^n k^{-(\log p_1/\log m)-1}$. Thus, $p_1^n e^{-l}\leq\mathbb{E}\big[\mathrm{e}^{-l|Z_n|^{\beta}}\big]\leq cp_1^n\sum_{k\geq 1}k^{-(\log p_1/\log m)-1}\mathrm{e}^{-lk^{\beta}}$, which implies the desired result. □

The following two lemmas are analogous results to [32, Lemma 2.4]. In the following, we denote by $\mathcal{M}$ the collection of all local finite point measures on $\mathbb{R}$. Recall that for $\xi\in\mathcal{M}$, $x\in\xi$ means x is an atom of ξ, and S_x is the position of x (i.e. $\xi=\sum_{x\in\xi}\delta_{S_x}$). $|\xi|$ represents the total number of its atoms. Let Z_n^{ξ} be the branching random walk started from $Z_0^{\xi}=\xi$; for simplicity, we write $Z_n^x:=Z_n^{\delta_{S_x}}$. Denote by $\bar{Z}_n^{\xi}(\cdot)$ the corresponding empirical distribution of Z_n^{ξ}. Let $W_n^x:=|Z_n^x|/m^n$.

Lemma 4. *Assume $\mathbb{E}[|Z_1|^{\beta}]<\infty$ or $|Z_1|\sim\mathrm{Pareto}(\beta)$ for some $\beta>1$. Then, for every $\epsilon\in\left(0,\frac{1}{3}\right)$, there exists a constant $C_1>0$ depending on ϵ and β such that, for any $\xi\in\mathcal{M}$, $n\geq 1$, and $A\subset\mathbb{R}$,*

$$\mathbb{P}\left(\bar{Z}_n^{\xi}(A)\geq\frac{1}{|\xi|}\sum_{x\in\xi}\nu_n\left(A-S_x\right)+\epsilon\right)\leq C_1|\xi|^{1-\beta}. \tag{6}$$

The same holds if $>$ and $+\epsilon$ are replaced by $<$, $-\epsilon$.

Proof. We first consider the case $\mathbb{E}\big[|Z_1|^{\beta}\big]<\infty$ for some $\beta>1$. By the branching property, for any $n\geq 1$ and $\epsilon\in(0,\frac{1}{3})$, we have

$$\begin{aligned}
&\mathbb{P}\left(\bar{Z}_n^{\xi}(A)\geq\frac{1}{|\xi|}\sum_{x\in\xi}\nu_n\left(A-S_x\right)+\epsilon\right)\\
&\quad=\mathbb{P}\left(\frac{\frac{1}{|\xi|}\sum_{x\in\xi}Z_n^x(A)/m^n}{\frac{1}{|\xi|}\sum_{x\in\xi}Z_n^x/m^n}\geq\frac{1}{|\xi|}\sum_{x\in\xi}\nu_n(A-S_x)+\epsilon\right)\\
&\quad\leq\mathbb{P}\left(\sum_{x\in\xi}W_n^x<|\xi|(1-\epsilon/2)\right)+\mathbb{P}\left(\sum_{x\in\xi}\left(W_n^x(A)-\nu_n\left(A-S_x\right)\right)>|\xi|\epsilon/3\right)\\
&\quad\leq\mathbb{P}\left(\sum_{x\in\xi}(1-W_n^x)>\frac{\epsilon}{3}|\xi|\right)+\mathbb{P}\left(\sum_{x\in\xi}\left(W_n^x(A)-\nu_n\left(A-S_x\right)\right)>|\xi|\epsilon/3\right)\\
&\quad=:I_1+I_2,
\end{aligned}\tag{7}$$

where $\epsilon\in\left(0,\frac{1}{3}\right)$ is used in the first inequality.

We first consider I_2. By [34, Corollary 1.6], we know that if Y_x, $x\in\xi$, are independent random variables with zero mean, and $A_t^+=\sum_{x\in\xi}\mathbb{E}\big[Y_x^t\mathbf{1}_{\{Y_x\geq 0\}}\big]<\infty$ for some $1\leq t\leq 2$, then, for $y^t\geq 4A_t^+$ and $z>y$,

$$\mathbb{P}\left(\sum_{x\in\xi}Y_x\geq z\right)\leq\sum_{x\in\xi}\mathbb{P}(Y_x>y)+\left(\mathrm{e}^2A_t^+/zy^{t-1}\right)^{z/2y}. \tag{8}$$

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So, if we choose $t=1+\frac{\beta-1}{2\beta}$, $y=\frac{1}{4\beta}z$, and $z=|\xi|\epsilon/3$, then $t\in(1, 2\wedge\beta]$. Let $Y_x=W_n^x(A)-\nu_n(A-S_x)$. Note that by (3), there exists a constant $C_\beta>0$ such that

$$\sup_n \mathbb{E}\big[W_n^t\big]<C_\beta<\infty. \tag{9}$$

Therefore, $A_t^+<\infty$. Furthermore, by (9), there exists a constant $C_{\epsilon,\beta}>0$ such that, for all $|\xi|>C_{\epsilon,\beta}$ and $n\geq 1$,

$$y^t=\left(\frac{\epsilon|\xi|}{12\beta}\right)^t\geq 4|\xi|C_\beta\geq 4|\xi|\mathbb{E}[W_n^t]\geq 4\sum_{x\in\xi}\mathbb{E}[Y_x^t\mathbf{1}_{\{Y_x\geq 0\}}].$$

So, by (8), it follows that, for all $|\xi|>C_{\epsilon,\beta}$, $n\geq 1$, and $A\subset\mathbb{R}$,

$$\begin{aligned} I_2 &\leq |\xi|\mathbb{P}\left(W_n^x(A)-\nu_n(A-S_x)>\frac{\epsilon|\xi|}{12\beta}\right)+\left(\frac{\mathrm{e}^2C_\beta 3^t}{(4\beta)^{1-t}\epsilon^t}\right)^{2\beta}|\xi|^{(1-t)2\beta} \\ &\leq |\xi|\mathbb{P}\left(W_n>\frac{\epsilon|\xi|}{12\beta}\right)+\left(\frac{\mathrm{e}^2C_\beta 3^t}{(4\beta)^{1-t}\epsilon^t}\right)^{2\beta}|\xi|^{1-\beta}. \end{aligned} \tag{10}$$

By (3) and the Markov inequality, there exists a constant $C'_{\epsilon,\beta}>0$ such that, for all $n\geq 1$,

$$\mathbb{P}\left(W_n>\frac{\epsilon|\xi|}{12\beta}\right)\leq\left(\frac{12\beta}{\epsilon|\xi|}\right)^\beta\mathbb{E}\left[\sup_n W_n^\beta\right]\leq C'_{\epsilon,\beta}|\xi|^{-\beta}. \tag{11}$$

Plugging (11) into (10) yields, for all $|\xi|>C_{\epsilon,\beta}$, $n\geq 1$, and $A\subset\mathbb{R}$,

$$I_2\leq C'_{\epsilon,\beta}|\xi|^{1-\beta}+\left(\frac{\mathrm{e}^2C_\beta 3^t}{(4\beta)^{1-t}\epsilon^t}\right)^{2\beta}|\xi|^{1-\beta}.$$

For I_1, using (8) again and letting $Y_x=1-W_n^x$, we have, for all $|\xi|>\frac{12\beta}{\epsilon}$ and $n\geq 1$,

$$I_1\leq|\xi|\mathbb{P}\left(1-W_n^x>\frac{\epsilon|\xi|}{12\beta}\right)+\left(\frac{\mathrm{e}^2 3^t}{(4\beta)^{1-t}\epsilon^t}\right)^{2\beta}|\xi|^{(1-t)2\beta}=\left(\frac{\mathrm{e}^2 3^t}{(4\beta)^{1-t}\epsilon^t}\right)^{2\beta}|\xi|^{1-\beta}.$$

Plugging the above two inequalities into (7) means that there exists a constant $T(\epsilon,\beta)>0$ such that, for $|\xi|>C_{\epsilon,\beta}\vee\frac{12\beta}{\epsilon}$,

$$\mathbb{P}\left(\bar{Z}_n^\xi(A)\geq\frac{1}{|\xi|}\sum_{x\in\xi}\nu_n\,(A-S_x)+\epsilon\right)\leq T(\epsilon,\beta)|\xi|^{1-\beta}.$$

Hence, to obtain (6), we can take $C_1:=\big(C_{\epsilon,\beta}\vee\frac{12\beta}{\epsilon}\big)^{\beta-1}\vee T(\epsilon,\beta)$. For the case $\mathbb{P}(|Z_1|>x)\sim\Theta(1)x^{-\beta}$, as a consequence of Lemma 1 we only need to replace (11) by

$$\mathbb{P}\left(W_n>\frac{\epsilon|\xi|}{12\beta}\right)\leq c_2\left(\frac{\epsilon}{12\beta}\right)^{-\beta}|\xi|^{-\beta}.$$

Replacing A with A^{c}, we obtain (6) with $>$ and $+\epsilon$ replaced by $<$, $-\epsilon$. □

Remark 7. We can check that if ξ is a point process with a fixed number of atoms the result is similar, with $\mathbb{E}[\nu_n(A-S_x)]$ replacing $\nu_n(A-S_x)$.

Lemma 5. *If* $|Z_1| \sim \text{Weibull}(\beta)$ *for some* $\beta \in (0, 1)$ *then, for every* $\epsilon > 0$ *small enough, there exist positive constants* C_2 *and* C_3 *depending on* ϵ *and* β *such that, for any* $\xi \in \mathcal{M}$, $n \geq 1$, *and any* $A \subset \mathbb{R}$,

$$\mathbb{P}\left(\bar{Z}_n^{\xi}(A) \geq \frac{1}{|\xi|} \sum_{x \in \xi} \nu_n(A - S_x) + \epsilon\right) \leq C_2 \mathrm{e}^{-C_3|\xi|^{\beta}}. \tag{12}$$

The same holds if $>$ *and* $+\epsilon$ *are replaced by* $<$, $-\epsilon$.

Proof. From (7), for every $\epsilon \in \left(0, \frac{1}{3}\right)$ we have

$$\begin{aligned}
&\mathbb{P}\left(\bar{Z}_n^{\xi}(A) \geq \frac{1}{|\xi|} \sum_{x \in \xi} \nu_n(A - S_x) + \epsilon\right) \\
&\quad \leq \mathbb{P}\left(\sum_{x \in \xi} (1 - W_n^x) > \frac{\epsilon}{3}|\xi|\right) + \mathbb{P}\left(\sum_{x \in \xi} W_n^x(A) > \sum_{x \in \xi} \nu_n(A - S_x) + |\xi|\epsilon/3\right) \\
&\quad =: I_1 + I_2.
\end{aligned} \tag{13}$$

For I_1, note that $\sup_{x,n} \mathbb{E}\left[\mathrm{e}^{1-W_n^x}\right] < \infty$. Furthermore, by Lemma 1, $\mathbb{E}\left[((1 - W_n^x)^-)^2\right] \leq 3 + \sup_n \mathbb{E}\left[W_n^2\right] < \infty$. Hence, by [32, Lemma 2.3], there exists a constant $c_5 > 0$ such that, for all n, ξ, and ϵ small enough,

$$I_1 = \mathbb{P}\left(\sum_{x \in \xi} (1 - W_n^x) > \frac{\epsilon}{3}|\xi|\right) \leq \mathrm{e}^{-c_5\epsilon^2|\xi|}. \tag{14}$$

For I_2, by Lemma 1 there exists a constant $c_6 > 0$ such that, for all $y > 0$ and $n \geq 1$, $\mathbb{P}(W_n^x(A) > y) \leq \mathbb{P}(W_n > y) \leq c_6 \exp\left[-l(m-1)^{\beta} y^{\beta}\right]$. Let X_i, $i \geq 1$, be i.i.d. copies of $W_n^x(A)$, and

$$a_i(|\xi|) = \begin{cases} \left(|\xi|\epsilon/3 + \sum_{x \in \xi} \nu_n(A - S_x)\right)^{-1}, & 1 \leq i \leq |\xi|; \\ 0, & i > |\xi|. \end{cases}$$

Then, by slight modifications of the proof of the upper bound [5, Theorem 2.1], there exist positive constants T, T' depending on ϵ and β such that, for any $|\xi| > T$, $n \geq 1$, and any $A \subset \mathbb{R}$,

$$I_2 = \mathbb{P}\left(\sum_{x \in \xi} W_n^x(A) > \sum_{x \in \xi} \nu_n(A - S_x) + |\xi|\epsilon/3\right) \leq \mathrm{e}^{-T'|\xi|^{\beta}}. \tag{15}$$

Plugging (14) and (15) into (13) concludes the proof of this lemma with $C_2 := \mathrm{e}^{T'T^{\beta}}$ and $C_3 := T' \wedge (c_5\epsilon^2)$. □

The following lemma concerns LDP of sums of i.i.d. Weibull-tail random variables.

Lemma 6. *Suppose* $\{X_i\}_{i \geq 1}$ *is a sequence of i.i.d. random variables having the same distribution as* X. *Assume that* $\mathbb{P}(X > x) = \Theta(1)\mathrm{e}^{-\lambda x^{\alpha}}$ *as* $x \to \infty$ *with some* $\alpha \in (1, \infty)$ *and* $\lambda > 0$, $t_n = o(n^{1/3})$, *and* $t_n \to \infty$. *For any* $0 < a < b \leq +\infty$, *we have*

$$\lim_{n \to \infty} \frac{{t_n}^{\alpha-1}}{n^{\alpha/2}} \log \mathbb{P}\left(\sum_{i=1}^{t_n} X_i \in (a\sqrt{n}, b\sqrt{n})\right) = -\lambda a^{\alpha}.$$

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Proof. The upper bound can be found in [13, Lemma B.1]. For the lower bound, since $C\mathrm{e}^{-\lambda x^{\alpha}} \le \mathbb{P}(X > x) \le C'\mathrm{e}^{-\lambda x^{\alpha}}$ for some constants $C' > C > 0$, we have, for n large enough,

$$\mathbb{P}\left(\sum_{i=1}^{t_n} X_i \in (a\sqrt{n}, b\sqrt{n})\right) \ge \mathbb{P}\left(X_i \in \left(\frac{a\sqrt{n}}{t_n}, \frac{b\sqrt{n}}{t_n}\right), \text{ for all } 1 \le i \le t_n\right)$$

$$\ge \left(C\exp\left[-\lambda\left(\frac{a\sqrt{n}}{t_n}\right)^{\alpha}\right] - C'\exp\left[-\lambda\left(\frac{b\sqrt{n}}{t_n}\right)^{\alpha}\right]\right)^{t_n}$$

$$\ge C^{t_n}\exp\left[-\lambda a^{\alpha}\frac{n^{\alpha/2}}{t_n^{\alpha-1}}\right].$$

Then, the desired lower bound follows, provided $t_n = o(\sqrt{n})$. □

3. Proof of Theorem 1

In this section we assume that $|Z_1| \sim \text{Pareto}(\beta)$ for some $\beta \in (1, +\infty)$, and $\mathbb{E}[\mathrm{e}^{\theta X}] < \infty$ for some $\theta > 0$. We also assume that $I_A(p) < \infty$ and $I_A(\cdot)$ is continuous at p.

Proof. We start with the lower bound. Fix $\epsilon > 0$. By the continuity of $I_A(\cdot)$ at p, there exist some $\delta > 0$ and $|x| < I_A(p) + \epsilon$ such that, for any small $\eta > 0$, $\inf_{y\in[x-\eta,x+\eta]} \nu(A - y) \ge p + \delta$. Consequently, we can choose $M > 1$ large enough that $\frac{1}{1+M^{-1}}\inf_{y\in[x-\eta,x+\eta]} \nu(A - y) \ge p + \frac{\delta}{2}$. Let $t_n := \lfloor h\sqrt{n}\rfloor$ with some $h > 0$, and $|v|$ be the generation of particle v. Set $Z_k^v := \sum_{u\in Z_{|v|+k}, u \text{ is a descent of } v} \delta_{S_u - S_v}$; $H(u) := \{w \in Z_{t_n+1} : w \notin Z_1^u\}$ for $u \in Z_{t_n}$; $\mathcal{E} := \{(\xi, k, r) \in \mathcal{M} \times \mathbb{N}^+ \times \mathbb{N}^+ : \xi$ has exactly one atom z such that $S_z \in [(x-\eta)\sqrt{n}, (x+\eta)\sqrt{n}]$; $k > 2Mr\}$; $E := \big\{Z_{t_n}$ has exactly one particle u such that $S_u \in [(x-\eta)\sqrt{n}, (x+\eta)\sqrt{n}]$ and $|Z_1^u| > 2M\sum_{v\ne u, v\in Z_{t_n}} |Z_1^v|\big\}$. Namely, Z_k^v is the kth generation of the sub-BRW emanating from particle v, and $H(u)$ represents a collection of particles at time $t_n + 1$ who are not the children of u. The proof of the lower bound is divided into two main steps.

For the first step, we will show that there exists a constant $C_M > 0$ such that, for n large enough, $\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \ge p) \ge C_M\mathbb{P}(E)$. Let $\{Z_n^i\}_{n\ge 0}$, $i \ge 1$, be i.i.d. copies of $\{Z_n\}_{n\ge 0}$ and $W_{n-t_n-1}^v := |Z_{n-t_n-1}^v| m^{-(n-t_n-1)}$, and let x_v be the displacement of particle v. By the Markov property, we have

$$\mathbb{P}\big(\bar{Z}_n(\sqrt{n}A) \ge p\big)$$

$$\ge \mathbb{P}\left(\begin{array}{l}\text{there exists } u \in Z_{t_n} \text{ such that} \\ \dfrac{\sum\limits_{v\in Z_1^u} Z_{n-t_n-1}^v(\sqrt{n}A - x_v - S_u) + \sum\limits_{w\in H(u)} Z_{n-t_n-1}^w(\sqrt{n}A - S_w)}{\sum\limits_{v\in Z_1^u} |Z_{n-t_n-1}^v| + \sum\limits_{w\in H(u)} |Z_{n-t_n-1}^w|} \ge p \text{ and} \\ \dfrac{\sum\limits_{v\in Z_1^u} |Z_{n-t_n-1}^v|}{\sum\limits_{w\in H(u)} |Z_{n-t_n-1}^w|} > M\end{array}\right)$$

$$\geq \int_{\mathcal{E}} \mathbb{P}\left(\frac{\sum_{i=1}^{k} Z_{n-t_n-1}^{i}(\sqrt{n}A - x_v - S_z)}{\sum_{i=1}^{k} |Z_{n-t_n-1}^{i}| + \sum_{j=1}^{r} |Z_{n-t_n-1}^{j}|} \geq p \text{ and } \right.$$

$$\left. \frac{\sum_{i=1}^{k} |Z_{n-t_n-1}^{i}|}{\sum_{j=1}^{r} |Z_{n-t_n-1}^{j}|} > M \right) \mathbb{P}(Z_{t_n} \in \mathrm{d}\xi,\ |Z_1^z| \in \mathrm{d}k,\ |H(z)| \in \mathrm{d}r)$$

$$\geq \int_{\mathcal{E}} \mathbb{P}\left(\frac{\sum_{i=1}^{k} Z_{n-t_n-1}^{i}(\sqrt{n}A - x_v - S_z)}{(1+M^{-1}) \sum_{i=1}^{k} |Z_{n-t_n-1}^{i}|} \geq p \text{ and } \right.$$

$$\left. \frac{\sum_{i=1}^{k} W_{n-t_n-1}^{i}}{\sum_{j=1}^{r} W_{n-t_n-1}^{j}} > M \right) \mathbb{P}\left(Z_{t_n} \in \mathrm{d}\xi,\ |Z_1^z| \in \mathrm{d}k,\ |H(z)| \in \mathrm{d}r\right)$$

$$=: \int_{\mathcal{E}} P_n(\xi, k, r)\mathbb{P}(Z_{t_n} \in \mathrm{d}\xi,\ |Z_1^z| \in \mathrm{d}k,\ |H(z)| \in \mathrm{d}r), \tag{16}$$

where $\mathbb{P}(Z_{t_n} \in \mathrm{d}\xi)$ represents the distribution of the point process Z_{t_n} (for a serious definition of a point process's distribution, see [8, Section 2.1]). To finish the first step, it suffices to show that

$$\lim_{n\to\infty} \inf_{(\xi,k,r)\in\mathcal{E}} P_n(\xi, k, r) > 0. \tag{17}$$

Since $A \in \mathcal{A}$ (see Assumption 1), we can write $A = \sum_{i=1}^{l} (a_i, b_i]$ for some natural number l, where $-\infty \leq a_i < b_i \leq +\infty$. Let $A(x, \eta) := \sum_{i=1}^{l} (a_i - x + \eta, b_i - x - \eta)$. Since $S_z \in [(x-\eta)\sqrt{n}, (x+\eta)\sqrt{n}]$, we can choose η small enough such that

$$\frac{1}{1+M^{-1}} \nu(A(x, \eta)) > p + \frac{\delta}{4} \quad \text{and} \quad \sqrt{n}A(x, \eta) \subset \sqrt{n}A - S_z. \tag{18}$$

Thus, by the central limit theorem, there exists a constant $C(M, h, x, \eta, \delta) > 0$ such that, for $n > C(M, h, x, \eta, \delta)$,

$$\frac{1}{1+M^{-1}} \nu_{n-t_n}(\sqrt{n}A(x, \eta)) > p + \frac{\delta}{8}. \tag{19}$$

Recall that $Z_{n-t_n-1}^{i}$, $W_{n-t_n-1}^{i}$, and X_i, $1 \leq i \leq k$, are respectively i.i.d. copies of Z_{n-t_n-1}, W_{n-t_n-1}, and X. Since $\sqrt{n}A(x, \eta) \subset \sqrt{n}A - S_z$, by (16), we have

$$P_n(\xi, k, m) \geq \mathbb{P}\left(\frac{\sum_{i=1}^{k} Z_{n-t_n-1}^{i}(\sqrt{n}A(x, \eta) - X_i)}{(1+M^{-1}) \sum_{i=1}^{k} |Z_{n-t_n-1}^{i}|} \geq p \text{ and } \frac{\sum_{i=1}^{k} W_{n-t_n-1}^{i}}{\sum_{j=1}^{\lfloor \frac{k}{2M} \rfloor} W_{n-t_n-1}^{j}} > M \right)$$

$$=: \mathbb{E}\left[\mathbf{1}_{\{A_{n,k} \geq p\}} \mathbf{1}_{\{B_{n,k} \geq M\}}\right].$$

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Fix $k > 2M$ (since $(\xi, k, r) \in \mathcal{E}$) and $\epsilon' \in (0, \eta)$. A random variable $N(\epsilon', k) > 0$ exists such that, for $n > N(\epsilon', k)$ and $1 \le i \le k$, $\sqrt{n}A(x, \eta + \epsilon') \subset \sqrt{n}A(x, \eta) - X_i \subset \sqrt{n}A(x, \eta - \epsilon')$. Thus,

$$\frac{\sum_{i=1}^{k} m^{-(n-t_n-1)} Z^i_{n-t_n-1}(\sqrt{n}A(x, \eta + \epsilon'))}{(1 + M^{-1}) \sum_{i=1}^{k} m^{-(n-t_n-1)} |Z^i_{n-t_n-1}|} \le A_{n,k}$$

$$\le \frac{\sum_{i=1}^{k} m^{-(n-t_n-1)} Z^i_{n-t_n-1}(\sqrt{n}A(x, \eta - \epsilon'))}{(1 + M^{-1}) \sum_{i=1}^{k} m^{-(n-t_n-1)} |Z^i_{n-t_n-1}|}.$$

Since $\bar{Z}_n(\sqrt{n}A) \to \nu(A)$ and $W_n \to W$, the above implies that $\lim_{n\to\infty} A_{n,k} = \frac{1}{1+M^{-1}} \nu(A(x, \eta))$, $\mathbb{P}$-a.s. On the other hand, it is easy to see that

$$\lim_{n\to\infty} B_{n,k} = \sum_{i=1}^{k} W^i \Big/ \sum_{j=1}^{\lfloor \frac{k}{2M} \rfloor} W^j =: B_k \qquad \mathbb{P}\text{-a.s.},$$

where W^i and W^j are i.i.d. copies of $W := \lim_{n\to\infty} |Z_n|/m^n$. Hence, by the dominated convergence theorem and (18), for any fixed $k > 2M$ we have

$$\lim_{n\to\infty} \mathbb{E}\left[\mathbf{1}_{\{A_{n,k} \ge p\}} \mathbf{1}_{\{B_{n,k} \ge M\}}\right] = \mathbb{P}\left(B_k \ge M\right). \tag{20}$$

So, to achieve (17), it suffices to show that

$$\lim_{n\to\infty} \sup_{k>2M} \left|\mathbb{E}\left[\mathbf{1}_{\{A_{n,k} \ge p\}} \mathbf{1}_{\{B_{n,k} \ge M\}}\right] - \mathbb{P}\left(B_k \ge M\right)\right| = 0 \tag{21}$$

and

$$\inf_{k>2M} \mathbb{P}\left(B_k > M\right) > 0. \tag{22}$$

By the strong law of large numbers, we can easily obtain $\inf_{k>2M} \mathbb{P}\left(B_k > M\right) > 0$. For (20), observe that

$$\begin{aligned} &\left|\mathbb{E}\left[\mathbf{1}_{\{A_{n,k} \ge p\}} \mathbf{1}_{\{B_{n,k} \ge M\}}\right] - \mathbb{P}\left(B_k \ge M\right)\right| \\ &\le \mathbb{P}(A_{n,k} < p) + \mathbb{P}(B_{n,k} \ge M, B_k < M) + \mathbb{P}(B_{n,k} < M, B_k \ge M). \end{aligned} \tag{23}$$

For the first term on the right-hand side of (23), let $\xi = \sum_{i=1}^{k} \delta_{X_i}$ be a point process (recall that the X_i, $i \ge 1$, are i.i.d. copies of X). By Remark 7 and (19), there exists a constant $C_1 > 0$ depending on δ and β such that, for $n > C(M, h, x, \eta, \delta)$,

$$\begin{aligned} \mathbb{P}(A_{n,k} < p) &= \mathbb{P}\left(\frac{\sum_{i=1}^{k} Z^i_{n-t_n-1}(\sqrt{n}A(x, \eta) - X_i)}{(1 + M^{-1}) \sum_{i=1}^{k} |Z^i_{n-t_n-1}|} < p\right) \\ &= \mathbb{P}\left(\bar{Z}^{\xi}_{n-t_n-1}(\sqrt{n}A(x, \eta)) < p\big(1 + M^{-1}\big)\right) \\ &\le \mathbb{P}\left(\bar{Z}^{\xi}_{n-t_n-1}(\sqrt{n}A(x, \eta)) < \nu_{n-t_n}(\sqrt{n}A(x, \eta)) - \frac{\delta}{8}\right) \end{aligned}$$

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$$= \mathbb{P}\left(\bar{Z}^{\xi}_{n-t_n-1}(\sqrt{n}A(x,\eta)) < \mathbb{E}\left[\frac{1}{|\xi|}\sum_{z\in\xi}\nu_{n-t_n-1}(\sqrt{n}A(x,\eta)-S_z)\right]-\frac{\delta}{8}\right)$$
$$\leq C_1k^{1-\beta}, \tag{24}$$

where the third equality holds since

$$\nu_{n-t_n}\left(\sqrt{n}A(x,\eta)\right) = \mathbb{P}\left(X_1+X_2+\cdots+X_{n-t_n}\in\sqrt{n}A(x,\eta)\right)$$
$$= \mathbb{E}[\nu_{n-t_n-1}(\sqrt{n}A(x,\eta)-S_z)].$$

For the second term on the right-hand side of (23), by the strong law of large numbers we have

$$\mathbb{P}(B_{n,k}\geq M, B_k<M)\leq \mathbb{P}(B_k<M)\to 0 \qquad \text{as } k\to\infty. \tag{25}$$

For the third term on the right-hand side of (23), there exists a constant $C(M, \beta) > 0$ such that, for any $k > 2M$ and $n \geq 1$,

$$\mathbb{P}(B_{n,k}<M, B_k\geq M)$$
$$\leq \mathbb{P}(B_{n,k}<M)$$
$$= \mathbb{P}\left(\frac{\sum_{i=1}^{k} W^i_{n-t_n-1}}{\sum_{j=1}^{\lfloor \frac{k}{2M}\rfloor} W^j_{n-t_n-1}} < M\right)$$
$$\leq \mathbb{P}\left(\sum_{i=1}^{k} W^i_{n-t_n-1} < M\sum_{j=1}^{\lfloor \frac{k}{2M}\rfloor} W^j_{n-t_n-1}, \sum_{j=1}^{\lfloor \frac{k}{2M}\rfloor} W^j_{n-t_n-1} < \frac{k}{2M}\frac{3}{2}\right)$$
$$+\mathbb{P}\left(\sum_{j=1}^{\lfloor \frac{k}{2M}\rfloor} W^j_{n-t_n-1} \geq \frac{k}{2M}\frac{3}{2}\right)$$
$$\leq \mathbb{P}\left(\sum_{i=1}^{k} W^i_{n-t_n-1} < \frac{3k}{4}\right) + \mathbb{P}\left(\sum_{j=1}^{\lfloor \frac{k}{2M}\rfloor} W^j_{n-t_n-1} \geq \frac{k}{2M}\frac{3}{2}\right)$$
$$\leq \mathbb{P}\left(\sum_{i=1}^{k}\left(1-W^i_{n-t_n-1}\right) > \frac{k}{4}\right) + \mathbb{P}\left(\sum_{j=1}^{\lfloor \frac{k}{2M}\rfloor}\left(W^j_{n-t_n-1}-1\right) \geq \frac{k}{2M}\frac{1}{2}\right)$$
$$\leq C(M,\beta)k^{1-\beta}, \tag{26}$$

where, for the last inequality, we used exactly the same arguments as for bounding I_1 and I_2 in Lemma 4. Plugging (25), (26), and (27) into (23), we obtain that for every $\epsilon > 0$ there exists a constant $C(\epsilon, M, \beta, \delta)$ such that, for $k > C(\epsilon, M, \beta, \delta)$ and $n > C(M, h, x, \eta, \delta)$, $\left|\mathbb{E}\left[\mathbf{1}_{\{A_{n,k}\geq p\}}\mathbf{1}_{\{B_{n,k}\geq M\}}\right] - \mathbb{P}\left(B_k \geq M\right)\right| < \epsilon$. Combining this with (20), there exists some large constant $C(\epsilon, M, \beta, h, x, \eta, \delta, p)$ such that, for $n > C(\epsilon, M, \beta, h, x, \eta, \delta, p)$ and $k > 2M$, $\left|\mathbb{E}\left[\mathbf{1}_{\{A_{n,k}\geq p\}}\mathbf{1}_{\{B_{n,k}\geq M\}}\right] - \mathbb{P}\left(B_k \geq M\right)\right| < \epsilon$. Thus, (21) holds. This, combined with

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(22) and (16), implies that there exist positive constants C_M and $C(M, \beta, h, x, \eta, \delta, p)$ such that, for $n > C(M, \beta, h, x, \eta, \delta, p)$, $\mathbb{P}\big(\bar{Z}_n(\sqrt{n}A) \geq p\big) \geq C_M \mathbb{P}(E)$, which completes the first step.

For the second step, we will give a lower bound of $\mathbb{P}(E)$. Define $\mathcal{F}_{t_n} := \sigma(Z_i, 1 \leq i \leq t_n)$. By the first step, for $n > C(M, \beta, h, x, \eta, \delta, p)$,

$$\begin{aligned}
&\mathbb{P}\left(\bar{Z}_n(\sqrt{n}A) \geq p\right) \\
&\geq C_M \mathbb{P}\left(\text{there exists } u \in Z_{t_n} \text{ such that } S_u \in [(x-\eta)\sqrt{n}, (x+\eta)\sqrt{n}], |Z_1^u| > 2M \sum_{v \neq u\, v \in Z_{t_n}} |Z_1^v|\right) \\
&= C_M \mathbb{E}\left[\sum_{u \in Z_{t_n}} \mathbf{1}_{\{S_u \in [(x-\eta)\sqrt{n}, (x+\eta)\sqrt{n}]\}} \mathbf{1}_{\{|Z_1^u| > 2M \sum_{v \neq u, v \in Z_{t_n}} |Z_1^v|\}}\right] \\
&= C_M \mathbb{E}\left[\sum_{u \in Z_{t_n}} \mathbf{1}_{\{S_u \in [(x-\eta)\sqrt{n}, (x+\eta)\sqrt{n}]\}} \mathbb{E}\left[\mathbf{1}_{\{|Z_1^u| > 2M \sum_{v \neq u, v \in Z_{t_n}} |Z_1^v|\}} \,\Big|\, \mathcal{F}_{t_n}\right]\right], \qquad (27)
\end{aligned}$$

where the first equality follows from the fact that the random variable inside the expectation can only be 0 or 1 (since there exists at most one individual satisfying $|Z_1^u| > 2M \sum_{v \neq u, v \in Z_{t_n}} |Z_1^v|$ for $u \in Z_{t_n}$). Let k_i, $i \geq 0$, be i.i.d. copies of $|Z_1|$, and independent of Z_{t_n}. Since $|Z_1| \sim \text{Pareto}(\beta)$, there exists a constant $C_4 > 0$ such that $\mathbb{P}(|Z_1| > x) > C_4 x^{-\beta}$ for all $x > 1$. Recall the well-known fact that if U and V are independent random variables then, for any bounded measurable function $F(x,y)$, we have $\mathbb{E}[F(U, V) \mid \sigma(V)] = \mathbb{E}[F(U, v)]\big|_{v=V}$. Using this fact, we have

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_{\{|Z_1^u| > 2M \sum_{\substack{v \neq u \\ v \in Z_{t_n}}} |Z_1^v|\}} \,\Big|\, \mathcal{F}_{t_n}\right] &= \mathbb{E}\left[\mathbf{1}_{\{k_0 > 2M \sum_{i=1}^{|Z_{t_n}|-1} k_i\}} \,\Big|\, \mathcal{F}_{t_n}\right] \\
&= \mathbb{E}\left[\mathbf{1}_{\{k_0 > 2M \sum_{i=1}^{j} k_i\}}\right]\Big|_{j=|Z_{t_n}|-1} \\
&= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{k_0 > 2M \sum_{i=1}^{j} k_i\}} \Big| \sigma(k_1, \ldots, k_j)\right]\right]\Big|_{j=|Z_{t_n}|-1} \\
&\geq C_4 \mathbb{E}\left[\left(2M \sum_{i=1}^{j} k_i\right)^{-\beta}\right]\Bigg|_{j=|Z_{t_n}|-1} \\
&= C_4 \mathbb{E}\left[\left(2M \sum_{i=1}^{|Z_{t_n}|-1} k_i\right)^{-\beta} \,\Bigg|\, \sigma(|Z_{t_n}|)\right] \\
&\geq C_4 (2M)^{-\beta} \mathbb{E}\left[|Z_{t_n+1}|^{-\beta} \,\big|\, \mathcal{F}_{t_n}\right]. \qquad (28)
\end{aligned}$$

Plugging (28) into (27) yields, for $n > C(M, \beta, h, x, \eta, \delta, p)$,

$$\begin{aligned}
&\mathbb{P}\left(\bar{Z}_n(\sqrt{n}A) \geq p\right) \\
&\geq C_M C_4 (2M)^{-\beta} \mathbb{E}\left[\mathbb{E}\left[\sum_{u \in Z_{t_n}} \mathbf{1}_{\{S_u \in [(x-\eta)\sqrt{n}, (x+\eta)\sqrt{n}]\}} |Z_{t_n+1}|^{-\beta} \,\Big|\, \mathcal{F}_{t_n}\right]\right] \\
&= C_M C_4 (2M)^{-\beta} \mathbb{E}\left[\frac{|Z_{t_n}|}{|Z_{t_n+1}|^{\beta}}\right] \nu_{t_n}\left([(x-\eta)\sqrt{n}, (x+\eta)\sqrt{n}]\right), \qquad (29)
\end{aligned}$$

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where the last equality follows from the fact that the branching and motion are independent. By Fatou's lemma, for n large enough,

$$\mathbb{E}\left[\frac{|Z_{t_n}|}{|Z_{t_n+1}|^{\beta}}\right] = m^{-\beta} m^{-(\beta-1)t_n} \mathbb{E}\left[\frac{W_{t_n}}{(W_{t_{n+1}})^{\beta}}\right] \geq 0.9\, \mathbb{E}\left[W^{1-\beta}\right] m^{-\beta} m^{-(\beta-1)t_n}. \tag{30}$$

Plugging (30) into (29) yields the existence of a constant $C'(\epsilon, M, \beta, h, x, \eta, \delta, p)$ such that, for $n > C'(\epsilon, M, \beta, h, x, \eta, \delta, p)$,

$$\begin{aligned}
&\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) \\
&\quad \geq C_M C_4 (2M)^{-\beta} 0.9 m^{-\beta} \mathbb{E}\left[W^{1-\beta}\right] m^{-(\beta-1)t_n} \nu_{t_n}\left([(x-\eta)\sqrt{n}, (x+\eta)\sqrt{n}]\right) \\
&\quad \geq C_M C_4 (2M)^{-\beta} 0.9 m^{-\beta} \mathbb{E}\left[W^{1-\beta}\right] m^{-(\beta-1)h\sqrt{n}} \exp\left[-\left(\Lambda\left(\frac{x-\eta}{h}\right) + \epsilon\right) h\sqrt{n}\right],
\end{aligned}$$

where the last inequality follows by Cramér's theorem. Hence, for every ϵ, η small enough and $h > 0$, the above yields

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) \geq -\left\{h(\beta-1)\log m + \left(\Lambda\left(\frac{x-\eta}{h}\right) + \epsilon\right) h\right\}.$$

First, let $\epsilon \to 0$, and then maximize the lower bound with h, to finally obtain

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) \geq -\inf_{h>0}\left\{h(\beta-1)\log m + \Lambda\left(\frac{I_A(p)}{h}\right) h\right\},$$

which concludes the desired lower bound for the case of $\beta - 1 < \frac{-\log p_1}{\log m}$. For $\beta - 1 \geq \frac{-\log p_1}{\log m}$, [13, Lemma 3.1] gives

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) \geq -\inf_{h>0}\left\{h \log \frac{1}{p_1} + \Lambda\left(\frac{I_A(p)}{h}\right) h\right\}.$$

We next consider the upper bound in Theorem 1. By the definition of $I_A(p)$, for every $\eta \in (0, I_A(p))$ there exists $\delta > 0$ such that

$$\sup_{|y| \leq I_A(p)-\eta} \nu(A-y) \leq p - \delta. \tag{31}$$

Set $B_n := [(-I_A(p)+\eta)\sqrt{n}, (I_A(p)-\eta)\sqrt{n}]$, $\mathcal{M}_1 := \left\{\xi \in \mathcal{M} : \frac{\xi(B_n^c)}{|\xi|} \leq \frac{\delta}{2}\right\}$, and $t_n := \lfloor h\sqrt{n} \rfloor$ for some $h > 0$. For every $\xi \in \mathcal{M}_1$ and n large enough, we have

$$\begin{aligned}
&\frac{1}{|\xi|} \sum_{z\in\xi} \nu_{n-t_n}(\sqrt{n}A - S_z) + \frac{\delta}{4} \\
&\leq \frac{1}{|\xi|} \sum_{\substack{z\in\xi \\ z\in B_n^c}} \nu_{n-t_n}(\sqrt{n}A - S_z) + \frac{1}{|\xi|} \sum_{\substack{z\in\xi \\ z\in B_n}} \nu_{n-t_n}(\sqrt{n}A - S_z) + \frac{\delta}{4} \\
&\leq \frac{\xi(B_n^c)}{|\xi|} + \sup_{|z|\leq I_A(p)-\eta} \nu_{n-t_n}(\sqrt{n}(A-z)) + \frac{\delta}{4} \\
&\leq \frac{\delta}{2} + p - \frac{3\delta}{4} + \frac{\delta}{4} \\
&= p,
\end{aligned}$$

where the third inequality follows by the generalized central limit theorem (see [32, Lemma 2.2]) and (31). Hence, by Lemma 4 and the Markov inequality, for n large enough,

$$\begin{aligned}
&\mathbb{P}\left(\bar{Z}_n(\sqrt{n}A)\ge p\right)\\
&\le \mathbb{P}\left(\bar{Z}_{t_n}(B_n^{\mathrm{c}})\ge \frac{\delta}{2}\right)+\mathbb{P}\left(\bar{Z}_{t_n}(B_n^{\mathrm{c}})\le \frac{\delta}{2},\,\bar{Z}_n(\sqrt{n}A)\ge p\right)\\
&\le \mathbb{E}\left[\frac{2}{\delta}\frac{Z_{t_n}(B_n^{\mathrm{c}})}{|Z_{t_n}|}\right]+\int_{\mathcal{M}_1}\mathbb{P}\left(\bar{Z}^{\xi}_{n-t_n}(\sqrt{n}A)\ge \frac{1}{|\xi|}\sum_{z\in\xi}\nu_{n-t_n}(\sqrt{n}A-S_z)+\frac{\delta}{4}\right)\mathbb{P}(Z_{t_n}\in \mathrm{d}\xi)\\
&\le \mathbb{E}\left[\frac{2}{\delta}\frac{Z_{t_n}(B_n^{\mathrm{c}})}{|Z_{t_n}|}\right]+C_1\mathbb{E}\big[|Z_{t_n}|^{-(\beta-1)}\big]. \qquad (32)
\end{aligned}$$

For the first term on the right-hand side of (32), define $\mathcal{G}_{t_n}:=\sigma(|Z_i|,\,1\le i\le t_n)$. Since the branching and motion are independent we have, for n large enough,

$$\begin{aligned}
\mathbb{E}\left[\frac{2}{\delta}\frac{Z_{t_n}(B_n^{\mathrm{c}})}{|Z_{t_n}|}\right]&=\mathbb{E}\left[\mathbb{E}\left[\frac{2}{\delta}\frac{Z_{t_n}(B_n^{\mathrm{c}})}{|Z_{t_n}|}\mid \mathcal{G}_{t_n}\right]\right]\\
&=\frac{4}{\delta}\nu_{t_n}\left((I_A(p)-\eta)\sqrt{n}\right)\\
&\le \frac{4}{\delta}\exp\left[-\left(\Lambda\left(\frac{I_A(p)-\eta}{h}\right)-\epsilon\right)h\sqrt{n}\right], \qquad (33)
\end{aligned}$$

where the second equality comes from the symmetry of the step size, and the last inequality follows by Cramér's theorem. For the second term on the right-hand side of (32), by Lemma 2, for n large enough,

$$\mathbb{E}\left[|Z_{t_n}|^{-(\beta-1)}\right]\le \begin{cases} 2C_0 m^{-(\beta-1)h\sqrt{n}}, & \beta-1\ge \dfrac{-\log p_1}{\log m};\\[2ex] 2C_0 p_1^{h\sqrt{n}}, & 0<\beta-1<\dfrac{-\log p_1}{\log m}.\end{cases} \qquad (34)$$

Plugging (33) and (34) into (32) yields that if $\beta-1\ge \frac{-\log p_1}{\log m}$ then

$$\mathbb{P}\left(\bar{Z}_n(\sqrt{n}A)\ge p\right)\le \frac{4}{\delta}\exp\left[-\left(\Lambda\left(\frac{I_A(p)-\eta}{h}\right)-\epsilon\right)h\sqrt{n}\right]+2C_0C_1(p,\delta,h)m^{-(\beta-1)h\sqrt{n}},$$

and if $0<\beta-1<\frac{\log \frac{1}{p_1}}{\log m}$ then

$$\mathbb{P}\left(\bar{Z}_n(\sqrt{n}A)\ge p\right)\le \frac{4}{\delta}\exp\left[-\left(\Lambda\left(\frac{I_A(p)-\eta}{h}\right)-\epsilon\right)h\sqrt{n}\right]+2C_0C_1(p,\delta,h)p_1^{h\sqrt{n}}.$$

So, the upper bound follows by optimizing h on $(0,+\infty)$. □

4. Proof of Theorem 2

In this section we consider the case that the offspring law has a Weibull tail, i.e. $\mathbb{P}(|Z_1|>x)\sim l_1\mathrm{e}^{-lx^{\beta}}$ as $x\to\infty$ for some constants $\beta\in(0,1)$ and $l_1, l\in(0,+\infty)$. We assume that

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$I_A(p) < \infty$ and $I_A(\cdot)$ is continuous at p. Comparing with the case where the offspring law has exponential moment, the results and proofs do not change in the Schröder case. However, in the Böttcher case things become different: the Böttcher constant will appear in the rate function. So, in this section we further assume $p_1 = 0$. Moreover, unlike the Pareto case, the tail of the step size matters for the decay scale of LDP. To show this, we will investigate two types of step size: bounded step size and Weibull-tail step size.

4.1. Proof of (1)

In this subsection we assume that $p_1 = 0$ and $0 < \text{ess sup}\, X = L < \infty$. We are going to show that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \log\left[-\log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p)\right] = \frac{I_A(p)}{L} \frac{\rho\beta}{\beta+\rho-\beta\rho} \log m.$$

Proof. We start with the lower bound. By the continuity of $I_A(\cdot)$ at p, for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|I_A(p+2\delta) - I_A(p)| < \varepsilon$. Furthermore, it is easy to see that there exists $x \in \mathbb{R}$ such that $\nu(A-x) = p + 2\delta$. So, by the continuity of $\nu(A-\cdot)$, there exists some $\eta > 0$ such that $\inf_{z\in\left(x,x+\frac{2\eta x}{L}\right)} \nu(A-z) \geq p+\delta$, where $(b, a) := (a, b)$, if $a < b$. Let $t_n := \left\lceil \frac{|x|\sqrt{n}}{(L-\eta)} \right\rceil$ and

$$E := \left\{ \text{there exists } u \in Z_{t_n} \text{ such that } S_u \in \left(x\sqrt{n}, x\sqrt{n} + \frac{2\eta x\sqrt{n}}{L}\right), |Z_1^u| > 2M \sum_{\substack{v\neq u\\ v\in Z_{t_n}}} |Z_1^v| \right\}.$$

Since $|Z_1| \sim \text{Weibull}(\beta)$, there exist constants $c_7 > 0$ and $l > 0$ such that $\mathbb{P}(|Z_1| > y) > c_7 \mathrm{e}^{-ly^\beta}$ for all $y > 0$. By similar arguments for the lower bound in Theorem 1, there exist constants $C_M > 0$ and $T(\eta, x) > 0$ such that, for n large enough,

$$\begin{aligned}\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) &\geq C_M \mathbb{P}(E)\\ &= C_M \mathbb{P}\left(S_{t_n} \in \left(x\sqrt{n}, x\sqrt{n} + \frac{2\eta x\sqrt{n}}{L}\right)\right) c_7 \mathbb{E}\left[|Z_{t_n}| \mathrm{e}^{-l(2M)^\beta |Z_{t_n+1}|^\beta}\right]\\ &\geq C_M \mathrm{e}^{-T(\eta,x)t_n} \mathbb{E}\left[\mathrm{e}^{-l(2M)^\beta |Z_{t_n+1}|^\beta}\right], \end{aligned} \tag{35}$$

where the last inequality follows by Cramér's theorem and the fact that $|x\sqrt{n}| \leq \left\lceil \frac{|x|\sqrt{n}}{(L-\eta)} \right\rceil L$. Hence, by Lemma 3, we have

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n}} \log\left[-\log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p)\right] \leq \frac{|x|}{L-\eta} \frac{\rho\beta}{\beta+\rho-\beta\rho} \log m.$$

Finally, by letting $\eta \to 0$ and $\varepsilon \to 0$, we obtain the desired lower bound.

Then, for the upper bound, for any $\varepsilon \in (0, I_A(p))$ set $t_n := \lfloor (I_A(p) - \varepsilon)\sqrt{n}/L \rfloor$. By Lemma 5 and arguments from [32, (2.30)–(2.33)], there exists $\delta > 0$ such that, for n large enough,

$$\begin{aligned}\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) &\leq \int_{\mathcal{M}} \mathbb{P}\left(\bar{Z}^{\xi}_{n-t_n}(\sqrt{n}A) \geq \frac{1}{|\xi|} \sum_{z\in\xi} \nu_{n-t_n}(\sqrt{n}A - S_z) + \frac{\delta}{3}\right) \mathbb{P}(Z_{t_n} \in \mathrm{d}\xi)\\ &\leq C_2 \mathbb{E}\left[\mathrm{e}^{-C_3|Z_{t_n}|^\beta}\right].\end{aligned}$$

Thus, by Lemma 3, we have

$$\liminf_{n\to\infty} \frac{1}{\sqrt{n}} \log\left[-\log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p)\right] \geq \frac{I_A(p)-\varepsilon}{L} \frac{\rho\beta}{\beta+\rho-\beta\rho} \log m.$$

Finally, the desired upper bound follows by letting $\varepsilon \to 0$. □

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4.2. Proof of (2)

In this subsection we consider the case where the step size X satisfies $\mathbb{P}(X > x) = \Theta(1)\mathrm{e}^{-\lambda x^\alpha}$ as $x \to \infty$ for some $\alpha \in (0, \infty)$, $\lambda > 0$. We are going to show that if $\alpha > 1$, then

$$\lim_{n\to\infty} \frac{(\log n)^{\alpha-1}}{n^{\alpha/2}} \log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) = -\lambda I_A(p)^\alpha \left(\frac{2\beta\rho \log m}{\alpha(\beta+\rho-\beta\rho)}\right)^{\alpha-1}.$$

For $\alpha \in (0, 1]$, using the same arguments as [13, Section 4.1.1] and Lemma 5, we can easily obtain $\lim_{n\to\infty} \frac{1}{n^{\alpha/2}} \log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) = -\lambda I_A(p)^\alpha$, so we feel free to omit its proof here.

Proof. For the lower bound, we start by fixing $\epsilon > 0$. There exist some $\eta, \delta > 0$ and $|x| < I_A(p) + \epsilon$ such that $\inf_{y\in(x-\eta,x+\eta)} \nu(A - y) \geq p + \delta$. Let $E := \big\{$there exists $u \in Z_{t_n}$ such that $S_u \in \big((x-\eta)\sqrt{n}, (x+\eta)\sqrt{n}\big)$, $|Z_1^u| > 2M \sum_{v\neq u, v\in Z_{t_n}} |Z_1^v|\big\}$, where $t_n := \lfloor t \log n \rfloor$ for some $0 < t < \frac{\alpha(\beta+\rho-\beta\rho)}{2\log(m+\epsilon)\beta\rho}$. Similar to (35), we have, for n large enough,

$$\begin{aligned}
\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \geq p) &\geq C_M \mathbb{P}(E) \\
&= C_M \mathbb{P}\left(S_{t_n} \in \big((x-\eta)\sqrt{n}, (x+\eta)\sqrt{n}\big)\right) c_7 \mathbb{E}\left[|Z_{t_n}| \mathrm{e}^{-l(2M)^\beta |Z_{t_n+1}|^\beta}\right] \\
&\geq c_7 C_M \exp\left\{-(\lambda+\epsilon)|x-\eta|^\alpha \frac{n^{\alpha/2}}{(t\log n)^{\alpha-1}}\right\} \mathbb{E}\left[\mathrm{e}^{-l(2M)^\beta |Z_{t_n+1}|^\beta}\right] \\
&\geq c_7 C_M \exp\left\{-(\lambda+\epsilon)|x-\eta|^\alpha \frac{n^{\alpha/2}}{(t\log n)^{\alpha-1}}\right\} \exp\left(-n^{\frac{\beta\rho t \log(m+\epsilon)}{\beta+\rho-\beta\rho}}\right) \\
&\geq c_7 C_M \exp\left\{-(\lambda+2\epsilon)|x-\eta|^\alpha \frac{n^{\alpha/2}}{(t\log n)^{\alpha-1}}\right\},
\end{aligned}$$

where the second inequality follows from Lemma 3, the third inequality comes from Lemma 3, and the last inequality follows from the fact that $t < \frac{\alpha(\beta+\rho-\beta\rho)}{2\log(m+\epsilon)\beta\rho}$. Hence, for any $\epsilon > 0$, some $\eta > 0$, $|x| < I_A(p) + \epsilon$, and any $0 < t < \frac{\alpha(\beta+\rho-\beta\rho)}{2\log(m+\epsilon)\beta\rho}$, we have

$$\liminf_{n\to\infty} \frac{(\log n)^{\alpha-1}}{n^{\alpha/2}} \log \mathbb{P}\big(\bar{Z}_n(\sqrt{n}A) \geq p\big) \geq -(\lambda+\epsilon)\frac{|x-\eta|^\alpha}{t^{\alpha-1}}.$$

Finally, by letting $\epsilon \to 0$, $t \to \frac{\alpha(\beta+\rho-\beta\rho)}{2\log m\beta\rho}$ gives the desired lower bound.

For the upper bound, set $t_n := \lfloor t \log n \rfloor$ for $t > \frac{\alpha(\beta+\rho-\beta\rho)}{2\beta\rho\log(m-\epsilon)}$, and $B_n := [(-I_A(p) + \eta)\sqrt{n}, (I_A(p) - \eta)\sqrt{n}]$. Using the arguments from (31) to (32), there exists some $\delta > 0$ such that, for n large enough,

$$\begin{aligned}
&\mathbb{P}\left(\bar{Z}_n(\sqrt{n}A) \geq p\right) \\
&\leq \mathbb{E}\left[\frac{2}{\delta}\frac{Z_{t_n}(B_n^{\mathrm{c}})}{|Z_{t_n}|}\right] + \int_{\mathcal{M}_1} \mathbb{P}\left(\bar{Z}^\xi_{n-t_n}(\sqrt{n}A) \geq \frac{1}{|\xi|}\sum_{z\in\xi} \nu_{n-t_n}(\sqrt{n}A - S_z) + \frac{\delta}{4}\right)\mathbb{P}(Z_{t_n} \in \mathrm{d}\xi) \\
&\leq \frac{2}{\delta}\mathbb{P}(S_{t_n} \in B_n^{\mathrm{c}}) + C_2 \mathbb{E}\left[\mathrm{e}^{-C_3|Z_{t_n}|^\beta}\right],
\end{aligned}$$

where the last inequality follows from Lemma 5. As a consequence, by Lemmas 6 and 3, for n large enough,

$$\mathbb{P}\left(\bar{Z}_n(\sqrt{n}A) \ge p\right) \le \exp\left\{-\lambda(I_A(p)-\eta-\epsilon)^{\alpha}\frac{n^{\alpha/2}}{(t\log n)^{\alpha-1}}\right\} + C_2\exp\left(-n^{\frac{\beta\rho t\log(m-\epsilon)}{\beta+\rho-\beta\rho}}\right)$$
$$\le \exp\left\{-(\lambda-2\epsilon)(I_A(p)-\eta)^{\alpha}\frac{n^{\alpha/2}}{(t\log n)^{\alpha-1}}\right\},$$

where the last inequality follows from $t > \frac{\alpha(\beta+\rho-\beta\rho)}{2\beta\rho\log(m-\epsilon)}$. So, for any $\epsilon, \eta > 0$ small enough and $t > \frac{\alpha(\beta+\rho-\beta\rho)}{2\beta\rho\log(m-\epsilon)}$, we have

$$\limsup_{n\to\infty}\frac{(\log n)^{\alpha-1}}{n^{\alpha/2}}\log\mathbb{P}\left(\bar{Z}_n(\sqrt{n}A)\ge p\right) \ge -(\lambda-2\epsilon)\frac{(I_A(p)-\eta)^{\alpha}}{t^{\alpha-1}},$$

which implies the upper bound by letting $\epsilon, \eta \to 0$ and $t \to \frac{\alpha(\beta+\rho-\beta\rho)}{2\beta\rho\log m}$. □

5. Proof of Theorem 3

In this section we assume $\mathbb{E}[|Z_1|^{\beta}] < \infty$ for some $\beta > 1$, and $I_A(\cdot)$ is continuous at p for some $p \in (\nu(A), 1-\nu(A))$. Here, we consider the step size to have a Pareto tail, i.e. $\mathbb{P}(X > x) \sim \kappa x^{-\alpha}$ as $x \to \infty$ for some constants $\kappa > 0$ and $\alpha > 2$. We are going to show that if $b < B$ then $\lim_{n\to\infty}\frac{1}{\log n}\log\mathbb{P}(\bar{Z}_n(\sqrt{n}A)\ge p) = -\frac{\alpha}{2}$. Furthermore, if $b = B$ and A is an unbounded set, the above still holds provided that $0 < p-\nu(A) < (1-\nu(A))/b$.

Proof. We start with the lower bound. By the continuity of $I_A(\cdot)$ at p, for every $\epsilon > 0$ there exist $\eta, \delta > 0$ such that, for some $|x| \le I_A(p)+\epsilon$, $\inf_{y\in[x-\eta,x+\eta]}\nu(A-y)\ge p+\delta$. Without loss of generality, we write $A = \sum_{j=1}^{h}(a_j, b_j]$. Set $A(x,\eta) := \sum_{j=1}^{h}\left(a_j-(x-\eta), b_j-(x+\eta)\right)$, $A(\epsilon) := \sum_{j=1}^{h}\left(a_j+\epsilon, b_j-\epsilon\right)$. Obviously, we can choose ϵ, η small enough such that $\nu(A(x,\eta)) > p+\frac{\delta}{2}$, $\nu(A(\epsilon)) > \nu(A)-\frac{\delta}{2}$. Set $\mathcal{E} := \left\{\xi\in\mathcal{M} : \xi = \sum_{i=1}^{b}\delta_{x_i\sqrt{n}}, \text{ where } x_1 \in (x-\eta, x+\eta),\ x_i\in(-\epsilon,\epsilon),\ i=2,\ldots,b\right\}$. By the Markov property, $\mathbb{P}(\bar{Z}_n(\sqrt{n}A)\ge p) \ge \int_{\mathcal{E}}\mathbb{P}\left(\bar{Z}^{\xi}_{n-1}(\sqrt{n}A)\ge p\right)\mathbb{P}(Z_1\in d\xi)$. For every $\xi\in\mathcal{E}$, it is easy to see that $A-x_1 \supset A(x,\eta)$, $A-x_i\supset A(\epsilon)$, $i=2,\ldots,b$. Hence,

$$\mathbb{P}(\bar{Z}_n(\sqrt{n}A)\ge p) \ge \int_{\mathcal{E}}\mathbb{P}\left(\bar{Z}^{\xi}_{n-1}(\sqrt{n}A)\ge p\right)\mathbb{P}(Z_1\in d\xi)$$
$$\ge \int_{\mathcal{E}}\mathbb{P}\left(\frac{Z^1_{n-1}(\sqrt{n}A(x,\eta))+\sum_{i=2}^{b}Z^i_{n-1}(\sqrt{n}A(\epsilon))}{\sum_{i=1}^{b}|Z^i_{n-1}|}\ge p\right)\mathbb{P}(Z_1\in d\xi)$$
$$= \mathbb{P}\left(\frac{Z^1_{n-1}(\sqrt{n}A(x,\eta))+\sum_{i=2}^{b}Z^i_{n-1}(\sqrt{n}A(\epsilon))}{\sum_{i=1}^{b}|Z^i_{n-1}|}\ge p\right)\mathbb{P}(Z_1\in\mathcal{E}), \tag{36}$$

where $Z^i_{n-1}(\cdot)$, $1\le i\le b$, are i.i.d. copies of $Z_{n-1}(\cdot)$. Since $\bar{Z}_n(\sqrt{n}A)\to\nu(A)$ and $|Z_n|m^{-n}\to W$ almost surely, we have, as $n\to\infty$,

$$\frac{Z^1_{n-1}(\sqrt{n}A(x,\eta))+\sum_{i=2}^{b}Z^i_{n-1}(\sqrt{n}A(\epsilon))}{\sum_{i=1}^{b}|Z^i_{n-1}|} \longrightarrow \frac{\sum_{i=2}^{b}\nu(A(\epsilon))W_i+\nu(A(x,\eta))W_1}{\sum_{i=1}^{b}W_i} \quad \mathbb{P}\text{-a.s.}, \tag{37}$$

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where W_i, $1 \le i \le b$, are i.i.d. copies of W. If $b < B$, then W_i has a continuous density on $(0, +\infty)$ (see [4, Chapter II, Lemma 2]). In this case, by the dominated convergence theorem and (37),

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{Z^1_{n-1}(\sqrt{n}A(x,\eta)) + \sum_{i=2}^b Z^i_{n-1}(\sqrt{n}A(\epsilon))}{\sum_{i=1}^b |Z^i_{n-1}|} \ge p\right) =: C(\epsilon, \eta, x) > 0. \tag{38}$$

Since $\mathbb{P}(X > x) \sim \kappa x^{-\alpha}$, there exists a constant $C(\kappa, x, \epsilon, \eta) =: C_7 > 0$ such that, for n large enough,

$$\mathbb{P}(Z_1 \in \mathcal{E}) \ge C_7 p_b n^{-\alpha/2}. \tag{39}$$

Plugging (38) and (39) into (36) yields, for n large enough, $\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \ge p) \ge 0.9C(\epsilon, \eta, x)C_7 p_b n^{-\alpha/2}$, which implies the desired lower bound if $b < B$.

If $b = B$ and A is unbounded, without loss of generality we assume $A = (a, +\infty)$. Set $\mathcal{E}' := \left\{\xi \in \mathcal{M} : \xi = \sum_{i=1}^b \delta_{x_i\sqrt{n}}, \text{ where } x_1 \in [t\sqrt{n}, +\infty), x_i \in (-\epsilon, \epsilon), i = 2, \ldots, b\right\}$, where t is some positive constant. Using similar arguments to above, we obtain

$$\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \ge p) \ge \mathbb{P}\left(\frac{Z^1_{n-1}(\sqrt{n}(a-t, +\infty)) + \sum_{i=2}^b Z^i_{n-1}(\sqrt{n}A(\epsilon))}{\sum_{i=1}^b Z^i_{n-1}} \ge p\right) \mathbb{P}(Z_1 \in \mathcal{E}').$$

For any $p \in \left(\nu(A), \nu(A) + \frac{1-\nu(A)}{b}\right)$, there exists $\delta' > 0$ such that

$$p < \frac{(b-1)(\nu(A) - \delta') + 1 - \delta'}{b}. \tag{40}$$

If we choose ϵ small enough and t large enough, then $\nu(A(\epsilon)) \ge \nu(A) - \delta'$, $\nu((a-t, +\infty)) > 1 - \delta'$. This, together with (37) and (40), shows that, almost surely,

$$\frac{Z^1_{n-1}(\sqrt{n}(a-t, +\infty)) + \sum_{i=2}^b Z^i_{n-1}(\sqrt{n}A(\epsilon))}{\sum_{i=1}^b |Z^i_{n-1}|} \to \frac{(b-1)\nu(A(\epsilon)) + \nu((a-t, +\infty))}{b} > p,$$

which implies the desired lower bound.

For the upper bound, set $t_n := \lfloor c \log n \rfloor$ for some $c > 0$. By copying the arguments from (31) to (32), there exist constants $\delta, \eta > 0$ such that, for n large enough, $\mathbb{P}\left(\bar{Z}_n(\sqrt{n}A) \ge p\right) \le \frac{4}{\delta}\mathbb{P}\left(S_{t_n} \ge (I_A(p) - \eta)\sqrt{n}\right) + C_1\mathbb{E}\left[|Z_{t_n}|^{-(\beta-1)}\right]$, where the last inequality follows from Lemma 4. For the first term of the right-hand side of the latter inequality, by (8), there exists a constant $C_8 > 0$ depending on c, κ, α, and η such that, for n large enough, $\frac{4}{\delta}\mathbb{P}\left(S_{t_n} \ge (I_A(p) - \eta)\sqrt{n}\right) \le C_8 n^{-\frac{\alpha}{2}} \log n$. For the second term, by Lemma 2, for n large enough, $\mathbb{E}\left[|Z_{t_n}|^{-(\beta-1)}\right] \le 2C_0\left(m^{-t_n(\beta-1)} + p_1^{t_n}\right)$. Hence, for n large enough, $\mathbb{P}\left(\bar{Z}_n(\sqrt{n}A) \ge p\right) \le C_8 n^{-\frac{\alpha}{2}} \log n + 2C_0\left(m^{-t_n(\beta-1)} + p_1^{t_n}\right)$. Since $t_n = \lfloor c \log n \rfloor$, if we choose $c > \frac{\alpha}{2(\beta-1)\log m} \vee \frac{\alpha}{-2\log p_1}$ then, for large n, $\mathbb{P}\left(\bar{Z}_n(\sqrt{n}A) \ge p\right) \le 2C_8 n^{-\frac{\alpha}{2}} \log n$. Thus, we have

$$\limsup_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \ge p) \le -\frac{\alpha}{2}.$$

□

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6. Proof of Theorem 4

In this subsection we assume that $I_A(p)=\infty$, $J_A(p)$ is continuous at p, and $\mathbb{E}\big[X^2\big]<\infty$. We are going to prove that if $|Z_1|\sim \text{Pareto}(\beta)$ with some $\beta>1$, then

$$\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\bar{Z}_n(\sqrt{n}A)\geq p)=\begin{cases}-J_A(p)(\beta-1)\log m, & 0<\beta-1<\dfrac{-\log p_1}{\log m};\\[2ex] J_A(p)\log p_1, & \beta-1\geq\dfrac{-\log p_1}{\log m},\end{cases}$$

and if $|Z_1|\sim \text{Weibull}(\beta)$ with some $\beta\in(0,1)$, then $\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\bar{Z}_n(\sqrt{n}A)\geq p)=J_A(p)\log p_1$ for $p_1>0$, and

$$\lim_{n\to\infty}\frac{1}{n}\log\big[-\log\mathbb{P}(\bar{Z}_n(\sqrt{n}A)\geq p)\big]=J_A(p)\frac{\rho\beta}{\beta+\rho-\beta\rho}\log m \qquad \text{for } p_1=0.$$

Proof. Considering the lower bound, we first examine $|Z_1|$ with a Pareto tail. If $\beta-1\geq\frac{-\log p_1}{\log m}$ then from [13, Lemma 3.4] we have $\liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\bar{Z}_n(\sqrt{n}A)\geq p)\geq J_A(p)\log p_1$. Now we shall consider the case of $0<\beta-1<\frac{-\log p_1}{\log m}$. Since $J_A(p)$ is continuous at p, for any $\epsilon>0$, there exists $\delta>0$ such that, for $r:=J_A(p+2\delta)\in(0,1)$, we have $|r-J_A(p)|<\epsilon$. Moreover, by the definition of $J_A(p)$, there exists $x\in\mathbb{R}$ such that

$$\nu\left(\frac{A-x}{\sqrt{1-r}}\right)\geq p+\frac{3}{2}\delta.$$

So, for any small $\eta>0$,

$$\nu\left(\bigcap_{y\in[x-\eta,x+\eta]}\frac{A-y}{\sqrt{1-r}}\right)\geq p+\delta. \tag{41}$$

Set $t_n:=\lfloor rn\rfloor$, and for large M set $E:=\big\{\text{there exists } u\in Z_{t_n} \text{ such that } S_u\in[(x-\eta)\sqrt{n},(x+\eta)\sqrt{n}],\ |Z_1^u|>2M\sum_{v\neq u,v\in Z_{t_n}}|Z_1^v|\big\}$. Similar to (18), from (41) and $S_u\in[(x-\eta)\sqrt{n},(x+\eta)\sqrt{n}]$, we can choose η small enough such that

$$\frac{1}{1+M^{-1}}\nu\left(\frac{A(x,\eta)}{\sqrt{1-r}}\right)>p+\frac{\delta}{4},\qquad \sqrt{n}A(x,\eta)\subset\sqrt{n}A-S_u, \tag{42}$$

where $A(x,\eta):=\sum_{i=1}^{l}(a_i-x+\eta,b_i-x-\eta)$. By [32, Lemma 2.2], it follows that

$$\lim_{n\to\infty}\nu_{n-t_n}(\sqrt{n}A(x,\eta))=\nu\left(\frac{A(x,\eta)}{\sqrt{1-r}}\right).$$

This, combined with (42), shows that there exists a constant $C(M,r,x,\eta,\delta)>0$ such that, for $n>C(M,r,x,\eta,\delta)>0$, $\frac{1}{1+M^{-1}}\nu_{n-t_n}(\sqrt{n}A(x,\eta))>p+\frac{\delta}{8}$, which plays the same role in this proof as (19) in Theorem 1. Using similar arguments to the proof of the lower bound in Theorem 1, there exists a constant $C_M>0$ such that, for n large enough,

$$\begin{aligned}\mathbb{P}(\bar{Z}_n(\sqrt{n}A)\geq p)&\geq C_M\mathbb{P}\,(E)\\ &\geq C_M0.9C_0m^{-t_n(\beta-1)}\nu_{t_n}\big([(x-\eta)\sqrt{n},(x+\eta)\sqrt{n}]\big)\\ &\geq 0.9^2C_MC_0C_9m^{-t_n(\beta-1)},\end{aligned} \tag{43}$$

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where the last inequality holds since $\lim_{n\to\infty} \nu_{t_n}\left([(x-\eta)\sqrt{n}, (x+\eta)\sqrt{n}]\right) = C(r, x, \eta) =: C_9 > 0$. Taking limits in (43) yields $\liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\bar{Z}_n(\sqrt{n}A) \ge p) \ge -(J_A(p)+\epsilon)(\beta-1)\log m$. Then, the desired lower bound follows by letting $\epsilon \to 0$.

We now consider the upper bound. For $\epsilon \in (0, J_A(p))$, set $t_n := \lfloor (J_A(p)-\epsilon)n \rfloor$. By the definition of $J_A(p)$ there exists $\delta > 0$ such that, for $\epsilon' \in [\epsilon, 2\epsilon]$,

$$\sup_{y\in\mathbb{R}} \nu\left(\frac{A-y}{\sqrt{1-J_A(p)+\epsilon'}}\right) \le p-\delta. \tag{44}$$

Thus, for any $\xi \in \mathcal{M}$,

$$\begin{aligned}\frac{1}{|\xi|}\sum_{y\in\xi} \nu_{n-t_n}(\sqrt{n}A - S_y) + \frac{\delta}{2} &\le \frac{1}{|\xi|}\sum_{y\in\xi} \nu\left(\frac{\sqrt{n}}{\sqrt{n-t_n}}A - \frac{S_y}{\sqrt{n-t_n}}\right) + \frac{3\delta}{4} \\ &\le p - \frac{3\delta}{4} + \frac{3\delta}{4} = p,\end{aligned}$$

where the first inequality follows from the generalized central limit theorem, and the second inequality follows from (44) and the fact that $g(u, v) = \nu(uA+v)$ is a continuous function on $\mathbb{R}^2$. Thus, for n large enough,

$$\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \ge p) \le \int_{\mathcal{M}} \mathbb{P}\left(\bar{Z}^{\xi}_{n-t_n}(\sqrt{n}A) \ge \frac{1}{|\xi|}\sum_{y\in\xi} \nu_{n-t_n}(\sqrt{n}A - S_y) + \frac{\delta}{2}\right)\mathbb{P}(Z_{t_n} \in \mathrm{d}\xi).$$

Hence, by Lemma 4, there exists $C_1 > 0$ such that, for large n, $\mathbb{P}(\bar{Z}_n(\sqrt{n}A) \ge p) \le C_1\mathbb{E}\left[|Z_{t_n}|^{-(\beta-1)}\right]$. As a consequence, the upper bound follows by Lemma 2.

If $|Z_1|$ has a Weibull tail, similar arguments to those above, together with Lemmas 3 and 5, get the desired results. □

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References

[1] AÏDÉKON, E., HU, Y. AND SHI, Z. (2019). Large deviations for level sets of branching Brownian motion and Gaussian free fields. *J. Math. Sci.* **238**, 348–365.
[2] ASMUSSEN, S. AND KAPLAN, N. (1976). Branching random walks I. *Stoch. Process. Appl.* **4**, 1–13.
[3] ATHREYA, K. B. AND HONG, J. (2013). An application of the coalescence theory to the branching random walks. *J. Appl. Prob.* **50**, 893–899.
[4] ATHREYA, K. B. AND NEY, P. E. (1972). *Branching Processes*. Springer, Berlin.
[5] AURZADA, F. (2020). Large deviations for infinite weighted sums of stretched exponential random variables. *J. Math. Anal. Appl.* **485**, 123814.

[6] BHATTACHARYA, A. (2018). Large deviation for extremes in branching random walk with regularly varying displacements. Preprint, arXiv:1802.05938.
[7] BIGGINS, J. D. (1990). The central limit theorem for the supercritical branching random walk, and related results. *Stoch. Process. Appl.* **34**, 255–274,
[8] BOVIER, A. (2016). *Gaussian Processes on Trees: From Spin Glasses to Branching Brownian Motion*. Cambridge University Press.
[9] BRAMSON, M. D., DING, J. AND ZEITOUNI, O. (2015). Convergence in law of the maximum of the two-dimensional discrete Gaussian free field. *Comm. Pure Appl. Math.* **69**, 62–123.
[10] BURACZEWSKI, D. AND MAŚLANKA, M. (2019). Large deviation estimates for branching random walks. *ESAIM Prob. Statist.* **23**, 823–840.
[11] CHAUVIN, B. AND ROUAULT, A. (1988). KPP equation and supercritical branching Brownian motion in the subcritical speed area: Application to spatial trees. *Prob. Theory Relat. Fields* **80**, 299–314.
[12] CHEN, X. (2001). Exact convergence rates for the distribution of particles in branching random walks. *Ann. Appl. Prob.* **11**, 1242–1262.
[13] CHEN, X AND HE, H. (2019). On large deviation probabilities for empirical distribution of supercritical branching random walks with unbounded displacements. *Prob. Theory Relat. Fields* **175**, 255–307.
[14] CHEN, X. AND HE, H. (2020). Lower deviation and moderate deviation probabilities for maximum of a branching random walk. *Ann. Inst. H. Poincaré Prob. Statist.* **56**, 2507–2539.
[15] DEMBO, A. AND ZEITOUNI, O. (1998). *Large Deviation Techniques and Applications*, 2nd edn. Springer, Berlin.
[16] DENISOV, D. E., KORSHUNOV, D. A. AND WACHTEL, V. I. (2013). On the asymptotics of the tail of distribution of a supercritical Galton–Watson process in the case of heavy tails. *Proc. Steklov Inst. Math.* **282**, 273–297.
[17] DERRIDA, B. AND SHI, Z. (2016). Large deviations for the branching Brownian motion in presence of selection or coalescence. *J. Statist. Phys.* **163**, 1285–1311.
[18] DERRIDA, B. AND SHI, Z. (2017). Large deviations for the rightmost position in a branching Brownian motion. In Modern Problems of Stochastic Analysis and Statistics. MPSAS 2016, V. Panov (ed.) (Springer Proc. Math. Statist. **208**). Springer, Cham.
[19] DERRIDA, B. AND SHI, Z. (2017). Slower deviations of the branching Brownian motion and of branching random walks. *J. Phys. A* **50**, 344001.
[20] FLEISCHMANN, K. AND WACHTEL, V. (2007). Lower deviation probabilities for supercritical Galton–Watson processes. *Ann. Inst. H. Poincaré Prob. Statist.* **43**, 233–255.
[21] FLEISCHMANN, K. AND WACHTEL, V. (2008). Large deviations for sums indexed by the generations of a Galton–Watson process. *Prob. Theory Relat. Fields* **141**, 445–470.
[22] GANTERT, N. AND HÖFELSAUER, T. (2018). Large deviations for the maximum of a branching random walk. *Electron. Commun. Prob.* **23**, 1–12.
[23] GAO, Z. AND LIU, Q. (2016). Exact convergence rates in central limit theorems for a branching random walk with a random environment in time. *Stoch. Process. Appl.* **126**, 2634–2664.
[24] GRÜBEL, R. AND KABLUCHKO, Z. (2017). Edgeworth expansions for profiles of lattice branching random walks. *Ann. Inst. H. Poincaré Prob. Statist.* **53**, 2103–2134.
[25] HARRIS, T. E. (1963). *The Theory of Branching Processes*. Springer, Berlin.
[26] HU, Y. (2016). How big is the minimum of a branching random walk? Ann. Inst. H. Poincaré Prob. Statist. **52**, 233–260.
[27] HU, Y. AND SHI, Z. (2007). A subdiffusive behaviour of recurrent random walk in random environment on a regular tree. *Prob. Theory Relat. Fields* **138**, 521–549.
[28] KLEBANER, C. F. (1982). Branching random-walk in varying environments. *Adv. Appl. Prob.* **14**, 359–367.
[29] LIANG, X. G. AND LIU, Q. (2013). Weighted moments of the limit of a branching process in a random environment. *Proc. Steklov Inst. Math.* **282**, 127–145.
[30] LIU, Q. (1995). Fixed points of a generalised smoothing transformation and applications to branching processes. *Adv. Appl. Prob.* **30**, 85–112.
[31] LIU, Q. (2006). On generalised multiplicative cascades. *Stoch. Process. Appl.* **86**, 263–286.
[32] LOUIDOR, O. AND PERKINS, W. (2015). Large deviations for the empirical distribution in the branching random walk. *Electron. J. Prob.* **18**, 1–19.
[33] LOUIDOR, O. AND TSAIRI, E. (2017). Large deviations for the empirical distribution in the general branching random walk. Preprint, arXiv:1704.02374.
[34] NAGAEV, S.V. (1979). Large deviations of sums of independent random variables. *Ann. Prob.*, **7**, 745–789.
[35] NEY, P. E. AND VIDYASHANKAR, A. N. (2003). Harmonic moments and large deviation rates for supercritical branching processes. *Ann. Appl. Prob.* **13**, 475–489.

[36] ÖZ, M. (2020). Large deviations for local mass of branching Brownian motion. *ALEA Lat. Am. J. Prob. Math. Statist.* **17**, 711–731.
[37] ROUAULT, A. (1993). Precise estimates of presence probabilities in the branching random walk. *Stoch. Process. Appl.* **44**, 27–39.
[38] SHI, Z. (2015). *Branching Random Walks* (Lecture Notes Math. 2151). Springer, Cham.